Sufficient conditions for a digraph to be Hamiltonian

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This paper is dedicated to the memory of Henri Meyniel

Abstract

We describe a new type of sufficient condition for a digraph to be Hamiltonian. Conditions of this type combine local structure of the digraph with conditions on the degrees of non-adjacent vertices. The main difference from earlier conditions is that we do not require a degree condition on all pairs of non-adjacent vertices. Our results generalize the classical conditions by Ghouila-Houri and Woodall.

1 Introduction

For convenience of the reader we provide all necessary terminology and notation in one section, Section 2.

There are many conditions which guarantee that an undirected graph is Hamiltonian. Almost all of them deal with degrees or degree sums of certain subsets of vertices, or with sizes of unions of neighbourhoods of certain subsets. Some conditions involve only certain pairs of non-adjacent vertices.

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One example is Fan's theorem which says that if G is a 2-connected undirected graph on n vertices such that $\max\{d(x), d(y)\} \geq \frac{n}{2}$ for all pairs of vertices x and y of distance two, then G is Hamiltonian [11].

For directed graphs the situation is quite different. Very few degree conditions are known to guarantee Hamiltonicity in strong digraphs (in all four theorems D is a strongly connected digraph on n vertices):

Theorem 1.1 *[12, (Ghouila-Houri)] If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is Hamiltonian.*

Theorem 1.2 *[16, (Woodall)] If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices x and y such that there is no arc from x to y, then D is Hamiltonian.*

Theorem 1.3 *[15, (Meyniel)] If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices in D then D is Hamiltonian.*

It is easy to see that Theorem 1.3 implies Theorems 1.1 and 1.2. For a short proof of Theorem 1.3 see [10].

Theorem 1.4 *[14, (Manoussakis)] Suppose that D satisfies the following condition for every triple $x, y, z \in V(D)$ such that x and y are non-adjacent: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2$. If there is no arc from z to x then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3n - 2$. Then D is Hamiltonian.*

Each of these theorems imposes a degree condition on all pairs of non-adjacent vertices. In this paper, we show that it is possible to weaken the rather strong demand of high degree for every pair of non-adjacent vertices, by requiring this only for some pairs of non-adjacent vertices. We do so by extending the following theorem on Hamiltonian locally semicomplete and out-semicomplete digraphs (the locally semicomplete case was first proved in [2], the out-semicomplete case was obtained in [6]):

Theorem 1.5 *A strongly connected locally semicomplete (out-semicomplete, respectively) digraph is Hamiltonian.*

Locally semicomplete digraphs include tournaments, and share many nice properties of tournaments (see, for example, [2, 3, 13]).

Besides generalizing Theorem 1.5, our results, Theorems 4.1 and 4.2, generalize Theorems 1.1 and 1.2, respectively.

In Section 2, we give necessary definitions and notation. Section 3 contains preliminary lemmas. We prove Theorems 4.1 and 4.2 in Section 4. These theorems neither imply nor are implied by the theorems of Meyniel and Manoussakis. In Section 5, we give examples to show that this is so and to demonstrate the sharpness of our results. We conclude with some remarks and conjectures.

2 Terminology and notation

We shall assume that the reader is familiar with the standard terminology on digraphs and refer to [9] for terminology not discussed here.

Every *cycle* and *path* is assumed simple and directed; its *length* is the number of its arcs. D always denotes a digraph with n vertices; it is *Hamiltonian* if it contains a *Hamiltonian cycle*, namely a cycle of length n.

Let x, y be distinct vertices in D. If there is an arc from x to y then we say that x *dominates* y and write $x \rightarrow y$. $\{x, y\}$ is *dominated* by a vertex z if $z \rightarrow x$ and $z \rightarrow y$. Likewise, $\{x, y\}$ *dominates* a vertex z if $x \rightarrow z$ and $y \rightarrow z$; in this case, we call the pair $\{x, y\}$ *dominating*.

D is an *out-semicomplete digraph* (*in-semicomplete digraph*) if D has no pair of non-adjacent dominated (dominating, respectively) vertices. D is a *locally semicomplete digraph* if D is both out-semicomplete and in-semicomplete.

If $x \in V(D)$ and H is a subgraph of D, the *in-degree* $d_H^-(x)$ (*out-degree* $d_H^+(x)$) of x with respect to H is the number of vertices in H dominating x (dominated by x, respectively). The *degree* of x with respect to H is $d_H(x) = d_H^-(x) + d_H^+(x)$. When $H = D$, the subscript H will be omitted.

If x and y are distinct vertices of D and P is a path from x to y, we say that P is an $(x, y)-$*path*. If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y.

Let C be a cycle in D. An $(x, y)-$path P is a C-*bypass* if $|V(P)| \geq 3$, $x \neq y$ and $V(P) \cap V(C) = \{x, y\}$. The length of the path $C[x, y]$ is the *gap* *of P with respect to C.*

D is *strongly connected* (or just *strong*) if there exists an $(x,y)-$path in D for every ordered pair of distinct vertices $\{x,y\}$ of D.

Let $P = u_1u_2...u_s$ be a path in D (possibly, $s = 1$) and let $Q = v_1v_2...v_t$ be a path in $D-V(P)$. P has a *partner* on Q if there is an arc (the *partner of* P) $v_i{\rightarrow}v_{i+1}$ on Q such that $v_i{\rightarrow}u_1$ and $u_s{\rightarrow}v_{i+1}$. In this case the path P can be inserted into Q to give a new (v_1,v_t)-path $Q[v_1,v_i]PQ[v_{i+1},v_t]$. P has a *collection of partners* on Q if there are integers $i_1 = 1 < i_2 < ... < i_m = s+1$ such that, for every $k = 2,3,...,m$, the subgraph $P[u_{i_{k-1}}, u_{i_k-1}]$ has a partner on Q. The notion of partners has already proved its usefulness in a number of papers (see, for example, [1, 4, 5]).

3 Lemmas

Our first lemma is a generalization of Lemma 3.2 in [4].

Lemma 3.1 *Let P be a path in D and let $Q = v_1v_2...v_t$ be a path in $D-V(P)$. If P has a collection of partners on Q, then there is a (v_1,v_t)-path R in D so that $V(R) = V(P) \cup V(Q)$.*

Proof: Let $P = u_1u_2...u_s$. Suppose that integers $i_1 = 1 < i_2 < ... < i_m = s+1$ satisfy the 'collection of partners' property, that is, for every $k = 2,3,...,m$, the subgraph $P[u_{i_{k-1}}, u_{i_k-1}]$ has a partner on Q.

We proceed by induction on m. If $m = 2$ then the claim is obvious, hence assume that $m \geq 3$. Let $x{\rightarrow}y$ be a partner of the subpath $P[u_{i_1}, u_{i_2-1}]$ on Q. Choose r as large as possible such that $u_{i_r-1}{\rightarrow}y$. Clearly, $P[u_{i_1}, u_{i_r-1}]$ can be inserted into Q to give a (v_1,v_t)-path Q^*. Thus, if $r = m$ we are done. Otherwise apply induction to the paths $P[u_{i_r}, u_s]$ and Q^*. □

The following lemma is a slight modification of the lemma in [10]; its proof is almost the same.

Lemma 3.2 *Let $Q = v_1v_2\dots v_t$ be a path in D and let w, w' be vertices of $V(D) - V(Q)$ (possibly $w = w'$). If there do not exist consecutive vertices v_i, v_{i+1} on Q such that $v_i{\rightarrow}w, w'{\rightarrow}v_{i+1}$ are arcs of D, then $d^-_Q(w) + d^+_Q(w') \leq t+\xi$, where $\xi = 1$ if $v_t{\rightarrow}w$ and 0, otherwise.*

In the special case when $w' = w$ above, we get the following interpretation of the statement of Lemma 3.2.

Lemma 3.3 *Let $Q = v_1 v_2 ... v_t$ be a path in D, and let $w \in V(D) - V(Q)$. If w has no partner on Q, then $d_Q(w) \leq t + 1$. If, in addition, v_t does not dominate w, then $d_Q(w) \leq t$.*

4 Main Results

Note that the proof of Theorem 4.1 initially follows the line of reasoning in the proof of Meyniel's theorem in [10] (see also [8]).

Theorem 4.1 *Let D be a strong digraph. Suppose that, for every dominated pair of non-adjacent vertices $\{x, y\}$, either $d(x) \geq n$ and $d(y) \geq n - 1$ or $d(x) \geq n - 1$ and $d(y) \geq n$. Then D is Hamiltonian.*

Proof: Assume that D is non-Hamiltonian and $C = x_1 x_2 ... x_m x_1$ is a longest cycle in D. As in the proof of Theorem 2 in [10], we can show that D contains a C-bypass $P = u_1 u_2 ... u_s$ ($s \geq 3$). Without loss of generality, let $u_1 = x_1$, $u_s = x_{\gamma+1}$, $0 < \gamma < m$. Suppose also that the gap γ of P is minimum among the gaps of all C-bypasses.

Since C is a longest cycle of D, $\gamma \geq 2$. Let $C' = C[x_2, x_\gamma]$, $C'' = C[x_{\gamma+1}, x_1]$, $R = D - V(C)$, and let x_j be any vertex in C' such that $x_1 \rightarrow x_j$. Let also x_k be an arbitrary vertex in C'.

We first prove that

$$d_{C''}(x_j) \geq |V(C'')| + 2. \tag{1}$$

Since C is a longest cycle and P has the minimum gap with respect to C, u_2 is not adjacent to any vertex on C', and there is no vertex $y \in V(R) - \{u_2\}$ such that either $u_2 \rightarrow y \rightarrow x_k$ or $x_k \rightarrow y \rightarrow u_2$. Therefore,

$$d_{C'}(x_k) + d_{C'}(u_2) \leq 2(|V(C')| - 1) \tag{2}$$

and

$$d_R(x_k) + d_R(u_2) \leq 2(n - m - 1). \tag{3}$$

By the maximality of C, u_2 has no partner on C'', so by Lemma 3.3,

$$d_{C''}(u_2) \leq |V(C'')| + 1. \tag{4}$$

The fact that the pair of non-adjacent vertices $\{x_j, u_2\}$ is dominated by x_1 along with (3),(2) and (4), implies that

$$2n - 1 \leq d(x_j) + d(u_2) \leq d_{C''}(x_j) + 2n - |V(C'')| - 3.$$

This implies (1).

By (1) and Lemma 3.3, x_2 has a partner on C''. Since C is a longest cycle, there exists $\beta \in \{3, ..., \gamma\}$ so that the subpath $C[x_2, x_{\beta-1}]$ has a collection of partners on C'', but $C[x_2, x_\beta]$ does not have such a collection. In particular, x_β has no partner on C''. Thus, by (1) and Lemma 3.3, x_1 does not dominate x_β and $d_{C''}(x_\beta) \leq |V(C'')|$. This along with (2)-(4) gives $d(x_\beta) + d(u_2) \leq 2n - 3$. Observe that $d(u_2) \geq n - 1$. Hence,

$$d(x_\beta) \leq n - 2. \tag{5}$$

By the definition of a collection of partners, there are $\alpha \in \{2, 3, ..., \beta - 1\}$ and $i \in \{\gamma + 1, ..., m\}$ such that $x_i \rightarrow x_\alpha$ and $x_{\beta-1} \rightarrow x_{i+1}$. Observe that the pair $\{x_\beta, x_{i+1}\}$ is dominated by $x_{\beta-1}$. Thus, by (5) and the assumption of the theorem, either $x_\beta \rightarrow x_{i+1}$ or $x_{i+1} \rightarrow x_\beta$. If $x_\beta \rightarrow x_{i+1}$, then the path $P[x_2, x_\beta]$ has a collection of partners on C'' which contradicts our assumption. Hence, $x_{i+1} \rightarrow x_\beta$. Considering the pair x_β, x_{i+2}, we conclude analogously that $x_{i+2} \rightarrow x_\beta$. Continuing this process, we finally conclude that $x_1 \rightarrow x_\beta$, contradicting the conclusion above that this arc does not exist.

□

Theorem 4.2 *Let D be a strong digraph. Suppose that* $\min\{d^+(x)+d^-(y),\ d^-(x)+d^+(y)\} \geq n$ *for every pair of dominating non-adjacent and dominated non-adjacent vertices* $\{x, y\}$*. Then D is Hamiltonian.*

Proof: Assume that D is not Hamiltonian and $C = x_1x_2...x_mx_1$ is a longest cycle in D. Set $R = D - V(C)$. We first prove that D has a C-bypass with 3 vertices.

Since D is strong, there is a vertex y in R and a vertex x in C such that $y \rightarrow x$. If y dominates every vertex on C, then C is not a longest cycle, since a path P from a vertex x_i on C to y such that $V(P) \cap V(C) = \{x_i\}$ together with the arc $y \rightarrow x_{i+1}$ and the path $C[x_{i+1}, x_i]$ form a longer cycle in D. Hence, either there exists a vertex $x_r \in V(C)$ such that $x_r \rightarrow y \rightarrow x_{r+1}$, in which case we have the desired bypass, or there exists a vertex $x_j \in V(C)$ so that y and x_j are non-adjacent, but $y \rightarrow x_{j+1}$. As the pair $\{y, x_j\}$ dominates x_{j+1}, $d^+(x_j) + d^-(y) \geq n$. This implies the existence of a vertex $z \in V(D) - \{x_j, x_{j+1}, y\}$ such that $x_j \rightarrow z \rightarrow y$. Since C is a longest cycle, $z \in V(C)$. So, $B = zyx_{j+1}$ is the desired bypass.

Without loss of generality, assume that $z = x_1$ and the gap j of B with respect to C is minimum among the gaps of all C-bypasses with three vertices. Clearly, $j \geq 2$.

Let $C' = C[x_2, x_j]$ and $C'' = C[x_{j+1}, x_1]$. Since C is a longest cycle, C' has no partner on C''. It follows from Lemma 3.2 that $d^+_{C''}(x_j) + d^-_{C''}(x_2) \leq |V(C'')| + 1$. By Lemma 3.3 and the maximality of C, $d_{C''}(y) \leq |V(C'')| + 1$. Analogously to the way we derived (2) in the proof of the previous theorem, we get that $d_R(y) + d^+_R(x_j) + d^-_R(x_2) \leq 2(n-m-1)$. Clearly, $d^+_{C'}(x_j) + d^-_{C'}(x_2) \leq 2|V(C')| - 2$. Since $d_{C'}(y) = 0$, the last four inequalities imply

$$d(y) + d^+(x_j) + d^-(x_2) \leq 2n - 2. \tag{6}$$

Since y is adjacent to neither x_2 nor x_j, the assumption of the theorem implies that $d^+(y) + d^-(x_2) \geq n$ and $d^-(y) + d^+(x_j) \geq n$, which contradicts (6). □

5 Examples of digraphs

Consider the following infinite family of digraphs $D = D(n, s, t)$ ($n \geq 2s + 2$, $t \leq (n - s)/2$, $s \geq 5$): $V(D) = \{x_1, x_2 ..., x_{n-s}, u_1, u_2, ..., u_s\}$, $A(D) = \{x_i \rightarrow x_j : 1 \leq i \neq j \leq n-s\} \cup \{u_i \rightarrow u_{i+1} : i = 1, 2, ..., s-1\} \cup \{x_i \rightarrow u_1, u_s \rightarrow x_i : i = 1, 2, ..., n - s\} - \{x_{2i-1} \rightarrow x_{2i}, x_{2i} \rightarrow x_{2i-1} : i = 1, 2, ..., t\}$. Note that $D(n, s, t)$ has at least three vertices of degree 2 and at least two non-adjacent vertices of degree 2. So, neither Meyniel's theorem nor Manoussakis' theorem applies to $D(n, s, t)$. Moreover, if t is positive, $D(n, s, t)$ is neither out-semicomplete nor in-semicomplete. However, $D(n, s, t)$ satisfies the conditions of both Theorems 4.1 and 4.2.

The following example shows the sharpness of the conditions of Theorems 4.1 and 4.2. Let G and H be two disjoint transitive tournaments such that $|V(G)| \geq 2, |V(H)| \geq 2$. Let w be the sink in G and w' the source in H. Form a new digraph by identifying w and w' to one vertex z. Add four new vertices x, y, u, v and the arcs $\{x \to v, y \to v, u \to x, u \to y\} \cup \{r \to g : r \in \{x, y, v\}, g \in V(G)\} \cup \{h \to s : h \in V(H), s \in \{u, x, y\}\}$. Call the resulting digraph D. It is easy to check that D is strong and non-Hamiltonian. Also x, y is the only pair of non-adjacent vertices which is dominating (dominated, respectively). An easy computation shows that

$$d(x) = d(y) = n - 1 = d^+(x) + d^-(y) = d^-(x) + d^+(y).$$

Note also that a slight change of the last example shows that, even for oriented graphs, the conditions of Theorems 4.1 and 4.2 cannot be weakened by more than a constant.

6 Remarks and conjectures

Having seen Theorems 1.5, 4.1 and 4.2, the reader may ask whether there might not also be a third type of local condition on pairs of vertices of distance two in the underlying graph, namely a condition on the degrees of non-adjacent vertices x and y for which there exists a vertex z with $x \to z \to y$, or $y \to z \to x$. However no such condition would be sufficient, even together with some requirement on the connectivity. This follows from the existence of arbitrarily highly strongly connected non-Hamiltonian digraphs D in which no pair of non-adjacent vertices x and y is joined by a (directed) path of length two (see [7]).

We believe that Theorem 4.2 can be generalized to the following, which would also generalize Meyniel's Theorem.

Conjecture 6.1 *Let D be a strong digraph. Suppose that $d(x)+d(y) \geq 2n-1$ for every pair of dominating non-adjacent and every pair of dominated non-adjacent vertices $\{x, y\}$. Then D is Hamiltonian.*

Perhaps this can even be generalized to

Conjecture 6.2 *Let D be a strong digraph. Suppose that, for every pair of dominated non-adjacent vertices $\{x, y\}$, $d(x) + d(y) \geq 2n - 1$. Then D is Hamiltonian.*

Let D be the digraph obtained from the complete digraph K^*_{n-3} (all pairs of vertices are on a cycle of length two) on $n-3$ vertices by adding three new vertices $\{x, y, z\}$ and the following arcs $\{x{\rightarrow}y, y{\rightarrow}x, y{\rightarrow}z, z{\rightarrow}y, z{\rightarrow}x\} \cup \{x{\rightarrow}u, u{\rightarrow}x, y{\rightarrow}u : u \in V(K^*_{n-3})\}$. Clearly D is strongly connected and the underlying undirected graph of D is 2-connected. The only pairs of non-adjacent vertices in D are z and any vertex $u \in V(K^*_{n-3})$ and here we have $d(z) + d(u) = 2n - 2$. Thus both conjectures above would be best possible. From the above example we also see that for any pair of non-adjacent vertices z and u, we have a path of length two between them in the underlying undirected graph and $\max\{d(z), d(u)\} = 2n - 5$. This shows that there is no digraph analogue of Fan's theorem (see page one) if we only impose a degree condition for non-adjacent vertices of distance two in the underlying undirected graph.

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