A sufficient condition for a semicomplete multipartite digraph to be Hamiltonian

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Abstract

A multipartite tournament is an orientation of a complete k-partite graph for some $k \geq 2$. A factor of a digraph D is a collection of vertex disjoint cycles covering all the vertices of D. We show that there is no degree of strong connectivity which together with the existence of a factor will guarantee that a multipartite tournament is Hamiltonian. Our main result is a sufficient condition for a multipartite tournament to be Hamiltonian. We show that this condition is general enough to provide easy proofs of many existing results on paths and cycles in multipartite tournaments. Using this condition, we obtain a best possible lower bound on the length of a longest cycle in any strongly connected multipartite tournament.

1 Introduction

In this paper we shall consider a well-known generalization of tournaments, multipartite tournaments. A *multipartite tournament* [4, 14] is a an orientation of a complete $k-$partite graph, for some $k \geq 2$. Special cases of multipartite tournaments are tournaments, where $k = n$, the number of vertices, and *bipartite tournaments*, where $k = 2$. Bipartite tournaments have been studied intensively in the pursuit for tournament-like properties.

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Many properties have been shown to extend to bipartite tournaments, see e.g. [2, 11].

Even for bipartite tournaments, strong connectivity is not sufficient to guarantee a Hamiltonian cycle. In fact, there is no s such that every $s-$connected bipartite tournament has a Hamiltonian path [12]. The important structure turns out to be the existence of a factor, a spanning 1-diregular subgraph: A bipartite tournament B has a Hamiltonian cycle if and only if it is strong and has a factor [6, 12] and a Hamiltonian path if and only if it has an almost factor – a path plus a disjoint collection of cycles, covering the vertices of B [7, 12]. Furthermore it was shown in [10] that the size of a longest cycle in a bipartite tournament B is equal to the size of the largest 1-diregular subdigraph of any strong component of B.

The author of [8, 9] proved that in the case of a Hamiltonian path, the characterization is the same for general multipartite tournaments. He also showed that a factor and strong connectivity is not sufficient to guarantee a Hamiltonian cycle in a general multipartite tournament [10, 11]. He introduced a subclass of the multipartite tournaments, called *ordinary multipartite tournaments* and showed that for this class the existence of a factor together with strong connectivity is necessary and sufficient [10, 11].

The example in [10, 11] showing that a factor and strong connectivity are not sufficient to guarantee a Hamiltonian cycle is not 2-connected. Hence, we may ask whether there is any degree of strong connectivity, which together with a factor is sufficient to guarantee a Hamiltonian cycle in a general multipartite tournaments. The answer is no, in fact, there is no s such that every $s-$connected multipartite tournament with a factor has a Hamiltonian cycle. Figure 1 shows a non-Hamiltonian multipartite tournament which is $s-$connected (s is the number of vertices in each of the sets A, B, C, D and X, Y, Z), and has a factor. We leave it to the reader to verify that there is no Hamiltonian cycle.

The Hamiltonian cycle problem for general multipartite tournaments seems much harder than in the special cases $k = 2$ and $k = n$. While there are polynomial algorithms for the Hamiltonian cycle problem in the two special cases above, the existence of a polynomial algorithm for Hamiltonian cycle problem in general multipartite tournaments remains an open problem [11].

Our main theorem in this paper is a sufficient condition for a general multipartite tournament to be Hamiltonian. Since there are no appropriate sufficient and necessary conditions yet, the main result, Theorem 4.4, is fairly

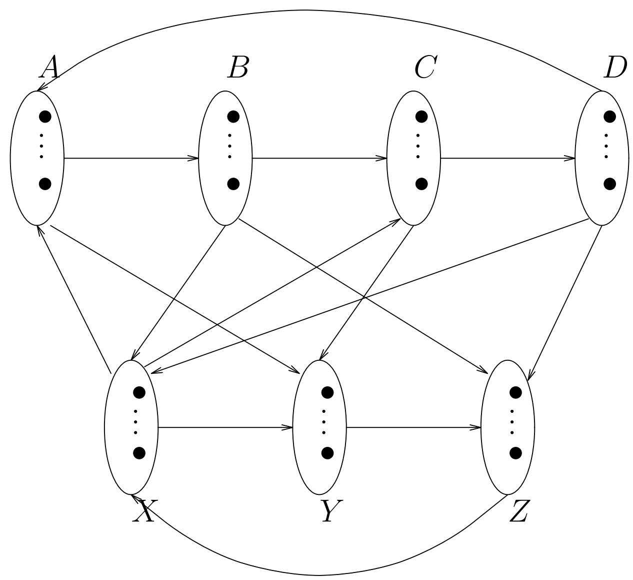

Figure 1: An $s-$connected non-Hamiltonian multipartite tournament with a factor. Each of the sets A, B, C, D and X, Y, Z induce independent sets with exactly s vertices. All arcs between two sets have the direction shown

useful from theoretical point of view. Indeed, in Section 5, we show that our condition is general enough to provide easy proofs of many existing results on multipartite tournaments. We also give a best possible lower bound on the length of a longest cycle in any strongly connected multipartite tournament (see Theorem 5.4).

Taking as a starting point Theorem 4.4 and using the partner technique developed in our paper, A. Yeo [15] has very recently managed to extend our main result (Theorem 4.4) to an even stronger sufficient condition for a multipartite tournament to be Hamiltonian[1]. Yeo's condition implies the following results: every regular multipartite tournament is Hamiltonian (conjectured in [16]), every k-connected multipartite tournament with at most k vertices in each colour class is Hamiltonian (conjectured by Y. Guo and L. Volkmann, personal communication, 1993).

In this paper we also study the problem of finding a cycle through a given set of vertices. We solve this problem completely for ordinary multipartite tournaments.

[1] A. Yeo [15] also uses Lemma 5.2.

We shall prove all the results (except in Section 3) for a slightly more general class of digraphs than multipartite tournaments – semicomplete multipartite digraphs (see below).

2 Terminology and notation

A digraph obtained by replacing each edge of a complete k-partite ($k \geq 2$) graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a *semicomplete k-partite digraph* or *semicomplete multipartite digraph* (abbreviated to SMD, and for $k = 2$ - to SBD). A semicomplete multipartite digraph is a *multipartite tournament* if it has no (directed) cycles of length two. Whenever we consider a SMD D, we use the term 'colour classes' to denote the uniquely determined partition classes of D. A SMD D is called an *ordinary* SMD if for every pair X, Y of colour classes all the arcs between X and Y are oriented from X to Y or oriented from Y to X or for any pair of adjacent vertices $x \in X$, $y \in Y$ both arcs xy and yx are in D. We use n to denote the number of vertices of the digraph studied.

Let D be a digraph. If there is an arc from a vertex x to a vertex y in D we say that x *dominates* y and use the notation $x \rightarrow y$ to denote this. If A and B are disjoint subsets of vertices of D we use the notation $A{\rightarrow}B$ to denote that $a{\rightarrow}b$ for any pair of adjacent vertices $a \in A$ and $b \in B$. $A{\Rightarrow}B$ means that $A{\rightarrow}B$ and no vertex of B dominates a vertex of A.

By a *cycle (path)* we mean a simple directed cycle (path, respectively). If x and y are vertices of D and P is a directed path from x to y, we say that P is an $(x, y)-$path. If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y. If X is a subset of vertex set $V(D)$ of D then $D < X >$ is the subgraph of D induced by X. If H is a subgraph of D then $D < H >$ means $D < V(H) >$.

A digraph D is *strongly connected* (or just *strong*) if there exists an $(x, y)-$path and a $(y, x)-$path in D for any choice of distinct vertices x, y of D. A digraph D is $k-$connected if for any $S \subset V(D)$ of at most $k - 1$ vertices, $D - S$ is strong.

A digraph D is called 1-*diregular* if every vertex of D has in- and out-degree 1. A digraph D is called *almost* 1-*diregular* if every vertex of D has in- and out-degree 1, except either *(i)* one of them having both in-degree and out-degree 0 or *(ii)* two of them where the first one has in-degree 0 and out-degree 1 and the second one has in-degree 1 and out-degree 0. Obviously, a 1-diregular digraph F is a collection of disjoint cycles, and an almost 1-diregular digraph L is a path and a collection of cycles all mutually disjoint. We shall denote this fact as follows: $F = C_1 \cup C_2 \cup ... \cup C_t$ ($t \geq 1$, C_i are cycles) and $L = P \cup C_1 \cup C_2 \cup ... \cup C_t$ ($t \geq 0$, P is a path and C_i are cycles).

Let D be a digraph. A 1-diregular (almost 1-diregular) spanning subgraph of D is called a *factor* (an *almost factor*). Let $F = C_1 \cup C_2$ be a factor or an almost factor in a digraph D, where C_i is a cycle or a path in D ($i = 1, 2$). A vertex $v \in V(C_i)$ is called *out-singular* (*in-singular*) with respect to C_{3-i} if $v \Rightarrow C_{3-i}$ ($C_{3-i} \Rightarrow v$), v is *singular* if it is either out-singular or in-singular.

Let x be a vertex on a path (cycle) Q. Then we shall denote the predecessor (successor) of x on Q by x^- (x^+).

Let P be a (x, y)-path in a digraph D and let $Q = v_1 v_2 ... v_t$ be a path or a cycle in $D - P$. Then we say that P has a *partner* on Q if there is an arc (*the partner of* P) $v_i \rightarrow v_{i+1}$ on Q such that $v_i \rightarrow x$ and $y \rightarrow v_{i+1}$. In this case the path P can be inserted to Q to give a new path (or cycle) $Q[v_1, v_i]PQ[v_{i+1}, v_t]$. We shall often consider partners for paths of length 0 or 1, i. e. for vertices and arcs.

For terminology not defined here, we refer the reader to [3, 5].

3 General lemmas

In this section we prove some lemmas that are valid for general digraphs. The essence of the results is that the existence of certain partners is sufficient to guarantee that a path and a cycle, or two cycles, can be merged into one cycle.

Lemma 3.1 *Let D be a digraph. Suppose that $P = u_1 u_2 ... u_r$ is a path in D and C is a cycle in $D - P$. Suppose that for each $i = 1, 2, \ldots, r - 1$, either the arc $u_i \rightarrow u_{i+1}$ has a partner or the vertex u_i has a partner on C, and, in addition, assume that u_r has a partner on C. Then D contains a cycle with the vertex set $V(P) \cup V(C)$.*

Proof: We proceed by induction on r. If $r = 1$ then the claim is obvious, hence assume that $r \geq 2$. Let $x \rightarrow y$ be a partner of the arc $u_1 \rightarrow u_2$ or of the vertex u_1 on C. Choose i as large as possible such that $u_i \rightarrow y$. Clearly,

$P[u_1, u_i]$ can be inserted in C to give a cycle C^*. Thus, if $i = r$ we are done. Otherwise apply induction to the path $P[u_{i+1}, u_r]$ and to the cycle C^*. $\square$

Lemma 3.2 *Let D be a digraph. Suppose that $P = u_1u_2...u_r$ is a path of odd length in D and C is a cycle in $D - P$. Suppose also that for each odd i $u_i \rightarrow u_{i+1}$ has a partner on C. Then D contains a cycle with the vertex set $V(P) \cup V(C)$.*

Proof: We proceed again by induction on r. If $r = 2$ then the claim is obvious, hence assume that $r \geq 4$. Let $x \rightarrow y$ be a partner of the arc $u_1 \rightarrow u_2$ on C. Choose maximum even i such that $u_i \rightarrow y$ and construct C^* as in Lemma 3.1. To complete the proof observe that for each odd $j \geq i+1$ $u_j \rightarrow u_{j+1}$ has a partner on C^* and apply an induction to C^* and $P[u_{i+1}, u_r]$. $\square$

Lemma 3.3 *Let D be a digraph. Suppose that C is a cycle of even length in D and Q is a cycle in $D - C$. Suppose also that for each arc $u \rightarrow v$ of C either the arc $u \rightarrow v$ or the vertex u has a partner on Q. Then D contains a cycle with the vertex set $V(Q) \cup V(C)$.*

Proof: If there is a vertex x on C having a partner on Q then apply Lemma 3.1 to $C[x^+, x]$ and Q. Otherwise, all the arcs of C have partners on Q and we can apply Lemma 3.2 to $C[y^+, y]$ and Q, where y is any vertex of C. $\square$

4 Main results

The following lemma allows us to use the general lemmas for SMDs.

Lemma 4.1 *Let $Q \cup C$ be a factor in a SMD D. Suppose that the cycle Q has no singular vertices (with respect to C) and D has no Hamiltonian cycle, then for every arc $x \rightarrow y$ of Q either it has a partner on C, or both vertices x and y have partners on C.*

Proof: Assume w.l.o.g. that there is some arc $x \rightarrow y$ on Q such that neither x nor $x \rightarrow y$ have partners on C. Since D is a SMD and x is non-singular and has no partner there exists a vertex v on C which is not adjacent to x and $v^- \rightarrow x \rightarrow v^+$. Since v is adjacent to y and $x \rightarrow y$ has no partner, $v \rightarrow y$. Then D contains a Hamiltonian cycle $Q[y, x]C[v^+, v]y$ which is impossible. $\square$

Lemma 4.2 *Let D be a SMD containing a factor $C_1 \cup C_2$ such that C_i has no singular vertices with respect to C_{3-i}, $i = 1,2$; then D is Hamiltonian.*

Proof: Assume that D is not Hamiltonian. Then by Lemmas 3.3 and 4.1 we conclude that both of C_1, C_2 are odd cycles. By Lemmas 3.1 and 4.1, no vertex in C_i has a partner on C_{3-i} $(i = 1,2)$. So, by Lemma 4.1, every arc of C_i has a partner on C_{3-i}.

Now we show that, in fact, every arc of C_i has at least two partners on C_{3-i} for $i = 1,2$. Consider an arc $x_1{\rightarrow}x_2$ of C_1. Since both x_1 and x_2 are non-singular and have no partners on C_2, there exist vertices v_1 and v_2 on C_2 such that v_i is not adjacent to x_i and $v_i^-{\rightarrow}x_i{\rightarrow}v_i^+$, $i = 1,2$. Using the fact that D is non-Hamiltonian SMD we conclude that the only arc between x_2 and v_1 is $x_2{\rightarrow}v_1$. For the same reason, v_2 dominates x_1 but is not dominated by x_1. Now $v_1^-{\rightarrow}v_1$ and $v_2{\rightarrow}v_2^+$ are partners of $x_1{\rightarrow}x_2$. Hence, $x_1{\rightarrow}x_2$ can have no two partners only in the case that $v_1^- = v_2$ and $v_1 = v_2^+$. We show that in this case D is Hamiltonian, contradicting the assumption above. Construct, at first, a cycle $C^* = C_1[x_2, x_1]C_2[v_1^+, v_2^-]x_2$ which contains all the vertices of D but v_1^-, v_1. The arc $v_1^-{\rightarrow}v_1$ has a partner on C_1, by the remark at the beginning of the proof. But $x_1{\rightarrow}x_2$ is not a partner for $v_1^-{\rightarrow}v_1$, since v_1 does not dominate x_2 and $v_1^- = v_2$ is not dominated by x_1. Hence, the arc $v_1^-{\rightarrow}v_1$ has a partner on C^*. Hence, the vertices v_1^-, v_1 can be inserted in C^* to give a Hamiltonian cycle of D. This completes the proof that every arc on C_i has at least two distinct partners on C_{3-i}.

Assume w.l.o.g. that the length of C_2 is not greater than that of C_1. Then C_1 has two arcs $x_i{\rightarrow}y_i$ $(i = 1,2)$ with a common partner $u{\rightarrow}v$ on C_2. As C_1 is odd, one of the paths $Q = C_1[y_1^+, x_2^-]$ and $C_1[y_2^+, x_1^-]$ has odd length. W.l.o.g. suppose that Q is odd. Obviously, $C^* = C_2[v, u]C_1[x_2, y_1]v$ is a cycle of D. By the fact shown above each arc of the path Q has a partner on C_2 different from $u{\rightarrow}v$. Therefore, each arc of Q has a partner on C^*. Hence, by Lemma 3.2 we conclude that D has a Hamiltonian cycle, contradicting the assumption. □

Let D be a SMD, $F = C_1 \cup C_2 \cup ... \cup C_t$ a 1-diregular subgraph of D. F is called *good* if it has no pair of cycles C_i, C_j $(i \neq j)$ such that C_i contains singular vertices with respect to C_j and they all are out-singular, and C_j has singular vertices with respect to C_i and they all are in-singular.

The following lemma gives the main result of the paper in case of a factor containing two cycles.

Lemma 4.3 *If D is a SMD containing a good factor $C_1 \cup C_2$, then D is Hamiltonian.*

Proof: The first case is that at least one of the cycles C_1 and C_2 has no singular vertices. If both C_1, C_2 have no singular vertices then D is Hamiltonian by Lemma 4.2. Assume now that only one of them has no singular vertices. Suppose w.l.o.g. that C_1 contains an out-singular vertex x and C_2 has no singular vertices. Since C_2 contains non-singular vertices, C_1 has at least one vertex which is not out-singular. Suppose that $x \in V(C_1)$ was chosen such that x^+ is not out-singular. Hence there is a vertex y on C_2 dominating x^+. If $x \rightarrow y$, then y has a partner on C_1 and hence by Lemmas 4.1, 3.1 D is Hamiltonian (consider $C_2[y^+, y]$ and C_1). Otherwise, x is not adjacent to y. In this case, $x \rightarrow y^+$ and D has a Hamiltonian cycle.

Consider the second case: each of C_1, C_2 have singular vertices. Assume w.l.o.g. that C_1 has an out-singular vertex x_1. If C_2 also contains an out-singular vertex x_2 then x_1 is not adjacent to x_2 and $x_i \rightarrow x^+_{3-i}$ for $i = 1, 2$. Hence D is Hamiltonian. If C_2 contains no out-singular vertices then it has in-singular vertices. Since $C_1 \cup C_2$ is a good factor, C_1 contains both out-singular and in-singular vertices. Since both C_1 and C_2 has in-singular vertices, the digraph D' obtained from D by reversing the orientations of the arcs of D has two cycles C'_1 and C'_2 containing out-singular vertices. We conclude that D' (and hence D) is Hamiltonian. □

The main result of our paper is the following

Theorem 4.4 *If D is a strong SMD containing a good factor $F = C_1 \cup C_2 \cup \ldots \cup C_t$ ($t \geq 1$), then D is Hamiltonian. Furthermore, given F one can find a Hamiltonian cycle in D in time $O(n^2)$.*

Proof: We proceed by induction on t. The claim is trivial for $t = 1$ and it is shown above for $t = 2$. Hence, assume that $t \geq 3$. By induction hypothesis, the digraph $D < C_1 \cup C_2 \cup \ldots \cup C_{t-1} >$ has a Hamiltonian cycle H. If $H \cup C_t$ is a good factor in D then we are done. Assume that $H \cup C_t$ is not good. Then, by the definition of a good factor and by the fact that a digraph containing a good factor is strong, $V(H)$ consists of following non-empty sets: a set O of out-singular vertices and a set N of non-singular vertices (with respect to C_t). $V(C_t)$ consists of following non-empty sets: a set I of in-singular vertices and a set S of non-singular vertices (with respect to H). By induction hypothesis, the digraph $D < C_2 \cup C_3 \cup ... \cup C_t >$ has a Hamiltonian cycle Q. If C_1 contains only vertices of N then all the vertices of C_1 are non-singular with respect to Q, a hence, by Lemma 4.3 D is Hamiltonian. Suppose, now, that C_1 contains also vertices of O. Since $C_1 \cup C_t$ is a good factor in $D < C_1 \cup C_t >$, S has a vertex x which is out-singular with respect to C_1. Therefore, Q has at least one in-singular vertex (a vertex of I) and at least one out-singular vertex (the vertex x) with respect to C_1. Again, by Lemma 4.3 we conclude that D is Hamiltonian.

It is easy to see that the proof above gives a recursive $O(n^2)$-algorithm.
$\square$

It is easy to construct Hamiltonian SMDs containing no good factor with at least two cycles. On the other hand, the SMD in Figure 1 shows that there exist non-Hamiltonian SMDs which are strong and have factors. Although it seems to be difficult to check if a digraph has a good factor, Theorem 4.4 is fairly useful from theoretical point of view.

5 Consequences of the main results

We shall show that several previously published results mentioned in the introduction are simple corollaries of Theorem 4.4, in fact they are consequences of its special case – Lemma 4.3.

Theorem 5.1 *[8, 9] A SMD D has a Hamiltonian path if and only if it has an almost factor. There exists an algorithm for finding a longest path in a SMD D in time $O(n^3)$.*

Proof: It is sufficient to prove that if P is a path and C a cycle of D such that $V(P) \cap V(C) = \emptyset$, then D has a path P' with $V(P') = V(P) \cup V(C)$. Let P and C be such a pair, and let u be the initial and v the terminal vertex of P. If u is non-singular or in-singular with respect to C, then obviously the path P' exists. Similarly if v is non-singular or out-singular with respect to C. Assume now that u is out-singular and v is in-singular with respect to C.

Add a new vertex w to D and the arcs $z \rightarrow w$, for all $z \neq u$ and the arc $w \rightarrow u$ to obtain the SMD D'. Then w forms a cycle C' with P in D' and

$C \cup C'$ is a good factor of D'. Therefore, by Lemma 4.3, D' has a Hamiltonian cycle. Then D contains a hamiltonian path.

It is easy to see that the proof above supplies a recursive $O(n^2)$-algorithm for finding a Hamiltonian path in D given an almost factor F. On the other hand, a maximum almost 1-diregular subgraph L of a SMD H can be constructed in time $O(n^3)$ (see [9]). Obviously, a Hamiltonian path of $H < F >$ is a longest path of H. $\square$

To obtain the rest of the theorems in this section, we need the following

Lemma 5.2 *Let D be a strong SMD containing a 1-diregular subgraph $F = C_1 \cup C_2 \cup ... \cup C_t$ such that for every pair i, j $(1 \leq i \leq j \leq t)$ $C_i \Rightarrow C_j$ or $C_j \Rightarrow C_i$ holds. Then D has a cycle of length at least $|V(F)|$ and one can find such a cycle in time $O(n^2)$ for a given F.*

Proof: Define a tournament $T(F)$ as follows: $C_1, ..., C_t$ forms the vertex set of $T(F)$ and $C_i \rightarrow C_j$ in $T(F)$ if and only if $C_i \Rightarrow C_j$ in D. Let H be the subgraph induced by the vertices of F and W a colour class of D having a representative in C_1.

First consider the case that $T(F)$ is strong. Then it has a Hamiltonian cycle. W.l.o.g., assume that $C_1C_2...C_tC_1$ is a Hamiltonian cycle in $T(F)$. If each of C_i $(i = 1, 2, ..., t)$ has a vertex from W then for every $i = 1, 2, ..., t$ pick any vertex w_i of $V(C_i) \cap W$. Then $C_1[w_1, w_1^-]C_2[w_2, w_2^-]...C_t[w_t, w_t^-]w_1$ is a Hamiltonian cycle in H. If there exists a cycle C_i containing no vertices of W, then we can assume w.l.o.g. that C_t has no vertices from W. Obviously, H has a Hamiltonian path starting at a vertex $w \in W \cap V(C_1)$ and finishing at some vertex v of C_t. Since $v \rightarrow w$, H is Hamiltonian.

Now consider the case where $T(F)$ is not strong. Replacing in F every collection X of cycles which induce a strong component in $T(F)$ by a Hamiltonian cycle in the subgraph induced by X, we obtain a new 1-diregular subgraph L of D such that $T(L)$ has no cycles. $T(L)$ contains a unique Hamiltonian path $Z_1Z_2...Z_s$, where Z_i is a cycle of L. Since D is strong there exists a path P in D with the first vertex in Z_s and the last vertex in Z_q $(1 \leq q < s)$ and the other vertices not in L. Assume that q is as small as possible. Then we can replace the cycles $Z_q, ..., Z_s$ by a cycle consisting of all the vertices of $P \cup Z_q \cup \ldots \cup Z_s$ except maybe one and derive a new 1-diregular subgraph with less cycles. Continuing in this manner, we obtain finally a single cycle.

Using the above proof together with an $O(n^2)$-algorithm for constructing a Hamiltonian cycle in a strong tournament [13] and obvious data structures one can obtain an $O(n^2)$-algorithm. $\square$

Lemma 5.3 *Let C and C' be disjoint cycles covering all vertices of a strong SMD D. Then D has a cycle of length at least $n-1$ containing all vertices of C.*

Proof: Suppose that the claim is not true. By Lemma 4.3, this means that each of C and C' has singular vertices with respect to the other cycle, and all singular vertices on one cycle are out-singular and all singular vertices on the other cycle are in-singular. Assume w.l.o.g. that C has only out-singular vertices with respect to C'. Since D is strong C has a non-singular vertex x. Furthermore we can choose x such that its predecessor x^- on C is singular. Let y be some vertex of C' such that $y \rightarrow x$. If x^- is adjacent to y^+, the successor of y on C', then D has a Hamiltonian cycle. Otherwise $x^- \rightarrow y^{++}$ and D has a cycle of length $n-1$ containing all vertices of C. $\square$

The next result was originally obtained by the second author [10] in a weaker form.

Theorem 5.4 *If a strong SMD D has a 1-diregular subgraph $F = C_1 \cup \ldots \cup C_t$ with $p (\leq n)$ vertices, then, for every i, D has a cycle of length at least $p-t+1$ covering all vertices of C_i.*

Proof: If any pair of cycles in F form a strong digraph, then we can use Lemma 5.3 above to reduce the set of cycles by one at the cost of loosing at most one vertex, and we can decide on which cycle to loose it if necessary. Continue this until we either have just one cycle, which clearly satisfies our claim, or we have cycles $C'_1, \ldots, C'_k$, such that all arcs between C'_i and C'_j $(i < j)$ go from C'_i to C'_j. Now we can apply Lemma 5.2. $\square$

One can apply this theorem to obtain some long cycle (more than a half of the length of a longest cycle) in a SMD D in time $O(n^3)$. The bound on the complexity is determined by that of an algorithm for finding a maximum 1-diregular subgraph in a digraph described in [9].

The following example shows that, for general SMD, the result in Theorem 5.4 is best possible: Consider the following k-partite $(k \geq 3)$ tournaments

$G = G(c,t),\ c \geq 2,\ t \geq 1$ with colour sets $W_1 \dots W_k$. $G(c,t)$ contains a factor $C_1 \cup C_2 \cup \dots \cup C_t$, where $C_i = x_1^i x_2^i ... x_c^i x_1^i$, $i = 1, 2, ..., t$. Moreover, for each $i = 1, ..., t$ the vertices x_2^i, x_c^i are contained in W_3, and if i is even then $x_1^i \in W_2$, otherwise $x_1^i \in W_1$. For each $i = 1, ..., t$, the arc set of $G < C_i >$ is a subset of $\{x_q^i \to x_s^i : 1 \leq q < s \leq c\} \cup \{x_c^i \to x_1^i\} \backslash \{x_1^i \to x_c^i\}$. For every $1 \leq i < j \leq t$, all arcs between C_i and C_j are oriented from C_i to C_j, except when $j = i+1$ in which case there exists the arc $x_1^{i+1} \to x_1^i$ instead of $x_1^i \to x_1^{i+1}$. It is easy to see that for every $i = 1, ..., t$ there is a longest cycle of $G(c,t)$ having exactly $t(c-1)+1$ vertices and containing all vertices of some C_i.

Corollary 5.5 *If a strong SMD D has an almost 1-diregular subgraph $F = P \cup C_1 \cup \ldots \cup C_t$ with p vertices and x is the first vertex of P, then D has a path of length at least $p - t - 1$ starting at x.*

Proof: Add a new vertex w to D and the arcs $z \to w$, for all $z \neq x$ and the arc $w \to x$ to obtain the SMD D'. By Theorem 5.4 D' has a cycle C of length at least $p + 1 - (t+1) + 1$ containing x and w. Remove w from C. $\square$

Theorem 5.6 *[6, 12] A semicomplete bipartite digraph B is Hamiltonian if and only if it is strong and has a factor.*

Proof: By Lemma 5.2, it is enough to prove that if C and C' are disjoint cycles covering all vertices of a SBD B, then B is Hamiltonian. This follows from Lemma 5.3 and the fact that B does not contain any cycle of length $n-1$ since B is bipartite. $\square$

Theorem 5.7 *[10] An ordinary SMD is Hamiltonian if and only if it is strong and has a factor.*

Proof: By Lemma 5.2, it is enough to show that if C and C' are disjoint cycles which induce a strong ordinary SMD D, then D is Hamiltonian. If C and C' have a pair x, y of non-adjacent vertices ($x \in V(C),\ y \in V(C')$) then obviously $x \to y^+,\ y \to x^+$ and D is Hamiltonian. Assuming that any pair of vertices from C and C' is adjacent, we complete the proof as in Lemma 5.3. $\square$

6 Cycles through k vertices in ordinary SMDs

In this section we provide a complete characterization of those ordinary SMDs that have a cycle through any set of k vertices. We call such digraphs *k-cyclic*.

Theorem 6.1 *An ordinary SMD D is $k-$cyclic if and only if it is strong and for every set Z of k vertices, there exists a 1-diregular subgraph of D which contains all the vertices of Z.*

Proof: One direction is trivial. Now suppose that D is strong and let Z be a set of k vertices of D and $C_1, \ldots, C_t$ a collection of cycles of D covering Z, chosen such that t is as small as possible.

Suppose that $t \geq 2$. By Theorem 5.7 and the minimality of t, we may assume that $C_1, \ldots, C_t$ form the strong components of the graph $D < C_1 \cup \ldots \cup C_t >$ and that there is no arc from C_j to C_i for $i < j$. It is easy to see that every vertex on C_i dominates every vertex on C_j for $i < j$. Because D is a strong digraph, there exists a path P starting at some vertex u on C_t and ending on some cycle C_i, $i < t$, such that P has only u and v in common with $D < C_1 \cup \ldots \cup C_t >$. Obviously, D contains a cycle C with vertex set precisely the vertices of $C_i, \ldots, C_t$ and P. This is a contradiction to the minimality of t.

Hence $t = 1$ and D has a cycle containing all the vertices of Z. $\square$

Corollary 6.2 *There exists an $O(n^{\frac{5}{2}})$ algorithm to decide if there is a cycle through a given set Z of k vertices in an ordinary SMD D on n vertices and finds one if it exists.*

Proof: First we show how to decide the existence of the 1-diregular subgraph F covering the vertices of Z. From D we construct the following bipartite graph B. The vertex set of B consists of two copies x, x' of every vertex x of D. The edge set is the following. For each arc $x \rightarrow y$ of D we have the edge xy'. In addition we add the edges xx' for all x which is not in Z. It is easy to see that D has a 1-diregular subgraph covering Z if and only if B has a perfect matching. Hence in time $O(n^{\frac{5}{2}})$ we can decide the existence of the required subgraph F and find one if it exists.

Suppose, we have found F. Next we throw away cycles from F which do not contain vertices of Z. Now we have a collection of cycles $C'_1, \ldots, C'_s$ covering the vertices of Z, such that each C'_i contains a vertex from Z. Using the proof of Theorem 5.7 we can reduce this to a collection of cycles $C_1, \ldots, C_t$ such that if $t \geq 2$, then $C_i \Rightarrow C_j$ $(i < j)$. Now we use the strong connectivity of D to find a path form some vertex u on C_t to some vertex v on C_i for some $i < t$, such that P has only u and v in common with $D < C_1 \cup \ldots \cup C_t >$. Using P we reduce the number of cycles and repeat the last step.

The complexity of the algorithm is dominated by the time it takes to check the existence of F. $\square$

In [1] it was shown that for general multipartite tournaments there is a polynomial algorithm to decide the existence of a cycle through any special pair of vertices. The case of k given vertices $k \geq 3$ remains open.

Corollary 6.3 *Every $k-$connected ordinary multipartite tournament D is $k-$cyclic.*

Proof: This follows from Theorem 6.1 by noting that, by Menger's Theorem, D has a set of cycles covering Z for any set of k vertices. $\square$

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