On the number of quasi-kernels in digraphs

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Abstract

A vertex set X of a digraph $D = (V, A)$ is a *kernel* if X is independent (i.e., all pairs of distinct vertices of X are non-adjacent) and for every $v \in V - X$ there exists $x \in X$ such that $vx \in A$. A vertex set X of a digraph $D = (V, A)$ is a *quasi-kernel* if X is independent and for every $v \in V - X$ there exist $w \in V - X, x \in X$ such that either $vx \in A$ or $vw, wx \in A$. In 1974, Chvátal and Lovász proved that every digraph has a quasi-kernel. In 1996, Jacob and Meyniel proved that, if a digraph D has no kernel, then D contains at least three quasi-kernels. We characterize digraphs with exactly one and two quasi-kernels, and, thus, provide necessary and sufficient conditions for a digraph to have at least three quasi-kernels. In particular, we prove that every strong digraph of order at least three, which is not a 4-cycle, has at least three quasi-kernels.

1 Introduction, terminology and notation

A vertex set X of a digraph $D = (V, A)$ is a *kernel* if X is independent (i.e., all pairs of distinct vertices of X are non-adjacent) and for every $v \in V - X$ there exists $x \in X$ such that $vx \in A$. A vertex set X of a digraph $D = (V, A)$ is a *quasi-kernel* if X is independent and for every $v \in V - X$ there exist $w \in V - X, x \in X$ such that either $vx \in A$ or $vw, wx \in A$. A digraph $T = (V, A)$ is a *tournament* if for every pair x, y of distinct vertices in V, either $xy \in A$ or $yx \in A$, but not both. A vertex of out-degree zero is called a *sink*.

While not every digraph has a kernel (e.g., a directed cycle $\vec{C}_n$ has a kernel if and only if n is even), Chvátal and Lovász [3] (see also Chapter 12 in [2]) proved that every digraph has a quasi-kernel. Jacob and Meyniel [6] proved that, if a digraph D has no kernel, then D contains at least three quasi-kernels. While the assertion of Chvátal and Lovász generalizes the fact that every tournament has a 2-serf, i.e., a quasi-kernel of cardinality 1, the Jacob-Meyniel theorem extends the result of Moon [7] that every tournament with no sink has at least three 2-serfs.

While the Jacob-Meyniel theorem provides sufficient conditions for a digraph to have at least three quasi-kernels, in Section 2, we characterize digraphs with exactly one and two quasi-kernels, and, thus, provide necessary and sufficient conditions for a digraph to have at least three quasi-kernels (see Theorem 2.6). In particular, we prove that every strong digraph, of order at least three, different from the 4-cycle $\vec{C}_4$ has at least three quasi-kernels. Note that, in our proofs, we naturally use the Chvátal-Lovász theorem, but not the more powerful Jacob-Meyniel theorem.

We use the standard terminology and notation on digraphs as given in [2]. We still provide most of the necessary definitions for the convenience of the reader.

For a digraph D, the vertex (arc) set is denoted by $V(D)$ $(A(D))$. Let x, y be a pair of vertices in D. If $xy \in A(D)$, we say x *dominates* y, and y *is dominated by* x, and denote it by $x{\rightarrow}y$. A digraph D is *strong* if, for every ordered pair x, y of distinct vertices in D, there is a path from x to y. An *orientation* of a digraph D is an oriented graph obtained from D by deleting exactly one arc from each 2-cycle in D. A *biorientation* of D is a digraph, which is a subdigraph of D and superdigraph of an orientation of D. The *closed in-neighbourhood* (*closed out-neighbourhood*) of a set X of vertices of a digraph $D = (V, A)$ is defined as follows.

$$N_D^-[X] = X \cup \{y \in V:\ \exists x \in X, y{\rightarrow}x\} \quad (N_D^+[X] = X \cup \{y \in V:\ \exists x \in X, x{\rightarrow}y\}).$$

For disjoint subsets X and Y of $V(D)$, let $X \times Y = \{xy:\ x \in X, y \in Y\}$, $(X, Y)_D = (X \times Y) \cap A(D)$; $D[X]$ is the subdigraph of D induced by X. If the digraph under consideration is clear from the context, then we will omit the subscript D.

2 Digraphs with exactly one and two quasi-kernels

We start with the following:

Lemma 2.1 *Let x be a vertex in a digraph D. If x is a non-sink, then D has a quasi-kernel not including x.*

Proof: Let $y \in N^+[x] - \{x\}$ be arbitrary. If $N^-[y] = V(D)$, then y is the required quasi-kernel. If $N^-[y] \neq V(D)$, let Q' be a quasi-kernel in $D - N^-[y]$. If y dominates a vertex in Q', then Q' is a quasi-kernel in D, which does not contain x. If y does not dominate a vertex in Q', then $Q' \cup \{y\}$ is a quasi-kernel in D, which does not include x.□

The following is an easy characterization of digraphs with merely one quasi-kernel.

Theorem 2.2 *A digraph D has only one quasi-kernel if and only if D has a sink and every non-sink of D dominates a sink of D. If a digraph D has only one quasi-kernel Q, then Q is a kernel and consists of the sinks of D.*

Proof: Assume that D has a sink and every non-sink of D dominates a sink of D. Let S be the set of sinks in D. To see that S is a unique quasi-kernel of D, it is enough to observe that every sink must be in a quasi-kernel.

Let D have only one quasi-kernel Q. To see that Q is the set of sinks in D, observe that Q contains all sinks in D and, by Lemma 2.1, Q does not have non-sinks. If x is a non-sink and x does not dominate a vertex in Q, then $Q \cup \{x\}$ is another quasi-kernel of D, a contradiction. Thus, we have proved that D has a sink and every non-sink of D dominates a sink of D. □

In view of Theorem 2.2, the following assertion is a strengthening of the Jacob-Meyniel theorem for the case of digraphs with no sinks.

Theorem 2.3 *Let D be a digraph with no sink. Then D has precisely two quasi-kernels if and only if D has an induced 4-cycle or 2-cycle, C, such that no vertex of C dominates a vertex in $D - V(C)$ and every vertex in $D - V(C)$ dominates at least two adjacent vertices in C.*

To prove Theorem 2.3, we will extensively use the following:

Lemma 2.4 *Let a digraph D have exactly two quasi-kernels, R and Q. Then the following claims hold:*

(i) If a vertex x in R dominates some vertex y such that $V(D) \neq N^-[y]$, then $Q - y$ is the only quasi-kernel in $D - N^-[y]$;

(ii) $\{R, Q\}$ is the set of quasi-kernels of every biorientation of D, in which both R and Q contain non-sinks.

Proof: Let $R_1, R_2, \ldots, R_k$ be the quasi-kernels in $D - N^-[y]$. Then $R'_1, R'_2, \ldots, R'_k$ are quasi-kernels in D, where $R'_i = R_i$ if $(y, R_i) \neq \emptyset$ and $R'_i = R_i \cup \{y\}$, otherwise, $i = 1, 2, \ldots, k$. Since D has only two quasi-kernels, $k \leq 2$. Since $x \in N^-[y]$ and $x \in R$, we conclude that $R - y$ is not a quasi-kernel in $D - N^-[y]$. By the Chvátal- Lovász theorem, every digraph has a quasi-kernel, so $Q - y$ is the unique quasi-kernel in $D - N^-[y]$.

Let D' be a biorientation of D, in which both R and Q contain non-sinks. Clearly, every quasi-kernel in D' is a quasi-kernel in D. However, by Theorem 2.2, neither R nor Q can be the only quasi-kernel in D'. Thus $\{R, Q\}$ is the set of quasi-kernels of D'. $\square$

Proof of Theorem 2.3: We first show that, if D has precisely two quasi-kernels, then D has the above-described structure. We will prove this assertion by induction on $|V(D)|$. The assertion is clearly true when $|V(D)| \leq 2$, so we may assume that it is true for all digraphs, D^*, with $|V(D^*)| < |V(D)|$. Let Q_1 and Q_2 be the only two quasi-kernels in D. Note that by Lemma 2.1, Q_1 and Q_2 must be disjoint (if $x \in Q_1 \cap Q_2$ then use Lemma 2.1 for x). We now prove the following claims.

Claim A: If $(Q_i, Q_j) \neq \emptyset$ ($\{i, j\} = \{1, 2\}$), then for every $w \in Q_i$, $(w, Q_j) \neq \emptyset$.

Proof of Claim A: Let $xy \in (Q_i, Q_j)$ and let w be a vertex in Q_i which has no arc into Q_j. By Lemma 2.4(i), $Q_j - y$ is the unique kernel in $D - N^-[y]$ and, thus, by Theorem 2.2, we must have an arc from w to $Q_j - y$ since $w \in V(D) - N^-[y]$, a contradiction.

Claim B: Both (Q_1, Q_2) and (Q_2, Q_1) are non-empty.

Proof of Claim B: Clearly $Q_1 \cup Q_2$ is not an independent set, as then it would be a quasi-kernel. Hence, without loss of generality we may assume that $(Q_1, Q_2) \neq \emptyset$. Suppose that $(Q_2, Q_1) = \emptyset$. Since Q_1 is a quasi-kernel, there exists a 2-path from any given $x \in Q_2$ to Q_1, say xzy ($z \notin Q_1 \cup Q_2$ and $y \in Q_1$).

We now show that every vertex in Q_2 must dominate z. Suppose that this is not the case, and let w be a vertex not dominating z. By Lemma 2.4, Q_1 is the only quasi-kernel in $D - N^-[z]$. However, by Theorem 2.2, this is a contradiction against the fact that w dominates no vertex in Q_1 ($w \in V(D) - N^-[z]$). Thus, $Q_2 \subseteq N^-[z]$.

Let D' be any orientation of D for which $(z, Q_2)_{D'} = \emptyset$, and let ab be an arc in $(Q_1, Q_2)_{D'}$. Since $z \in V(D') - N^-_{D'}[b]$, we have $V(D') \neq N^-_{D'}[b]$. By Lemma 2.4, $Q_2 - b$ is the only quasi-kernel in $D' - N^-_{D'}[b]$. By Theorem 2.2, $Q_2 - b$ is a kernel in $V(D') - N^-_{D'}[b]$. However, $Q_2 - b$ is not a kernel in $D' - N^-_{D'}[b]$ as z dominates no vertex in $Q_2 - b$, a contradiction.

Claim C: Let $\{a, b\}$ be a set of two distinct vertices from Q_1 and let $\{c, d\}$ be a set of two distinct vertices from Q_2. Then we cannot have both $a \to c$ and $d \to b$.

Proof of Claim C: Assume that $a \rightarrow c$ and $d \rightarrow b$. Suppose first that $c \not\rightarrow b$. By Lemma 2.4, $Q_1 - b$ is the only quasi-kernel in $V(D) - N^-[b]$. However, since the arc $ac \in D - N^-[b]$ we see that $Q_1 - b$ contains a non-sink in $V(D) - N^-[b]$ in contradiction with Theorem 2.2. Suppose now that $c \rightarrow b$, and let D' equal $D - bc$ (if $bc \notin D$, then $D' = D$). By Lemma 2.4, $Q_2 - c$ is the only quasi-kernel in $V(D') - N^-[c]$. However, since the arc $db \in D' - N^-_{D'}[c]$ we see that $Q_2 - c$ contains a non-sink in contradiction with Theorem 2.2.

Claim D: Either $D[Q_1 \cup Q_2]$ is a 2-cycle or $D[Q_1 \cup Q_2]$ contains an induced 4-cycle.

Proof of Claim D: If either Q_1 or Q_2 has only one vertex, then without loss of generality we may assume that $|Q_1| = 1$. If $|Q_2| = 1$ then by Claim B, $D[Q_1 \cup Q_2]$ is a 2-cycle, so assume that $|Q_2| \geq 2$. Let $Q_1 = \{x\}$ and observe that by Claims A and B there exists a pair a, b of distinct vertices in Q_2 such that $ax, xb \in A(D)$. Let D' be any orientation of D with $ax, xb \in A(D')$. By Lemma 2.4, $Q_1 - x$ is the only quasi-kernel in the non-empty digraph $D' - N^-_{D'}[x]$, which contradicts the fact that $Q_1 = \{x\}$.

Therefore, we may now assume that both Q_1 and Q_2 have cardinality at least two. By Claim B, there exists an arc x_2x_1 in $(Q_2, Q_1)_D$. Let $y_1 \in Q_1 - \{x_1\}$ be arbitrary, and observe that $(y_1, Q_2) \neq \emptyset$, by Claims A and B. By Claim C, $y_1x_2 \in (y_1, Q_2)$. Let $y_2 \in Q_2 - \{x_2\}$ be arbitrary. Analogously, we have $y_2y_1 \in A(D)$. Finally, Claims A and C imply that $x_1y_2 \in A(D)$. Therefore, $C = x_2x_1y_2y_1x_2$ is a 4-cycle. Observe that C is an induced 4-cycle, by Claim C and the fact that $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are independent sets (they are subsets of quasi-kernels).

Claim E: If $abcda$ is a 4-cycle such that $\{a, c\} \subseteq Q_1$ and $\{b, d\} \subseteq Q_2$, then there is no arc from $\{a, b, c, d\}$ to any vertex in $D - \{a, b, c, d\}$.

Proof of Claim E: Assume that the claim is false and that there exists a vertex $z \in V(D) - \{a, b, c, d\}$ such that there is an arc from $\{a, b, c, d\}$ to z. Without loss of generality, assume that $az \in A(D)$, and consider the following two cases.

Case 1: $z \rightarrow c$. Let D' be any orientation of D with $zc, az \in A(D')$. By Lemma 2.4, $Q_2 - z$ is the only quasi-kernel in $D' - N^-_{D'}[z]$. However, the existence of the arc $bc \in D'$ contradicts Theorem 2.2.

Case 2: $z \not\rightarrow c$. By Lemma 2.4(i), $Q_1 - c$ is the only quasi-kernel in $D - N^-_D[c]$. However, the existence of the arc $az \in D - N^-[c]$ contradicts Theorem 2.2.

Claim F: If $abcda$ is a 4-cycle such that $\{a, c\} \subseteq Q_1$ and $\{b, d\} \subseteq Q_2$, then every vertex in $D - \{a, b, c, d\}$ dominates two adjacent vertices on $abcda$.

Proof of Claim F: Let $x \in V(D) - \{a, b, c, d\}$ be arbitrary. If x has no arc into $\{a, b, c, d\}$, then consider the digraph $D^* = D - N^-[x]$. Clearly, $Q_1 - N^-[x]$ and $Q_2 - N^-[x]$ are distinct quasi-kernels in D^*; D^* cannot have another quasi-kernel as D has only two quasi-kernels. Therefore there are exactly two quasi-kernels in D^*, and by our induction hypothesis, these quasi-kernels are precisely $\{a, c\}$ and $\{b, d\}$. Observe that, by Claim E, x is adjacent to no vertex from the set $\{a, b, c, d\}$. However, this means that both $\{x, a, c\}$ and $\{x, b, d\}$ are quasi-kernels in D, contradicting the fact that Q_1 and Q_2 are disjoint. Therefore, x must have an arc into $\{a, b, c, d\}$. Observe that since x is arbitrary, this implies that $\{a, c\}$ and $\{b, d\}$ are quasi-kernels in D.

Without loss of generality, assume that $x \rightarrow a$ in D. Suppose also that $x \not\rightarrow b$ and $x \not\rightarrow d$, as otherwise we would be done. However, these assumptions imply that $\{x, b, d\}$ also is a quasi-kernel, along with $\{a, c\}$ and $\{b, d\}$, a contradiction.

Claim G: If $C = D[Q_1 \cup Q_2]$ is a 2-cycle, then no vertex of C dominates a vertex in $D - V(C)$ and every vertex in $D - V(C)$ dominates both vertices in C.

Proof of Claim G: Let $C = xyx$. Assume there exists an arc xz, $z \neq y$. Consider an orientation, D', of D such that $D' - N^-_{D'}[x]$ contains z and does not contain y. On one hand, D' has no quasi-kernels other than $\{x\}$ and $\{y\}$; on the other hand, either Q or $Q \cup \{x\}$ is a quasi-kernel in D', where Q is a quasi-kernel in $D' - N^-_{D'}[x]$. We have arrived at a contradiction. Therefore $(V(C), V(D) - V(C)) = \emptyset$. Furthermore, every vertex $v \in V(D) - V(C)$ must dominate both vertices on C since otherwise there would be a quasi-kernel containing v.

Claims D,E, F and G prove the assertion on the structure of D.

Now assume that D has the structure described in this theorem, and C is the cycle in D. If C is a 2-cycle, then it is easy to see that each of the two vertices on C is a quasi-kernel (and kernel) in D, and that there are no other quasi-kernels in D. So now assume that $C = abcda$ is an induced 4-cycle in D. Observe that $\{a, c\}$ and $\{b, d\}$ are quasi-kernels in D. Since $(\{a, b, c, d\}, V(D) - \{a, b, c, d\}) = \emptyset$, any quasi-kernel in D must contain a vertex, x, in C. Since the successor x^+ of x in C has to be able to reach the quasi-kernel with a path of length at most two, $(x^+)^+$ must also belong to the quasi-kernel. Since all other vertices are adjacent to one of these vertices, the only quasi-kernels are $\{a, c\}$ and $\{b, d\}$. □

As corollaries we obtain the following two theorems.

Theorem 2.5 *A strong digraph D of order at least three has at least three quasi-kernels, unless D is $\vec{C}_4$.*

Proof: Immediate from the previous theorems, Theorems 2.2 and 2.3. □

Theorem 2.6 *Let D be a digraph, S the set of sinks in D, R the set of vertices that have* *an arc into S, and $H = D - S - R$. Then D has precisely two quasi-kernels, if and only if one of the following holds:*

(a) There is a 2-cycle C in H such that at most one of the vertices in C has an arc into R, no vertex of C dominates a vertex in $H - V(C)$, and every vertex in $H - V(C)$ dominates both vertices in C.

(b) There is an induced 4-cycle, C, in H such that no vertex of C dominates a vertex in $D - V(C)$ and every vertex in $H - V(C)$ dominates two adjacent vertices in C.

(c) The digraph H has at least two vertices. There is a vertex x in H such that no vertex of H is dominated by x, all the vertices of $H - x$ dominate x, i.e., $(V(H) - \{x\}, x) = (V(H) - \{x\}) \times \{x\}$, and there is a kernel Q in $H - x$, consisting only of sinks in $H - x$. Moreover, there is no arc from Q to R.

(d) The digraph H has exactly one vertex and this vertex dominates a vertex in R.

Proof: We first show that, if D has precisely two quasi-kernels, then D has the above-described structure. Let D be a digraph with exactly two quasi-kernels. If D has no sinks, then by Theorem 2.3, D has the structure described in part (a) or (b) with $R \cup S = \emptyset$. Hence, we may assume that D contains some sinks, and let S, R and H be as defined in the formulation of this theorem. Let us first prove that H has at most one sink.

Suppose that there are at least two sinks in H. Let x and y be two distinct sinks in H. Note that both x and y have arcs into R, since otherwise they would belong to S or R. Let Q_1 be a quasi-kernel in H, Q_2 a quasi-kernel in $H - x$, and Q_3 a quasi-kernel in $H - y$. Since $\{x, y\} \subseteq Q_1$, $\{x, y\} \cap Q_2 = \{y\}$ and $\{x, y\} \cap Q_3 = \{x\}$ we see that $Q_1 \cup S$, $Q_2 \cup S$ and $Q_3 \cup S$ are 3 different quasi-kernels in D, a contradiction. Hence, H has at most one sink.

Suppose that there is exactly one sink x in H. Since the case of H having exactly one vertex is trivial, we may assume that H contains at least two vertices. Let Q_1 be a quasi-kernel in H, and let Q_2 be a quasi-kernel in $H - x$. Note that $S \cup Q_1$ and $S \cup Q_2$ are different quasi-kernels in D (as $x \in Q_1$ and x has an arc into R). Therefore, Q_2 must be the unique quasi-kernel in $H - x$, and, by Theorem 2.2, Q_2 is a kernel in $H - x$ consisting only of sinks in $H - x$. Since x is the only sink in H, every vertex in Q_2 dominates x. Therefore, $\{x\}$ is a quasi-kernel in H. Since x must be the unique quasi-kernel in H and x is a sink, we must have $(V(H) - \{x\}, x) = (V(H) - \{x\}) \times \{x\}$. Thus, $S \cup \{x\}$ and $S \cup Q_2$ are quasi-kernels in D. If there is a vertex $w \in Q_2$ which dominates a vertex in R, then let Q_3 be a quasi-kernel in $H - w - x$, and observe that $Q_3 \cup S$ is a third quasi-kernel, a contradiction. Therefore, D has the structure described in part (c).

Suppose now that H has no sink. (Since D has more than one quasi-kernel, H is non-empty.) By Theorem 2.2, there are at least two quasi-kernels, Q_1 and Q_2, in H. If Q is a quasi-kernel in H, then $S \cup Q$ is a quasi-kernel in D. Hence, Q_1 and Q_2 are the only quasi-kernels in H, and, thus, the structure of H is provided by Theorem 2.3. Let C be the 2-cycle or induced 4-cycle given in Theorem 2.3.

If C is a 2-cycle, xyx, then, by Theorem 2.3, to show that D has the structure described in part (a) it suffices to prove that at most one of the vertices x and y has an arc into R. Assume that both x and y have arcs into R. Let Q_3 be a quasi-kernel in $H - x - y$, if $V(H) \neq \{x, y\}$, and the empty set, otherwise. However, $S \cup x$, $S \cup y$ and $S \cup Q_3$ are three different quasi-kernels in D, a contradiction.

If C is an induced 4-cycle, $abcda$, then, by Theorem 2.3, to show that D has the structure described in part (b) it suffices to prove that no vertex in $V(C)$ dominates a vertex in R. Without loss of generality, assume that a dominates a vertex in R. By Lemma 2.1, there exists a quasi-kernel, Q, in $H - a$, which does not contain b, as b is not a sink in $H - a$. However, $Q \cup S$, $\{a, c\} \cup S$ and $\{b, d\} \cup S$ are three different quasi-kernels in D, a contradiction.

This proves that, if D has exactly two quasi-kernels, then D has the structure described in the formulation of this theorem. If D has the structure provided in part (a), (b), (c) or (d), then it is not too difficult to check that there are exactly two quasi-kernels in D. $\square$

3 Disjoint quasi-kernels

If a digraph D has a sink x, then every quasi-kernel in D must contain x. Hence, a digraph with sinks has no disjoint quasi-kernels. However, one may suspect that every digraph with no sink has a pair of disjoint quasi-kernels. By Lemma 2.1, this is true for digraphs with exactly two quasi-kernels: see the first paragraph in the proof of Theorem 2.3. One can show that this is also true for every digraph which possesses a quasi-kernel of cardinality at most two.

Unfortunately, in general, the above claim does not hold. Consider the following construction suggested to us by the referee. Let T be a tournament having the property that for every pair x, y of vertices there exists a vertex z such that $x \rightarrow z$ and $y \rightarrow z$. (The existence of such tournaments was first proved by Erdős [4], see also Section 1.2 in [1]. It was shown by Graham and Spencer [5] that some quadratic residue tournaments are such tournaments, see also Section 9.1 in [1].) Extend T to a digraph D by adding, for every vertex x in T, a new vertex x' together with the arc $x'x$.

Clearly, D has no sink and every quasi-kernel of D contains exactly one vertex in T. If Q_x and Q_y are a pair of quasi-kernels of D containing the vertices x and y, respectively, then they are not disjoint because they both have to contain z', where $x \rightarrow z$ and $y \rightarrow z$.

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References

[1] N. Alon and J.H. Spencer, *The Probabilistic Method*, 2nd edition, Wiley, New York, 2000.

[2] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2000.

[3] V. Chvátal and L. Lovász, Every directed graph has a semi-kernel, Lecture Notes in Math. 411 (1974) 175, Springer, Berlin.

[4] P. Erdős, On a problem in graph theory, Math. Gaz. 47 (1963) 220–223.

[5] R.L. Graham and J.H. Spencer, A constructive solution to a tournament problem. Canad. Math. Bull. 14 (1971) 45–48.

[6] H. Jacob and H. Meyniel, About quasi-kernels in a digraph, Discrete Math. 154 (1996) 279–280.

[7] J.W. Moon, Solution to problem 463, Math. Mag. 35 (1962) 189.