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Zeta functions related to the pro-p group $\mathrm{SL}_1(\Delta_p)$

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Abstract

Let $\mathbb{D}_p$ be a central simple $\mathbb{Q}_p$-division algebra of index 2, with maximal $\mathbb{Z}_p$-order Δ_p. We give an explicit formula for the number of subalgebras of any given finite index in the $\mathbb{Z}_p$-Lie algebra $\mathcal{L} := \mathfrak{sl}_1(\Delta_p)$. From this we obtain a closed formula for the zeta function $\zeta_{\mathcal{L}}(s) := \sum_{M \leqslant \mathcal{L}} |\mathcal{L} : M|^{-s}$. The results are extended to the p-power congruence subalgebras of $\mathcal{L}$, and as an application we obtain the zeta functions of the corresponding congruence subgroups of the uniform pro-p group $\mathrm{SL}_1^2(\Delta_p)$.

1. *Introduction*

Let G be a finitely generated pro-p group, and for every $n \in \mathbb{N}_0$ let $\hat{a}_n(G)$ denote the number of subgroups of index p^n in G. The subgroup growth of G is determined by the sequence $\hat{a}_n(G)$, $n \in \mathbb{N}$. The group G is said to have *polynomial subgroup growth* if there exists $\alpha \in \mathbb{R}_{\geqslant 0}$ such that for all $n \in \mathbb{N}_0$ we have $\hat{a}_n(G) \leqslant p^{\alpha n}$.

It is known that G has polynomial subgroup growth if and only if it is p-adic analytic; see [**1**, corollary 8·34]. The aim of current research is to provide a more detailed description of pro-p groups with “slow” polynomial growth. The present paper forms a small detour from the author's characterization of pro-p groups with linear subgroup growth [**8**], which solves a problem posed by Shalev in [**9**].

Working in the context of Lie groups, it is natural to look for a link between the subgroup growth of a p-adic analytic group and the subalgebra growth of an associated Lie algebra. Indeed, every compact p-adic analytic group contains an open uniform pro-p subgroup, and there is an equivalence between the category of uniform pro-p groups and the category of powerful $\mathbb{Z}_p$-Lie algebras; see [**1**, chapter 9]. Suppose that G is a uniform pro-p group and let $L = L_G$ denote the powerful $\mathbb{Z}_p$-Lie algebra corresponding to G. Ilani [**6**] has shown that, if $p \geqslant \dim(L)$, then there is an index-preserving isomorphism between the open subalgebra lattice of L and the open subgroup lattice of G. In general, the subgroup growth of G and the subalgebra growth of L have at least the same asymptotic behaviour; see [**8**]. We are thus led to consider the subalgebra growth of Lie algebras over the p-adic integers $\mathbb{Z}_p$.

Let L be a finite dimensional $\mathbb{Z}_p$-Lie algebra. For every $n \in \mathbb{N}_0$ let $\hat{a}_n(L)$ denote the number of subalgebras of index p^n in L. The arithmetic sequence $\hat{a}_n(L)$, $n \in \mathbb{N}_0$, can be encoded in a generating function

$$\zeta_L(s) := \sum_{n=0}^{\infty} \hat{a}_n(L) p^{-ns} = \sum_{\substack{M \leqslant L \text{ with} \\ |L:M| < \infty}} |L\colon M|^{-s}.$$

In the first instance, this is just a formal Dirichlet series; however, Grunewald, Segal and Smith [**5**] have shown that $\zeta_L(s)$ is in fact a rational function over $\mathbb{Q}$ of p^{-s}. These rational functions are surprisingly difficult to compute, even in small dimensions. Essentially three explicit examples have been calculated; the resulting zeta functions can be neatly expressed in terms of $\zeta_p(s) := \sum_{k=0}^{\infty} p^{-ks}$, the local p-factor of the Riemann zeta function.

(1) If $L = \mathbb{Z}_p^d$ is abelian of dimension d, then

$$\zeta_L(s) = \zeta_p(s)\zeta_p(s-1)\cdots\zeta_p(s-d+1).$$

(2) If $L = \mathbb{Z}_p\mathbf{x} + \mathbb{Z}_p\mathbf{y} + \mathbb{Z}_p\mathbf{z}$, with defining relations $[\mathbf{x},\mathbf{y}] = \mathbf{z}$ and $[\mathbf{x},\mathbf{z}] = [\mathbf{y},\mathbf{z}] = 0$, is the so-called Heisenberg algebra, then

$$\zeta_L(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-3)\zeta_p(3s-3)^{-1}.$$

(3) If $L = \mathfrak{sl}_2(\mathbb{Z}_p)$ and $p > 2$, then

$$\zeta_L(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-1)\zeta_p(2s-2)\zeta_p(3s-1)^{-1}.$$

Examples (1) and (2) were calculated in [**5**], together with some more complicated Dirichlet series counting ideals rather than subalgebras. Example (3) is due to Ilani [**7**] and du Sautoy [**2**]. More recently, the zeta function of $\mathfrak{sl}_2(\mathbb{Z}_2)$ has also been calculated [**4**].

Our list of examples is beautiful, but rather short – much too short to provide a reasonable starting point for a more general theory. The main purpose of this paper is to investigate the subalgebra growth of the $\mathbb{Z}_p$-Lie algebra $\mathfrak{sl}_1(\Delta_p)$, which affords some interesting new features. As indicated above, our main motivation for studying zeta functions of $\mathbb{Z}_p$-Lie algebras comes from the subject of subgroup growth, and applications of our results in that area will be pointed out. More on zeta functions of groups, and the connection with $\mathbb{Z}_p$-Lie algebras, can be found in [**3**].

Let $\mathbb{D}_p$ be a central simple $\mathbb{Q}_p$-division algebra of index 2; recall that up to isomorphism there is precisely one such object. Let Δ_p denote the (unique) maximal $\mathbb{Z}_p$-order in $\mathbb{D}_p$ and write $\mathfrak{P}$ for the maximal ideal of Δ_p. The set $\mathfrak{sl}_1(\Delta_p)$ of elements of reduced trace zero in Δ_p forms a $\mathbb{Z}_p$-Lie algebra. For every $m \in \mathbb{N}_0$ let $\mathfrak{sl}_1^m(\Delta_p) := \mathfrak{sl}_1(\Delta_p) \cap \mathfrak{P}^m$ denote the mth congruence subalgebra of $\mathfrak{sl}_1(\Delta_p)$.

THEOREM 1·1. *Let $\mathbb{D}_p$ be a central simple $\mathbb{Q}_p$-division algebra of index 2, with maximal $\mathbb{Z}_p$-order Δ_p. Consider the $\mathbb{Z}_p$-Lie algebra $\mathcal{L} := \mathfrak{sl}_1(\Delta_p)$.*

(1) *Suppose that $p = 2$. Then for every $n \in \mathbb{N}_0$ we have*

$$\hat{a}_n(\mathcal{L}) = \begin{cases} \frac{1}{3}\left(13 \cdot 2^{n+1} - 3 \cdot 2^{(n+6)/2} + 1\right) & \text{if } n \equiv_2 0, \\ \frac{1}{3}\left(11 \cdot 2^{n+1} - 3 \cdot 2^{(n+5)/2} + 1\right) & \text{if } n \equiv_2 1. \end{cases}$$

The zeta function of $\mathcal{L}$ is given by

$$\zeta_{\mathcal{L}}(s) = \Phi(2^{-s})\zeta_2(s)\zeta_2(2s-1)\zeta_2(2s-2),$$

where $\Phi(T) = 1 + 6T + 6T^2 - 12T^3 \in \mathbb{Z}[T]$. *Moreover, for every* $m \in \mathbb{N}_0$ *the zeta function of* $2^m\mathcal{L} = \mathfrak{sl}_1^{2m}(\Delta_2)$ *is given by*

$$\zeta_{2^m\mathcal{L}}(s) = \zeta_{\mathbb{Z}_2{}^3}(s) - 2^{(-s+2)(m+2)}(1 + 3 \cdot 2^{-s})\zeta_2(s-2)\zeta_2(2s-1)\zeta_2(2s-2).$$

(2) *Suppose that* $p > 2$. *Then for every* $n \in \mathbb{N}_0$ *we have*

$$\hat{a}_n(\mathcal{L}) = \frac{A(n)}{(p-1)^2(p+1)},$$

where

$$A(n) = \begin{cases} p^{n+3} - (p+1)p^{(n+2)/2} + 1 & \text{if } n \equiv_2 0, \\ p^{n+2} - (p+1)p^{(n+1)/2} + 1 & \text{if } n \equiv_2 1. \end{cases}$$

The zeta function of $\mathcal{L}$ *is given by*

$$\zeta_{\mathcal{L}}(s) = \zeta_p(s)\zeta_p(2s-1)\zeta_p(2s-2),$$

and it satisfies the functional equation

$$\zeta_{\mathcal{L}}(s)\mid_{p\mapsto p^{-1}} = -p^{-5s+3}\zeta_{\mathcal{L}}(s).$$

More generally, for every $m \in \mathbb{N}_0$ *the zeta function of* $p^m\mathcal{L} = \mathfrak{sl}_1^{2m}(\Delta_p)$ *is given by*

$$\zeta_{p^m\mathcal{L}}(s) = \zeta_{\mathbb{Z}_p{}^3}(s) - p^{(-s+2)m}\Psi(p, p^{-s})\zeta_p(s-2)\zeta_p(2s-1)\zeta_p(2s-2),$$

where $\Psi(T_1, T_2) = T_1T_2(1 + T_1 + T_1T_2) \in \mathbb{Z}[T_1, T_2]$.

Remarks 1·2. If we exclude the case $p = 2$, then for every $m \in \mathbb{N}_0$ the zeta function of $p^m\mathcal{L} = \mathfrak{sl}_1^{2m}(\Delta_p)$ varies uniformly in p and p^{-s}. The corresponding formulae for $p = 2$ are similar, but do not fit strictly into the general pattern. For the following remarks let us focus on the case $p > 2$.

(1) From [**2**, **7**] it is easily seen that the sequence $\hat{a}_n(\mathfrak{sl}_2(\mathbb{Z}_p))$, $n \in \mathbb{N}_0$, grows asymptotically like np^n; in contrast to this, the sequence $\hat{a}_n(\mathfrak{sl}_1(\Delta_p))$, $n \in \mathbb{N}_0$, only grows like p^n. In other words the rate of growth "distinguishes" between the split and the non-split form of p-adic Lie algebras of type A_1. In [**8**] it is shown that, up to commensurability, $\mathfrak{sl}_1(\Delta_p)$ is the only non-soluble $\mathbb{Z}_p$-Lie algebra with linear subalgebra growth.

(2) Again, from [**2**, **7**] it follows that

$$\lim_{n\to\infty} \frac{\hat{a}_n(\mathfrak{sl}_2(\mathbb{Z}_p))}{np^n} = \frac{p+1}{2(p-1)}.$$

In contrast to this, the sequence $p^{-n}\hat{a}_n(\mathfrak{sl}_1(\Delta_p))$, $n \in \mathbb{N}_0$, has two distinct limit points, namely p^3 and p^2. A general "explanation" of this phenomenon is given in [**8**].

(3) Suppose that L is a powerful $\mathbb{Z}_p$-Lie algebra and let $G = G_L$ denote the uniform pro-p group corresponding to L. Ilani [**6**] has shown that, if $p \geqslant \dim(L)$, then there is an index-preserving isomorphism between the open subalgebra lattice of L and the open subgroup lattice of G.

In our case, still assuming $p > 2$, this means that for every $m \in \mathbb{N}_{\geqslant 2}$ the zeta function of the uniform pro-p group $\mathrm{SL}_1^m(\Delta_p)$ is equal to the zeta function of the powerful $\mathbb{Z}_p$-Lie algebra $\mathfrak{sl}_1^m(\Delta_p)$. Hence Theorem 1·1 also determines the zeta functions of the p-power congruence subgroups of $\mathrm{SL}_1^2(\Delta_p)$.

(4) There is an interesting connection between counting subalgebras in the $\mathbb{Z}_p$-Lie algebra $\mathfrak{sl}_2(\mathbb{Z}_p)$ and counting $\mathbb{F}_p[\![t]\!]$-subalgebras in the $\mathbb{F}_p[\![t]\!]$-Lie algebra $\mathfrak{sl}_2(\mathbb{F}_p[\![t]\!])$; see [**2**].
Let $\mathbb{E}_p$ be a central simple $\mathbb{F}_p(\!(t)\!)$-division algebra of index 2, with maximal $\mathbb{F}_p[\![t]\!]$-order Σ_p. Then, still assuming $p > 2$ and translating our proof of Theorem 1·1 word for word, it can be seen that the zeta function counting $\mathbb{F}_p[\![t]\!]$-subalgebras in the $\mathbb{F}_p[\![t]\!]$-Lie algebra $\mathfrak{sl}_1(\Sigma_p)$ is equal to the (ordinary) zeta function of $\mathfrak{sl}_1(\mathbb{Z}_p)$.

These comments suggest a number of interesting questions; we formulate two problems explicitly.

Problem 1·3. Investigate the subalgebra growth of $\mathbb{Z}_p$-Lie algebras, which are full $\mathbb{Z}_p$-sublattices in central simple $\mathbb{Q}_p$-division algebras of higher index.

Problem 1·4. Let $\mathbb{E}_p$ be a central simple $\mathbb{F}_p(\!(t)\!)$-division algebra of index 2, with maximal $\mathbb{F}_p[\![t]\!]$-order Σ_p. Investigate and compare the subgroup growth of the pro-p groups $\mathrm{SL}_1^1(\Sigma_p)$ and $\mathrm{SL}_2^1(\mathbb{F}_p[\![t]\!])$.

There is also some interest in zeta functions related to counting ideals instead of subalgebras. Again, let L be a finite dimensional $\mathbb{Z}_p$-Lie algebra. For every $n \in \mathbb{N}_0$ let $\hat{a}_n^{\triangleleft}(L)$ denote the number of ideals of index p^n in L. Define

$$\zeta_L^{\triangleleft}(s) := \sum_{n=0}^{\infty} \hat{a}_n^{\triangleleft}(L) p^{-ns} = \sum_{\substack{I \trianglelefteq L \text{ with} \\ |L:I|<\infty}} |L\colon I|^{-s}.$$

In a short addendum we prove that $\zeta_L^{\triangleleft}(s)$ can be expressed in a rather concise form, whenever $\mathbb{Q}_p \otimes L$ is a simple p-adic Lie algebra. This generalizes [**2**, proposition 4·1]. As a "working example" we obtain:

PROPOSITION 1·5. *Suppose that* $p > 2$ *and let* $\mathcal{L}$ *be as in Theorem* 1·1. *Then we have*

$$\zeta_{\mathcal{L}}^{\triangleleft}(s) = (1 + p^{-s})\zeta_p(3s) = \frac{\zeta_p(s)\zeta_p(3s)}{\zeta_p(2s)}.$$

The paper is organized as follows. Section 2 deals with preliminary definitions and not-so-standard notation. In Section 3 we prove Theorem 1·1. Section 4 is the short addendum, already referred to above.

Notation. Throughout the paper p denotes a prime number. The p-adic numbers are denoted by $\mathbb{Q}_p$, the p-adic integers by $\mathbb{Z}_p$. By convention, a $\mathbb{Z}_p$-Lie algebra is torsion-free with respect to addition and finite dimensional. We write $\mathbb{N}$ for the set of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Further notations are introduced in the text as needed.

2. *Preliminaries*

There are two simple p-adic Lie algebras of type A_1. One of these is the familiar Lie algebra $\mathfrak{sl}_2(\mathbb{Q}_p)$, the other can be obtained as follows. Let

$$\begin{array}{ll} \rho := -3, & \text{if } p = 2, \\ \rho \in \{1, 2, \ldots, p-1\}, \text{ not a square modulo } p, & \text{if } p > 2. \end{array}$$

Then the *quaternion algebra*

$$\mathbb{D}_p := \mathbb{Q}_p + \mathbb{Q}_p\mathbf{u} + \mathbb{Q}_p\mathbf{v} + \mathbb{Q}_p\mathbf{uv},$$

defined by the multiplication rules

$$\mathbf{u}^2 = \rho, \quad \mathbf{v}^2 = p, \quad \mathbf{uv} = -\mathbf{vu},$$

is non-split; it is a central simple $\mathbb{Q}_p$-division algebra of index 2. The *reduced norm* and *reduced trace* of $\mathbf{x} = \alpha + \beta\mathbf{u} + \gamma\mathbf{v} + \delta\mathbf{uv} \in \mathbb{D}_p$ are given by

$$\mathrm{N}(\mathbf{x}) = \alpha^2 - \rho\beta^2 - (\gamma^2 - \rho\delta^2)p \quad \text{and} \quad \mathrm{T}(\mathbf{x}) = 2\alpha.$$

The valuation of $\mathbb{Q}_p$ extends uniquely to $\mathbb{D}_p$, and the ring of integers Δ_p of $\mathbb{D}_p$ forms the unique *maximal* $\mathbb{Z}_p$*-order* in $\mathbb{D}_p$. It can be described as

$$\Delta_p = \begin{cases} \left\{0, \frac{1}{2} + \frac{1}{2}\mathbf{u}, \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{uv}, \frac{1}{2} + \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{uv}\right\} + \mathbb{Z}_2 + \mathbb{Z}_2\mathbf{u} + \mathbb{Z}_2\mathbf{v} + \mathbb{Z}_2\mathbf{uv} & \text{if } p = 2, \\ \mathbb{Z}_p + \mathbb{Z}_p\mathbf{u} + \mathbb{Z}_p\mathbf{v} + \mathbb{Z}_p\mathbf{uv} & \text{if } p > 2. \end{cases}$$

The *maximal ideal* $\mathfrak{P}$ of Δ_p is generated by $\mathbf{v}$, that is $\mathfrak{P} = \mathbf{v}\Delta_p$.

The set $\mathrm{SL}_1(\mathbb{D}_p)$ of elements of reduced norm one in $\mathbb{D}_p$ forms a compact p-adic analytic group of dimension three, and coincides with $\mathrm{SL}_1(\Delta_p) := \mathrm{SL}_1(\mathbb{D}_p) \cap \Delta_p$. Let $m \in \mathbb{N}_0$. Then the mth *congruence subgroup* of $\mathrm{SL}_1(\Delta_p)$ is defined as $\mathrm{SL}_1^m(\Delta_p) := \mathrm{SL}_1(\Delta_p) \cap (1 + \mathfrak{P}^m)$.

In a similar way, the set $\mathfrak{sl}_1(\mathbb{D}_p)$ of elements of reduced trace zero in $\mathbb{D}_p$ forms a simple $\mathbb{Q}_p$-Lie algebra of dimension three and has a full $\mathbb{Z}_p$-sublattice $\mathfrak{sl}_1(\Delta_p) := \mathfrak{sl}_1(\mathbb{D}_p) \cap \Delta_p$. Writing $\mathbf{i} := \frac{1}{2}\mathbf{u}$, $\mathbf{j} := \frac{1}{2}\mathbf{v}$ and $\mathbf{k} := \frac{1}{2}\mathbf{uv}$, we have

$$\mathfrak{sl}_1(\Delta_p) = \begin{cases} \{0, \mathbf{j} + \mathbf{k}\} + 2\mathbb{Z}_2\mathbf{i} + 2\mathbb{Z}_2\mathbf{j} + 2\mathbb{Z}_2\mathbf{k} & \text{if } p = 2, \\ \mathbb{Z}_p\mathbf{i} + \mathbb{Z}_p\mathbf{j} + \mathbb{Z}_p\mathbf{k} & \text{if } p > 2, \end{cases}$$

where

$$[\mathbf{i}, \mathbf{j}] = \mathbf{k}, \qquad [\mathbf{i}, \mathbf{k}] = \rho\mathbf{j}, \qquad [\mathbf{j}, \mathbf{k}] = -p\mathbf{i}. \tag{2·1}$$

The mth *congruence subalgebra* of $\mathfrak{sl}_1(\Delta_p)$ is $\mathfrak{sl}_1^m(\Delta_p) := \mathfrak{sl}_1(\Delta_p) \cap \mathfrak{P}^m$, and it satisfies

$$\mathfrak{sl}_1^m(\Delta_p) = \begin{cases} \left\{0, 2^{\lfloor m/2 \rfloor}(\mathbf{j} + \mathbf{k})\right\} + 2^{\lceil m/2 \rceil + 1}\mathbb{Z}_2\mathbf{i} + 2^{\lfloor m/2 \rfloor + 1}\mathbb{Z}_2\mathbf{j} + 2^{\lfloor m/2 \rfloor + 1}\mathbb{Z}_2\mathbf{k} & \text{if } p = 2, \\ p^{\lceil m/2 \rceil}\mathbb{Z}_p\mathbf{i} + p^{\lfloor m/2 \rfloor}\mathbb{Z}_p\mathbf{j} + p^{\lfloor m/2 \rfloor}\mathbb{Z}_p\mathbf{k} & \text{if } p > 2. \end{cases}$$

Note that for $p > 2$ and $m \geqslant 2$ the Lie algebra $\mathfrak{sl}_1^m(\Delta_p)$ is powerful; indeed it corresponds to the uniform pro-p group $\mathrm{SL}_1^m(\Delta_p)$ via the exponential map.

3. *Zeta functions counting subalgebras*

In this section we derive explicit formulae for the subalgebra growth of the $\mathbb{Z}_p$-Lie algebra $L := \mathbb{Z}_p\mathbf{i} + \mathbb{Z}_p\mathbf{j} + \mathbb{Z}_p\mathbf{k}$. If $p > 2$, then $L = \mathfrak{sl}_1(\Delta_p)$ is precisely the Lie algebra we are interested in. If $p = 2$, then $\mathfrak{sl}_1(\Delta_2)$ is a subalgebra of index 4 in L and its zeta function is closely related to the one of L.

For every $\mathbf{x} \in L$ let $v(\mathbf{x}) := \sup\{k \in \mathbb{N}_0 \mid \mathbf{x} \in p^k L\}$. We write $(L, +)$ to denote the additive group of the Lie algebra L, and for every subgroup H of $(L, +)$ we define the *lower level* of H in L as

$$\ell(H) := \ell_L(H) := \inf\{k \in \mathbb{N}_0 \mid H \supseteq p^k L\}.$$

It will be convenient to consider the *"defect zeta function"*

$$\zeta_L^-(s) = \sum_{n=0}^{\infty} \hat{d}_n(L)p^{-ns} := \zeta_{\mathbb{Z}_p^3}(s) - \zeta_L(s),$$

counting subgroups of $(L, +)$ which are not closed under the Lie bracket. Let $n \in \mathbb{N}_0$, and for every $k \in \mathbb{N}_0$ define

$$\begin{aligned} b_{k,n} &:= b_{k,n}(L) := \#\{M \mid M \text{ a subalgebra of } L \text{ with } \ell(M) = k \text{ and } |L\colon M| = p^n\},\\ c_{k,n} &:= c_{k,n}(L) := \#\{H \mid H \text{ a subgroup of } (L, +) \text{ with } \ell(H) = k \text{ and } |L\colon H| = p^n\},\\ d_{k,n} &:= d_{k,n}(L) := c_{k,n} - b_{k,n}. \end{aligned}$$

Then clearly, we have

$$\hat{a}_n(L) = \hat{a}_n\left(\mathbb{Z}_p^3\right) - \hat{d}_n(L), \quad \text{and} \quad \hat{d}_n(L) = \sum_{k=0}^{\infty} d_{k,n} = \sum_{k=\lceil n/3 \rceil}^{n} d_{k,n}. \tag{3·1}$$

LEMMA 3·1. *Let $n, k \in \mathbb{N}_0$ with $n/2 < k \leqslant n$. Then we have*

$$\hat{a}_n\left(\mathbb{Z}_p^3\right) = \frac{p^{2n+3} - (p+1)p^{n+1} + 1}{(p-1)^2(p+1)}$$

and

$$c_{k,n}(L) = c_{k,n}\left(\mathbb{Z}_p^3\right) = \frac{(p^3-1)(p^{2k-1} - p^{3k-n-2})}{(p-1)^2}.$$

Proof. The first formula can be computed easily from example (1) in the Introduction. For the second formula see [**7**, proposition 3·1], or adapt the proof of Lemma 3·3 below.

Let H be a finite index subgroup of $(L, +)$. For $j \in \{1, 2, 3\}$ define

$$s_j(H) := \min\{k \in \mathbb{N}_0 \mid |(H \cap p^k L) + p^{k+1}L\colon p^{k+1}L| \geqslant p^j\}.$$

Note that $|L\colon H| = p^{s_1(H)+s_2(H)+s_3(H)}$ and $s_3(H) = \ell(H)$.

A *good basis* for H is a triple $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in L^3$ such that

$$H = \mathbb{Z}_p\mathbf{a}_1 + \mathbb{Z}_p\mathbf{a}_2 + \mathbb{Z}_p\mathbf{a}_3 \quad \text{and} \quad v(\mathbf{a}_j) = s_j(H) \text{ for all } j \in \{1, 2, 3\}. \tag{GB}$$

Note that H has good bases. Moreover, if $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is a good basis for H, then $H = \mathbb{Z}_p\mathbf{a}_1 \oplus \mathbb{Z}_p\mathbf{a}_2 \oplus \mathbb{Z}_p\mathbf{a}_3$.

Let $\mathbf{a}, \mathbf{b} \in L \setminus \{0\}$. Writing $\mathbf{a} = p^{v(\mathbf{a})}(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k})$ and $\mathbf{b} = p^{v(\mathbf{b})}(\lambda\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k})$ with coefficients in $\mathbb{Z}_p$, we define

$$x(\mathbf{a}, \mathbf{b}) := \det\begin{pmatrix} \beta & \gamma \\ \mu & \nu \end{pmatrix}, \quad y(\mathbf{a}, \mathbf{b}) := \det\begin{pmatrix} \alpha & \gamma \\ \lambda & \nu \end{pmatrix}, \quad z(\mathbf{a}, \mathbf{b}) := \det\begin{pmatrix} \alpha & \beta \\ \lambda & \mu \end{pmatrix}.$$

LEMMA 3·2. *Let H be a finite index subgroup of $(L, +)$, and write $s_j := s_j(H)$ for $j \in \{1, 2\}$. Let $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be a good basis for H.*

(1) *Suppose that $s_1 + s_2 \geqslant \ell(H)$. Then H is a Lie subalgebra of L.*
(2) *Suppose that $s_1 + s_2 < \ell(H)$ and $p = 2$. Then H is a Lie subalgebra of L if and only if one of the following holds*:

(*a*) $s_1 + s_2 = \ell(H) - 1$, $x(\mathbf{a},\mathbf{b}) \equiv_2 1$ *and* $y(\mathbf{a},\mathbf{b}) \equiv_2 z(\mathbf{a},\mathbf{b}) \equiv_2 0$;
(*b*) $s_1 + s_2 = \ell(H) - 1$ *and* $y(\mathbf{a},\mathbf{b}) \equiv_2 z(\mathbf{a},\mathbf{b}) \equiv_2 1$;
(*c*) $s_1 + s_2 = \ell(H) - 2$, $x(\mathbf{a},\mathbf{b}) \equiv_2 0$ *and* $y(\mathbf{a},\mathbf{b}) \equiv_2 z(\mathbf{a},\mathbf{b}) \equiv_2 1$.

(3) *Suppose that* $s_1 + s_2 < \ell(H)$ *and* $p > 2$. *Then* H *is a Lie subalgebra of* L *if and only if* $s_1 + s_2 = \ell(H) - 1$, $x(\mathbf{a},\mathbf{b}) \not\equiv_p 0$ *and* $y(\mathbf{a},\mathbf{b}) \equiv_p z(\mathbf{a},\mathbf{b}) \equiv_p 0$.

Proof. Put $a := v(\mathbf{a})$, $b := v(\mathbf{b})$. As before we write $\mathbf{a} = p^a(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k})$ and $\mathbf{b} = p^b(\lambda\mathbf{i}+\mu\mathbf{j}+\nu\mathbf{k})$ with coefficients in $\mathbb{Z}_p$. Also put $x := x(\mathbf{a},\mathbf{b})$, $y := y(\mathbf{a},\mathbf{b})$ and $z := z(\mathbf{a},\mathbf{b})$. Then

$$[\mathbf{a},\mathbf{b}] = p^{a+b}(-px\mathbf{i} + \rho y\mathbf{j} + z\mathbf{k}).$$

We note that H is a Lie subalgebra of L if and only if $[\mathbf{a},\mathbf{b}] \in H$.

(1) By assumption, we have $a + b \geqslant \ell(H)$, so $[\mathbf{a},\mathbf{b}] \in p^{a+b}L \subseteq H$.

(2) Since $H = \mathbb{Z}_p\mathbf{a} \oplus \mathbb{Z}_p\mathbf{b} \oplus \mathbb{Z}_p\mathbf{c}$, the vectors (α,β,γ) and (λ,μ,ν) are linearly independent modulo p. In particular, at least one of x, y, z is not congruent to zero modulo p.

Put $k := \ell(H) - s_1 - s_2 = \ell(H) - (a+b) \in \mathbb{N}$. The elements

$$p^b\mathbf{a} = p^{a+b}(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}),$$
$$p^a\mathbf{b} = p^{a+b}(\lambda\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k})$$

generate the group $H \cap p^{a+b}L$ modulo $p^{\ell(H)}L$. So $[\mathbf{a},\mathbf{b}] \in H$ if and only if $(-px, \rho y, z)$ is a $\mathbb{Z}_p$-linear combination of (α,β,γ) and (λ,μ,ν) modulo p^k. So $[\mathbf{a},\mathbf{b}] \in H$ if and only if

$$-px^2 - \rho y^2 + z^2 = \det\begin{pmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ -px & \rho y & z \end{pmatrix} \equiv_{p^k} 0. \tag{3·2}$$

Recall that ρ is not a square modulo p. For $k = 1$, this implies: $[\mathbf{a},\mathbf{b}] \in H$ if and only if $x(\mathbf{a},\mathbf{b}) \not\equiv_p 0$ and $y(\mathbf{a},\mathbf{b}) \equiv_p z(\mathbf{a},\mathbf{b}) \equiv_p 0$. For $k \geqslant 2$, the congruence (3·2) is never satisfied.

(3) The argument is similar to case (2). Put $k := \ell(H) - (a+b)$. Then $[\mathbf{a},\mathbf{b}] \in H$ if and only if $-2x^2 + 3y^2 + z^2 \equiv_{2^k} 0$, leading to conditions (*a*)–(*c*). We leave the reader to fill in the details.

Lemma 3·3. *Let* $a, k \in \mathbb{N}_0$ *with* $a \leqslant (k-1)/2$.

(1) *Let* $\mathcal{M}_1$ *be the set of all finite index subalgebras* M *of* L *with* $s_1(M) = a$, $s_2(M) = k - 1 - a$, $\ell(M) = k$. *Then*

$$\#\mathcal{M}_1 = \begin{cases} 9 \cdot 2^{2(k-a)-3} & \text{if } p = 2 \text{ and } a < (k-1)/2, \\ 3 \cdot 2^{k-1} & \text{if } p = 2 \text{ and } a = (k-1)/2, \\ (p+1)p^{2(k-a)-3} & \text{if } p > 2 \text{ and } a < (k-1)/2, \\ p^{k-1} & \text{if } p > 2 \text{ and } a = (k-1)/2. \end{cases}$$

(2) *Suppose that* $p = 2$ *and let* $\mathcal{M}_2$ *be the set of all finite index subalgebras* M *of* L *with* $s_1(M) = a$, $s_2(M) = k - 2 - a$, $\ell(M) = k$. *Then*

$$\#\mathcal{M}_2 = \begin{cases} 3 \cdot 2^{2(k-a)-3} & \text{if } a < (k-2)/2, \\ 2^k & \text{if } a = (k-2)/2. \end{cases}$$

Proof. We prove the assertions for $p > 2$. The formulae stated in (1) and (2) for $p = 2$ are obtained by a similar argument.

Suppose that $p > 2$. We write $b := k - 1 - a$ and $c := k$. Let $\mathcal{T}_1$ be the set of all triples $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in L^3$ such that:

(i) $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a good basis for some finite index subgroup H of $(L, +)$ with $s_1(H) = a$, $s_2(H) = b$, $s_3(H) = c$;

(ii) $x(\mathbf{a}, \mathbf{b}) \not\equiv_p 0$, and $y(\mathbf{a}, \mathbf{b}) \equiv_p z(\mathbf{a}, \mathbf{b}) \equiv_p 0$.

Simple counting yields

$$\begin{aligned} t_1 &:= \#\{(\mathbf{a} + p^c L, \mathbf{b} + p^c L) \mid (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathcal{T}_1\} \\ &= (p^2 - 1)p^{3(c-a-1)} \cdot (p^2 - p)p^{3(c-b-1)}. \end{aligned}$$

Let H be finite index subgroup of $(L, +)$ with $s_1(H) = a$, $s_2(H) = b$ and $s_3(H) = c$. Let $\mathcal{T}_2$ be the set of all good bases for H. Then counting yields

$$\begin{aligned} t_2 &:= \#\{(\mathbf{a} + p^c L, \mathbf{b} + p^c L) \mid (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathcal{T}_2\} \\ &= \begin{cases} (p-1)p^{(c-a-1)+(c-b)} \cdot (p^2 - p)p^{2(c-b-1)} & \text{if } a < b, \\ (p^2 - 1)p^{2(c-a-1)} \cdot (p^2 - p)p^{2(c-b-1)} & \text{if } a = b; \end{cases} \end{aligned}$$

and from Lemma 3·2 (3) we infer that

$$\#\mathcal{M}_1 = t_1/t_2 = \begin{cases} (p+1)p^{2(c-a)-3} & \text{if } a < b, \\ p^{2(c-a)-2} & \text{if } a = b. \end{cases}$$

Rewriting this in terms of k gives the formulae stated in (1) for $p > 2$.

LEMMA 3·4. *Let $n, k \in \mathbb{N}_0$ with $k \leqslant n$. Then we have*

$$b_{k,n} = \begin{cases} c_{k,n} & \text{if } k \leqslant n/2, \\ 3 \cdot (2^{2k-1} - 2^{k-1}) & \text{if } p = 2 \text{ and } k = (n+1)/2, \\ 2^{2k-1} - 2^k & \text{if } p = 2 \text{ and } k = (n+2)/2, \\ 0 & \text{if } p = 2 \text{ and } k \geqslant (n+3)/2, \\ (p-1)^{-1}(p^{2k-1} - p^{k-1}) & \text{if } p > 2 \text{ and } k = (n+1)/2, \\ 0 & \text{if } p > 2 \text{ and } k \geqslant (n+2)/2. \end{cases}$$

Proof. We prove the lemma for $p > 2$. The formulae stated for $p = 2$ are obtained by a similar argument.

Suppose that $p > 2$. Let H be a subgroup of $(L, +)$ with $\ell(H) = k$ and $|L{:}\,H| = p^n$. Write $s_j := s_j(H)$ for $j \in \{1, 2\}$. If $k \leqslant n/2$, then $s_1 + s_2 = n - k \geqslant k = \ell(H)$, so H is a subalgebra of L by Lemma 3·2. If $k \geqslant (n+2)/2$, then $s_1 + s_2 = n - k \leqslant k - 2 = \ell(H) - 2$, so H is not a subalgebra of L by Lemma 3·2.

It remains to consider the case where $k = (n+1)/2$. Lemma 3·3 shows that under this assumption

$$b_{k,n} = \begin{cases} \sum_{j=0}^{(k-2)/2} (p+1)p^{2k-2j-3} & \text{if } k \equiv_2 0, \\ \sum_{j=0}^{(k-3)/2} (p+1)p^{2k-2j-3} + p^{k-1} & \text{if } k \equiv_2 1. \end{cases}$$

After summing, this simplifies to $b_{k,n} = (p-1)^{-1}(p^{2k-1} - p^{k-1})$.

LEMMA 3·5. *Let* $n, k \in \mathbb{N}_0$ *with* $k \leqslant n$. *Then we have*

$$d_{k,n} = \begin{cases} 0 & \text{if } k \leqslant n/2, \\ 2 \cdot \left(2^{n+1} - 2^{(n+1)/2}\right) & \text{if } p = 2 \text{ and } k = (n+1)/2, \\ 6 \cdot \left(2^{n+1} - 2^{(n+2)/2}\right) & \text{if } p = 2 \text{ and } k = (n+2)/2, \\ 7 \cdot (2^{2k-1} - 2^{3k-n-2}) & \text{if } p = 2 \text{ and } k \geqslant (n+3)/2, \\ (p-1)^{-1}(p+1)\left(p^{n+1} - p^{(n+1)/2}\right) & \text{if } p > 2 \text{ and } k = (n+1)/2, \\ (p-1)^{-2}(p^3-1)(p^{2k-1} - p^{3k-n-2}) & \text{if } p > 2 \text{ and } k \geqslant (n+2)/2. \end{cases}$$

Proof. This follows from Lemmata 3·1 and 3·4 by direct calculation. Again, we give full details only for $p > 2$.

Suppose that $p > 2$. The formulae for $k \leqslant n/2$ and $k \geqslant (n+2)/2$ follow trivially from the quoted lemmata, and for $k = (n+1)/2$ we obtain

$$\begin{aligned} d_{k,n} &= \frac{(p^3-1)(p^{2k-1} - p^{3k-n-2})}{(p-1)^2} - \frac{p^{2k-1} - p^{k-1}}{p-1} \\ &= \frac{((p^3-1) - (p-1))(p^{2k-1} - p^{k-1})}{(p-1)^2} \\ &= \frac{(p+1)\left(p^{n+1} - p^{(n+1)/2}\right)}{(p-1)}. \end{aligned}$$

LEMMA 3·6. *Let* $n \in \mathbb{N}_0$.

(1) *Suppose that* $p = 2$. *Then we have*

$$\hat{d}_n(L) = \begin{cases} \frac{1}{3}\left(2^{2n+3} - 5 \cdot 2^{n+2} + 3 \cdot 2^{(n+4)/2}\right) & \text{if } n \equiv_2 0, \\ \frac{1}{3}\left(2^{2n+3} - 2^{n+4} + 3 \cdot 2^{(n+3)/2}\right) & \text{if } n \equiv_2 1. \end{cases}$$

(2) *Suppose that* $p > 2$. *Then we have* $\hat{d}_n(L) = D(n)/(p-1)^2(p+1)$, *where*

$$D(n) = \begin{cases} p^{2n+3} - (p^2+p+1)p^{n+1} + (p+1)p^{(n+2)/2} & \text{if } n \equiv_2 0, \\ p^{2n+3} - (2p+1)p^{n+1} + (p+1)p^{(n+1)/2} & \text{if } n \equiv_2 1. \end{cases}$$

Proof. Using Lemma 3·5 and equation (3·1), the proof becomes purely computational. Again, we assume that $p > 2$ and leave the reader to check the $p = 2$ case. Recall that $\hat{d}_n(L) = \sum_{k=\lceil n/3 \rceil}^{n} d_{k,n}$. If $n \equiv_2 0$, we get

$$\begin{aligned} \hat{d}_n(L) &= \sum_{k=(n+2)/2}^{n} \frac{p^3-1}{(p-1)^2}(p^{2k-1} - p^{3k-n-2}) \\ &= \frac{p^3-1}{(p-1)^2}\left(\frac{p^{2n+2} - p^{n+2}}{p(p^2-1)} - \frac{p^{3n+3} - p^{3(n+2)/2}}{p^{n+2}(p^3-1)}\right) \\ &= \frac{(p^2+p+1)(p^{2n+1} - p^{n+1})}{(p-1)^2(p+1)} - \frac{(p+1)\left(p^{2n+1} - p^{(n+2)/2}\right)}{(p-1)^2(p+1)} \\ &= \frac{p^{2n+3} - (p^2+p+1)p^{n+1} + (p+1)p^{(n+2)/2}}{(p-1)^2(p+1)}. \end{aligned}$$

If $n \equiv_2 1$, we get

$$\begin{aligned}
\hat{d}_n(L) &= \frac{p+1}{p-1}\left(p^{n+1} - p^{(n+1)/2}\right) + \sum_{k=(n+3)/2}^{n} \frac{p^3-1}{(p-1)^2}(p^{2k-1} - p^{3k-n-2}) \\
&= \frac{(p^2-1)\left(p^{n+1} - p^{(n+1)/2}\right)}{(p-1)^2} \\
&\quad + \frac{p^3-1}{(p-1)^2}\left(\frac{p^{2n+2} - p^{n+3}}{p(p^2-1)} - \frac{p^{3n+3} - p^{3(n+3)/2}}{p^{n+2}(p^3-1)}\right) \\
&= \frac{(p^3+p^2-p-1)\left(p^{n+1} - p^{(n+1)/2}\right)}{(p-1)^2(p+1)} \\
&\quad + \frac{(p^2+p+1)(p^{2n+1} - p^{n+2})}{(p-1)^2(p+1)} - \frac{(p+1)\left(p^{2n+1} - p^{(n+3)/2+1}\right)}{(p-1)^2(p+1)} \\
&= \frac{p^{2n+3} - (2p+1)p^{n+1} + (p+1)p^{(n+1)/2}}{(p-1)^2(p+1)}.
\end{aligned}$$

Proof of Theorem 1·1. The proof is now purely computational.

First suppose that that $p > 2$. Then $L = \mathcal{L} = \mathfrak{sl}_1(\Delta_p)$. The formula for $\hat{a}_n(L)$, $n \in \mathbb{N}_0$, follows from Lemma 3·6 and equation (3·1). Next we compute the defect zeta function $\zeta_L^-(s) = \sum_{n=0}^{\infty} \hat{d}_n p^{-ns}$. From Lemma 3·6 we obtain

$$\begin{aligned}
\zeta_L^-(s) \cdot (p-1)^2(p+1) &= \sum_{k=0}^{\infty} p^{2k+3}p^{-ks} - (p^2+p+1)\sum_{k=0}^{\infty} p^{2k+1}p^{-2ks} \\
&\quad + (p+1)\sum_{k=0}^{\infty} p^{(2k+2)/2}p^{-2ks} - (2p+1)\sum_{k=0}^{\infty} p^{2k+2}p^{-(2k+1)s} \\
&\quad + (p+1)\sum_{k=0}^{\infty} p^{(2k+2)/2}p^{-(2k+1)s} \\
&= p \cdot \Phi(p, p^{-s}),
\end{aligned}$$

where

$$\begin{aligned}
\Phi(p,T) &= p^2(1-p^2T)^{-1} + (p+1)(1+T)(1-pT^2)^{-1} \\
&\quad - \left((p^2+p+1) + (2p+1)pT\right)(1-p^2T^2)^{-1}.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&\Phi(p,T) \cdot (1-p^2T)(1-pT^2)(1-p^2T^2) \\
&\quad = p^2\left(1-(p^2+p)T^2 + p^3T^4\right) + (p+1)(1+T)(1-p^2T-p^2T^2+p^4T^3) \\
&\qquad - \left(p^2+p+1+(2p+1)pT\right)\left(1-p^2T-pT^2+p^3T^3\right) \\
&\quad = (p^4-2p^2+1)T + (p^4-p^3-p^2+p)T^2 \\
&\quad = (p^2-1)^2T + (p-1)^2(p+1)pT^2.
\end{aligned}$$

This shows that

$$\zeta_L(s) = \zeta_{\mathbb{Z}_p{}^3}(s) - \Psi(p,p^{-s})\zeta_p(s-2)\zeta_p(2s-1)\zeta_p(2s-2),$$

where

$$\Psi(T_1, T_2) = T_1T_2(1 + T_1 + T_1T_2).$$

This simplifies to

$$\begin{aligned}\zeta_L(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2) \\ &\quad - p^{-s+1}(1 + p + p^{-s+1})\zeta_p(s-2)\zeta_p(2s-1)\zeta_p(2s-2) \\ &= \frac{(1 + p^{-s+1})(1 - p^{-2s+1}) - p^{-s+1}(1 + p + p^{-s+1})(1 - p^{-s})}{(1 - p^{-s})(1 - p^{-s+2})(1 - p^{-2s+1})(1 - p^{-2s+2})} \\ &= \zeta_p(s)\zeta_p(2s-1)\zeta_p(2s-2),\end{aligned}$$

and it is easy to check the functional equation. The last claim, about the congruence subalgebras of L, follows from [**2**, theorem 2·1].

Now suppose that $p = 2$. Then $\mathcal{L} = \mathfrak{sl}_1(\Delta_2)$ is a subalgebra of index 4 in L. So for every $n \in \mathbb{N}_0$ we have

$$\hat{a}_n(\mathcal{L}) = \hat{a}_{n+2}(L) - \#\{M \leqslant L \mid |L : M| = 2^{n+2}, M \nsubseteq \mathcal{L}\}.$$

The subalgebra growth of L is calculated similarly as above; we obtain

$$\hat{a}_n(L) = \begin{cases} \frac{1}{3}\left(7 \cdot 2^{n+1} - 3 \cdot 2^{(n+4)/2} + 1\right) & \text{if } n \equiv_2 0, \\ \frac{1}{3}\left(5 \cdot 2^{n+1} - 3 \cdot 2^{(n+3)/2} + 1\right) & \text{if } n \equiv_2 1. \end{cases}$$

If $n \in \mathbb{N}_0$ and if M is a subalgebra of index 2^{n+2} in L with $M \nsubseteq \mathcal{L}$, then clearly $s_1(M) = 0$ and $s_3(M) = n - s_2(M) + 2$; moreover Lemma 3·2 provides the restriction $n/2 \leqslant s_2(M) \leqslant (n+2)/2$. Using Lemma 3·2 once again, it is therefore easy to show that for every $n \in \mathbb{N}_0$,

$$\#\{M \leqslant L \mid |L : M| = 2^{n+2}, M \nsubseteq \mathcal{L}\} = \begin{cases} 5 \cdot 2^{n+1} & \text{if } n \equiv_2 0, \\ 3 \cdot 2^{n+1} & \text{if } n \equiv_2 1. \end{cases}$$

From this we obtained the desired formulae for $\hat{a}_n(\mathcal{L})$ and a straightforward computation gives the corresponding zeta function $\zeta_{\mathcal{L}}(s)$. The last claim, about the congruence subalgebras of $\mathcal{L}$, follows again from [**2**, theorem 2·1].

In conclusion we record the following formulae, which describe explicitly the coefficients of the zeta functions associated to the p-power congruence subalgebras of $\mathfrak{sl}_1(\Delta_p)$; they can be computed easily from Theorem 1·1, and the resulting formulae underline our second comment in Remarks 1·2.

Proposition 3·7. *Let $\mathbb{D}_p$ be a central simple $\mathbb{Q}_p$-division algebra of index 2, with maximal $\mathbb{Z}_p$-order Δ_p. Suppose that $p > 2$, and let $m \in \mathbb{N}_0$. Then for every $n \in \mathbb{N}_0$ we have*

$$\hat{a}_n\left(\mathfrak{sl}_1^{2m}(\Delta_p)\right) = \frac{A(m,n)}{(p-1)^2(p+1)},$$

where

$$A(m,n) = \begin{cases} p^{2n+3} - (p+1)p^{n+1} + 1 & \text{if } m \geqslant n, \\ ((p^2+p+1)p^m - p - 1)p^{n+1} & \\ \quad - (p+1)p^{(n+3m+2)/2} + 1 & \text{if } m < n \text{ and } m+n \equiv_2 0, \\ ((2p+1)p^m - p - 1)p^{n+1} & \\ \quad - (p+1)p^{(n+3m+1)/2} + 1 & \text{if } m < n \text{ and } m+n \equiv_2 1. \end{cases}$$

4. *Addendum: zeta functions counting ideals*

Throughout this section let L be a $\mathbb{Z}_p$-Lie algebra of dimension d. Suppose that M is a subalgebra of L. The lower level $\ell_L(M)$ was introduced in the previous section; we now define the *upper level* of M in L,

$$u_L(M) := \sup\{k \in \mathbb{N}_0 \mid M \subseteq p^k L\}.$$

For every $\mathbf{x} \in L$ let $\langle \mathbf{x} \rangle_L$ denote the ideal generated by $\mathbf{x}$ in L. For every $k \in \mathbb{N}_0$ let $\mathcal{I}_k(L) := \{I \trianglelefteq L \mid u_L(I) = k \text{ and } |L{:}\,I| < \infty\}$. We observe:

LEMMA 4·1. *Let L be a $\mathbb{Z}_p$-Lie algebra of dimension d and let $k \in \mathbb{N}_0$. Then there is a bijection $\mathcal{I}_0(L) \to \mathcal{I}_k(L)$, given by $I \mapsto p^k I$, and for every $I \in \mathcal{I}_0(L)$ we have $|L{:}\,p^k I| = p^{kd}|L{:}\,I|$.*

From this almost trivial observation we obtain

$$\zeta_L^{\triangleleft}(s) = \sum_{k=0}^{\infty} \sum_{I \in \mathcal{I}_k(L)} |L{:}\,I|^{-s} = \zeta_p(ds) \sum_{I \in \mathcal{I}_0(L)} |L{:}\,I|^{-s}. \tag{4·1}$$

In general, the sum on the right-hand side of (4·1) may still be rather complicated; but there is a class of $\mathbb{Z}_p$-Lie algebras for which $\mathcal{I}_0(L)$ is finite, so that $\zeta_L^{\triangleleft}(s)$ takes a particularly simple shape. We define the *rigidity* of L,

$$r(L) := \sup\{\ell_L(I) - u_L(I) \mid I \trianglelefteq L \text{ with } |L{:}\,I| < \infty\}.$$

From (4·1) it is easy to observe:

LEMMA 4·2. *Let L be a $\mathbb{Z}_p$-Lie algebra of dimension d and suppose that $r := r(L)$ is finite. Then there exists a polynomial $\Phi \in \mathbb{Z}[T]$ with non-negative coefficients and of degree at most rd such that*

$$\zeta_L^{\triangleleft}(s) = \Phi(p^{-s})\zeta_p(ds).$$

In particular, for every $n \in \{1, 2, \ldots, d\}$ and all $j \in \mathbb{N}_{\geqslant r}$ we have

$$\hat{a}_{n+jd}^{\triangleleft}(L) = \hat{a}_{n+(r-1)d}^{\triangleleft}(L).$$

The next result states under which conditions $r(L)$ is finite, so that Lemma 4·2 becomes applicable; compare with [**2**, proposition 4·1].

PROPOSITION 4·3. *Let L be a $\mathbb{Z}_p$-Lie algebra. Then $r(L)$ is finite if and only if $\mathbb{Q}_p \otimes L$ is a simple p-adic Lie algebra.*

Proof. "$\leftarrow$". Assume that $\mathbb{Q}_p \otimes L$ is a simple p-adic Lie algebra. Let $K := L \setminus pL$. For every $n \in \mathbb{N}_0$ define $U_n := \{\mathbf{x} \in K \mid \ell_L(\langle \mathbf{x} \rangle_L) \leqslant n\}$. Then $\{U_n \mid n \in \mathbb{N}_0\}$ provides an open covering of the compact set K. Hence we find $r \in \mathbb{N}_0$ such that every $\mathbf{x} \in K$ satisfies $\ell_L(\langle \mathbf{x} \rangle_L) \leqslant r$.

Next suppose that I is a finite index ideal of L and write $m := u_L(I) \in \mathbb{N}_0$. Choose $\mathbf{y} \in I$ such that $\mathbf{y} \in p^m L \setminus p^{m+1} L$, and put $\mathbf{x} := p^{-m}\mathbf{y} \in K$. It follows that $\ell_L(\langle \mathbf{y} \rangle_L) = m + \ell_L(\langle \mathbf{x} \rangle_L) \leqslant m + r$, so $\ell_L(I) - u_L(I) \leqslant (m + r) - m = r$. This shows that $r(L) \leqslant r$ is finite.

"$\rightarrow$". Now assume that the p-adic Lie algebra $L^{(e)} := \mathbb{Q}_p \otimes L$ is not simple and let $r \in \mathbb{N}_0$. We have to find a finite index ideal I of L with $\ell_L(I) - u_L(I) > r$. Let $J^{(e)}$ be a non-trivial proper ideal of $L^{(e)}$. Then $J := L \cap J^{(e)}$ is a non-trivial ideal of infinite index in L. Put $m := u_L(J) + r + 1$. Then $I := J + p^m L$ is a finite index ideal of L with $\ell_L(I) - u_L(I) = r + 1 > r$.

Example 4·4. Suppose that $L = \mathbb{Z}_p\mathbf{i} + \mathbb{Z}_p\mathbf{j} + \mathbb{Z}_p\mathbf{k}$ as in the previous section. Then it is easily checked that for every $m \in \mathbb{N}_0$ we have $r(p^m L) = m + 1$, and for $m = 0$ one obtains Proposition 1·5.

In both situations, counting subalgebras and counting ideals, there is a probabilistic formula which relates the zeta functions of L and pL; see [**2**]. Moreover, if the dimension of L is no larger than three, this formula provides a practical procedure for calculating one of these zeta functions from the other. Du Sautoy has used this fact to compute $\zeta^{\triangleleft}_{L}(s)$ for $\mathcal{L} = \mathfrak{sl}_2^m(\mathbb{Z}_p)$, $m \in \mathbb{N}_0$; albeit there appears to be a mistake in [**2**, lemma 4·6]. With some care it is possible to correct those formulae and in a similar way we can calculate $\zeta^{\triangleleft}_{\mathcal{L}}(s)$ for $\mathcal{L} = \mathfrak{sl}_1^{2m}(\Delta_p)$, $m \in \mathbb{N}_0$. Unfortunately, the resulting formulae are not particularly illuminating.

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