Incorporating Encoding into Quantum System Design

Alastair Kay
Royal Holloway University of London, Egham, Surrey, TW20 0EX, UK[*]

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When creating a quantum system whose natural dynamics provide useful computational operations, designers have two key tools at their disposal: the (constrained) choice of both the Hamiltonian and the the initial state of the system (an encoding). Typically, we fix the design, and utilise encodings *post factum* to tolerate experimental imperfections. In this paper, we describe a vital insight that incorporates encoding into the design process, with radical consequences. This transforms the study of perfect state transfer from the unrealistic scenario of specifying the Hamiltonian of an entire system to the far more realistic situation of being given a Hamiltonian over which we had no choice in the design, and designing time control of just two parameters to still achieve perfect transfer.

I. INTRODUCTION

Quantum computers are at the stage of being able to perform some computations much faster than their classical counterparts, possibly even surpassing the requirements of quantum supremacy [1, 2]. Nevertheless, these are very specific instances of algorithms, and we are still far from implementing arbitrary algorithms. That will need to wait until the available resources are significantly increased, and fault-tolerant computation becomes a reality. In the near-term, we instead operate with noisy, intermediate scale devices (the so-called 'NISQ' era [3]). A critical goal, then, is to implement all the elements of a computation as quickly and accurately as possible in order to maximise the quantum advantage of our device before being overcome by the inevitable decoherence. This means working at the fundamental, 'machine code' level, which for many devices means describing the interactions between qubits with a Hamiltonian with time-controlled fields. For any given task, how quickly can it be implemented? How is that run-time affected by how much control we choose to implement? (Roughly speaking, the more control we use, the more potential to introduce error, so we want operations to be as fast as possible, but with as little control as possible.) How hard is it to find the controls that implement our desired operation?

One limit of this scenario is that of no control whatsoever, allowing the natural Hamiltonian dynamics to achieve the desired task. This requires a different, specific Hamiltonian for each task. One well-studied benchmark task in this context is known as perfect state transfer [4–8], where one transfers an unknown quantum state between two remote regions of a quantum computer. Perhaps surprisingly, the results here give a faster transfer than that offered by the gate model of quantum computation [9]. The two-fold speed-up may be attributed to the use of multi-qubit interference instead of localised two-qubit operations. On the other hand, some minimal levels of control enable even faster transfer, saturating the limit imposed by the system's group velocity [10, 11].

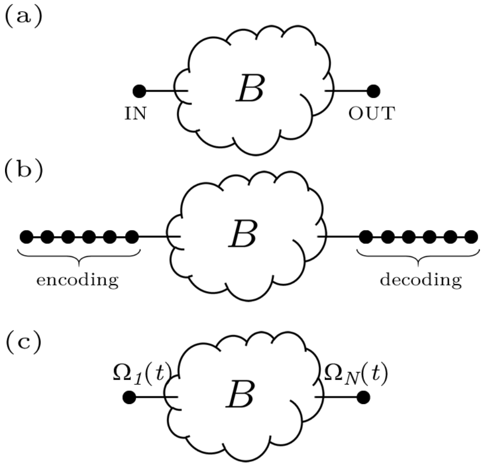

Figure 1. A symmetric system, described by Hamiltonian B is extended by spins chains to fix some of the overall system's eigenvalues (b), permitting a task such as perfect encoded transfer between the two additions. (c) The extensions are simulated by time-varying control of two coupling strengths.

Studies of perfect state transfer have, since their inception, been charged with the major shortcoming of requiring precisely engineered conditions in order to achieve their results, being unable to adapt for manufacturing imperfections etc. High quality transfer [12, 13] requires less variation in the coupling strengths, but still requires precision. This constraint has only been reduced in exchange for an uncertain arrival time [14] or mitigated via the use of an encoding process [10, 15–17].

Our aim is to reject all such impositions, and instead demonstrate how an *existing* system can be altered or controlled to achieve a task such as perfect state transfer. In this paper, we take as a proxy for no engineering requirements the desire for a uniformly coupled system. A subsequent study will generalise the present results to near-universal applicability, the key difference being the need for a highly technical proof of the existence of solutions, which we are side-stepping by considering the uniform case. Thus, we are interested in a perfect quantum state transfer system where the central region is uniformly coupled, but we can choose the couplings at either

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[*] alastair.kay@rhul.ac.uk

end. One extreme, where one or two couplings at the end of the chain are chosen, is already known to give finite quality, but imperfect transfer [12, 13]. The opposite extreme is where all couplings may be chosen [5] and results in perfect transfer. We bridge these two extremes, proving that when the central third of the chain is fixed, the error in the transfer is exponentially small in the chain length. This result is similar in nature to that of [18], but with an exponential improvement in error behaviour. We will show that the same chain achieves perfect transfer if one uses the first and last thirds of the chain as encoding/decoding regions. Our method permits the creation of extensions of arbitrary lengths, and we numerically investigate the performance of these different extensions, and their trade off between transfer time and accuracy.

In Section II, we review the required background of perfect state transfer on spin chains, while also introducing our insight about how to use encoding methods. Section III shows how to symmetrically extend a pre-existing spin chain, fixing some of the eigenvalues of the overall system. These two methods combine to give perfect transfer. We explore the case of a chain, initially of 40 qubits, uniformly coupled, in Section IV. We extend this to a chain of 124 qubits on which perfect transfer can be observed. We consider the same chain from the perspective of end-to-end transfer, proving that high quality transfer results, approaching perfection exponentially quickly in the chain length. Finally, we numerically study the performance of shorter extensions to the original system, and find that they can also be extremely effective. In Section V A, we extend the results beyond those of state transfer to the creation of useful, entangled, states.

We also apply a result of Haselgrove's [15], as depicted in Fig. 1, which immediately demonstrates how to replace the extensions with time control of a single coupling strength at either end of the chain. Employing these results yields perfect communication through an imperfectly prepared system just by modifying the end couplings, and essentially maps to a constructive, analytic method of time control for perfect communication between two pendant vertices on a network, similar to the study of [19], studied from a control-theory perspective. The advantage of choosing the special case of a uniform chain to extend is that we have a good basis for comparison to results such as [10, 20].

II. SPIN CHAINS

Consider a system Hamiltonian

$$H=\tfrac{1}{2}\sum_{n=1}^{N}B_nZ_n+\tfrac{1}{2}\sum_{n=1}^{N-1}J_n(X_nX_{n+1}+Y_nY_{n+1}),\quad (1)$$

in which Z_n specifies a Pauli-Z matrix applied to qubit n, and $\mathbb{1}$ elsewhere. This describes a coupled chain of length N with tunable coupling strengths J_n and magnetic fields B_n. We will generally choose to set $B_n = 0$ for simplicity. We group the N qubits into three distinct sets, Λ_{in}, Λ_{bulk} and Λ_{out}. Couplings between two qubits that are both in Λ_{bulk} are assumed to be equal, and taken to be 1 without loss of generality, while we retain the ability to choose all the others. We also take $\hbar = 1$ so that all energies, transfer times etc. are dimensionless.

The Hamiltonian H decomposes into a set of subspaces characterised by the number of $|1\rangle$s in the basis elements. We focus primarily on the single-excitation subspace, spanned by states

$$|n\rangle=|0\rangle^{\otimes(n-1)}|1\rangle|0\rangle^{\otimes(N-n)}.$$

Within this subspace, we introduce the projectors onto the three different regions. For example,

$$\Pi_{\text{in}}=\sum_{n\in\Lambda_{\text{in}}}|n\rangle\langle n|.$$

Perfect encoded state transfer identifies a single-excitation state $|\Psi_{\text{in}}\rangle$ localised to the input region ($\Pi_{\text{in}}|\Psi_{\text{in}}\rangle=|\Psi_{\text{in}}\rangle$) which evolves in the transfer time t_0 to $|\Psi_{\text{out}}\rangle=e^{-iHt_0}|\Psi_{\text{in}}\rangle$ where $\Pi_{\text{out}}|\Psi_{\text{out}}\rangle=|\Psi_{\text{out}}\rangle$. This gives perfect transfer of a quantum state because an arbitrary superposition $\alpha|0\rangle^{\otimes N}+\beta|\Psi_{\text{in}}\rangle$ can be created on the input region, evolving to the output state $\alpha|0\rangle^{\otimes N}+\beta|\Psi_{\text{out}}\rangle$.

Our primary goal is to discover how to choose the J_n on input and output regions such that we achieve perfect encoded state transfer. To assess the quality of transfer, we evaluate σ, the maximum singular value of $\Pi_{\text{out}}e^{-iHt_0}\Pi_{\text{in}}$, and define the fidelity to be $F=\sigma^2$ [15] or transfer error $\varepsilon=1-\sigma^2$ [21]. The left- and right-singular vectors in this case correspond to the states $|\Psi_{\text{out}}\rangle$ and $|\Psi_{\text{in}}\rangle$ repectively. End-to-end transfer is a special case with $\Pi_{\text{in}}=|1\rangle\langle 1|$ and $\Pi_{\text{out}}=|N\rangle\langle N|$ [22].

Throughout this paper, we will assume symmetry: $J_n=J_{N-n}$. The reason for this is that the necessary and sufficient conditions for perfect state transfer between opposite ends of a chain are well known [7], and include the requirement of this symmetry. While the symmetry is not necessary when one moves away from the end of the chain, it vastly reduces further requirements to a simple spectral condition. If we define the symmetry operator

$$S=\sum_{i=1}^{N}|N+1-i\rangle\langle i|,$$

then $SHS=H$ and hence, if transfer is perfect, $|\Psi_{\text{out}}\rangle\sim S|\Psi_{\text{in}}\rangle$ up to a phase on each site. A subscript S_A specifies the symmetry operator for a specific matrix A.

A. Making use of Encoding

Let the eigenvalues and eigenvectors of H be λ_n and $|\lambda_n\rangle$ respectively, in decreasing order, $\lambda_n>\lambda_{n+1}$. In a symmetric system, the Hamiltonian decomposes into two subspaces $H = H_+ \oplus H_-$, with each eigenvalue being associated with one of these. Indeed, for a chain, $\lambda_{2n-1} \in \text{spec}(H_+)$ and $\lambda_{2n} \in \text{spec}(H_-)$ for all n. In such a symmetric system, the perfect transfer conditions using an encoded state $|\Psi_{\text{in}}\rangle$ at time t_0 are readily stated [5]:

$$\exists \phi : e^{-i\lambda_n t_0} = \pm e^{i\phi} \qquad \forall \lambda_n \in \text{spec}(H_\pm) : \langle \lambda_n | \Psi_{\text{in}} \rangle \neq 0$$

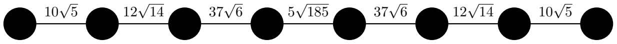
Up to an arbitrary scale factor and offset, the spectrum for a perfect state transfer system is a set of integers where the even (odd) integers are assigned to H_+ (H_-) and the perfect transfer time is π. In the field-free case of $B_n = 0$, ϕ is an integer multiple of $\frac{\pi}{2}$.

For end-to-end transfer, where $|\Psi_{\text{in}}\rangle = |1\rangle$, then $\langle \lambda_n | \Psi_{\text{in}} \rangle$ is non-zero for *all* eigenvectors, and hence every eigenvalue must satisfy the integral condition. Transfer between nodes in the bulk of a chain has, to date, defied such easy description because it is possible for a given $\langle \lambda_n | m \rangle$ to be 0, meaning that the corresponding eigenvalue need not fulfil the spectral conditions. However, we will now use this to our advantage. Imagine that we have a fixed Hamiltonian H whose eigenvalues we know. These may be categorised as the set Γ_P which satisfy the perfect transfer conditions at time t_0, and $\Gamma_{\bar{P}}$, the imperfect ones which do not satisfy the perfect transfer conditions. If we can select an encoding $|\Psi_{\text{in}}\rangle$ such that for all $n \in \Gamma_{\bar{P}}$, $\langle \lambda_n | \Psi_{\text{in}} \rangle = 0$, we have perfect encoded state transfer. Our task is straightforward: find any state supported on Π_{in} that is in the null space of

$$\{\Pi_{\text{in}} |\lambda_n\rangle\}_{n \in \Gamma_{\bar{P}}} .$$

The existence of such a state is guaranteed provided the size of the encoding region is larger than $|\Gamma_{\bar{P}}|$.

Example 1 *Consider the following chain:*



where each circle is a qubit, a number over an edge is a coupling strength J between the specified pair of qubits, and a number over a qubit is a field strength B_n on that qubit. This system has eigenvalues $\sqrt{185} \times \{\pm 1, \pm 2, \pm 6, \pm 10\}$. We make the assignment

$$\Gamma_P = \sqrt{185} \times \{\pm 2, \pm 6, \pm 10\}$$
$$\Gamma_{\bar{P}} = \{\pm \sqrt{185}\}.$$

Using only the values in Γ_P, we have a perfect transfer time $t_0 = \pi/(4\sqrt{185})$ since this gives the set of values

$$\lambda_n t_0 = \frac{-5\pi}{2}, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2},$$

which have gaps of π between them.

So, if we choose to take an encoding region of size $|\Gamma_{\bar{P}}| + 1 = 3$, and evaluate the two eigenvectors of $\Gamma_{\bar{P}}$ restricted to the first 3 sites, we have

$$|1\rangle \mp \frac{\sqrt{37}}{10} |2\rangle - \frac{3}{8}\sqrt{\frac{7}{10}} |3\rangle .$$

There is a state

$$3\sqrt{7} |1\rangle + 8\sqrt{10} |3\rangle$$

that is orthogonal to both of these. In the time $t_0 = \pi/(4\sqrt{185})$, this transfers to

$$-i(3\sqrt{7} |8\rangle + 8\sqrt{10} |6\rangle),$$

which is just on the decoding region (also size 3). We have perfect transfer of a single encoded excitation, and hence perfect transfer of a single encoded qubit.

By interpreting the use of the encoding/decoding regions in this way, we have the opportunity to incorporate encoding into our analytic strategies for the first time, rather than just adding it in subsequently. If the null space is of dimension k, one can encode k qubits, and they will all be perfectly received (up to a suitable decoding sequence upon arrival) [7]. Moving beyond the perfect transfer regime, a good use of the encoding strategy (although not necessarily optimal) is to find the eigenvalues that are the 'worst offenders' (e.g. from the un-encoded case of just using the input $|1\rangle$) and set them to 0.

III. CHAIN EXTENSIONS

The application of this encoding strategy is now clear — while perfect state transfer requires engineering a system such that every eigenvalue satisfies a precise condition, we can forgo fixing M eigenvalues in exchange for encoding/decoding regions of size $M + 1$. We are now tasked with solving this problem: for a fixed central region, how do we symmetrically extend that chain such that is has certain eigenvalues of our choosing?

Our strategy is inspired by [23], which showed how to create a one-sided extension of a chain, fixing some of the eigenvalues. In the single excitation subspace, H can be written in the form

$$H = \left(\begin{array}{c|c|c} S_A A S_A & J & \\ \hline J & B & J \\ \hline & J & A \end{array} \right)$$

where $B = S_B B S_B$ is the fixed central region (i.e. a uniformly coupled chain), and A is the output region that we will be able to select. The symmetry structure of H becomes

$$H_\pm = \left(\begin{array}{c|c} S_A A S_A & J \\ \hline J & B_\pm \end{array} \right).$$

Requiring a target eigenvalue λ for H with symmetry $\sigma \in \pm$ imposes that

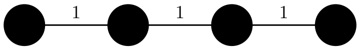
$$\det(H_\sigma - \lambda \mathbb{1}) = 0.$$

This can be expressed in terms of polynomials such as $Q_A(x)$, the characteristic polynomial of A, and $P_B^\sigma(x)$, the characteristic polynomial of the principal submatrix of B_σ, i.e. B_σ with its first row and column removed.

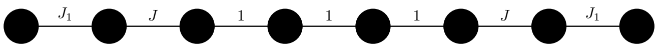
$$Q_A(\lambda) Q_B^\sigma(\lambda) = J^2 P_A(\lambda) P_B^\sigma(\lambda). \tag{2}$$

If A comprises a set of M sites, then Q_A and P_A are monic polynomials of degree M and $M-1$ respectively. We don't know the coefficients of the polynomials, but there are $2M-1$ of them, and each instance of a λ in Eq. (2) is just a linear equation for those coefficients. Given $2M-1$ target eigenvalues, if a solution exists, $P_A(x)$ and $Q_A(x)$ are uniquely determined. Moreover, if we know $P_A(x)$ and $Q_A(x)$, we can uniquely reconstruct A [24] (up to signs in the coupling strengths). In this argument, we have assumed that J is known. Little change is required if J is unknown, we just need one more parameter as, effectively, P_A is no longer monic.

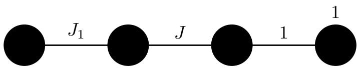
Example 2 *Starting from chain of 4 qubits,*



we demand a symmetric extension such that eigenvalues of the overall system include ±1 *and* ±2*, the positive values being associated with the symmetric subspace.*



The symmetric subspace is also equivalent to a chain, but with a non-zero field on the final spin:



We impose that the symmetric subspace should contain the eigenvalues $1, 2$. *Dividing this into two sections*

$$A = \begin{pmatrix} 0 & J_1 \\ J_1 & 0 \end{pmatrix}, \qquad B_+ = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

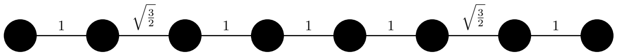
then we can readily evaluate

$$\frac{Q_B^+(x)}{P_B^+(x)} = \frac{x(x-1)-1}{x-1}$$

which must equal

$$\frac{J^2 P_A(x)}{Q_A(x)} = \frac{J^2 x}{x^2 - J_1^2}$$

at $x = 1, 2$. *In this case, we directly solve the two simultaneous equations to find* $J_1^2 = 1$ *and* $J^2 = \frac{3}{2}$. *We thus see that the chain*



has the desired eigenvalues.

In this section, we have shown how to take a fixed central region contains M_B qubits, extending it to have $N = 2M + M_B$ qubits, with the ability to fix $2M$ eigenvalues (in the J unknown case). Once we incorporate our conclusions of Section II A about the use of encoding, perfect encoded transfer is possible provided $M > M_B$.

IV. EXAMPLES

For the purposes of numerical examples, it is convenient to use the field-free case, i.e. where the matrices A and B have 0 on the diagonal, halving the number of parameters we have to work with. We shall assume that B comprises an even number of qubits, such that the chain as a whole has an even number of qubits. As a result, both B and H will have all eigenvalues occurring in $\pm\lambda$ pairs. Note that $|\lambda\rangle$ and $|-\lambda\rangle$ have opposite symmetries. Instead of solving Eq. (2) for both, we can build this feature into the structure of the polynomials that we're solving for, specifically that $Q_A(x)$ comprises only even (odd) powers if M is even (odd), while $P_A(x)$ is the opposite.

Solving the linear equations (2) directly is challenging as the structures involved closely resemble Vandermonde matrices, including terms such as λ^M, which rapidly lead to numerical instabilities. Instead, we recognise that the problem is that of finding a rational function of specific degrees which fits known values at specific points. There are several existing techniques for solving this such as Thiele's continued fraction routine [25]. In all our numerical tests, we have used Algorithm 1 presented in [26] (see also [27]) as a particularly efficient algorithm whose iterative structure will be familiar to those who work with tridiagonal matrices or orthogonal polynomials. However, this only works in the unknown J case. It is not correctly set up to be able to trade the loss of one parameter (a target eigenvalue) for the gain of another (the known J), as it chooses to reduce the degree of P by 1 instead of causing it to be monic.

To incorporate the field-free assumption into the rational interpolation algorithm [26], we need the function $f(x) = \frac{P(x)}{Q(x)}$ to be anti-symmetric in the case of A being of even length. If we have positive points (target eigenvalues) x_i for which the rational function must have values f_i (and hence also values $-x_i$ such that $f(-x_i) = -f_i$), then instead we attempt to find a rational function $g(x)$ which satisfies $\{g(x_i^2) = f_i/x_i\}$. This means that we will have determined

$$g(x) = \frac{p(x)}{q(x)},$$

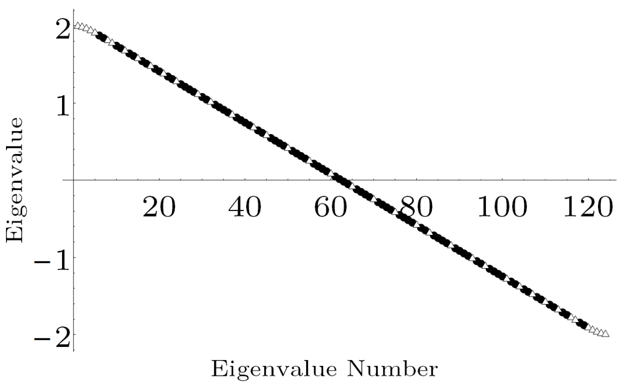


Figure 2. Extending a uniformly coupled chain of 40 qubits to 124 qubits, achieving perfect encoded transfer. The eigenvalues of H that were fixed to satisfy the PST conditions (circles) and not fixed (triangles).

from which we can construct

$$f(x)=\frac{xp(x^2)}{q(x^2)}=\frac{P(x)}{Q(x)}. \tag{3}$$

It is straightforward to verify the required relations

$$\begin{aligned} f(x_i)&=x_i g(x_i^2)=f_i\\ f(-x_i)&=-x_i g(x_i^2)=-f_i. \end{aligned}$$

A. Perfect Encoded Transfer

Consider the example where the central region comprises 40 qubits, uniformly coupled, strength 1. We will introduce M qubits (even) at either end of the chain, and fix a value δ, corresponding to a state transfer time of $t_0=\pi/\delta$. We choose the couplings of the extension such that the overall system has eigenvalues

$$\delta\left\{M-2k-\tfrac{1}{2}\right\}_{k=0}^{M-1} \tag{4}$$

in the symmetric subspace. As we aim for a field-free system, there will also be eigenvalues $\delta\left\{M-2k-\frac{3}{2}\right\}$ in the antisymmetric subspace. Setting $M=42$ is sufficient to guarantee perfect transfer, assuming a solution exists. While we have no guarantee about the existence of a solution (which is beyond the scope of this paper), this particular model is quite forgiving. Recall that the spectrum for a uniformly coupled chain of N qubits is

$$2\cos\left(\frac{\pi n}{N+1}\right).$$

In the central region $n\sim\frac{N+1}{2}$, the spectrum is near-linear. Thus, imposing that it should be exactly linear in its central region is not a big deviation. One can choose a range of gradients δ for the linear section, from which two stand out:

- A uniformly coupled system has gradient $\frac{2\pi}{N+1}$ in the central region of its spectrum, and this has the fastest possible group velocity (i.e. transfer speed) of any system with maximum coupling strength $J_{\max}=1$. Given that we are fixing about 2/3 of the eigenvalues, which is much more than the typical linear region, it seems unlikely that we will be able to match this gradient.
- The fastest perfect transfer system [5] has gradient $\frac{4}{N}$ throughout its spectrum. It should be possible that we can match this gradient. Selecting a gradient at this extreme may suggest the possibility of getting even the eigenvalues that we don't choose close to the linear pattern as well. Indeed, this was particularly striking when we chose not to just focus on the central region. See Fig. 2 for an example.

Figure 3(a) depicts our chosen example where we have focussed on fixing the eigenvalues in the central region, as specified by Eq. (4). This achieves perfect encoded transfer in time 94.5 (as compared to the PST gradient giving transfer time ∼ 99). However, by using the optimal encoding [15], we see in Fig. 3(b) that extremely high fidelity transfer is achieved in a much shorter time, essentially coinciding with the first arrival peak of a wavepacket travelling at the maximum group velocity of the system, i.e. as fast as transfer could possibly occur. We will comment further on why this should be the case in Section IV C.

We should compare this solution to the best known previous solution, in which one simply extends the uniform chain with another uniform chain, and uses the optimal encoding, which is inspired by creating wavepackets that travel through the system at the group velocity [10, 11]. Visual inspection reveals that the solutions are comparable. For the uniform chain, however, transfer is never perfect, and as soon as the central region is not uniformly coupled, we don't know how to proceed. Nevertheless, this comparison suggests that it might be interesting to reduce the size of the controlled regions. In Fig. 4, we see that even modestly sized extensions, supplemented by encoding, are extremely effective in improving transfer fidelity [28], indeed far more effective than uniform extensions.

B. Imperfect State Transfer Families without Encoding

Figure 3 shows one further striking feature. It plots the weight of each eigenvector on the first site, and this is strongly weighted in the central region where the eigenvalues have been tuned to have the linear relation required of perfect transfer. Thus high quality transfer will result *without any encoding.*

To study this in greater detail, let us assume that a family of solutions of the form described in Fig. 3

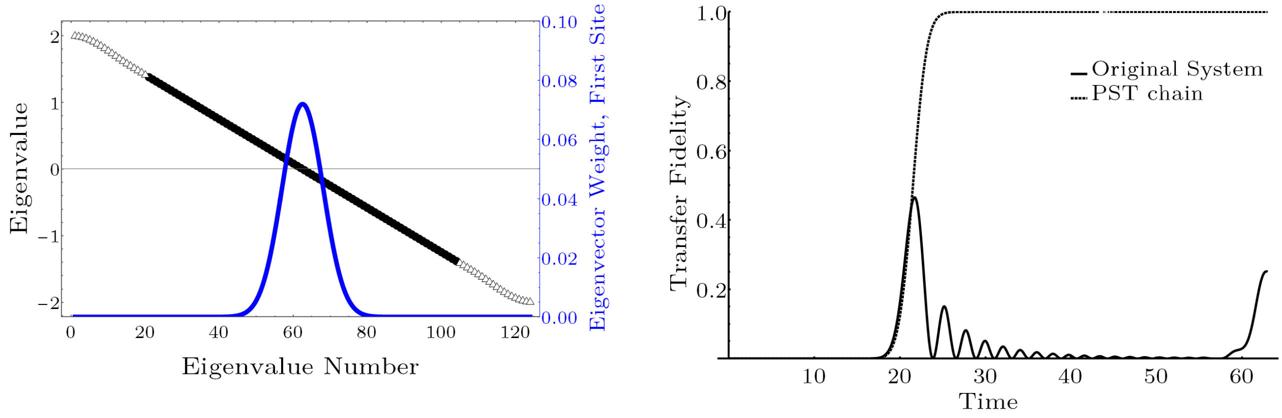


Figure 3. (a) Eigenvalues of a chain extended from 40 qubits to give PST. The central eigenvalues satisfy the linear condition (circles), with the others (triangles) being uncontrolled. The continuous (blue) line depicts the weight of each eigenvector on the first site of the chain. (b) Plot of the encoded state transfer fidelity of the extended perfect transfer chain (dashed) compared to original chain (continuous). Transfer using a 142 qubit uniformly coupled chain with encoding/decoding is indistinguishable from the dashed case.

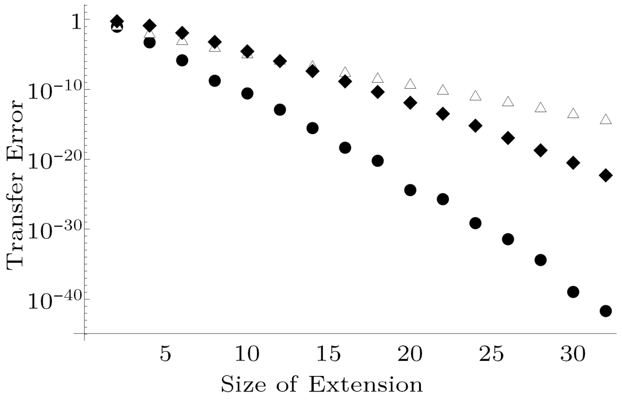


Figure 4. For a central region of 40 qubits, we find symmetric extensions and assess the transfer error at $t = \pi/\delta$ where δ is fixed. We compare end-to-end transfer (triangles) and optimal encoding over the entire encoding/decoding region (circles). We also include the time-optimised transfer error of a uniformly coupled chain of the same length (diamonds) using encoding over the entire encoding/decoding region.

and Eq. (4) exist. Let $|\psi\rangle = \sum_n a_n |\lambda_n\rangle$ be our encoding, giving a decoding of $|\phi\rangle = \sum_n a_n(-1)^{n+1} |\lambda_n\rangle$. Now let $\Gamma_{\bar{P}}$ be the set of indices n for which the λ_n do not satisfy the perfect state transfer conditions. At worst case, the transfer fidelity would be

$$F_{\text{min}} = 1 - 2 \sum_{n\in\Gamma_{\bar{P}}} |a_n|^2.$$

In Fig. 3, the spectrum is very close to linear throughout its range (and exactly linear in the central region). The energy gap must be approximately $\delta = 4J/(N-1)$, giving a transfer time of π/δ. Since the spectrum entirely determines the values a_n on a symmetric chain, this conveys that the a_n will be extremely close to those of the perfect transfer chain [5, 29], so we can take the analytic solutions for those eigenvectors as excellent approximations. Thus, for a chain of total length N,

$$|a_n|^2 = \frac{1}{2^{N-1}} \binom{N-1}{n-1}.$$

We take the large N limit, so the summation for F_{min} becomes an integral. With $\Gamma_{\bar{P}}$ being all those eigenvalues apart from the central $2N/3$ of them, we get an error of

$$1 - F^{\text{PST}}_{\text{min}} = \frac{8}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{N}/6} e^{-2\theta^2} d\theta < \frac{12}{\sqrt{2N\pi}} e^{-N/18}.$$

This yields asymptotically perfect transfer between opposite ends of the chain. This is an exponential improvement in approach compared to [18]. It might be considered to be taking the studies of [12, 13] to their ultimate limit, demonstrating how many couplings it is sufficient to fix in order to get asymptotically perfect transfer with a uniformly coupled central region and end-to-end transfer, not just a fidelity over some finite threshold.

From our proof of end-to-end transfer, it is also clear that for any $(N - |\Gamma_{\bar{P}}|)/2$ that grows faster than $\sim \sqrt{N}$, the integral will also vanish. So perhaps we only need to fix $O(\sqrt{N})$ eigenvalues? This is a different proposition as, with control over fewer eigenvalues, it is less likely that we approximate the linear spectrum for its full range. The spectrum will be much closer to that of the uniform chain, more of whose eigenvectors have non-negligible support on the first site. One might expect to compensate with encoding. Fig. 4 shows how the transfer fidelity varies with the length of the extension, with the error dropping exponentially. For example if we have only extended by 8 spins, rather than the ~ 40 required to achieve perfect transfer, we still achieve a transfer with error approximately 2×10^{-9}, while the uniformly extended chain only achieves an error of 10^{-4}.

C. Time for Encoded Transfer

The fact that these chains permit extremely high quality end-to-end transfer, being closely related to a perfect state transfer chain, yields some insight about the speed of the high accuracy encoded state transfer, observed in Fig. 3(b). Consider a perfect state transfer chain of length N such as in [5], with a state transfer time t_0 (which grows linearly in N if we scale the system such that its maximum coupling strength is 1). Starting localised at the first site, the motion is essentially the ballistic motion of a wavepacket, centred on the position

$$(N-1)\sin^2\left(\frac{\pi t}{2t_0}\right)+1$$

with a spread

$$\sigma=\frac{\sqrt{N-1}}{2}\sin\left(\frac{\pi t}{t_0}\right).$$

In fact, the distribution of probabilities per site is exactly that of an $(N-1)$-sample Bernoulli distribution with $p(t)=\sin^2\left(\frac{\pi t}{2t_0}\right)$. Thus, the wavepacket is almost entirely restricted to the encoding region until a time t_{in} (such that $p(t_{\text{in}})\sim\frac{1}{3}$) and almost entirely restricted to the decoding region after a time t_0-t_{in}. When the wavepacket is in those regions, we can recreate it with encoding and decoding. The resulting error for encoding may be bounded by a Chernoff bound as

$$\varepsilon\leq e^{-(N-1)p(3p-1)^2/(21-9p)}$$

with a symmetric equivalent for the decoding error. Hence, we only need a transfer time of approximately $0.22t_0+O(1/\sqrt{N})$ with an error that is exponentially small in N. This time, $0.17N$, compares extremely well with the limit of $N/6$ imposed by the maximum group velocity being 2 [30]. While this strategy may not be optimal, it provides a lower bound for the performance. We should note however that the spread of the wavepacket, $O(N^{1/2})$, is broader than the optimal wavepacket for the equivalent uniformly extended chain, $O(N^{1/3})$ [11].

V. SIMULATING EXTENSIONS WITH TIME CONTROL

The work of [15] contains two useful strategies. We have made extensive use of one here — the encoding/decoding of the state, giving it a new interpretation for how it can be used to achieve perfect transfer. The second, how certain sections of a spin system can be replaced by time control, is just as useful. It can be used directly, without alteration. Instead of adding many qubits to the initial fixed system, we just control (varying in time) two of the coupling strengths. This is depicted conceptually in Fig. 1. Transfer at fidelity F between encoding and decoding regions translates directly to transfer between the two extremal sites in the virtualised system at fidelity F.

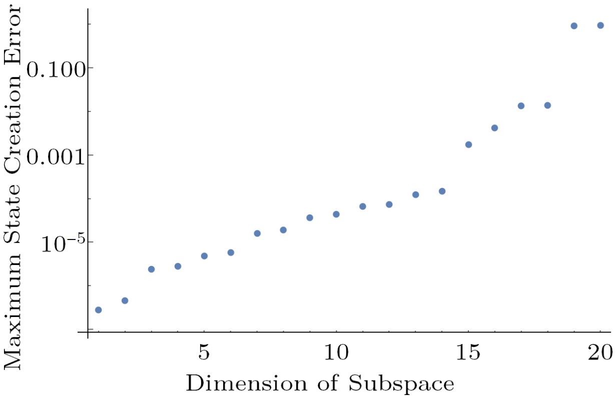


Figure 5. For a 40 qubit chain as specified in Fig. 2, states can be created on the first 20 qubits of the bulk with a maximum error for a given dimension of space.

A. Creation of States

In fact, the ability of the virtualisation technique of [15] to simulate a perfect transfer chain in which every site perfectly transfers to its mirror site in the perfect transfer time t_0 has some extremely powerful consequences.

Let's say, for instance, that we're given a uniformly coupled chain of M_B qubits. Before we append engineered chains for the purpose of tuning the spectrum, we will append a further M_B uniformly coupled qubits. We will refer to these as the "mirror system". Then we add the extra chains at both ends to tune the spectrum. The longer the chains, the more accurate the protocol that we'll realise, at the cost of longer time. Once we've solved for that system, we will virtualise everything that we've added. Note that the virtualisation procedure is state dependent: it depends on the system's initial state. It is this initial state that we will now utilise. In particular, if we create *any* single excitation state that we like in the mirror system, then in the perfect state transfer time, it arrives perfectly on the original system. Translated into the virtualised system, you start with a single excitation on one of the two extremal spins, and the time control determines any single-excitation output state that you desire on the bulk system!

Of course, our strategy will only ever approximate a perfect transfer chain. To that end, we take the state $|\Psi\rangle$ that we want to create, and run the Hamiltonian evolution backwards to find the best possible starting state. If Π_{out} is the projector onto the output region, including also the mirror system, then up to normalisation, the best possible starting state is

$$|\Psi_{\text{in}}\rangle=\Pi_{\text{out}}e^{iHt_0}|\Psi\rangle,$$

and the transfer fidelity is

$$\langle\Psi| e^{-iHt_0}\Pi_{\text{out}}e^{iHt_0} |\Psi\rangle .$$

In order to understand the efficacy of our system, let us calculate the eigenvalues

$$\Pi_{\text{bulk}}e^{-iHt_0}\Pi_{\text{out}}e^{iHt_0}\Pi_{\text{bulk}}.$$

For the chain specified by Fig. 2, these are plotted in Fig. 5. We see that there's a large space from which states can be created with high fidelity.

VI. CONCLUSIONS & FUTURE WORK

We have shown how a uniformly coupled chain can be symmetrically extended by M qubits on either side, fixing $2M-1$ (or $2M$) of the eigenvalues to those that we specify. By also implementing an encoding/decoding procedure over the M qubits at either end, we can avoid populating up to $M-1$ eigenvectors whose eigenvalues do not satisfy the perfect transfer condition. We can thus create a perfect encoded transfer chain where the central third is fixed to being uniformly coupled. Operating close to the speed limit of the system yields a transfer whose error is exponentially small in the chain length. Moreover, thanks to [15], all the additional spins can be 'virtualised', i.e. replaced simply by time control of a single coupling strength at either end of the uniform chain. We have demonstrated numerically that even with shorter extensions, extremely high fidelity transfer can be achieved. Equally, if one wants to dispense with encoding, high quality transfer is possible, with an error that decreases exponentially in the chain length. A small modification of the protocol allows for the creation of a wide range of single-excitation states. The algorithms for computing the extensions, and corresponding time control in the virtualisation, are extremely efficient.

None of the formalism developed here is limited to the initial system being a uniformly coupled chain. *Any* chain is equally amenable. However, the challenge is ensuring that solutions to the set of linear equations (2) exist. That, and the consequences for transfer speed, are topics for a future paper. Indeed, aside from addressing a few minor technicalities, there is no reason why we need to restrict to the pre-specified region being a chain. Any coupling topology should be possible, such as those describing the interactions between qubits in an IBMQ device. These use precisely a Hamiltonian of the form Eq. (1). Moreover, since we are already using encodings, these encodings can be optimally updated to incorporate knowledge of the system noise [16]. As such, this methodology heralds a new era for quantum state transfer, and related studies, in which we can adapt to a provided system rather than having to request specific properties.

That said, there remain limitations. The most obvious ones are that (i) systems such as IBMQ do not directly provide access to time control of coupling strengths, only the local magnetic fields, (ii) if the extended system is a chain, multiple excitations behave well [7]. However, if the virtualisation procedure of [15] is used, the single excitation subspace no longer provides a good description of the behaviour in higher excitation subspaces, and (iii) we don't yet know how to incorporate the treatment of noise such as [16] into the virtualisation procedure. These are issues that we hope may be addressed in the future.

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