ACTA ARITHMETICA
XCIII.4 (2000)

Families of Pisot numbers with negative trace

by

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1. Introduction. A *Pisot number* is a real algebraic integer, θ, such that $\theta > 1$ and all conjugates of θ (other than θ itself) have modulus less than 1. The set of all Pisot numbers is usually denoted S (after Salem).

Suppose that r, k, $a_1, \ldots, a_{r-k}$ are all integers with $r \geq 2$, $0 \leq k \leq r$, $a_i \geq 2$ $(1 \leq i \leq r-k)$. If $r = 2$ and $k = 0$, then we exclude $a_1 = a_2 = 2$. Then it was shown in [1] that the only roots of the equation

$$\sum_{i=1}^{r-k} \frac{z^{a_i-1}-1}{z^{a_i}-1} + \frac{k}{z} = 1 \tag{1}$$

are a certain Pisot number $\theta_{r,k}(a_1, \ldots, a_{r-k})$, say, and its conjugates. Let U be the set of all such Pisot numbers (T being used for Salem numbers!). Then (see [1])

- U is a proper subset of S;
- trace : $U \to \mathbb{Z}$ is surjective.

In particular, there exist Pisot numbers of negative trace.

Indeed a construction was given in [1] which could produce Pisot numbers of any desired trace. Unfortunately, to produce negative trace the construction required that the degree of the Pisot number should be huge. An example was given (not claimed to be best-possible!) with trace -5 and degree 141 731 565 070 951.

In this paper it is shown how to construct Pisot numbers with negative trace and much smaller degree: the current record is 23 837. This cannot be too far from minimal for elements of U, in that a key result of this paper is that for minimality we may assume that each a_i is a product of at least four distinct prime factors. It is hoped that a second, more computational, paper will establish several other extremal results. The ultimate goal is to find the smallest degree of any element of S with negative trace, and finding

2000 *Mathematics Subject Classification*: Primary 11R06.

the minimal degree for elements of U with negative trace would be a step along the way.

2. Formulas for the degree and trace. To compute the minimal polynomial of $\theta_{r,k}(a_1,\ldots,a_{r-k})$ we need to clear denominators in (1). If we multiply (1) by the dth cyclotomic polynomial for every $d>1$ dividing one of the a_i, and multiply by z if $k>0$, then the denominators will have been cleared. It was shown in [1], by computing residues at all relevant dth roots of 1, that nothing less will do. Hence we can read off formulas for the degree and trace:

$$\text{degree}(\theta_{r,k}) = \sum_{1<d\mid a_i \text{ for some } i} \varphi(d) + \varepsilon, \tag{2}$$

where

$$\varepsilon = \begin{cases} 0 & \text{if } k=0, \\ 1 & \text{if } k>0 \end{cases}$$

and

$$\text{trace}(\theta_{r,k}) = r + \sum_{1<d\mid a_i \text{ for some } i} \mu(d)\,. \tag{3}$$

Here φ and μ are Euler's totient function and the Möbius function respectively.

From (2) and (3) we see immediately that in seeking negative trace with minimal degree we must have $k=0$. From now on we restrict to $k=0$, and write $\theta_r(a_1,\ldots,a_r)$ for $\theta_{r,0}(a_1,\ldots,a_r)$. We shall now always have $\varepsilon=0$ in (2).

Applying inclusion-exclusion to (2) gives an alternative degree formula, which may be easier to use for computations:

$$\begin{aligned}\text{degree}(\theta_r) = {} & -1 + \sum_i a_i - \sum_{i<j} \gcd(a_i,a_j) \\ & + \sum_{i<j<k} \gcd(a_i,a_j,a_k) - \ldots\end{aligned} \tag{4}$$

Similarly we get a second formula for the trace:

$$\text{trace}(\theta_r) = \sum_{S\subseteq\{a_1,\ldots,a_r\},\,|S|\ge 2,\,\gcd(S)>1} (-1)^{|S|}, \tag{5}$$

where $\gcd(\{x,y,z,\ldots\})$ means $\gcd(x,y,z,\ldots)$.

Suppose that there are n distinct primes dividing $a_1,\ldots,a_r$. Then (3) may be written

$$\text{trace}(\theta_r) = r - n + \sum_{d\mid a_i \text{ for some } i,\,\omega(d)\ge 2} \mu(d), \tag{6}$$

where $\omega(d)$ is the number of distinct prime factors of d.

We glean two obvious minimality conditions from these formulas. Suppose that $\theta_r(a_1,\dots,a_r)$ has minimal degree amongst elements of U with negative trace. Then from (3) and (2) we see that

$$\text{Each } a_i \text{ is squarefree.} \tag{7}$$

Moreover (3) shows that the trace depends only on the pattern of the primes dividing $a_1,\dots,a_r$, not on the primes themselves. Hence from (2) we find that for minimality

$$\begin{aligned}&\text{If } p_1,\dots,p_n \text{ are the distinct primes dividing } a_1,\dots,a_r,\\ &\text{then } p_1,\dots,p_n \text{ are the first } n \text{ primes in some order.}\end{aligned} \tag{8}$$

In what follows, we shall suppose that (7) and (8) always hold.

3. Dual Pisot numbers. Suppose that $\theta_r(a_1,\dots,a_r) \in U$, with $\theta_r(a_1,\dots,a_r)$ satisfying (7) and (8). Thus $p_1,\dots,p_n$ are the first n primes in some order. Let $q_1,\dots,q_r$ be the first r primes. We define the *dual* of $\theta_r(a_1,\dots,a_r)$ to be $\theta_n(b_1,\dots,b_n)$, where b_j $(1 \le j \le n)$ is the product of those q_i for which a_i is divisible by p_j. Note that the dual Pisot number does not depend on the ordering of $p_1,\dots,p_n$: changing the order merely has the affect of permuting $b_1,\dots,b_n$, which does not change $\theta_n(b_1,\dots,b_n)$.

For example, if $r = 3$, $n = 4$, then the dual of $\theta_3(p_1p_2p_3, p_1p_2, p_3p_4)$ is $\theta_4(q_1q_2, q_1q_2, q_1q_3, q_3)$. (We have $b_1 = q_1q_2$ since p_1 divides a_1 and a_2, etc. If we chose to swap the labels p_3 and p_4, then the dual would be $\theta_4(q_1q_2, q_1q_2, q_3, q_1q_3) = \theta_4(q_1q_2, q_1q_2, q_1q_3, q_3)$.)

The dual of the dual of $\theta_r(a_1,\dots,a_r)$ is just $\theta_r(a_1,\dots,a_r)$.

Under duality, a subset S of $\{a_1,\dots,a_r\}$ for which $\gcd(S) > 1$ corresponds to a divisor $d > 1$ of one of the b_j, and $(-1)^{|S|} = (-1)^{\omega(d)} = \mu(d)$. Hence

$$\sum_{S\subseteq\{a_1,\dots,a_r\},\,|S|\ge 2,\,\gcd(S)>1} (-1)^{|S|} = \sum_{d\mid b_j \text{ for some } j,\,\omega(d)\ge 2} \mu(d).$$

Comparing (5) and (6) we have

$$\operatorname{trace}(\theta_n(b_1,\dots,b_n)) = n - r + \operatorname{trace}(\theta_r(a_1,\dots,a_r)). \tag{9}$$

For an example of the usefulness of this, one can check from, e.g., (6) that for negative trace we must have $r \ge 6$. Duality immediately tells us that for negative trace (assuming (7) and (8)) we must have $n \ge 6$. This will be pursued further in a second paper, where it will be shown that we need $n \ge 8$, which is a best-possible bound.

4. More minimality conditions. Deleting all appearances of any p_i $(1 \le i \le n)$ from $a_1,\dots,a_r$ will reduce the degree, by (2). In seeking minimal degree with negative trace, we may therefore impose a third minimality condition, in addition to (7) and (8):

(10) Deleting all appearances of any p_i will produce a Pisot number with larger trace.

Of course, we cannot delete all appearances of p_i if this would reduce r to 0 or 1, or to 2 with $a_1 = a_2 = 2$. Any such primes p_i will be excluded from consideration in checking (10): we only delete deletable primes.

Dually we insist that

(11) Deleting any a_i will produce a Pisot number with larger trace.

Again, if $r = 2$ then we deem that (11) is satisfied although we cannot delete any a_i; or if $r = 3$ then we only consider deletions which do not result in $r = a_1 = a_2 = 2$ (if any).

Note that deleting an a_i may not decrease the degree. Certainly the degree is never increased, but it will be unchanged if a_i divides some other a_j. In this case the trace would be decreased by 1, using (3). Thus (11) implies that

(12) No a_i divides any other a_j. (Unless $r = 2$, or $r = 3$ and $a_1 = a_2 = a_3 = 2$.)

We label this as a new condition, for convenience, although as remarked it follows from (11). In effect we are insisting that amongst elements of U with negative trace and minimal degree, we seek those with smallest (most negative) trace.

As a final minimality condition, we consider the effect of permuting the primes $p_1, \ldots, p_n$ which divide any of $a_1, \ldots, a_r$. This leaves the trace unchanged, but may change the degree, so we insist that:

(13) No permutation of $p_1, \ldots, p_n$ will lower the degree.

Definition. If $\theta_r(a_1, \ldots, a_r)$ satisfies (7), (8), (10) and (11) (and hence also (12)), and has negative trace, then we say that $\theta_r(a_1, \ldots, a_r)$ has a *locally minimal pattern of primes*. If also (13) holds, then we say that $\theta_r(a_1, \ldots, a_r)$ is *locally minimal*.

In seeking minimal degree amongst elements of U with negative trace, we may restrict to locally minimal elements.

5. The two main theorems. The following result is extremely useful, and immediately provides a nontrivial lower bound on the degrees of elements of U with negative trace, although we shall not pursue this here.

THEOREM 1. *If $\theta_r(a_1,\ldots,a_r)$ has a locally minimal pattern of primes, then each a_i is divisible by at least four primes.*

We shall see that four is best-possible: indeed there are locally minimal elements of U for which *each* a_i is divisible by *exactly* four primes.

From the proof, we isolate the following lemma, which will prove useful in constructing families of Pisot numbers with negative trace.

LEMMA. *Let $\theta_r(a_1,\ldots,a_r)\in U$. Let $p_1,\ldots,p_s$ be the distinct primes dividing a_1. For each $T\subseteq\{1,\ldots,s\}$, let S_T be the subset of $\{a_2,\ldots,a_r\}$ containing those a_i $(2\le i\le r)$ which are divisible by all the primes in T:*

$$S_T=\{a_i : 2\le i\le r,\ p_j\mid a_i \text{ for all } j\in T\}. \tag{14}$$

Then

$$\operatorname{trace}(\theta_r(a_1,\ldots,a_r)) = \sum_{\emptyset\ne T\subseteq\{1,\ldots,s\},\, S_T\ne\emptyset}(-1)^{|T|+1}+\sum_{S\subseteq\{a_2,\ldots,a_r\},\,|S|\ge2,\,\gcd(S)>1}(-1)^{|S|}.$$

Proof. We use (5), and split the sum as $\Sigma_1+\Sigma_2$, where

$$\Sigma_1=\sum_{S\subseteq\{a_1,\ldots,a_r\},\,a_1\in S,\,|S|\ge2,\,\gcd(S)>1}(-1)^{|S|},$$
$$\Sigma_2=\sum_{S\subseteq\{a_2,\ldots,a_r\},\,|S|\ge2,\,\gcd(S)>1}(-1)^{|S|}.$$

We may suppose that $a_1=p_1\ldots p_s$, since repeated prime factors in a_1 change neither Σ_1 nor Σ_2.

For each subset $S\subseteq\{a_1,\ldots,a_r\}$ such that $a_1\in S$, $|S|\ge2$ and $\gcd(S)>1$, we consider those nonempty T contained in $\{1,\ldots,s\}$ such that $\prod_{i\in T}p_i$ divides $\gcd(S)$ (equivalently, $S-\{a_1\}\subseteq S_T$). For such S we have

$$\sum_{T\ne\emptyset,\,\prod_{i\in T}p_i\mid\gcd(S)}(-1)^{|T|+1}=1,$$

hence

$$\begin{aligned}\Sigma_1&=\sum_S(-1)^{|S|}\sum_{T\ne\emptyset,\,\prod_{i\in T}p_i\mid\gcd(S)}(-1)^{|T|+1}\\&=\sum_{T\ne\emptyset}(-1)^{|T|+1}\sum_{S-\{a_1\}\subseteq S_T,\,|S|\ge2,\,a_1\in S}(-1)^{|S|}\\&=\sum_{T\ne\emptyset,\,S_T\ne\emptyset}(-1)^{|T|+1},\end{aligned}$$

as desired.

Note that Σ_2 is the trace of the Pisot number obtained by deleting a_1. Of course there is nothing special about a_1, and the Lemma tells us how to compute the change of trace if we add or delete any a_i.

For example, with $\theta_3(6,10,15)$, we have $a_1 = 6 = 2\times 3$, $S_{\{2\}} = \{10\}$, $S_{\{3\}} = \{15\}$, $S_{\{2,3\}} = \emptyset$, and

$$\sum_{T\neq\emptyset,\, S_T\neq\emptyset} (-1)^{|T|+1} = (-1)^{|\{2\}|+1} + (-1)^{|\{3\}|+1} = 1+1 = 2,$$

so we have

$$\mathrm{trace}(\theta_3(6,10,15)) = 2 + \mathrm{trace}(\theta_2(10,15)) = 3.$$

Proof of Theorem 1. Suppose that $\theta_r(a_1,\ldots,a_r)$ has a locally minimal pattern of primes. By (7) we have $a_1 = p_1\ldots p_s$ for distinct primes $p_1,\ldots,p_s$. For negative trace, one readily checks from (3) or (5) that $r>3$, so that $\theta_{r-1}(a_2,\ldots,a_r)\in U$ and has larger trace. By the Lemma,

$$\Sigma_1 = \sum_{\emptyset\neq T\subseteq\{1,\ldots,s\},\, S_T\neq\emptyset} (-1)^{|T|+1} < 0,$$

where S_T is defined by (14).

Note that by (10), $S_{\{p_i\}}\neq\emptyset$ for any i, else we could delete p_i without changing the trace.

If $s=1$, then

$$\Sigma_1 = \sum_{T=\{p_1\}} (-1)^{|T|+1} = 1.$$

If $s=2$, then either

$$\Sigma_1 = \sum_{T\in\{\{p_1\},\{p_2\}\}} (-1)^{|T|+1} = 2,$$

or

$$\Sigma_1 = \sum_{T\in\{\{p_1\},\{p_2\},\{p_1,p_2\}\}} (-1)^{|T|+1} = 1.$$

If $s=3$, then if $S_{\{p_1,p_2,p_3\}}\neq\emptyset$ we have $\Sigma_1 = 3-3+1 = 1$, else $\Sigma_1 \geq 3-3 = 0$.

Hence we must have $s\geq 4$. There is nothing special about a_1, so each a_i must be divisible by at least four primes.

One can also prove Theorem 1 using (3), rather than (5). I have chosen to go via (5) because the Lemma will be useful later.

If $a_1 = p_1p_2p_3p_4$, then we can be rather precise about the possible ways in which the primes p_1, p_2, p_3, p_4 appear amongst the other a_i:

Theorem 2. *Suppose that $\theta_r(a_1,\ldots,a_r)$ has a locally minimal pattern of primes, and that $a_1 = p_1p_2p_3p_4$ with p_1, p_2, p_3, p_4 primes. Then, after relabelling the primes and the a_i if necessary, we have one of three cases*:

Case A:

$$\begin{aligned}
a_1 &= p_1 \cdot p_2 \cdot p_3 \cdot p_4,\\
a_2 &= p_1 \cdot p_2 \cdot b_2,\\
a_3 &= p_1 \cdot p_3 \cdot b_3,\\
a_4 &= p_1 \cdot p_4 \cdot b_4,\\
a_5 &= p_2 \cdot p_3 \cdot b_5,\\
a_6 &= p_2 \cdot p_4 \cdot b_6,\\
a_7 &= p_3 \cdot p_4 \cdot b_7,
\end{aligned}$$

with $b_2, \ldots, b_7$ *coprime to* a_1. *We may have* $r > 7$, *but no further* a_i *can be divisible by any three of* p_1, p_2, p_3, p_4.

Case B:

$$\begin{aligned}
a_1 &= p_1 \cdot p_2 \cdot p_3 \cdot p_4,\\
a_2 &= p_1 \cdot p_2 \cdot b_2,\\
a_3 &= p_1 \cdot p_3 \cdot b_3,\\
a_4 &= p_1 \cdot p_4 \cdot b_4,\\
a_5 &= p_2 \cdot p_3 \cdot b_5,\\
a_6 &= p_2 \cdot p_4 \cdot b_6,
\end{aligned}$$

with $b_2, \ldots, b_6$ *coprime to* a_1. *Each further* a_i *is divisible by at most two of* p_1, p_2, p_3, p_4, *and is not divisible by* $p_3 p_4$.

Case C:

$$\begin{aligned}
a_1 &= p_1 \cdot p_2 \cdot p_3 \cdot p_4,\\
a_2 &= p_1 \cdot p_2 \cdot p_3 \cdot b_2,\\
a_3 &= p_1 \cdot p_4 \cdot b_3,\\
a_4 &= p_2 \cdot p_4 \cdot b_4,\\
a_5 &= p_3 \cdot p_4 \cdot b_5,
\end{aligned}$$

with $b_2, \ldots, b_5$ *coprime to* a_1. *Each further* a_i *is divisible by at most two of* p_1, p_2, p_3, p_4, *except that we allow divisibility by* $p_1 p_2 p_3$.

In Case A, *deleting* a_1 *increases the trace by* 2; *in Cases B and C*, *deleting* a_1 *increases the trace by* 1.

Proof. As in the proof of Theorem 1, we have $S_{\{p_i\}} \neq \emptyset$ for $i = 1, 2, 3, 4$.

Suppose first that $S_{\{p_i, p_j, p_k\}} \neq \emptyset$ for some triple of primes (p_i, p_j, p_k), with S_T defined by (14). Relabelling, we may assume that $S_{\{p_1, p_2, p_3\}} \neq \emptyset$. Then, with notation as in the Lemma,

$$\Sigma_1 = 3 - 3 + 1 + \sum_{p_4 \in T,\, S_T \neq \emptyset} (-1)^{|T|+1},$$

so we require

$$\sum_{p_4\in T,\, S_T\neq\emptyset} (-1)^{|T|+1} \le -2.$$

We can achieve this if and only if $S_{\{p_1,p_4\}}$, $S_{\{p_2,p_4\}}$, $S_{\{p_3,p_4\}}$ are all nonempty, and $S_{\{p_i,p_j,p_4\}} = \emptyset$ for any $1 \le i < j < 4$. This gives us Case C.

Next we have the possibility that no three of p_1, p_2, p_3, p_4 divide any one of $a_2,\ldots,a_r$. To achieve $\Sigma_1 < 0$ we need $S_{\{p_i,p_j\}} \neq \emptyset$ for either 5 or 6 of the possible pairs $1 \le i < j \le 4$, giving Case B or Case A respectively.

Corollary. *If $\theta_r(a_1,\ldots,a_r)$ has a locally minimal pattern of primes and if each p_i divides exactly three of the a_i, then each a_i is divisible by at least five primes.*

We shall see that "five" is best-possible.

Proof. Examining the patterns in Cases A, B, C of Theorem 2, we see that at least one of p_1, p_2, p_3, p_4 divides at least four of the a_i.

6. The symmetry group, and some locally minimal examples. Given a locally minimal pattern of n primes, we still have $n!$ permutations of the first n primes to consider in order to find a locally minimal Pisot number. Exploiting symmetry speeds this search.

Definition. The *symmetry group* of $\theta_r(a_1,\ldots,a_r)$ is the group consisting of those permutations of $p_1,\ldots,p_n$ which induce permutations of $a_1,\ldots,a_r$ (and so fix the Pisot number).

For example, let us consider the pattern given in [1] with trace -5. This took $r = 6$, $n = 20$, with each prime dividing exactly three of the a_i: all $\binom{6}{3} = 20$ triples being covered. By (5), trace $= \binom{6}{2} - \binom{6}{3} = -5$.

This pattern is locally minimal, but we can delete any four of the p_i and still have a negative trace.

It seems at first natural to delete four primes as symmetrically as possible, giving the following locally minimal pattern with $r = 6$, $n = 16$:

$$\begin{aligned}
a_1 &= p_2\cdot p_3\cdot p_6\cdot p_7\cdot p_8\cdot p_{11}\cdot p_{13}\cdot p_{14},\\
a_2 &= p_2\cdot p_4\cdot p_5\cdot p_6\cdot p_8\cdot p_{14}\cdot p_{15}\cdot p_{16},\\
a_3 &= p_2\cdot p_4\cdot p_5\cdot p_9\cdot p_{10}\cdot p_{11}\cdot p_{12}\cdot p_{13},\\
a_4 &= p_1\cdot p_4\cdot p_7\cdot p_8\cdot p_{10}\cdot p_{12}\cdot p_{13}\cdot p_{15},\\
a_5 &= p_1\cdot p_3\cdot p_6\cdot p_9\cdot p_{10}\cdot p_{11}\cdot p_{15}\cdot p_{16},\\
a_6 &= p_1\cdot p_3\cdot p_5\cdot p_7\cdot p_9\cdot p_{12}\cdot p_{14}\cdot p_{16}.
\end{aligned}$$

The symmetry group has order 24, and is isomorphic to S_4. It is generated by

$$(p_1\ p_{16})(p_2\ p_{13})(p_3\ p_9)(p_4\ p_8)(p_5\ p_7)(p_6\ p_{10})(p_{12}\ p_{14}),$$

$$(p_1\ p_8)(p_2\ p_9)(p_3\ p_{14})(p_4\ p_{10})(p_5\ p_{11})(p_6\ p_{16})(p_{12}\ p_{13}),$$

and

$$(p_1\ p_{12})(p_2\ p_6)(p_3\ p_{13})(p_4\ p_{16})(p_5\ p_{15})(p_8\ p_{14})(p_9\ p_{10}).$$

For this case, I shall give a proof that this is indeed the symmetry group: in later (simpler!) cases I shall leave the justification as an exercise.

Certainly the above three permutations of $p_1, \ldots, p_{16}$ induce permutations of $a_1, \ldots, a_6$, namely

$$(a_1\ a_3)(a_2\ a_4),$$
$$(a_1\ a_6)(a_2\ a_5),$$

and

$$(a_4\ a_6)(a_3\ a_5).$$

Also they induce permutations of $\{p_5, p_7, p_{11}, p_{15}\}$, namely $(p_5\ p_7)$, $(p_5\ p_{11})$, and $(p_5\ p_{15})$. These three transpositions generate the full symmetric group on $\{p_5, p_7, p_{11}, p_{15}\}$, so our symmetry group, G say, contains a subgroup isomorphic to S_4.

It is enough now to show that any element of G which fixes each of p_5, p_7, p_{11}, p_{15}, fixes every prime. First note that each a_i is divisible by a uniquely-determined pair of primes from $\{p_5, p_7, p_{11}, p_{15}\}$, so any element of G fixing each of p_5, p_7, p_{11}, p_{15} fixes each of $a_1, \ldots, a_6$. But each p_i divides a uniquely-determined triple of $a_1, \ldots, a_6$, so any element of G fixing $a_1, \ldots, a_6$ must fix $p_1, \ldots, p_{16}$, and we are done.

A pleasing geometrical interpretation of this symmetry group was supplied by Chris Smyth and Elmer Rees. We can view $a_1, \ldots, a_6$ as the edges of a tetrahedron. Each p_i appears in three edges, and the four missing triples can be taken to correspond to the four faces of the tetrahedron (or their complements). The symmetry group then corresponds to permutations of the vertices of the tetrahedron.

In principle, utilising this symmetry group reduces the search for a locally minimal Pisot number with this pattern of primes by a factor of 24. In practice, I found it easier to fix p_1 and p_2, then loop through all 14! possibilities for $p_3, \ldots, p_{16}$. Under the action of G there are twelve orbits for the ordered pair (p_1, p_2), so I searched through $12 \times 14!$ possibilities, gaining a factor of 20 rather than 24. The search took two weeks (rather than forty) on my home PC. The minimal degree is 34 250 586 162, achieved, for example, when

$$(p_1, \ldots, p_{16}) = (7, 17, 47, 29, 2, 13, 3, 23, 31, 19, 5, 53, 37, 43, 11, 41)\,.$$

In fact one can do a little better (also noted by Chris Smyth) by deleting four triples from the twenty asymmetrically, deleting all those dividing both a_1 and a_2. This leaves $\gcd(a_1, a_2) = 1$, allowing deletion of a further prime, giving trace $14 - 15 = -1$ (using (5)). There are two essentially distinct ways of deleting this fifth prime, the first of which shows that the "five" in the Corollary to Theorem 2 is best-possible.

The first possibility is:

$$\begin{aligned}
a_1 &= p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5,\\
a_2 &= p_6 \cdot p_7 \cdot p_8 \cdot p_9 \cdot p_{10} \cdot p_{11},\\
a_3 &= p_1 \cdot p_3 \cdot p_6 \cdot p_7 \cdot p_{11} \cdot p_{12} \cdot p_{13} \cdot p_{15},\\
a_4 &= p_4 \cdot p_5 \cdot p_6 \cdot p_8 \cdot p_{10} \cdot p_{12} \cdot p_{14} \cdot p_{15},\\
a_5 &= p_2 \cdot p_3 \cdot p_4 \cdot p_7 \cdot p_8 \cdot p_9 \cdot p_{13} \cdot p_{14} \cdot p_{15},\\
a_6 &= p_1 \cdot p_2 \cdot p_5 \cdot p_9 \cdot p_{10} \cdot p_{11} \cdot p_{12} \cdot p_{13} \cdot p_{14}.
\end{aligned}$$

The symmetry group has order 4, isomorphic to $C_2 \times C_2$. The minimal degree is 12 160 477 837.

Moving p_{15} from a_5 to a_1 gives the second possibility:

$$\begin{aligned}
a_1 &= p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_{15},\\
a_2 &= p_6 \cdot p_7 \cdot p_8 \cdot p_9 \cdot p_{10} \cdot p_{11},\\
a_3 &= p_1 \cdot p_3 \cdot p_6 \cdot p_7 \cdot p_{11} \cdot p_{12} \cdot p_{13} \cdot p_{15},\\
a_4 &= p_4 \cdot p_5 \cdot p_6 \cdot p_8 \cdot p_{10} \cdot p_{12} \cdot p_{14} \cdot p_{15},\\
a_5 &= p_2 \cdot p_3 \cdot p_4 \cdot p_7 \cdot p_8 \cdot p_9 \cdot p_{13} \cdot p_{14},\\
a_6 &= p_1 \cdot p_2 \cdot p_5 \cdot p_9 \cdot p_{10} \cdot p_{11} \cdot p_{12} \cdot p_{13} \cdot p_{14}.
\end{aligned}$$

There are more symmetries here: 12 of them, with the symmetry group isomorphic to $S_3 \times C_2$. The minimal degree is 7 627 134 993.

Even with $r = 6$ one can do much better, by allowing the p_j to divide more than three of the a_i. Consider the following ten-prime pattern:

$$\begin{aligned}
a_1 &= p_1 \cdot p_2 \cdot p_4 \cdot p_9 \cdot p_{10},\\
a_2 &= p_1 \cdot p_2 \cdot p_5 \cdot p_6 \cdot p_7 \cdot p_8,\\
a_3 &= p_1 \cdot p_3 \cdot p_5 \cdot p_6 \cdot p_7,\\
a_4 &= p_1 \cdot p_3 \cdot p_4 \cdot p_8 \cdot p_9 \cdot p_{10},\\
a_5 &= p_2 \cdot p_3 \cdot p_5 \cdot p_7 \cdot p_9,\\
a_6 &= p_2 \cdot p_3 \cdot p_4 \cdot p_6 \cdot p_8 \cdot p_{10}.
\end{aligned}$$

All 15 pairs (a_i, a_j) have gcd > 1, as do 19 of the 20 triples (the missing one being (a_1, a_3, a_5)). There are three primes appearing four times, giving trace $15 - 19 + 3 = -1$ (using (5)). The symmetry group is S_3, and the minimal degree is 1 106 669, achieved, for example, when $(p_1, \ldots, p_{10}) = (3, 5, 7, 29, 19, 13, 17, 2, 23, 11)$.

7. Constructing smaller examples. We can use the patterns of Theorem 2, and the result of the Lemma, to try to build locally minimal patterns with smaller degree.

To accommodate Case A of Theorem 2, we need $r \geq 7$, and the total number of primes, counting with multiplicity, must be at least 28. The following remarkable pattern would be locally minimal, if only it had negative trace!

$$\begin{aligned} a_1 &= p_1 \cdot p_2 \cdot p_3 \cdot p_4, \\ a_2 &= p_1 \cdot p_2 \cdot p_5 \cdot p_6, \\ a_3 &= p_1 \cdot p_3 \cdot p_5 \cdot p_7, \\ a_4 &= p_1 \cdot p_4 \cdot p_6 \cdot p_7, \\ a_5 &= p_2 \cdot p_3 \cdot p_6 \cdot p_7, \\ a_6 &= p_2 \cdot p_4 \cdot p_5 \cdot p_7, \\ a_7 &= p_3 \cdot p_4 \cdot p_5 \cdot p_6. \end{aligned}$$

This satisfies all the minimality conditions (7), (8), (10), (11), but has trace $21-28+7=0$ (using (5)). The pattern is self-dual. Each pair of the a_i is divisible by exactly two of the p_j. With only 28 of the 35 possible triples covered, we can modify this pattern to give trace -1 by adding an eighth prime, dividing, say, a_1, a_2 and a_7. Relabelling we get a locally minimal pattern with a total of only 31 primes (the record: presumably optimal):

$$\begin{aligned} a_1 &= p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_8, \\ a_2 &= p_1 \cdot p_2 \cdot p_5 \cdot p_6 \cdot p_8, \\ a_3 &= p_3 \cdot p_4 \cdot p_5 \cdot p_6 \cdot p_8, \\ a_4 &= p_1 \cdot p_3 \cdot p_5 \cdot p_7, \\ a_5 &= p_1 \cdot p_4 \cdot p_6 \cdot p_7, \\ a_6 &= p_2 \cdot p_3 \cdot p_6 \cdot p_7, \\ a_7 &= p_2 \cdot p_4 \cdot p_5 \cdot p_7. \end{aligned}$$

The trace is $21 - 29 + 7 = -1$ (using (5)). The symmetry group is isomorphic to S_4 (permuting a_4, a_5, a_6, a_7, and exhibiting S_3 as a quotient group, permuting a_1, a_2, a_3). The minimal degree for this pattern is 69 213, achieved when $a_1, \ldots, a_7$ is some permutation of 14 586, 15 470, 19 635, 570, 2 926, 5 187, 13 585.

It would be nice to have each a_i divisible by only 4 primes. With $r = 7$, 8, or 9, this is impossible. But for $r = 10$ we can achieve it. The simplest construction takes two copies of Case A in Theorem 2, based on primes $\{p_1, p_2, p_3, p_4\}$ and $\{p_5, p_6, p_7, p_8\}$, and glues them together to give a self-dual pattern with $r = n = 8$:

$$\begin{aligned} a_1 &= p_1 \cdot p_2 \cdot p_3 \cdot p_4, \\ a_2 &= p_1 \cdot p_2 \cdot p_5 \cdot p_6, \\ a_3 &= p_1 \cdot p_3 \cdot p_5 \cdot p_7, \\ a_4 &= p_1 \cdot p_4 \cdot p_6 \cdot p_7, \\ a_5 &= p_2 \cdot p_3 \cdot p_5 \cdot p_8, \\ a_6 &= p_2 \cdot p_4 \cdot p_6 \cdot p_8, \\ a_7 &= p_3 \cdot p_4 \cdot p_7 \cdot p_8, \\ a_8 &= p_5 \cdot p_6 \cdot p_7 \cdot p_8. \end{aligned}$$

Here the trace is $24 - 32 + 8 = 0$ (using (5)). Using the Lemma, we see that although there are no values of a_9 with $\omega(a_9) = 4$ which would make the trace negative, there are several that preserve trace 0, such as

$$a_9 = p_1 \cdot p_2 \cdot p_7 \cdot p_8,$$

giving $\Sigma_1 = 4 - 4 = 0$ ($S_{\{p_1,p_8\}} = S_{\{p_2,p_7\}} = \emptyset$), in the notation of the Lemma. And now if we add, for example,

$$a_{10} = p_1 \cdot p_3 \cdot p_6 \cdot p_8$$

we achieve $\Sigma_1 = 4 - 5 = -1$, and $\mathrm{trace}(\theta_{10}(a_1, \ldots, a_{10})) = -1$. In terms of (5) we have trace $= 41 - 80 + 52 - 16 + 2 = -1$. Although we have a total of forty primes, we can achieve a smaller degree than before, helped by each a_i being divisible by only 4 primes. The symmetry group is nonabelian of order 16, and the minimal degree is 25 125, achieved by

$$\begin{aligned} a_1 &= 2 \cdot 5 \cdot 7 \cdot 17, \\ a_2 &= 2 \cdot 5 \cdot 11 \cdot 19, \\ a_3 &= 2 \cdot 7 \cdot 13 \cdot 19, \\ a_4 &= 2 \cdot 11 \cdot 13 \cdot 17, \\ a_5 &= 3 \cdot 5 \cdot 7 \cdot 19, \\ a_6 &= 3 \cdot 5 \cdot 11 \cdot 17, \\ a_7 &= 3 \cdot 7 \cdot 13 \cdot 17, \\ a_8 &= 3 \cdot 11 \cdot 13 \cdot 19, \\ a_9 &= 2 \cdot 3 \cdot 7 \cdot 11, \\ a_{10} &= 2 \cdot 3 \cdot 5 \cdot 13. \end{aligned}$$

We can do marginally better. Consider the first locally minimal pattern of this section, with $r = 7$, $n = 8$, and 31 primes. The dual pattern has

$r = 8$, $n = 7$, and trace $8 - 7 + (-1) = 0$, using (9):

$$\begin{aligned} a_1 &= p_1 \cdot p_2 \cdot p_3 \cdot p_4, \\ a_2 &= p_1 \cdot p_2 \cdot p_5 \cdot p_6, \\ a_3 &= p_1 \cdot p_3 \cdot p_5 \cdot p_7, \\ a_4 &= p_1 \cdot p_4 \cdot p_6 \cdot p_7, \\ a_5 &= p_2 \cdot p_3 \cdot p_6 \cdot p_7, \\ a_6 &= p_2 \cdot p_4 \cdot p_5 \cdot p_7, \\ a_7 &= p_3 \cdot p_4 \cdot p_5 \cdot p_6, \\ a_8 &= p_1 \cdot p_2 \cdot p_7 \end{aligned}$$

(after some relabelling). We can replace a_8 by

$$a_8 = p_1 \cdot p_2 \cdot p_7 \cdot p_8$$

without changing the trace. Now adding

$$a_9 = p_1 \cdot p_4 \cdot p_5 \cdot p_8$$

preserves trace 0: $\Sigma_1 = 4 - 4 = 0$. Finally, adding

$$a_{10} = p_3 \cdot p_4 \cdot p_7 \cdot p_8$$

gives us trace -1: $\Sigma_1 = 4 - 5 = -1$. In terms of (5), we have trace $= 42 - 86 + 61 - 21 + 3 = -1$.

The gain over the previous pattern is that now we have three primes appearing 6 times, rather than two, and one prime appearing only 3 times. This allows us to have more smaller primes. The symmetry group is S_3, and the minimal degree is 23 837, achieved when $a_1, \ldots, a_{10}$ is some permutation of 390, 462, 1 190, 1 938, 1 995, 2 090, 2 805, 4 641, 4 862, 5 005.

Reference

[1] J. F. McKee, P. Rowlinson and C. J. Smyth, *Salem numbers and Pisot numbers from stars*, in: Number Theory in Progress, Vol. I, K. Győry, H. Iwaniec and J. Urbanowicz (eds.), de Gruyter, Berlin, 1999, 309–319.

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Received on 11.10.1999 (3699)