Doctoral Dissertation

Structure of directed graphs and hypergraphs

By

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Declaration of authorship

I hereby declare that this thesis is entirely my own. Where I have consulted the work of others, this is always clearly stated.

Signed: Jingdong Ai

Date: 2020/5/4

Abstract

In this thesis, we consider the following topics on directed graphs: proper orientation and biorientation numbers of outerplanar graphs, strong arc-disjoint decompositions of semicomplete compositions, proximity and remoteness in directed graphs, k-ary spanning trees contained in tournaments, and 4-kings in multipartite hypertournaments. In the chapter on the proper orientation number, we describe a result in support of a conjecture stated by Araujo et al. [10] that there is a constant c such that the proper orientation number of an outerplanar graph is at most c, which is fully solved by Chen et al [25] recently. In the chapter on the proper biorientation number, we discuss a new notion of the biorientation number of a graph and its relation to the proper orientation number. In the chapter on proximity and remoteness, we extend these two parameters from undirected to directed graphs and study properties of the two parameters. In the chapter on strong arc-disjoint decompositions of semicomplete compositions, we characterize a class of semicomplete compositions which can be decomposed into two arc-disjoint strong spanning subgraphs. In the chapter on k-ary spanning trees, we prove that every tournament on at least 10 vertices has a 4-ary spanning tree and obtain a lower bound for 5-ary spanning trees. In the chapter on 4-kings in multipartite hypertournaments, we give solutions to two conjectures of Petrovic [55].

Key Words: digraph, proper orientation, decomposition, proximity, remoteness, spanning tree, multipartite hypertournaments

Contents

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Chapter 1

Introduction

In this thesis, we deal with graphs and in particular directed graphs. Directed graphs have many applications which can be found in many areas, such as combinatorial optimization, network science, computer science and information security. For a recent collection of graph applications, see [65]. Graph Theory is a very active research topic [17, 21]. An undirected graph is a pair $G = (V, E)$, where V is a set whose elements are called vertices, and E is a set of pairs of vertices, whose elements are called edges. For any two vertices of G, say a and b, if (a, b) is an edge, we say a is adjacent to b. Many practical problems can be transformed into problems in graph theory. For example, consider different buildings, and the roads between them. Then we can construct an undirected graph in which each building is represented by a vertex and there is an edge between two vertices if the corresponding buildings are connected by a road. However, we may require more information. For example, there maybe a one-way road from place A to B, so that B is reachable from A, but not the other way round. Here a natural model is a digraph. A directed graph (or just digraph) D consists of a finite set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pairs of distinct vertices called arcs. If (a, b) is an arc, we say a dominates b (or b is dominated by a) and denote it by $a \to b$. We say a is adjacent to b if $a \to b$ or $b \to a$. It is well known that the methods developed for undirected graphs often cannot be used straight forwardly for directed graphs, which makes them an interesting topic of research in their own right.

1.1 Terminology and Notation

For a digraph $D = (V, A)$, we call $n = |V(D)|$ the *order* of D and $m = |A(D)|$ the *size* of D. For each $v \in V(D)$, the *in(out)-degree* of v, denoted by $d_D^-(v)$ ($d_D^+(v)$), is the number of vertices u such that $(u, v) \in A((v, u) \in A)$. The *in(out)-neighborhood* of v is $N_D^-(v) = \{u \in V(D) : \ uv \in A(D)\}$ ($N_D^+(v) = \{u \in V(D) : vu \in A(D)\}$.) We will often omit the subscript D when D is clear from the context. We denote by $\Delta^+(D)$ and $\Delta^-(D)$ the maximum out-degree and maximum in-degree of D respectively, similarly, by $\delta^+(D)$ and $\delta^-(D)$ the minimum out-degree and minimum in-degree of D, respectively. The maximum semi-degree and the minimum semi-degree of D are $\Delta^0(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ and $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$ respectively. A digraph D is *regular* if $\delta^0(D) = \Delta^0(D)$.

The *complement* $\overline{D}$ of a digraph D is a digraph with $V(\overline{D}) = V(D)$ and $A(\overline{D}) = \{uv \mid u \neq v \in V(D), uv \notin A(D)\}$. D is *strong* if for every pair u, v of vertices in D there are a dipath from u to v and a dipath from v to u.

Let $p = v_1, v_2, \ldots, v_n$ be a dipath from v_1 to v_n, we call P a *dicycle* if $v_1 = v_n$. The *distance* $d(u, v)$ from vertex u to vertex v in D is the length of a shortest dipath from u to v. An *orientation* of a graph G is a digraph obtained from G by replacing each edge by exactly one of the two possible arcs. We call an orientation of a complete graph and complete k-partite graph a *tournament* and a *k-partite tournament*, respectively. A 2-partite tournament is also called a *bipartite tournament.*

1.2 Outline of the Dissertation

In Chapter 2, we consider the proper orientation number of triangle-free bridgeless outerplanar graphs. An orientation of an undirected graph G is a digraph obtained from G by replacing each edge by exactly one of two possible arcs with the same endpoints. We call an orientation proper if neighbouring vertices have different in-degrees. The proper orientation number of a graph G, denoted by $\vec{\chi}(G)$ or pon(G), is the minimum over all proper orientations of G of maximum in-degree of a proper orientation of G. Araujo et al. [10] asked whether there is a constant c such that $\vec{\chi}(G) \leq c$ for every outerplanar graph G and showed that $\vec{\chi}(G) \leq 7$ for every cactus G. We prove that $\vec{\chi}(G) \leq 3$ if G is a triangle-free 2-connected outerplanar graph and $\vec{\chi}(G) \leq 4$ if G is a triangle-free bridgeless outerplanar graph, which support that the question has a positive answer. Recently, Chen er at. [25] prove that $\vec{\chi}(G)$ of any planar graph G is bounded. This is joint work with S. Gerke, G. Gutin, Y. Shi and Z. Taoqiu [5].

In Chapter 3, we introduce proper biorientations of G where an edge xy of G can be replaced by either arc xy or arc yx or both arcs xy and yx. This is a natural extension of proper orientation considered in Chapter 2. Similarly to pon(G), we can define the proper biorientation number pbon(G) of G using biorientations instead of orientations. We compare pbon(G) with pon(G) for various classes of graphs. We show that for trees T, the tight bound pon$(T) \leq 4$ extends to the tight bound pbon$(T) \leq 4$, and for cacti G, the tight bound pon$(G) \leq 7$ extends to the tight bound pbon$(G) \leq 7$. We also prove that there is an infinite number of trees T for which pbon(T) <pon(T). Let (H, w) be a weighted digraph with a weight function $w : A(H) \to \mathbb{Z}_+$. The in-weight $w^-_H(v)$ of a vertex v of H is the sum of the weights of arcs towards v. A semi-proper p-orientation (D, w) of an undirected graph G is an orientation D of G together with a weight function $w : A(D) \to \mathbb{Z}_+$, such that the in-weight of any adjacent vertices are distinct and $w^-_D(v) \leq p$ for every $v \in V(D)$. The semi-proper orientation number spon(G) of

a graph G (introduced by Dehghan in 2019) [26] is the minimum p such that G has a p-semi-proper orientation (D, w) of G. We prove that $\text{spon}(G) \leq \text{pbon}(G)$ and characterize graphs G for which $\text{spon}(G) = \text{pbon}(G)$. This is joint work with S. Gerke, G. Gutin, H. Lei, Y. Shi [6].

In Chapter 4, we focus on two parameters, namely remoteness and proximity. These parameters were introduced independently by Zelinka [64] and Aouchiche and Hansen [11] for undirected graphs and studied in several papers, see e.g. [11, 14, 28–30, 64]. It seems we are the first to consider them for digraphs. Let D be a strongly connected digraph. The average distance $\bar{\sigma}(v)$ of a vertex v of D is the arithmetic mean of the distances from v to all other vertices of D. The remoteness $\rho(D)$ and proximity $\pi(D)$ of D are the maximum and the minimum of the average distances of the vertices of D, respectively. We obtain sharp upper and lower bounds on $\pi(D)$ and $\rho(D)$ as functions of the order n of D and describe the extreme digraphs for all the bounds. We also obtain such bounds for strong tournaments. We show that for a strong tournament T, we have $\pi(T) = \rho(T)$ if and only if T is regular. Due to this result, one may conjecture that every strong digraph D with $\pi(D) = \rho(D)$ is regular. We present an infinite family of non-regular strong digraphs D such that $\pi(D) = \rho(D)$. We describe such a family for undirected graphs as well. This is joint work with S. Gerke, G. Gutin and S. Mafunda [4].

In Chapter 5, we work on arc decompositions of digraphs. This topic started with a conjecture posed by Thomassen [61] that there exists an integer N such that every N-arc-strong digraph D contains a pair of arc-disjoint in- and out-branchings. Bang-Jensen and Yeo generalized this by introducing the concept of a good decomposition. A digraph $D = (V, A)$ has a good decomposition if A has two disjoint sets A_1 and A_2 such that both (V, A_1) and (V, A_2) are strong. Let T be a digraph with t vertices $u_1, \ldots, u_t$ and let $H_1, \ldots H_t$ be digraphs such that H_i has vertices $u_{i,j_i},\ 1 \leq j_i \leq n_i$. Then the composition $Q = T[H_1, \ldots, H_t]$ is a digraph with vertex set $\{u_{i,j_i} \mid 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$ and arc set

$$A(Q) = \cup_{i=1}^{t} A(H_i) \cup \{u_{ij_i}u_{pq_p} \mid u_iu_p \in A(T), 1 \leq j_i \leq n_i, 1 \leq q_p \leq n_p\}.$$

For digraph composition $Q = T[H_1, \ldots, H_t]$, we obtain sufficient conditions for Q to have a good decomposition, and a characterization of Q with a good decomposition when T is a strong semicomplete digraph and each H_i is an arbitrary digraph with at least two vertices. For digraph products, we prove the following: (a) if $k \geq 2$ is an integer and G is a strong digraph which has a collection of arc-disjoint cycles covering all vertices, then the Cartesian product digraph $G^{\square k}$ (the kth powers with respect to Cartesian product) has a good decomposition; (b) for any strong digraphs G, H, the strong product $G \boxtimes H$ has a good decomposition. This is joint work with Y. Sun and G. Gutin [59].

In Chapter 6, we study unavoidable trees in tournament. A rooted tree is called a *k-ary tree*, if all non-leaf vertices have exactly k children, except possibly one non-leaf vertex has at most $k-1$ children. It is known that every tournament has a hamiltonian path [21], and a hamiltonian path is actually a 1-ary tree. It is therefore natural to study the general problem of whether a tournament contains a k-ary spanning tree. Denote by $h(k)$ the minimum integer such that every tournament of order at least $h(k)$ contains a k-ary spanning tree. Lu et al. [50] proved the existence of $h(k)$, and showed that $h(2) = 4$ and $h(3) = 8$. The exact values of $h(k)$ remain unknown for $k \geq 4$. We prove that $h(k) = \Omega(k \log k)$, especially, $h(4) = 10$ and $h(5) \geq 13$. This is joint work with H. Lei, Y. Shi, S. Yao and Z. Zhang [7].

In Chapter 7, we consider two conjectures stated by Petrovic [54], the first states that every multipartite k-uniform hypertournament ($k \geq 2$) with at most one transmitter contains a 4-king, the other states that every bipartite k-uniform hypertournament B ($k \geq 2$) without transmitters has at least two 4-kings in each partite set of B. We call a vertex v a transmitter if the in-degree of v is 0. A vertex v is a k-king of digraph D if for every vertex $u \in D$, D has a (u, v)-dipath of length at most k. We prove the first conjecture and give counterexamples for the other. This is joint work with S. Gerke, G. Gutin [3].

Chapter 2

Proper Orientation Number of Triangle-free Bridgeless Outerplanar Graphs

2.1 Introduction

An embedding of a planar graph into the plane is *outerplanar* if all vertices of G belong to some face of G'. A planar graph G is *outerplanar* if it has an outerplanar embedding. Outerplanar graphs are not only of interest in graph theory, they are also useful in applications, see, e.g., Gutin et al. [37], where it is proved that the so-called vertex horizontal graphs used in the analysis of time series are precisely outerplanar graphs with a Hamilton path.

Recall that an orientation of a graph $G = (V, E)$ is a digraph $D = (V, A)$ obtained from G by replacing each edge by exactly one of two possible arcs with the same end-vertices. An orientation D of G is proper if $d^-_D(v) \neq d^-_D(u)$, for all $uv \in E(G)$. An orientation with maximum in-degree at most k is called a *proper k-orientation*. The *proper orientation number* of a graph G, denoted by $\vec{\chi}(G)$, is the minimum integer k such that G admits a proper k-orientation.

This graph parameter was introduced by Ahadi and Dehghan [1]. They observed that this parameter is well-defined for any graph G since one can always obtain a proper $\Delta(G)$-orientation, where $\Delta(G)$ is the maximum degree of G. This fact can be proved by induction on the size of G by removing a vertex of maximum degree and orienting all edges towards this vertex. Every proper orientation of a graph G induces a proper vertex-coloring of G. Thus, $\omega(G) - 1 \leq \chi(G) - 1 \leq \vec{\chi}(G) \leq \Delta(G)$, where $\omega(G)$ is the number of vertices in a maximum clique of G. Ahadi and Dehghan [1] proved that it is NP-complete to compute $\vec{\chi}(G)$ even for planar graphs. Araujo et al. [9] strengthened this result by showing it holds for bipartite planar graphs of maximum degree 5.

A parameter α is *monotonic* if $\alpha(H) \leq \alpha(G)$ for every induced subgraph H of G. Unfortunately, the proper orientation number is not monotonic; an example of a tree T and its leaf x such as $\vec{\chi}(T) = 2$ but $\vec{\chi}(T - x) = 3$ is given in [10]. This makes it difficult to prove upper bounds on the parameter even for relatively narrow classes of graphs. Araujo et al. [10] asked whether there is a constant c such that $\vec{\chi}(G) \leq c$ for every outerplanar graph G. They proved that for any cactus G (where every 2-connected component is either an edge or a cycle), we have $\vec{\chi}(G) \leq 7$ and that for any tree T, $\vec{\chi}(T) \leq 4$ (see also [43] for a short algorithmic proof).

We prove the following results.

Theorem 2.1. *For any triangle-free, 2-connected, outerplanar graph G, we have $\vec{\chi}(G) \leq 3$. The bound is tight.*

Note that the intersection between cacti and 2-connected outerplanar graph consists only of cycles. The proof of Theorem 2.1 is non-trivial and requires a good understanding of difficult cases in bounding the parameter for a triangle-free 2-connected outerplanar graph G when, using an outerplanar embedding of G, G is constructed from a cycle by adding ears one by one. In essence, the proof consists of an appropriate algorithm and its correctness proof. The algorithm can be run in linear time.

By increasing the bound by one, we can widen the class of outerplanar graphs as follows. The proofs are based on Theorem 2.1. A *bridge* is an edge whose deletion increases the number of connected components of G. A graph is *bridgeless* if it has no bridges. We call a graph *tree-free* if any bridge connects two 2-connected components and each of the two 2-connected components has size at least 3. Here, we treat an edge as a 2-connected component.

Theorem 2.2. *For any triangle-free, bridgeless, outerplanar graph G, we have $\vec{\chi}(G) \leq 4$.*

Theorem 2.3. *For any triangle-free, tree-free, outerplanar graph G, $\vec{\chi}(G) \leq 4$.*

In the literature, there are some other interesting upper bounds for the parameter on planar graphs. In particular, Knox el al. [43] proved that $\vec{\chi}(G) \leq 5$ for a 3-connected planar bipartite graph G and Noguci [52] showed that $\vec{\chi}(G) \leq 3$ for any bipartite planar graph with $\delta(G) \geq 3$. Recently, Chen el al. [25] proved that $\vec{\chi}(G) \leq 14$ for a planar graph G.

This Chapter is organized as follows. In Section 2.2, we give some simple lemmas for proper orientations of a path. In Section 2.3, we construct triangle-free 2-connected outerplanar graphs G by choosing a finite face and adding to it finite faces one by one such that every newly added face shares precisely one edge; we can talk about adding paths instead of faces. Then we give an algorithm to obtain a proper 3-orientation of G by orienting each added path. In fact, the algorithm has to foresee what further paths will be added, in particular, by paying attention to an especially tricky situation of eventually constructing a k-fan (its definition is also given in Section 2.3). We conclude this section by giving an example of a 2-connected outerplanar graph G with $\vec{\chi}(G) = 3$. In Section 2.4, we provide proofs of Theorems 2.2 and 2.3 based on Theorem 2.1.

2.2 Orientations on paths

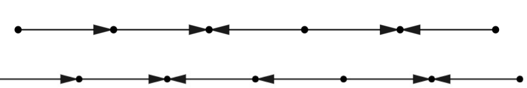
In this section, we collect some simple lemmas about proper orientations on paths. For a path P we denote by $|P|$ the number of edges of P.

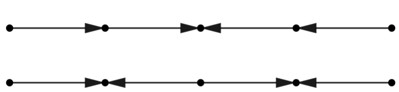
Lemma 2.4. *Let* $P = v_1 \dots v_n$ *be a path.*

1. *If* $|P| \geq 5$ *then there exists a proper orientation such that* $d^-(v_1) = 0$, $d^-(v_2) = 1$, $d^-(v_{n-2}) = 0$, $d^-(v_{n-1}) = 2$ *and* $d^-(v_n) = 0$.
2. *If* $|P| = 4$ *then there are two proper orientations such that* $d^-(v_1) = 0$ *and* $d^-(v_5) = 0$ *and*
   (a) $d^-(v_1) = 0$, $d^-(v_2) = 1$, $d^-(v_3) = 2$, $d^-(v_4) = 1$, $d^-(v_5) = 0$, *and*
   (b) $d^-(v_1) = 0$, $d^-(v_2) = 2$, $d^-(v_3) = 0$, $d^-(v_4) = 2$, $d^-(v_5) = 0$, *respectively.*
3. *If* $|P| = 3$ *then there exists a proper orientation with* $d^-(v_1) = 0$, $d^-(v_2) = 1$, $d^-(v_3) = 2$ *and* $d^-(v_4) = 0$.

Proof. 1. First, let us remark that every vertex of in-degree 0 in a proper orientation of a path with n vertices can be replaced by

 to obtain a proper orientation of a path with $n+2$ vertices. So it suffices to find an orientation of a path of length 5 and 6 as we can then extend these paths by repeatedly replacing v_{n-2} as above. These orientations can be easily found:



2. 

3.

□

Lemma 2.4 is simple, but very helpful. Let G be a graph with a proper orientation. If we add a path P with $|P| \neq 4$ to an edge of G, we can orient P by Lemma 2.4 to obtain a proper orientation of the new graph. To see this, consider an edge $e = \{a, b\}$ and a proper orientation D of G. If $d_D^-(a) = 2$, then we use the orientation of P such that $a = v_1$ and $b = v_n$. As D is a proper orientation, we know that $d_D^-(b) \neq 2$ and therefore the orientation of the new graph is proper. (In what follows, we omit subscripts on in-degrees when the orientation is clear from the context.) If $d^-(a) = 1$, then we use the orientation of P such that $a = v_n$ and $b = v_1$. As before, D is a proper orientation so $d^-(b) \neq 1$ and therefore the orientation of the new graph is proper. Now we can reverse the roles of a and b. Finally if $d^-(a), d^-(b) \notin \{1, 2\}$, then we can just set $a = v_1$ and $b = v_n$ (or the other way round). Note that the orientations of the edges in G and the in-degrees of any vertex in G do not change.

If $|P| = 4$, then the situation becomes more complicated if $d^-(a) = 1$ and $d^-(b) = 2$. (In all other cases we just choose the orientation of Lemma 2.4 that avoids the in-degrees.) To deal with this case, we need other proper orientations of graphs. Some orientations are repeated for easier reference later.

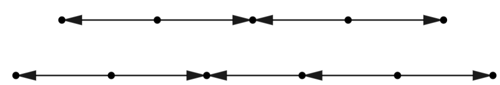
Lemma 2.5. *Let $P = v_1 \dots v_n$ be a path with $|P| \geq 4$. Then there exists a proper orientation such that $d^-(v_1) = 0$, $d^-(v_2) = 2$, $d^-(v_{n-1}) = 2$ and $d^-(v_n) = 0$.*

Proof. As in the proof of Lemma 2.4, we can replace in any path with a proper orientation a vertex of in-degree 0 by a path of length 2 with both edges oriented inwards to obtain a proper orientation for a path with 2 more edges. Observe that the second orientation of a path of length 4 satisfies the conditions of this lemma, it suffices to give an orientation of a path of length 5.

□

Lemma 2.6. *Let $P = v_1 \dots v_n$ be a path with $|P| \geq 4$. Then there exists a proper orientation such that $d^-(v_1) = 1$, $d^-(v_2) = 0$, $d^-(v_{n-1}) = 0$ and $d^-(v_n) = 1$.*

Proof. As in the proof of Lemma 2.4, we can replace in any path with a proper orientation a vertex of in-degree 0 by a path of length 2 with both edges oriented inwards to obtain a proper orientation for a path with 2 more edges. So we need to exhibit orientations for the paths of length 4 and 5.



□

Lemma 2.7. *Let $P = v_1 \dots v_n$ be a path.*

1. *If $|P| \geq 5$ and $|P|$ is odd, then there exists a proper orientation such that $d^-(v_1) = 0$, $d^-(v_2) = 2$, $d^-(v_{n-2}) = 0$, $d^-(v_{n-1}) = 2$ and $d^-(v_n) = 0$.*

2. *If $|P| \geq 6$ and $|P|$ is even, then there exists a proper orientation such that $d^-(v_1) = 0$, $d^-(v_2) = 2$, $d^-(v_{n-1}) = 1$ and $d^-(v_n) = 0$.*

3. *If $|P| = 4$, then there exists a proper orientation such that $d^-(v_1) = 0$, $d^-(v_2) = 2$, $d^-(v_3) = d^-(v_{n-2}) = 0$, $d^-(v_4) = d^-(v_{n-1}) = 2$ and $d^-(v_5) = 0$.*

4. *If $|P| = 3$, then there exists a proper orientation such that $d^-(v_1) = 0$, $d^-(v_2) = 2$, $d^-(v_3) = 1$, $d^-(v_4) = 0$.*

Proof. As in the proof of Lemma 2.4, we can replace in any path with a proper orientation a vertex of in-degree 0 by a path of length 2 with both edges oriented inwards to obtain a proper orientation for a path with 2 more edges. So we only have to consider the paths of length 5 and 6 respectively.

1. This is the same proper orientation on a path of length 5 as given in the proof of Lemma 2.5.
2.
3. This is the same statement as in Lemma 2.4 Case 2b.
4. This is the same statement about a path of length 3 as in Lemma 2.4 just in reverse order.

□

Lemma 2.8. *Let $P = v_1 \dots v_n$ be a path.*

1. *If $|P| \geq 5$ and $|P|$ is odd, then there exists a proper orientation such that $d^-(v_1) = 0$, $d^-(v_2) = 1$, $d^-(v_{n-2}) = 0$, $d^-(v_{n-1}) = 2$ and $d^-(v_n) = 0$.*
2. *If $|P| \geq 4$ and $|P|$ is even, then there exists a proper orientation such that $d^-(v_1) = 0$, $d^-(v_2) = 1$, $d^-(v_{n-1}) = 1$ and $d^-(v_n) = 0$.*

Proof. For $|P| = 4$, the orientation in Lemma 2.4 Case 2a satisfies the condition. For the remaining cases as in the proof of Lemma 2.4 we can replace in any path with a proper orientation a vertex of in-degree 0 by a path of length 2 with both edges oriented inwards to obtain a proper orientation for a path with 2 more edges. So we only have to consider the paths of length 5 and 6 respectively.

1.
2.

□

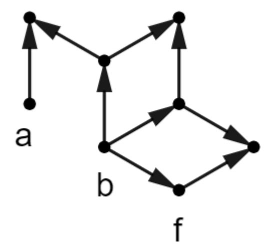
2.3 Proof of Theorem 2.1

2.3.1 Description of the algorithm

Let G be a 2-connected, triangle-free, outerplanar graph. Consider an outerplanar embedding G' of G into the plane such that all vertices of G belong to the infinite face of G'. We can construct G by choosing a finite face (which is a chordless cycle in G) and adding to it finite faces one by one. Since G is 2-connected and G' is an outerplanar embedding, each face added shares an edge with previous faces. In what follows, we will talk about adding paths, instead of faces, between the vertices of an edge: such a path and the corresponding edge form the boundary of the corresponding face. We will say that such a path is *attached* to the corresponding edge.

To describe an algorithm that provides a proper 3-orientation of G, we need the following terminology. We call an edge *active* if there will be a path attached to it in the future. Otherwise we call it *inactive*. Let D be an orientation of a subgraph H of G. We call a vertex of D with in-degree 2 adjacent to a vertex of in-degree 3 a *troublemaker*. An edge of H is called a 1-2 *edge* if the in-degrees of its end-vertices in D are 1 and 2.

It turns out that active 1-2 edges are problematic when the vertex with in-degree 2 is a troublemaker. So assume we have an active 1-2 edge $\{a, b\}$ so that $d^-(a) = 1$, $d^-(b) = 2$ and b is a troublemaker and the edge is oriented from b to a. (The case that the edge is oriented from a to b will be discussed later.) Assume we attach to it a path of length 3 oriented as in Lemma 2.4 part 3. This gives a new active 1-2 edge with b as a troublemaker. We can repeat this $k-1$ times (to get k paths of length 3 altogether), see the following picture for $k = 3$:



If we now want to add a path of length 4 to the active 1-2 edge $\{b, f\}$ without changing the existing orientation we would get a problem. If a path of length 4 is indeed added we call this structure a *k-fan* and use the orientation as in Figure 2.1. Later in our construction, we will always make sure that the path between a and b is oriented in a way that the in-degree of a will be 3 (that is, both edges incident to a will be oriented towards a) and b has in-degree 2. Note that a 0-fan is simply a path of length 4 and that we do not include the original 1-2 edge in our k-fan (we include only the k paths of length 3 and the path of length 4).

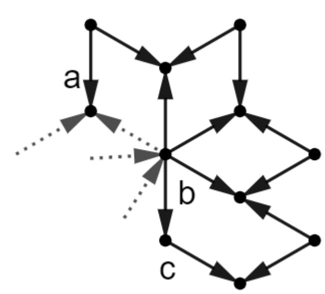


Figure 2.1 The orientation of a k-fan if no k'-fan is added to $\{b, c\}$

Note that the orientation in Figure 2.1 creates a new 1-2 edge $\{b, c\}$ which may be active. If another k'-fan is added to this edge then we use the orientation in Figure 2.2 where both edges in the path of length 4 are oriented towards c (otherwise the two orientations are identical). Note that c will have in-degree equal to 3 when a k'-fan is added and all the other in-degrees in the original k-fan (including b's) will not change.

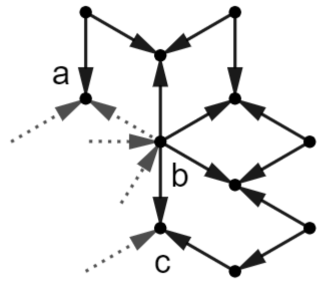


Figure 2.2 The orientation of a k-fan if another k'-fan is added to $\{b, c\}$

As mentioned before edges with a vertex of in-degree 1 and in-degree 2 where the latter is a troublemaker may cause problems. We will anticipate these problems using the following procedure (**Procedure-1**), where e is an edge and $x \in e$ is a troublemaker. Note that calling **Procedure-1** often creates a new active 1-2 edge and we deal with this by repeatedly calling **Procedure-1** until there is no active 1-2 edge stemming from the original edge e.

Procedure-1(e, x)

If no path gets attached to e **then** exit.

If a k-fan gets attached to e **then** let c be the vertex of the 5-cycle of degree 2 adjacent to x.

If $\{x, c\}$ will get attached a k'-fan **then** use the orientation of a k-fan as in Figure 2.2 and run **Procedure-1**$(\{x, c\}, x)$.

otherwise (no k'-fan) use the orientation of a k-fan as in Figure 2.1 and run **Procedure-1**$(\{x, c\}, x)$.

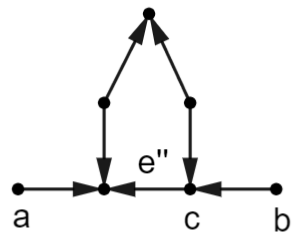
otherwise (no k-fan gets attached to e) let P be the path that gets attached to e and let c be the vertex adjacent to x in P.

If a k-fan is attached to $\{x, c\}$ **then** use Lemma 2.5 to orient P and call **Procedure-1**({x,c},x) [Note that c will have in-degree 3 after **Procedure-1** has finished and that $|P| \neq 3$ as otherwise P would be part of the fan.]
otherwise use Lemma 2.4 to orient P. Let e' be the edge of P incident to x. Call **Procedure-1**(e', x).

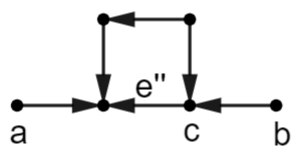
Now we can describe our algorithm, which uses the following **Procedure-2** to avoid too many nested if-then statements. This procedure is only used in one particular case if a path P of length 3 is added to an edge $\{a, b\}$ with trouble-maker a and with b having in-degree 3. In this case, if we use Lemma 2.4 we will create two active $1-2$ edges. One is oriented from $'2'$ to $'1'$, another is oriented from $'1'$ to $'2'$. Therefore we have to be a bit more careful. The edge e' will be the edge incident to a on P and we will assume that there is no k-fan on e' when calling this procedure. The reader may want to come back to this procedure when it is called in the algorithm.

Procedure-2(e',a,b)
If there will be a k-fan attached to the edge e'' of P that is not incident to a or b **then**

If it is a 0-fan we use the following orientation:



Otherwise we use the following orientation for P and the first path of length 3 of the k-fan:



[Note that this does not create a new 1-2 edge.]

otherwise use Lemma 2.4 to orient P and call **Procedure-1**(e', b) and then **Procedure-1**(e'', c) where e'' is the edge of P that is not incident to a or b and c is the vertex of this edge of in-degree 2 (so adjacent to b)).

Algorithm

(A0) Choose an embedding of G with all the vertices on the outer face. Choose a face and orient it using Lemma 2.4 identifying v_1 and v_n (or orient it in any other proper way).

While there exists an active edge e we do the following:

Let $e = \{a, b\}$. W.l.o.g. we may assume that $d^-(a) < d^-(b)$. Let P be the path that is attached to e.

(A1) **If** neither a nor b is a troublemaker and $|P| \neq 4$ **then** orient P as in Lemma 2.4.

(A2) **If** neither a nor b is a troublemaker and $|P| = 4$ **then**

if $d^{-}(a) = 1$ and $d^{-}(b) = 2$ **then** use the following orientation (from a to b):

[Note that this introduces a vertex of in-degree 3]

otherwise [not a 1-2 edge] use Lemma 2.4.

(A3) **If** b is a troublemaker, let e' be the edge on P incident to b

(We will see later that $d^{-}(a) \neq 1$.)

If a k-fan will be attached to e' we use Lemma 2.4 such that the in-degree 2 vertex c of P is adjacent to b (so $b = v_n$), and call **Procedure-1**(e', b). [Note that after we called **Procedure-1** the in-degree of b will be 2 and the in-degree of the other vertex c incident to e' will be 3.]

Otherwise we use Lemma 2.4 in the usual way and call **Procedure-1**(e', b).

(A4) **If** a is a troublemaker, let e' be the edge on P incident to a

If a k-fan will be attached to e' we use Lemma 2.7 with $v_1 = a$ and call **Procedure-1**(e', a).

Otherwise If $|P| = 3$ call **Procedure-2** (e',a,b)

otherwise we use Lemma 2.8 to orient P and call **Procedure-1**(e', a).

2.3.2 Correctness of the algorithm

Let D be an orientation of a subgraph H of G. A path Q of length 2 in H is an 1-2-3 *path* (with respect to H) if the internal vertex of Q is of in-degree 2 and the other two vertices are of in-degrees 1 and 3 in D.

Lemma 2.9. *The only way to get an active edge $e = \{a, b\}$ such that a is a troublemaker and $d^{-}(b) = 1$ is to attach a path or a k-fan to an edge incident to a troublemaker.*

Proof. We first look at ways to create vertices of in-degree 3. In step (A1) we cannot create a new vertex of in-degree 3. In step (A2) we create a vertex x of in-degree 3, but it cannot have a neighbour of in-degree 2 as we started with a proper orientation and x had in-degree 2 before we attached the path of length 4 and the in-degree of no other vertex changes. Therefore we cannot create an 1-2-3 path in this step. □

Lemma 2.10. *At each iteration of the main* **while***-loop there is no active* 1-2 *edge of an* 1-2-3 *path.*

Proof. We have seen in Lemma 2.9 that we cannot get an active 1-2 edge of an 1-2-3 path in (A1) and (A2). So now assume the lemma is true as we enter (A3). If a k-fan is attached, then the neighbour c of b in P will have in-degree 3 and the neighbour d of c in P will have in-degree at most 1. So c does not create a new troublemaker and no new 1-2-3 path. We then may repeatedly call Procedure-1. Note that we may create an active 1-2 edges in an 1-2-3 path but we immediately will call Procedure-1 on these edges. We only finish when no more paths or k-fans are attached and so the 1-2 edge is inactive.

We can argue similarly for (A4) but here we have some added complications because $d^-(b) = 3$ and we do not want to introduce a new 1-2-3 path. We do so by insuring that either the vertex adjacent to b in P has in-degree 1 (if P is even, $|P| \neq 4$) or has in-degree 2 but its neighbour has in-degree equal to 0 (P odd, $|P| = 4$). Also we may add the path of length 3 in which case we get two active 1-2 edges in a 1-2-3 paths if we use Lemma 2.4. Note, that adding a path of length 3 to a 2-3 edge (i.e. an edge with vertex in-degrees 2 and 3) is the only way to create more than one active 1-2 edge in a 1-2-3 path.

We deal with this case by first considering the 1-2-3 path that contains only one edge of the new path. If a k-fan is added then we choose a different orientation of the path (see Figure 2.3), which yields the in-degree sequence 2, 3, 1, 3, where the degree 2 vertex is (and remains) the troublemaker. In this case, we do not have to deal with a second active 1-2 edge. We may create a new active 1-2 edge

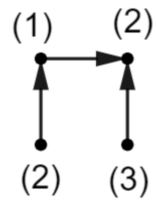


Figure 2.3 The required orientation of the path. The in-degrees are in brackets.

which is part of the k-fan but we repeatedly use Procedure-1 until no more paths or k-fans get added to such an edge.

If a k-fan is added to the edge not containing the troublemaker then we also change the orientation of the path and either add a path of length 4 (in case of a 0-fan) or one path of length 3 of the k-fan. In either case, we do not create a new 1-2 edge and do not have to use Procedure-1.

If no k fan is added to either of these edges then we just use Lemma 2.4 and then Procedure-1 on both edges until no active 1-2 edge is present. □

It is now straightforward to check that using our algorithm we create a proper 3-orientation of G at the end of each step (A1), (A2), (A3) and (A4). Thus Theorem 2.1 follows.

2.3.3 A 2-connected triangle-free outerplanar graph G with $\vec{\chi}(G) = 3$

Consider the 2-connected outerplanar graph consisting of one cycle C_5 of length 5 and a path of length 4 attached to each of its edges, see Figure 2.4.

Assume for a contradiction that we can orient the graph in such a way that the maximum in-degree is less than 3. The only way to orient the 5-cycle without creating a 1-2-edge is to orient the edges cyclicly so that every vertex in C_5 has in-degree equal to 1. Note that a 1-2-edge e in C_5 would mean that we cannot

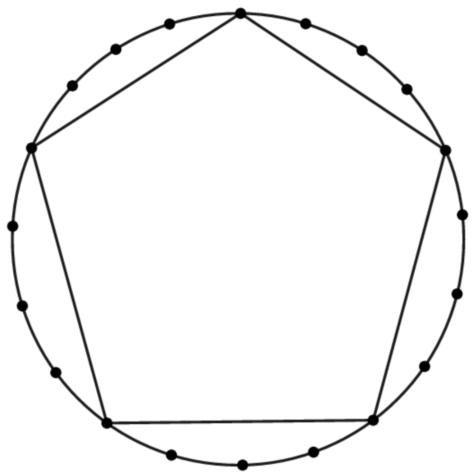


Figure 2.4 An example of a graph G with $\vec{\chi}(G) = 3$

orient the path of length 4 attached to e in a proper way without increasing the in-degree of one of the endpoints of e; and since the orientation should remain proper that means that the in-degree of one vertex has to become 3. But if all the in-degrees on C_5 are equal to one, then we do not have a proper orientation, so when we attach the first path of length 4 to an edge e, one of the vertices a of e will get an in-degree of 2. Observe that the other edge incident to a in C_5 is still active and this forces the orientation to give a in-degree 3 as above.

2.4 Proofs of Theorems 2.2 and 2.3

A *block* in a graph G is a maximal 2-connected subgraph of G. A *block tree* of a connected graph G denoted by $B(G)$ is a bipartite graph with bipartition (B, C), where B is the set of blocks of G and C is the set of cut-vertices of G, a block L and a cut-vertex v being adjacent in $B(G)$ if and only if L contains v. It is not hard to see that $B(G)$ is a tree [23].

To prove Theorem 2.2, it suffices to consider a connected, bridgeless, outerplanar graph G. Choose a block L_1 of G as a root of $B(G)$ and apply breadth-first search on $B(G)$ from L_1 to visit all blocks of G one by one. Now if we consider the blocks of G in the order $L_1, \ldots, L_p$ they were visited, we can orient the edges of one block and then extend it to another block sharing a cut-vertex as follows, without ever encountering a block we have already oriented.

Let G' be an outerplanar embedding of G into the plane. Consider L_1 and orient its edges as in Section 2.3. Assume that we have oriented the edges of blocks $L_1, \ldots, L_{j-1}$ and we wish to orient edges of L_j. Assume that a cut-vertex v is the parent of L_j on the rooted tree $B(G)$ and let H be the orientation of $L_1 \cup \cdots \cup L_{j-1}$. If $d_H^-(v) \neq 2$, then we choose a face of G' in L_j containing v and use Lemma 2.5, where we identify v_1 and v_n with v, to get the orientation of the border cycle of L_j in G'. If $d_H^-(v) = 2$, we just orient the edges towards v as in Lemma 2.6 so that v will have in-degree 4. Note that we only change an in-degree 2 vertex to an in-degree 4 vertex, so no in-degree 4 vertices will ever be adjacent. Now we can just proceed orienting L_j with our algorithm for the 2-connected case.

Now consider a connected, triangle-free, tree-free, outerplanar graph G. We proceed similarly to the bridgeless case, but now sometimes two blocks are connected by a bridge instead of a cut-vertex. So assume we have oriented the edges in $L_1, \ldots, L_i$ and we want to orient the edges of L_{i+1} connected to $L_1, \ldots, L_i$ via a bridge $e = \{a, b\}$ where $b \in L_{i+1}$. We orient e towards b. If the in-degree of a in the orientation F of $L_1 \cup \cdots \cup L_i$ does not equal 1, then we consider a face of G' containing b in L_{i+1} and orient its border cycle using Lemma 2.5 identifying v_1 and v_n with b. If the in-degree of a in F does equal 1, then we use Lemma 2.6 to obtain a vertex of in-degree 3. Note that the neighbours of b in L_{i+1} have in-degree 0 in F, so we do not create a 1-2-3 path. We can now proceed to orient the edges in L_{i+1} as in Section 2.3.

Chapter 3

Proper Orientation, Proper Biorientation and Semi-proper Orientation Numbers of Graphs

3.1 Introduction

In this chapter we introduce a new graph parameter, the proper biorientation number, and show some of its basic properties. The introduction of this parameter was motivated by a recent paper by Dehghan [26] on semi-proper orientations of graphs. To define these notions we need some basic notation.

Recall that an orientation D of G is *proper* if for every $xy \in E(G)$, we have $d_D^-(x) \neq d_D^-(y)$. An orientation D is a *p-orientation* if the maximum in-degree of a vertex in D is at most p. The minimum integer p such that G has a proper p-orientation is called the *proper orientation number* $\mathrm{pon}(G)$ of G.

Let (H, w) be a weighted digraph with a weight function $w : A(H) \to \mathbb{Z}_+$. The *in-weight* $w_H^-(v)$ of a vertex v of H is the sum of the weights of arcs towards v. A *semi-proper p-orientation* (D, w) of an undirected graph G is an orientation D of G together with a weight function $w : A(D) \to \mathbb{Z}_+$, such that the in-weight of any adjacent vertices are distinct and $w_D^-(v) \leq p$ for every $v \in V(D)$. The *semi-*

proper orientation number $\text{spon}(G)$ of a graph G is the minimum p such that G has a p-semi-proper orientation (D, w) of G. This parameter was first introduced by Dehghan [26] who proved that for every graph G there is a $\text{spon}(G)$-semi-proper orientation in which the weight of each edge in G is 1 or 2. This shows that there is an equivalent definition of a semi-proper orientation, where we can only replace an edge xy of G either by one arc with end-vertices x and y or by two arcs between x and y, both directed either from x to y or from y to x.

Dehghan's theorem and the equivalent definition of a semi-proper orientation above lead us to the following natural extension of a proper orientation. Let G be a graph. A *biorientation* of G is a digraph D obtained from G by replacing every edge xy by arc xy, arc yx, or two mutually opposite arcs xy, yx [17]. An arc xy of D is called *single* if there is no arc yx. Thus, a biorientation D is an *orientation* if all arcs of D are single. One can define a *proper biorientation* and *p-biorientation* in absolutely the same way as a proper orientation and a p-orientation. The minimum integer p such that G has a proper p-biorientation is called the *proper biorientation number* $\text{pbon}(G)$ of G. Note that for any graph G, $\text{pbon}(G) \leq \text{pon}(G)$.

In Section 5.2, we compare $\text{pbon}(G)$ with $\text{pon}(G)$ for various classes of graphs. In Subsection 3.2.1, we compare $\text{pbon}(T)$ with $\text{pon}(T)$ for trees T. Araújo et al. [9] proved that for every tree T, we have $\text{pon}(T) \leq 4$ and this bound is tight. It follows from $\text{pbon}(G) \leq \text{pon}(G)$ that $\text{pbon}(T) \leq 4$. We prove that the last bound is also tight. The fact that for trees the tight upper bound on proper orientation number coincides with that on proper biorientation number does not mean that the two numbers are equal on trees. We show that there is a tree T^* such that $\text{pbon}(T^*) = 3$ and $\text{pon}(T^*) = 4$. We extend this result by showing that there is an infinite number of trees for which the two numbers are not equal.

Araujo et al. [10] proved that $\text{pon}(G) \leq 7$ for every cactus G and the bound is tight. In Subsection 3.2.2 we prove that the bound remains tight for proper biorientation number on cacti as well. It is natural to ask when $\text{pbon}(G) =$

pon(G). While we are unable to give a complete answer, in Subsection 3.2.3 we show that pbon(G) = pon(G) for every graph with pon(G) ≤ 3. The previously discussed result on T^* shows that we cannot replace 3 by 4 in the last inequality.

In Section 3.3, we prove that spon(G) $\leq$ pbon(G) for every graph G. Thus, for every graph G,

$$\text{spon}(G) \leq \text{pbon}(G) \leq \text{pon}(G). \tag{3.1.1}$$

We also characterize when spon(G) = pbon(G) for a graph G.

We conclude Chapter 3 by discussing a number of open problems.

3.2 pbon(G) vs pon(G)

The following simple observation will be useful in some proofs below.

Observation 3.1. *Let D be a proper biorientation of G and let xy and yx be two mutually opposite arcs. If $d^-(y) = 1$, then we can always obtain a new proper biorientation such that $d^-(y) = 0$ by removing the arc xy.*

We will use the notion of an x-pendant subgraph. Let H be an induced proper subgraph of G and let $x \in V(H)$. The subgraph H is *x-pendant* if there are no edges between $V(H) - \{x\}$ and $V(G) - V(H)$.

3.2.1 pbon(G) vs pon(G) for trees

Theorem 3.2. *[9] If T is a tree, then pon(T) ≤ 4, and this bound is tight.*

Araujo et al. [9] proved that for the tree R_3 in Fig. 3.1, we have pon(R_3) = 3. In R_3, x is called its *root*.

Lemma 3.3. *[9] Let G be a graph with an x-pendant subgraph R_3. In any proper 3-orientation D of G, for every $z \in N_{G-R_3}(x)$ we have $xz \in A(D)$.*

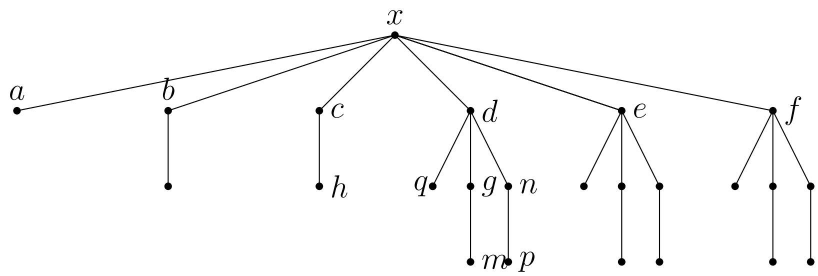


Figure 3.1 R_3

Theorem 3.4. *If T is a tree, then $pbon(T) \leq 4$, and this bound is tight.*

This theorem follows from the assertion that $\mathrm{pon}(T) \leq 4$ for every tree [9] and that $\mathrm{pbon}(T_3) = 4$ for the tree T_3 obtained from two copies of R_3 with roots x and x' by adding the edge xx'. Araujo et al. [9] showed that $\mathrm{pon}(T_3) = 4$. We will show a slightly stronger result.

Lemma 3.5. *We have $pbon(T_3) = 4$.*

Proof. We prove this lemma by contradiction. Suppose that there is a proper 3-biorientation D of T_3 and D has the minimum possible number of arcs among such biorientations. By Lemma 3.3 D has at least one non-single arc and by Observation 3.1 all non-single arcs have endpoints with in-degree 3 and 2, respectively. Assume first that $xx', x'x \in A(D)$ are the only non-single arcs. Then modify D by deleting xx' and adding a new vertex x'' and two new arcs xx'' and $x''x'$. Note that the new digraph is a proper 3-orientation of the undirected underlying graph. However, this contradicts Lemma 3.3. So we may assume one of the copies of R_3 has a non-single arc.

Without loss of generality, we assume that the copy of R_3 with root x contains a non-single arc yz. Since $d^-(y), d^-(z) \in \{2, 3\}$ and by symmetry, it suffices to consider the following cases.

Case 1: cx is a non-single arc. Then delete arc xc and replace any arc(s)

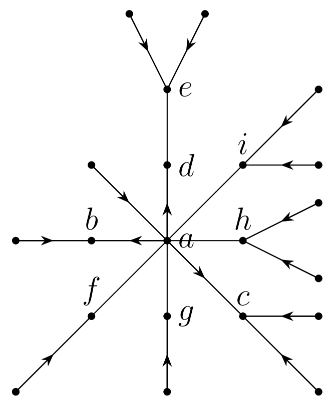


Figure 3.2 T_1

between c and h by arc ch. The new proper 3-biorientation has fewer arcs than D, a contradiction.

Case 2: dg is a non-single arc. Then delete dg and replace any arc(s) between g and m by arc gm. The new proper 3-biorientation has fewer arcs than D, a contradiction.

Case 3: xd is a non-single arc. Then replace all arcs incident to d by a (single) arcs pointing away from d, replace any arcs between g and m by mg, and replace any arcs between n and p by pn. The new proper 3-biorientation has fewer arcs than D, a contradiction. □

It is natural to ask if there exists a tree T such that $\text{pbon}(T) < \text{pon}(T)$, and below we give a positive answer to this question. Let us construct a graph T^* from the tree T_1 depicted in Figure 3.2 as follows. Let us identify each leaf vertex of T_1 with the root of a copy of R_3 such that the other vertices of the copies are not identified with any vertex of T_1.

Theorem 3.6. *We have* $pon(T^*) = 4$ *and* $pbon(T^*) = 3$.

We prove this theorem by showing the following three lemmas.

Lemma 3.7. *There is a proper 3-orientation D of R_3 such that $d^-_D(x) = 0$.*

Proof. We use the labelling of vertices of R_3 in Fig. 3.1. Orient from x all edges incident with x. Orient all edges not incident with x but incident with b or c towards b or c. Orient the subtree of R_3 induced by $\{d, q, g, m, n, p\}$ using the following arcs: dq, gd, mg, nd, pn. Finally orient similarly the subtrees rooted at e and f. It is not hard to verify that we have obtained a required orientation. □

Lemma 3.8. *pon*$(T^*) = 4$.

Proof. By Theorem 3.2, it suffices to prove that the pon$(T^*) \geq 4$. Suppose that there is a proper 3-orientation D of T^*, then by Lemma 3.3 for every leaf u of T_1 and its neighbor v, we have $uv \in A(D)$. Observe that there is an arc from a to at least one vertex of each set $\{b, f, g\}$ and $\{c, h, i\}$ respectively, say b and c, since $d^-(a) \leq 3$. Now we have $d^-(b) = 2$ and $d^-(c) = 3$, so $d^-(a) = 1$ and $ad \in A(D)$. Then no matter how de is oriented in D we have either $d^-(a) = d^-(d) = 1$ or $d^-(d) = d^-(e) = 2$ contradicting the assumptions that D is a proper orientation. Thus, we conclude that pon$(T^*) = 4$. □

Lemma 3.9. *pbon*$(T^*) = 3$.

Proof. In Fig. 3.2, orient from a all un-oriented edges incident with a and replace edge ed by two mutually opposite arcs. Proper 3-orient every copy of R_3 such that the root has in-degree zero (it is possible by Lemma 3.7). This proves that T^* has a proper 3-biorientation. Now we can show that pbon$(T^*) = 3$ using Observation 3.1 (as in the proof of Lemma 3.5). □

Theorem 3.10. *There is an infinite number of trees T_n with pbon$(T_n) <$ pon(T_n).*

Proof. For every $n \geq 0$, we construct a tree T^*_n from T^* by adding n paths of length 2 which share only vertex a with each other and T^* such that a is not a

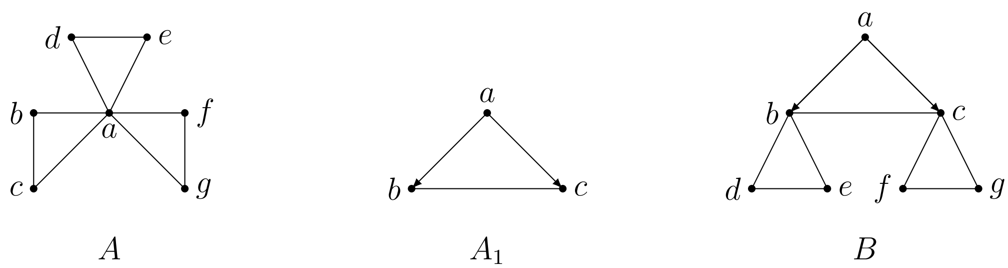


Figure 3.3 Graphs A, A_1 and B

central vertex of the paths. Consider the proper 3-biorientation of T^* described in the proof of Lemma 3.9. Note that $d^-(a) = 1$. Orient all edges of the n paths towards the central vertices. Observe that the biorientation of T_n^* is a proper 3-biorientation. As in Lemma 3.9, we can see that $\text{pbon}(T_n^*) = 3$. From the proof of Lemma 3.8, it follows that $\text{pon}(T_n) = 4$. $\square$

3.2.2 pbon(G) vs pon(G) for cacti G

Araujo et al. [10] prove the following bound for cacti.

Theorem 3.11. *[10] If G is a cactus, then pon$(G) \leq 7$ and this bound is tight.*

To prove the tightness of the bound in Theorem 3.11, Araujo et al. [10] constructed a graph G_1 such that $\text{pon}(G_1) = 7$. Let us describe G_1. Let F be a graph which is the union of sixteen triangles:

(i) K with vertex set $\{v_1, v_2, v_3\}$;

(ii) K_i^j with vertex set $\{v_i, y_i^j, z_i^j\}$ for $i \in \{1, 2, 3\}$ and $j \in \{1, \ldots, 5\}$.

Let G_1 be the graph obtained from F by adding a copy of A (see Fig. 3.3) and five copies of B (see Fig. 3.3) at every vertex $v \in F$, identifying the vertex a of both A and B with each vertex of F. We will use a proof similar to that in [10] to show that $\text{pbon}(G_1) = 7$. We first prove two lemmas.

Lemma 3.12. *Let G be a graph which contains A as an a-pendant subgraph ($V(G) \neq V(A)$). For every proper biorientation D of G, we have $d_D^-(a) \notin \{1, 2\}$.*

Proof. We prove this lemma by contradiction. Suppose that there is a proper biorientation of G such that $d^-(a) \in \{1, 2\}$. Since A has three triangles, one of them must be oriented as A_1 (see Fig. 3.3). Then if bc is replaced by a single arc then either $d^-(b) = 1$ and $d^-(c) = 2$ or $d^-(b) = 2$ and $d^-(c) = 1$, a contradiction for both cases. If bc is replaced by a pair of mutually opposite arcs, then $d^-(b) = d^-(c) = 2$, a contradiction. □

Lemma 3.13. *Let G be a graph which contains B as an a-pendant subgraph ($V(G) \neq V(B)$). Let D be a proper biorientation of G such that ab and ac are single arcs in D. Then $d^-(b) \notin \{1, 2\}$ and $d^-(c) \notin \{1, 2\}$ implying that $d^-(b), d^-(c) \in \{3, 4\}$.*

Proof. We prove this lemma by contradiction. Suppose that $d^-(b) = 1$ or $d^-(b) = 2$. If $d^-(b) = 1$, then bd and be are single arcs in D and no matter how we replace edge de by a single arc or two mutually opposite arcs, we arrive at a contradiction. If $d^-(b) = 2$, then without loss of generality $d^-(d) = 1$ and $d^-(e) = 0$ since bde is a triangle. Hence eb, ed and db are single arcs of D implying that $d^-(b) \geq 3$, a contradiction. Therefore, $d^-(b) \notin \{1, 2\}$. Similarly, we can prove that $d^-(c) \notin \{1, 2\}$. By the restrictions on $d^-(b)$ and $d^-(c)$, we conclude that $d^-(b), d^-(c) \in \{3, 4\}$. □

Theorem 3.14. *We have $pbon(G_1) \geq 7$.*

Proof. We prove this theorem by contradiction. Suppose that there is a proper 6-biorientation D of G_1. If there is a vertex u of F with $d_D^-(u) \in \{3, 4\}$, then in the proper 6-biorientation of one of the five copies of B corresponding to u we have two single arcs directed from $u = a$ (as B in Fig. 3.3). However, this contradicts Lemma 3.13. Then by Lemma 3.12, $d^-(u) \in \{0, 5, 6\}$ for all vertices $u \in V(F)$.

Since K is a triangle and D is a proper 6-biorientation, without loss of generality, we may assume that $d^-(v_1) = 5$ and $d^-(v_2) = 0$. Then v_2v_1 is a single arc in D. Since each K_1^j ($j \in \{1, \ldots, 5\}$) is a triangle, each of them has a vertex of in-degree zero. This implies that $d^-(v_1) = 6$, a contradiction. □

By Theorems 3.11 and 3.14 and inequality (3.1.1), we obtain the following:

Theorem 3.15. *If G is a cactus, then $pbon(G) \leq 7$, and this bound is tight.*

3.2.3 pbon(G) vs pon(G) for arbitrary graphs G

Theorem 3.16. *Let G be a graph, if $pon(G) \leq 3$, then $pbon(G) = pon(G)$.*

Proof. We prove this theorem by contradiction. Suppose that there is a graph G with $\text{pon}(G) \leq 3$, but $\text{pbon}(G) < \text{pon}(G)$. If $\text{pbon}(G) = 1$, then the corresponding biorientation must be an orientation, so $\text{pon}(G) = 1$, a contradiction. If $\text{pbon}(G) = 2$, then there is a proper 2-biorientation D of G and all mutually opposite arcs xy, yx of D satisfy $d^-(x) = 2$ and $d^-(y) = 1$. By Observation 3.1 we can delete xy to obtain a proper 2-biorientation and therefore a proper 2-orientation which contradicts our assumption. □

3.3 spon(G) vs pbon(G) for arbitrary graphs G

In this section, we will first prove that $\text{spon}(G) \leq \text{pbon}(G)$ for every graph G and then obtain a characterization of graphs G for which $\text{spon}(G) = \text{pbon}(G)$.

Theorem 3.17. *For every graph G,* $\text{spon}(G) \leq \text{pbon}(G)$*. Moreover, for every proper pbon(G)-biorientation D of G, there is a semi-proper orientation D' of G*

such that the in-weight of every vertex x in D' is no more than its in-degree of x in D.

Proof. Let D be a proper pbon(G)-biorientation of G. A vertex $v \in V(G)$ is called of *the first type* if there are arcs into v but they are all non-single, and of *the second type*, otherwise. Let v be a vertex of the first type. Then delete all (non-single) arcs into v. Note that in the new D, the in-degree of v equals zero and the in-degree of each of its neightbors in G is positive and has not changed. Thus, D remains a proper biorientation of G and the in-degree of every vertex has not increased. Note that v is now a vertex of the second type. If the new D has a vertex of the first type, continue as above.

Now we may assume that all vertices of D are of the second type. We will perform the following procedure. For every vertex u incident with $2p_u(> 0)$ non-single arcs in D (we count separately both members of a pair of arcs of the form uv, vu), delete every non-single arc into u and set the weight of some single arc into u to $p_u + 1$. Set the weight of every non-weighted arc to 1. Note that when the procedure ends we get a semi-proper orientation D' of G in which the in-weight of every vertex is no more than its in-degree in the initial proper pbon(G)-biorientation of G. Thus, we are done. □

Theorem 3.18. *For a graph G and integer k, we have* spon(G) = pbon(G) = k *if and only if spon(G) = k and there is a proper k-semi-orientation such that the in-weight of each vertex is no more than its degree.*

Proof. If spon(G) = pbon(G) = k, then clearly there is a proper k-biorientation of G. By Theorem 3.17, we can obtain a semi-proper k-orientation of G from a proper k-biorientation of G such that the in-weight of each vertex in the semi-proper orientation is no more than its in-degree in the biorientation. We are done.

Conversely, assume that spon(G) = k and there is a semi-proper k-orientation

D' of G such that the in-weight of each vertex in D' is no more than its degree in G. Then we can obtain a proper k-biorientation D of G in the following way. Since the in-weight of each vertex in D' is no more than its degree in G, for every vertex v of G add some arcs opposite to existing single arcs to make the number of arcs into each vertex equal to its in-weight. Now we can set the weight of every edge to 1 to obtain a proper k-biorientation. Since $k = \text{spon}(G) \leq \text{pbon}(G)$, we conclude that $\text{pbon}(G) = k$. □

There is an infinite number of graphs G with $\text{spon}(G) < \text{pbon}(G)$. Indeed, Dehghan [26] observed that for every tree T, $\text{spon}(T) \leq 2$ due to the following semi-proper 2-orientation. Choose a vertex v of T and for an edge xy of T call x the v-closer vertex of xy if the path from v to y includes x. Orient every edge xy from its v-closer vertex to the other vertex and assign weight 1 (2, respectively) to every edge xy with v-closer vertex x such that the distance from v to y is odd (even, respectively). However, the trees T_n constructed in the proof of Theorem 3.10 are of the proper biorientation number 3.

3.4 Open Problems

We have provided only a sufficient condition for $\text{pbon}(G) = \text{pon}(G)$. It would be interesting to establish a full characterization. All graphs G which we studied satisfy $\text{pon}(G) - \text{pbon}(G) \leq 1$. Is this true in general? If not, is there a constant c such that $\text{pon}(G) - \text{pbon}(G) \leq c$ for every graph G.

There is a large number of open problems on the proper orientation number of graphs listed in [1,2,5,9,10]. It would be interesting to investigate biorientation analogs of these problems. Among of the most studied proper orientation number problems are the following two posed in [8, 10]: Is there a constant c such that for every outerplanar (planar, respectively) graph G, $\text{pon}(G) \leq c$. While these problems remain unsolved [5], Dehghan and Havet [27] and independently Gu et al. [21] proved that $\mathrm{spon}(G) \leq 4$ for every outerplanar graph G and Dehghan and Havet [27] proved that $\mathrm{spon}(G) \leq 6$ for every planar graph G. It would be interesting to attack the two problems for the proper biorientation number.

Chapter 4

Proximity and Remoteness in Directed and Undirected Graphs

4.1 Introduction

The *distance* of a vertex $u \in V(D)$ is defined as $\sigma(u) = \sum_{v \in V(D)} d(u, v)$, and the *average distance* of a vertex as $\bar{\sigma}(u) = \frac{\sigma(u)}{n-1}$. The *eccentricity* $\mathrm{ecc}(u)$ of a vertex u in a digraph D is the distance from u to a vertex farthest from u, the *radius* $\mathrm{rad}(D)$ and *diameter* $\mathrm{diam}(D)$ are the minimum and maximum of all eccentricities of D, respectively. For a nonnegative integer p, let $[p] = \{i \in \mathbb{Z} \mid 1 \leq i \leq p\}$. Thus, $[0] = \emptyset$.

If i is a nonnegative integer and $v \in V(D)$, then $N^{+i}(v)$ is the set of all vertices at distance i from v, and n_i is its cardinality. In particular, $N^{+0}(v) = \{v\}$ and $N^{+1}(v) = N^{+}(v)$. Observe that we have that $n_i \geq 1$ if and only if $0 \leq i \leq \mathrm{ecc}(v)$. Let p be a positive integer. A vertex u of a digraph D is called a *p-king* if $V(D) = \bigcup_{i=0}^{p} N^{+i}(u)$.

We make use of the notion of *distance degree* $X(v)$ of v defined in [30] as the sequence $(n_0, n_1, \ldots, n_{\mathrm{ecc}(v)})$. For an arbitrary sequence $X = (x_0, x_1, \ldots, x_\ell)$ of integers, let $g(X) = \sum_{i=0}^{\ell} i x_i$. Observe that $\sigma(v) = g(X(v))$.

The *proximity* $\pi(D)$ and the *remoteness* $\rho(D)$ are $\min\{\bar{\sigma}(u)|\ u \in V(D)\}$ and $\max\{\bar{\sigma}(u)|\ u \in V(D)\}$, respectively. These parameters were introduced independently by Zelinka [64] and Aouchiche and Hansen [11] for undirected graphs and studied in several papers, see e.g. [11, 14, 28–30, 64]. We could not find any literature pertaining to proximity and remoteness for directed graphs. In this chapter, we extend the study of these two distance measures from undirected graphs to directed graphs. We also present a new result on the topic for undirected graphs.

There are several results in the literature on proximity and remoteness. Several authors have also investigated the relationships between these two distance measures and with other graph parameters such as diameter and radius. Zelinka [64] and Aouchiche and Hansen [11] independently showed that in a connected graph G of order n, the proximity and the remoteness are bounded from above as follows.

Theorem 4.1. *Let G be a connected graph of order $n \geq 2$. Then*

$$1 \leq \pi(G) \leq \begin{cases} \frac{n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{n+1}{4} + \frac{1}{4(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

The lower bound holds with equality if and only if G has a vertex of degree $n-1$. The upper bound holds with equality if and only if G is a path or a cycle. Also, $1 \leq \rho(G) \leq n/2$. The lower bound holds with equality if and only if G is a complete graph. The upper bound holds with equality if and only if G is a path.

In Section 4.2, we prove Theorem 4.3, which is an analog of Theorem 4.1 for strong digraphs. Note that the upper bound for proximity in Theorems 4.3 is different from the one in Theorem 4.1.

The difference between remoteness and proximity for graphs with given order was bounded by Aouchiche and Hansen [11] as follows.

Theorem 4.2. *Let G be a connected graph on $n \geq 3$ vertices with remoteness $\rho(G)$ and proximity $\pi(G)$. We have*

$$\rho(G) - \pi(G) \leq \begin{cases} \frac{n-1}{4} & \text{if } n \text{ is odd,} \\ \frac{n-1}{4} + \frac{1}{4(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

Equality holds if and only if G is a graph obtained from a path $P_{\lceil \frac{n}{2} \rceil}$ and any connected graph H on $\lfloor \frac{n}{2} + 1 \rfloor$ vertices by connecting an endpoint of the path with any vertex of H.

In Section 4.2, we obtain Theorem 4.4, which is an analog of Theorem 4.2 for strong digraphs. Note that the upper bound on the difference between remoteness and proximity given in Theorem 4.4 is different from that in Theorem 4.2. Theorem 4.4 describes all strong digraphs D for which the upper bound on $\rho(D) - \pi(D)$ holds with equality. A trivial lower bound for $\rho(D) - \pi(D)$ is 0. Neither Theorem 4.2 nor Theorem 4.4 provide characterizations on when $\rho(H) = \pi(H)$, where H is a graph and digraph, respectively. While it seems to be hard to characterize* all digraphs D with the property $\rho(D) = \pi(D)$, we start research in this direction in Section 4.3. We prove that a tournament satisfies such a property if and only if it is regular. To show this result we first prove an analog of Theorem 4.3 for strong tournaments, which may be of independent interest. One may conjecture that every strong digraph D with $\rho(D) = \pi(D)$ is regular. However, we demonstrate that this is already false for bipartite tournaments. In Section 4.4, we show that this is also false for undirected graphs.

We conclude our Chapter in Section 4.5 with a discussion about distance parameters for directed and undirected graphs. For more research on proximity and remoteness and related distance parameters for undirected graphs, see, e.g., [11, 14, 29, 30].

*It appears to be an interesting open questions to characterize such strong digraphs and connected graphs.

4.2 Bounds on Proximity and Remoteness for Strong Digraphs

Theorem 4.3. *Let D be a strong digraph on $n \geq 3$ vertices. We have*

$$1 \leq \pi(D) \leq \frac{n}{2}.$$

The lower bound holds with equality if and only if D has a vertex of out-degree $n-1$. The upper bound holds with equality if and only if D is a dicycle. And,

$$1 \leq \rho(D) \leq \frac{n}{2}.$$

The lower bound holds with equality if and only if D is a complete digraph. The upper bound holds with equality if and only if D is strong and contains a Hamiltonian dipath $v_1 v_2 \ldots v_n$ such that $\{v_i v_j \mid 2 \leq i+1 < j \leq n\} \subseteq A(\overline{D})$.

Proof. The lower bounds and extreme digraphs for them are trivial. For the upper bounds, since $\pi(D) \leq \rho(D)$, it suffices to prove only the upper bound for remoteness. Let u be an arbitrary vertex of D. It suffices to show that $\sigma(u) \leq \binom{n}{2}$. Let $X(u) = (n_0, n_1, \ldots, n_d)$ and $v \in N^{+d}(u)$, where $d = \mathrm{ecc}(u)$. Then there is a dipath from u to v implying that $n_i \geq 1$ for every $i \in [d]$ (clearly $n_0 = 1$). Let $X = X(u)$. Observe that $d \leq n-1$ and if $d = n-1$ then $n_i = 1$ for each $i \in [d]$ implying that $\sigma(u) = g(X) = \binom{n}{2}$. If $d < n-1$ then there is a maximum i such that $n_i > 1$. Observe that for $X' = (n_0, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_d, 1)$, we have $g(X) < g(X')$. Replacing X by X' and constructing a new X' if d is still smaller than $n-1$, we will end up with $X = (1, \ldots, 1)$ implying that $\sigma(u) < \binom{n}{2}$.

Observe that the upper bound of $\pi(D)$ is reached if D is a dicycle. Now let D be a strong digraph which is not a dicycle and let u be a vertex such that $d = d_D^+(u) = \Delta^+(D) \geq 2$. Then $(n-1)\pi(D) \leq \sigma(u) = g(X(u))$. Observe that $X(u) = (1, d, n_2, \ldots, n_{\mathrm{ecc}(u)})$ and $\mathrm{ecc}(u) \leq n - d$. Using the same argument as in the paragraph above, we conclude that $g(X(u)) < g(Y)$, where $Y = (1, y_1, \ldots, y_{n-1})$ and $y_i = 1$ for each $i \in [n-1]$. Thus, $(n-1)\pi(D) < \binom{n}{2}$ and $\pi(D) < n/2$.

The upper bound of $\rho(D)$ is reached if and only if D is strong and there is a vertex v_1 such that $\mathrm{ecc}(v_1) = n - 1$. The last condition is equivalent to D containing a Hamiltonian dipath $v_1v_2 \dots v_n$ such that $\{v_iv_j \mid 2 \le i+1 < j \le n\} \subseteq A(\overline{D})$ (as otherwise $\mathrm{ecc}(v_1) < n - 1$). $\square$

Theorem 4.3 allows us to easily prove the following:

Theorem 4.4. *Let D be a strong digraph. We have $\rho(D) - \pi(D) \le \frac{n}{2} - 1$. The upper bound holds with equality if and only if D contains a Hamiltonian dipath $v_1v_2 \dots v_n$ such that $\{v_iv_j \mid 2 \le i+1 < j \le n\} \subseteq A(\overline{D})$ and at least one of the vertices v_{n-1}, v_n has out-degree $n - 1$.*

Proof. The upper bound follows from the lower bound on $\pi(D)$ and upper bound on $\rho(D)$ in Theorem 4.3. Observe that if $\rho(D) - \pi(D) = \frac{n}{2} - 1$, then $\rho(D) = \frac{n}{2}$ and $\pi(D) = 1$. By Theorem 4.3, this is equivalent to D containing a Hamiltonian dipath $v_1v_2 \dots v_n$ such that $\{v_iv_j \mid 2 \le i+1 < j \le n\} \subseteq A(\overline{D})$ and at least one of the vertices v_{n-1}, v_n has out-degree $n - 1$. $\square$

4.3 Digraphs D with $\rho(D) = \pi(D)$

The definition of a regular digraph implies that in a regular tournament we have $d^+(u) = d^-(u) = \frac{n-1}{2}$ for every $u \in V(T)$. Note that this can happen only when n is odd. A tournament is *almost regular* if n is even and $d^+(u) = \frac{n}{2}$ or $d^+(u) = \frac{n-2}{2}$ for every $u \in V(T)$.

This section has two parts: the first proves that for a strong tournament T, $\rho(T) = \pi(T)$ if and only if T is regular, and the second provides an infinite family of strong bipartite tournaments T for which $\rho(T) \neq \pi(T)$.

4.3.1 Tournaments

We will use the following well-known result.

Proposition 4.5. *[45] Every tournament contains a 2-king, moreover, every vertex with maximum out-degree is a 2-king.*

The next theorem is an analog of Theorem 4.3 for tournaments and allows us to easily show that for a strong tournament T, $\rho(T) = \pi(T)$ if and only if T is regular.

Theorem 4.6. *Let D be a strong tournament on n vertices. We have*

$$\frac{n}{n-1} \leq \pi(D) \leq \begin{cases} \frac{3}{2} & \text{if } n \text{ is odd,} \\ \frac{3}{2} - \frac{1}{2(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

The lower bound holds with equality if and only if $\Delta^+(D) = n-2$. The upper bound holds with equality if and only if D is a regular or almost regular tournament. And,

$$\frac{n}{2} \geq \rho(D) \geq \begin{cases} \frac{3}{2} & \text{if } n \text{ is odd,} \\ \frac{3}{2} + \frac{1}{2(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

The lower bound holds with equality if and only if D is a regular or almost regular tournament. The upper bound holds with equality if and only if D is isomorphic to the tournament T_n with $V(T_n) = \{v_1, \dots, v_n\}$ and $A(T_n) = \{v_i v_{i+1} \mid i \in [n-1]\} \cup \{v_j v_i \mid 2 \leq i+1 < j \leq n\}$.

Proof. Since D is strong, $\Delta^+(D) \leq n-2$. Thus, $\frac{n}{n-1} \leq \pi(D)$. We have $\pi(D) = \frac{n}{n-1}$ if and only if there are vertices u, v, w such that $ux \in A(D)$ for all $x \in V(D) \setminus \{u, v\}$, $vu \in A(D)$ but $wv \in A(D)$, which occurs if and only if $\Delta^+(D) = n-2$ since by Proposition 4.5 a vertex of out-degree $n-2$ is a 2-king.

For the upper bound on $\pi(D)$, consider a vertex u of D such that $d^+(u) = \Delta^+(D)$. Then, by Proposition 4.5, u is a 2-king of D. Since the average out-degree is $\frac{n-1}{2}$, we have that

$$d^+(u) \geq \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Now,

$$\begin{aligned} \sigma(u) &= \sum_{i=0}^{d} i n_i \\ &= \Delta^+(D) + 2\left(n - \Delta^+(D) - 1\right) \\ &\leq \begin{cases} \frac{3}{2}(n-1) & \text{if } n \text{ is odd,} \\ \\ \frac{3}{2}n - 2 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

And taking the average completes the proof of this bound. Observe that the equality holds only when

$$d^+(u) = \Delta^+(D) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The upper bound of $\rho(D)$ and the uniqueness (up to isomorphism) of a tournament D with $\rho(D) = n/2$ follow directly from the corresponding bound in Theorem 4.3 and the characterization of digraphs H for which $\rho(H) = n/2$.

For the lower bound of $\rho(D)$, consider a vertex u of D such that $d^+(u) = \delta^+(D)$. We have that

$$d^+(u) \leq \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \\ \frac{n-2}{2} & \text{if } n \text{ is even.} \end{cases}$$

Now,

$$\begin{aligned} \sigma(u) &= \sum_{i=0}^{d} i n_i \\ &\geq \delta^+(D) + 2\left(n - \delta^+(D) - 1\right) \\ &\geq \begin{cases} \frac{3}{2}(n-1) & \text{if } n \text{ is odd,} \\ \\ \frac{3}{2}n - 1 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

And taking the average completes the proof of this bound. Observe that the equality holds only when

$$d^+(u) = \delta^+(D) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \\ \frac{n-2}{2} & \text{if } n \text{ is even.} \end{cases}$$

Thus the lower bound holds with equality if and only if D is a regular or almost regular tournament. □

Theorem 4.6 allows us to easily obtain such a characterization of strong tournaments T with $\rho(T) = \pi(T)$.

Theorem 4.7. *For any strong tournament T, we have $\rho(T) = \pi(T)$ if and only if T is a strong regular tournament.*

Proof. Since $\pi(D) \leq \rho(D)$ for every strong digraph and by the upper bounds on $\pi(H)$ and lower bounds for $\rho(H)$ for strong tournaments in Theorem 4.6 (including the characterizations of when the bounds hold), we have that for a strong tournament T of order n, $\rho(T) = \pi(T)$ if and only if T is regular. □

4.3.2 Bipartite Tournaments

Following Theorem 4.7, one may conjecture that every digraph D with $\rho(D) = \pi(D)$ is regular. However, in this subsection we will show that this is not true and such counterexamples can be found already among bipartite tournaments. To obtain such counterexamples, we will first study some properties of strong bipartite tournaments.

In what follows, let $T = T[A, B]$ be a strong bipartite tournament with partite sets A and B of sizes n and m, respectively.

Definition 4.8. A vertex x of a digraph D is called *bad* if the out-neighborhood of x is a proper subset of the out-neighborhood of another vertex of D. A vertex

z is *good* if it is not bad. A bipartite tournament is called *bad* if it has a bad vertex, otherwise it is called *good*.

Definition 4.9. For a vertex v, we denote by $M(v)$ the set of vertices with the same out-neighborhood as v, i.e., $M(v) = \{u|N^+(u) = N^+(v)\}$; and we denote $|M(v)|$ by $\mu(v)$.

Lemma 4.10. *If $T = T[A, B]$ is a bad strong bipartite tournament, then $\pi(T) \neq \rho(T)$.*

Proof. Let vertices u and v be such that $N^+(u) \subset N^+(v)$. Then there is a vertex $w \in N^+(v) \setminus N^+(u)$. Note that for every $x \in (A \cup B) \setminus \{u, v, w\}$, we have $d(v, x) \leq d(u, x)$. Since vwu is a path in T, we have $2 = d(v, u) \leq d(u, v)$. Finally, $1 = d(v, w) < d(u, w)$.

Thus, $\sigma(v) < \sigma(u)$ implying $\pi(T) \neq \rho(T)$. □

Thus, we may restrict ourselves only to good strong bipartite tournaments.

The following result is Lemma 1 in [21].

Proposition 4.11. *Let T be a bipartite tournament without vertices of in-degree 0 and let x be a good vertex. Then we have $d(x, y) \leq 3$ for every $y \in V(D) \setminus M(x)$.*

Lemma 4.12. *Let $T = T[A, B]$ be a good strong bipartite tournament. Then every vertex of T is a 4-king.*

Proof. Let x be an arbitrary vertex and, without loss of generality, let $x \in A$. By Lemma 4.11, $d(x, y) \leq 3$ for every $y \in V(D) \setminus M(x)$. Let $z \in M(x)$. Since T is strong, there an arc yz for some $y \in B$. Hence $d(x, u) \leq 4$ for every $u \in A \cup B$. □

Lemma 4.13. *Let $T = T[A, B]$ be good, $\rho(T) = \pi(T)$ and $u, v \in V(T)$ such that $u \neq v$, and let $n = |A|$, $m = |B|$. If both u and v are in A or in B, then*

$$\mu(u) - d^+(u) = \mu(v) - d^+(v). \tag{4.3.1}$$

If $v \in A$ and $u \in B$ then

$$2(\mu(v) - d^+(v)) + |B| = 2(\mu(u) - d^+(u)) + |A|. \tag{4.3.2}$$

Proof. For every vertex $v \in A$, $\sigma(v) = \sum_{i=0}^{4} in_i = d^+(v) + 2(n - \mu(v)) + 3(m - d^+(v)) + 4(\mu(v) - 1) = 2(\mu(v) - d^+(v)) + 2n + 3m - 4$. Similarly, if $v \in B$ then $\sigma(v) = 2(\mu(v) - d^+(v)) + 2m + 3n - 4$. The results of the lemma follow from $\sigma(u) = \sigma(v)$ when u and v are from the same partite set of T and when they are from different partite sets of T. □

Lemmas 4.12 and 4.13 imply the following:

Corollary 4.14. *For a strong bipartite tournament T, we have $\rho(T) = \pi(T)$ if and only if T is good and there is a constant c such that for every $v \in A$ and $u \in B$,*

$$2(\mu(v) - d^+(v)) + m = 2(\mu(u) - d^+(u)) + n = c.$$

In particular, for a strong bipartite tournament T with $n = m$, we have $\rho(T) = \pi(T)$ if and only if T is good and $d^+(u) - \mu(u)$ is the same for every vertex u.

To help us find bipartite tournaments T with $\rho(T) = \pi(T)$, one can use the following:

Corollary 4.15. *Let T be a good bipartite tournament and c a constant such that $\mu(x) = c$ for every $x \in V(T)$. We have $\rho(T) = \pi(T)$ if and only if $d^+(v) = m/2$ and $d^+(u) = n/2$ for every $v \in A$ and $u \in B$.*

Proof. Suppose that $\rho(T) = \pi(T)$. By (4.3.1) and $\mu(x) = c$ for every $x \in V(T)$, there exist constants c' and c'' such that $d^+(v) = c'$ and $d^+(u) = c''$ for every $v \in A$ and $u \in B$. Since T has nm arcs, $c'n + c''m = nm$. By (4.3.2) and $\mu(x) = c$ for every $x \in V(T)$, we have $m - 2c' = n - 2c''$ implying $c'' = c' - \frac{m-n}{2}$. After substitution of c'' by $c' - \frac{m-n}{2}$ in $c'n + c''m = nm$ and simplification, we obtain $c' = m/2$. This and $m - 2c' = n - 2c''$ imply $c'' = n/2$.

Now suppose that $d^+(v) = m/2$ and $d^+(u) = n/2$ for every $v \in A$ and $u \in B$. Since $\mu(x) = c$ for every $x \in V(T)$ and T is good, by Corollary 4.14, we have $\rho(T) = \pi(T)$. □

The following result shows, in particular, that there are non-regular digraphs D for which $\rho(D) = \pi(D)$.

Theorem 4.16. *For both $|A| = |B|$ and $|A| \neq |B|$, there is an infinite number of bipartite tournaments T with $\rho(T) = \pi(T)$.*

Proof. For the case of $|A| = |B|$, we can use the following simple bipartite tournament T. Let n be even and let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ be partitions of A and B into subsets of size $n/2$. Let $N^+(v_i) = B_i$ for every $v_i \in A_i$ for $i = 1, 2$. By Corollary 4.15, $\rho(T) = \pi(T)$.

For the case of $|A| \neq |B|$, first consider the following bipartite tournament T_1. Let $A = \{1, 2, 3, 4\}, B = \{1', 2', 3', 4', 5', 6'\}$, and $N^+(1) = \{1', 2', 3'\}, N^+(2) = \{1', 4', 5'\}, N^+(3) = \{2', 4', 6'\}$, $N^+(4) = \{3', 5', 6'\}$. Observe that $d^+(v) = 3$ and $\mu(v) = 1$ for every $v \in A$, $d^+(u) = 2$ and $\mu(u) = 1$ for every $u \in B$. Thus, by Corollary 4.15, T_1 has $\rho(T_1) = \pi(T_1)$. Now we obtain a new bipartite tournament T_t from T_1 by replacing every vertex x in T by $t \geq 2$ vertices $x_1, \dots, x_t$ such that $x_i y_j \in A(T_t)$ if and only if $xy \in A(T)$. Let A_t and B_t be the partite sets of T_t, $t \geq 1$. Observe that for every $t \geq 1$, T_t is good, $\mu(x) = t$ for every $x \in V(T_t)$, and $d^+(v) = |B_t|/2$ and $d^+(u) = |A_t|/2$ for every $v \in A$ and $u \in B$. Thus, by Corollary 4.15 $\rho(T) = \pi(T)$. □

4.4 Undirected Graphs G with $\rho(G) = \pi(G)$

Using computer search, we found two non-regular graphs G for which $\rho(G) = \pi(G)$. The graphs are depicted in Figures 4.1 and 4.2. Using the first of the graphs, we will prove the following:

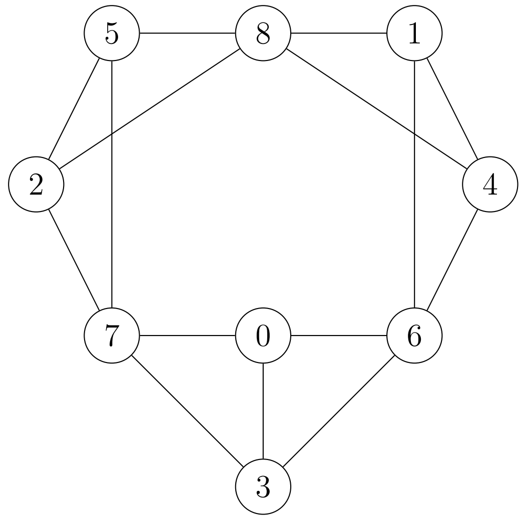


Figure 4.1 A non-regular graph G of order 9 with $\rho(G) = \pi(G)$

Theorem 4.17. *There is an infinite number of non-regular graphs G with $\rho(G) = \pi(G)$.*

Proof. Consider the following non-regular graph G (see Fig. 4.1) with $V(G) = \{v_i \mid i \in \{0\} \cup [8]\}$ and

$$E(G) = \{v_0v_3, v_0v_6, v_0v_7, v_1v_4, v_1v_6, v_1v_8, v_2v_5, v_2v_7, v_2v_8, v_3v_6, v_3v_7, v_4v_6, v_4v_8, v_5v_7, v_5v_8\}.$$

Observe that $X(v_0) = X(v_1) = \ldots = X(v_5) = (1, 3, 4, 1)$ and $X(v_6) = X(v_7) = X(v_8) = (1, 4, 2, 2)$. Thus, $\sigma(v_i) = 14$ for every $i \in \{0\} \cup [8]$ implying that $\rho(G) = \pi(G)$. Now obtain a new graph G_t from G by replacing every vertex x in G by $t \geq 2$ vertices $x_1, \ldots, x_t$ such that $x_iy_j \in E(G_t)$ if and only if $xy \in E(G)$. Hence for each x and each i and j with $i \neq j$, we have $d(x_i, x_j) = 2$. Thus, $\sigma(x_i) = 14t + 2(t - 1)$ for every vertex x_i of G_t implying that $\rho(G_t) = \pi(G_t)$. □

4.5 Discussion

Recall that the *radius* $\mathrm{rad}(D)$ and *diameter* $\mathrm{diam}(D)$ are the minimum and maximum of all eccentricities of D, respectively. The following inequalities hold

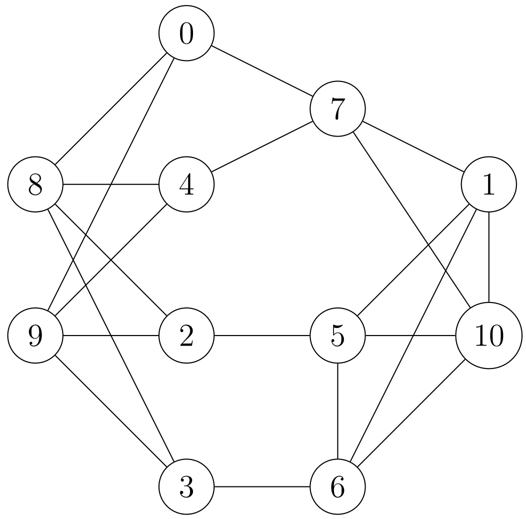


Figure 4.2 A non-regular graph G of order 11 with $\rho(G) = \pi(G)$

by definition for both connected graphs and strong digraphs:

$$\pi(D) \leq \rho(D), \quad \mathrm{rad}(D) \leq \mathrm{diam}(D), \quad \pi(D) \leq \mathrm{rad}(D), \quad \rho(D) \leq \mathrm{diam}(D), \quad 1 \leq \mathrm{diam}(D) \leq n-1.$$

For strong digraphs, the parameters $\mathrm{rad}(D)$ and $\rho(D)$ are incomparable as we can see from the following two examples.

(*i*) Let $c \in [n-1]$ be arbitrary. Denote by D_c the strong digraph with vertices $v_0, v_1, \ldots, v_{n-1}$ and arc set

$$A(D_c) = \{v_0v_j \mid j \in [n-1]\} \cup \{v_pv_{p+1} \mid p \in [n-2]\} \cup \{v_{n-1}v_1, v_cv_0\}.$$

It is not hard to see that $\mathrm{rad}(D_c) = 1$ and $\rho(D_c) = \frac{n}{2}$, hence, $\mathrm{rad}(D_c) < \rho(D)$ for $n \geq 3$.

(*ii*) If D is a dicycle $\overrightarrow{C}_n$ on n vertices, we have that $\mathrm{rad}(D) = n-1$ and $\rho(D) = \frac{n}{2}$, hence, $\mathrm{rad}(D) > \rho(D)$ for $n \geq 3$.

The following results hold for connected graphs but do not necessarily for strong digraphs.

1. For a connected graph G, $\mathrm{diam}(G) \leq 2\mathrm{rad}(G)$. A simple example showing that this inequality does not hold for strong digraphs is when D is the digraph D_c defined above. Then $\mathrm{rad}(D) = 1$ and $\mathrm{diam}(D) = n - 1$, hence, $\mathrm{diam}(D) > 2\mathrm{rad}(D)$.

2. For a connected graph G, if v is a vertex of G such that $\mathrm{ecc}(v) = \mathrm{rad}(G) = r$, then for each $i \in [r-1]$, there exist at least two vertices at distance i from v. A counterexample to this observation for strong digraphs is when D is a dicycle. Then $\mathrm{ecc}(v) = \mathrm{rad}(G) = n - 1$ for every vertex $v \in V(D)$, and for every vertex $v \in V(D)$ there is only one vertex of distance $i \in [n-2]$ from v.

3. For a connected graph G of order n, if $n \geq 2$ then $1 \leq \mathrm{rad}(G) \leq \lfloor \frac{n}{2} \rfloor$. As a counterexample for strong digraphs, again consider D to be a dicycle on n vertices. Indeed, $\mathrm{rad}(D) = n - 1 > \lfloor \frac{n}{2} \rfloor$ for $n \geq 3$.

Chapter 5

Arc-disjoint strong spanning subdigraphs in compositions and products of digraphs

5.1 Introduction

Recall that a digraph $D = (V, A)$ is *strongly connected* (or *strong*) if there exists a path from x to y and a path from y to x in D for every pair of distinct vertices x, y of D. A digraph D is *k-arc-strong* if $D - X$ is strong for every subset $X \subseteq A$ of size at most $k - 1$.

A digraph D is *semicomplete* if for every pair x, y of distinct vertices of D, there is at least one arc between x and y. In particular, a tournament is a semicomplete digraph, where there is exactly one arc between x, y for every pair x, y of distinct vertices. A digraph D is *locally semicomplete* if the out-neighborhood and in-neighborhood of every vertex of D induce semicomplete digraphs.

An *out-branching* B_s^+ (respectively, *in-branching* B_s^-) in a digraph $D = (V, A)$ is a connected spanning subdigraph of D in which each vertex $x \neq s$ has precisely one arc entering (leaving) it and s has no arcs entering (leaving) it. The vertex

s is the *root* of B_s^+ (respectively, B_s^-).

Edmonds [33] characterized digraphs having k arc-disjoint out-branchings rooted at a specified root s. Furthermore, there exists a polynomial algorithm for finding k arc-disjoint out-branchings from a given root s if they exist (see p. 346 of [21]). However, if we ask for the existence of a pair of arc-disjoint branchings B_s^+, B_s^- such that the first is an out-branching rooted at s and the latter is an in-branching rooted at s, then the problem becomes NP-complete (see Section 9.6 of [21]). In connection with this problem, Thomassen [61] posed the following conjecture: There exists an integer N so that every N-arc-strong digraph D contains a pair of arc-disjoint in- and out-branchings.

Bang-Jensen and Yeo generalized the above conjecture as follows.* A digraph $D = (V, A)$ has a *good decomposition* if A has two disjoint sets A_1 and A_2 such that both (V, A_1) and (V, A_2) are strong [19].

Conjecture 5.1. *[20] There exists an integer N so that every N-arc-strong digraph D has a good decomposition.*

For a general digraph D, it is a hard problem to decide whether D has a decomposition into two strong spanning subdigraphs.

Theorem 5.2. *[20] It is NP-complete to decide whether a digraph has a good decomposition.*

Clearly, every digraph with a good decomposition is 2-arc-strong. Bang-Jensen and Yeo characterized semicomplete digraphs with a good decomposition.

Theorem 5.3. *[20] A 2-arc-strong semicomplete digraph D has a good decomposition if and only if D is not isomorphic to S_4, where S_4 is obtained from the complete digraph with four vertices by deleting a cycle of length 4 (see Figure 5.1). Furthermore, a good decomposition of D can be obtained in polynomial time when it exists.*

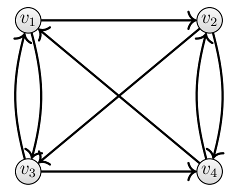


Figure 5.1 Digraph S_4

The following result extends Theorem 5.3 to locally semicomplete digraphs.

Theorem 5.4. *[19] A 2-arc-strong locally semicomplete digraph D has a good decomposition if and only if D is not the second power of an even cycle.*[†]

Let T be a digraph with vertices $u_1, \dots, u_t$ ($t \geq 2$) and let $H_1, \dots H_t$ be digraphs such that H_i has vertices u_{i,j_i}, $1 \leq j_i \leq n_i$. Then the *composition* $Q = T[H_1, \dots, H_t]$ is a digraph with vertex set $\{u_{i,j_i} \colon 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$ and arc set

$$A(Q) = \cup_{i=1}^{t} A(H_i) \cup \{u_{ij_i} u_{pq_p} \colon u_i u_p \in A(T), 1 \leq j_i \leq n_i, 1 \leq q_p \leq n_p\}.$$

In this Chapter, we continue research on good decompositions in classes of digraphs and consider digraph compositions and products.

In Section 5.2, for digraph compositions $Q = T[H_1, \dots H_t]$, we obtain sufficient conditions for Q to have a good decomposition (Theorem 5.6) and a characterization of Q with a good decomposition when T is a strong semicomplete digraph and each H_i is an arbitrary digraph with at least two vertices (Theorem 5.7). Remarkably, in Theorem 5.7 as in Theorem 5.3, there are only a finite number of exceptional digraphs, which for Theorem 5.7 is three. Thus, as Theorems 5.3 and 5.4, Theorem 5.7 confirms Conjecture 5.1 for a special class of digraphs.

In Section 5.3, for digraph products, we prove the following: (a) if $k \geq 2$ is an integer and G is a strong digraph which has a collection of arc-disjoint cycles

[*] Every strong digraph D has an out- and in-branching rooted at any vertex of D.

[†] The second power of a directed cycle $\{v_1, v_2, \dots, v_n, v_1\}$ is obtained by adding an arc from v_i to v_{i+2} for every $i \in [n]$, where $v_{n+1} = v_1$ and $v_{n+2} = v$.

covering all its vertices, then the Cartesian product digraph $G^{\Box k}$ (the kth power of G with respect to Cartesian product) has a good decomposition (Theorem 5.11); (b) for any strong digraphs G, H, the strong product $G \boxtimes H$ has a good decomposition (Theorem 5.14). Necessary definitions of the digraph products are given in Section 5.3.

Simple examinations of our constructive proofs show that all our decompositions can be found in polynomial time.

We conclude Chapte5 in Section 5.4, where we pose a number of open problems.

5.2 Compositions of digraphs

The composition of digraphs is a useful concept in digraph theory, see e.g. [21]. In particular, they are used in the Bang-Jensen-Huang characterization of quasi-transitive digraphs and its structural and algorithmic applications for quasi-transitive digraphs and their extensions; see e.g., [21, 22, 36].

Let us start from a simple observation, which will be useful in the proofs of the theorems of this section.

Lemma 5.5. *Let D be a digraph on t vertices ($t \geq 2$) and let $H'_1, \dots, H'_t$ be digraphs with no arcs. If an induced subdigraph Q^* of $Q' = D[H'_1, \dots, H'_t]$ with at least one vertex in each H'_i, $i \in [t]$ has a good decomposition, then so does Q'.*

Proof. Let $\{u_{i,1}, \dots, u_{i,n_i}\}$ be the set of vertices of H'_i for every $i \in [t]$. For every $i \in [t]$, let $H_i^{(m_i)}$ be the subdigraph of H'_i induced by $\{u_{i,1}, \dots, u_{i,m_i}\}$, where $1 \leq m_i \leq n_i$. Without loss of generality, let $Q^* = D[H_1^{(m_1)}, \dots, H_t^{(m_t)}]$ and let Q^* have a decomposition into arc-disjoint strong spanning subdigraphs Q^*_1, Q^*_2. To extend this decomposition to Q', for every i, j, where $i \in [t]$ and $j \in \{1, 2\}$, add to Q^*_j the vertices $u_{i,m_i+1}, \dots, u_{i,n_i}$ and let them have the same in- and out-neighbors as $u_{i,1}$. (This way the inserted vertices will keep Q_1^* and Q_2^* strongly connected.) □

The following theorem gives sufficient conditions for a digraph composition to have a good decomposition. As in Theorem 5.3, S_4 will denote the digraph obtained from the complete digraph with four vertices by deleting a cycle of length 4.

Theorem 5.6. *Let T be a digraph with vertices $u_1, \dots, u_t$ $(t \geq 2)$ and let $H_1, \dots, H_t$ be digraphs. Let the vertex set of H_i be $\{u_{i,j_i} \colon 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$ for every $i \in [t]$. Then $Q = T[H_1, \dots, H_t]$ has a good decomposition if at least one of the following conditions holds:*

(a) *T is a 2-arc-strong semicomplete digraph and $H_1, \dots, H_t$ are arbitrary digraphs, but Q is not isomorphic to S_4;*

(b) *T has a Hamiltonian cycle and one of the following conditions holds:*

- *t is even and $n_i \geq 2$ for every $i = 1, \dots, t$;*
- *t is odd, $n_i \geq 2$ for every $i = 1, \dots, t$ and at least two distinct subdigraphs H_i have arcs;*
- *t is odd and $n_i \geq 3$ for every $i = 1, \dots, t$ apart from one i for which $n_i \geq 2$.*

(c) *T and all H_i are strong digraphs with at least two vertices.*

Proof. For every $i \in [t]$, let H'_i be the digraph obtained from H_i by deleting all arcs. Let $Q' = T[H'_1, \dots, H'_t]$. We will prove parts of the theorem one by one.

Part (a)

If T is not isomorphic to S_4 then we are done by Theorem 5.3 and Lemma 5.5.

Now assume that T is isomorphic to S_4, but Q is not isomorphic to S_4. Let the vertices of T be u_1, u_2, u_3, u_4 and its arcs

$$u_1u_2, u_2u_1, u_3u_4, u_4u_3, u_1u_4, u_2u_3, u_4u_2, u_3u_1.$$

Since Q is not isomorphic to S_4, at least one of H_1, H_2, H_3, H_4 has at least two vertices. Without loss of generality, let H_1 have at least two vertices. Consider the subdigraph Q^* of Q' induced by $\{u_{1,1}, u_{1,2}, u_{2,1}, u_{3,1}, u_{4,1}\}$. Then Q^* has two arc-disjoint strong spanning subdigraphs: Q_1^* with arcs

$$\{u_{1,1}u_{2,1}, u_{2,1}u_{1,2}, u_{1,2}u_{4,1}, u_{4,1}u_{3,1}, u_{3,1}u_{1,1}\}$$

and Q_2^* with arcs

$$\{u_{2,1}u_{1,1}, u_{1,1}u_{4,1}, u_{4,1}u_{2,1}, u_{2,1}u_{3,1}, u_{3,1}u_{1,2}, u_{1,2}u_{2,1}\}.$$

It remains to apply Lemma 5.5 to obtain a good decomposition of Q.

Part (b) Without loss of generality, assume that $u_1u_2 \ldots u_tu_1$ is a Hamiltonian cycle of T. Let $U = \bigcup_{i=1}^{t}\{u_{i,1}, u_{i,2}\}$.

Case 1: t is even and $n_i \geq 2$ for every $i = 1, \ldots, t$. The following arc sets induce arc-disjoint strong spanning subdigraphs Q_1^*, Q_2^* of $Q'[U]$:

$$\{u_{i,j}u_{i+1,j} \colon\ 1 \leq i \leq t-1, 1 \leq j \leq 2\} \cup \{u_{t,1}u_{1,2}, u_{t,2}u_{1,1}\}; \tag{5.2.1}$$

$$\{u_{i,j}u_{i+1,j')} \colon\ 1 \leq i \leq t-1, 1 \leq j \leq 2\} \cup \{u_{t,1}u_{1,1}, u_{t,2}u_{1,2}\}, \tag{5.2.2}$$

where $j' = j + 1 \pmod 2$.

It remains to apply Lemma 5.5.

Case 2: t is odd, $n_i \geq 2$ for every $i = 1, \ldots, t$ and at least two distinct subdigraphs H_i have arcs. Let e_p, e_q be arcs in two distinct subdigraphs H_p

and H_q. We may assume that both end-vertices of e_p and e_q are in U. Observe that while Q_1^* (with arcs listed in (5.2.1)) is strong, Q_2^* (with arcs listed in (5.2.2)) forms two arc-disjoint cycles C and Z. We may assume that the tail (head) of e_p (e_q) is in C and and the head (tail) of e_p (e_q) is in Z (otherwise, relabel vertices in $\{u_{p,1}, u_{p,2}\}$ and/or $\{u_{q,1}u_{q_2}\}$). Thus, adding e_p and e_q to Q_2^* makes it strong. To obtain two arc-disjoint strong spanning subdigraphs of Q from Q_1^*, Q_2^*, let every vertex $u_{i,j}$ for $j \geq 3$ and $1 \leq i \leq t$ have the same out- and in-neighbors as $u_{i,1}$ in Q'.

Case 3: t is odd and $n_i \geq 3$ for every $i \in [t]$ apart from one i for which $n_i \geq 2$. Without loss of generality, assume that $n_1 \geq 2$ and $n_i \geq 3$ for all $i \in \{2, 3, \dots, t\}$.

First we consider the subcase in which $t = 3$, $n_1 = 2$, and $n_2 = n_3 = 3$. Then Q' has two arc-disjoint spanning subdigraphs Q_1^* and Q_2^* with arc sets

$$\{u_{1,1}u_{2,1}, u_{3,1}u_{1,1}, u_{1,2}u_{2,2}, u_{1,2}u_{2,3}, u_{3,2}u_{1,2}, u_{3,3}u_{1,2}, u_{2,1}u_{3,2}, u_{2,2}u_{3,1}, u_{2,3}u_{3,3}\},$$

$$\{u_{1,1}u_{2,2}, u_{1,1}u_{2,3}, u_{3,2}u_{1,1}, u_{3,3}u_{1,1}, u_{1,2}u_{2,1}, u_{3,1}u_{1,2}, u_{2,1}u_{3,3}, u_{2,2}u_{3,2}, u_{2,3}u_{3,1}\},$$

respectively. Observe that Q_1^* and Q_2^* are strong since they contain the closed walks through all vertices, respectively:

$$u_{1,2}u_{2,2}u_{3,1}u_{1,1}u_{2,1}u_{3,2}u_{1,2}u_{2,3}u_{3,3}u_{1,2};\, u_{1,1}u_{2,2}u_{3,2}u_{1,1}u_{2,3}u_{3,1}u_{1,2}u_{2,1}u_{3,3}u_{1,1}.$$

Now we extend the previous subcase to that in which $n_1 = 2$ and $n_i = 3$ for all $i \in \{2, 3, \dots, t\}$. First replace index 3 in every vertex of the form $u_{3,i}$ by t in the two arc sets of the previous subcase. Then replace every arc of the form $u_{2,i}u_{t,j}$ in Q_1^* by the path $u_{2,i}u_{3,i} \dots u_{t-1,i}u_{t,j}$. In Q_2^*, we replace $u_{2,1}u_{t,3}$ by the path $u_{2,1}u_{3,2}u_{4,1}u_{5,2} \dots u_{t-1,1}u_{t,3}$, replace $u_{2,2}u_{t,2}$ by the path $u_{2,2}u_{3,1}u_{4,2}u_{5,1} \dots u_{t-1,2}u_{t,2}$, replace $u_{2,3}u_{t,1}$ by the path $u_{2,3}u_{3,2}u_{4,3}u_{5,2} \dots u_{t-1,3}u_{t,1}$, and finally add the path $u_{2,2}u_{3,3}u_{4,2}u_{5,3} \dots u_{t-1,2}$.

Finally, we extend the previous subcase to the general one using Lemma 5.5.

Part (c) For $j \in \{1, 2\}$, let T_j be the subdigraph of Q induced by vertex set $\{u_{i,j} \colon 1 \leq i \leq t\}$. Clearly, $T_1 \cong T_2 \cong T$ and T_1 and T_2 are strong.

Let Q_1 be the spanning subdigraph of Q with arc set $A(Q_1) = A(T_1) \cup (\bigcup_{i=1}^{t} A(H_i))$. Observe that Q_1 is strong since T_1 and each H_i are strong, and T_1 has a common vertex with each H_i, where $1 \leq i \leq t$.

Let Q_2 be the spanning subdigraph of Q with arc set $A(Q_2) = A(Q) \setminus A(Q_1)$. To see that Q_2 is strong, we only need to find a strong subdigraph in Q_2 which contains x and y for each pair of distinct vertices x and y in Q_2.

Let $x = u_{1,1}$. Without loss of generality, assume that $y \in \{u_{1,2}, u_{2,1}, u_{2,2}\}$. We first consider the subcase that $y = u_{2,1}$. Observe that there is at least one arc entering and one arc leaving $u_{1,2}$ ($u_{2,2}$) in T_2, and so there are two arcs, say a and b (c and d), with opposite directions between x (y) and T_2 in Q_2. Then by adding the arcs a, b, c, d, and the vertices x, y to T_2, we obtain a strong subdigraph T_2' of Q_2 which contains both x and y, as desired. For the case that $y \in \{u_{1,2}, u_{2,2}\}$, we just add the arcs a, b, and the vertex x to T_2, and then obtain a strong subdigraph T_2'' of Q_2 which contains both x and y. Hence, Q_2 is strong. Then we conclude that Q has a good decomposition.

□

We will use Theorem 5.6 to prove the following characterization for certain compositions $T[H_1, \ldots, H_t]$, where T is a strong semicomplete digraph. In the characterization, $\overline{K_p}$ will stand for the digraph of order p with no arcs. Also, $\overrightarrow{C}_k$ and $\overrightarrow{P}_k$ will denote the cycle and path with k vertices, respectively.

Theorem 5.7. *Let T be a strong semicomplete digraph on $t \geq 2$ vertices and let $H_1, \ldots, H_t$ be arbitrary digraphs, each with at least two vertices. Then $Q = T[H_1, \ldots, H_t]$ has a good decomposition if and only if Q is not isomorphic to one* *of the following three digraphs:* $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_2}]$, $\overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K_2}, \overline{K_2}]$. $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$.

Proof. Let us first prove the 'only if' part of the theorem, i.e. $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_2}]$, $\overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K_2}, \overline{K_2}]$ and $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$ do not have good decompositions. By Lemma 5.5, it suffices to show that neither $\overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K_2}, \overline{K_2}]$ nor $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$ has a good decomposition. The proof is by reductio ad absurdum.

Suppose that $Q = \overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K_2}, \overline{K_2}]$ has a decomposition into two strong spanning subdigraphs Q_1, Q_2. Since Q has 13 arcs, without loss of generality, we may assume that Q_1 is a Hamiltonian cycle of Q. Since the arc of H_1 cannot be in a Hamiltonian cycle of Q, without loss of generality, let $Q_1 = u_{1,1}u_{2,1}u_{3,1}u_{1,2}u_{2,2}u_{3,2}u_{1,1}$. Then the remaining arcs of Q form two disjoint cycles $u_{1,1}u_{2,2}u_{3,1}u_{1,1}$ and $u_{1,2}u_{2,1}u_{3,2}u_{1,2}$ and a single arc between them, a contradiction to the assumption that Q_2 is strong.

Suppose that $Q = \overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$ has a decomposition into two strong spanning subdigraphs Q_1, Q_2. Since Q has 16 arcs and has no Hamiltonian cycle, each of Q_1, Q_2 has 8 arcs. Since Q has only cycles of lengths 3 and 6 and Q_1 is strong, without loss of generality, we may assume that Q_1 consists of a cycle $u_{1,1}u_{2,1}u_{3,1}u_{1,2}u_{2,2}u_{3,2}u_{1,1}$ and a path $u_{2,1}u_{3,3}u_{1,1}$. Then Q_2 consists of two cycles $u_{1,1}u_{2,2}u_{3,1}u_{1,1}$ and $u_{1,2}u_{2,1}u_{3,2}u_{1,2}$ and a path $u_{2,2}u_{3,3}u_{1,2}$. Observe that Q_2 is not strong, a contradiction.

Now we will show the 'if' part of the theorem by reductio ad absurdum as well. Assume that Q is not isomorphic to either of the three digraphs, but has no good decomposition.

By Camion's Theorem [24], T has a Hamiltonian cycle $C = u_1u_2 \dots u_tu_1$. Thus, Conditions (b) of Theorem 5.6 are applicable. By the conditions, t must be odd and for at least two distinct indexes $p, q \in \{1, 2, \dots, t\}$, we have $n_p = n_q = 2$.

Suppose $t \geq 5$. Then there will be arcs between H_i and H_{i+2} in Q for every $i = 1, 2, \dots, t-2$. Recall Case 2 of Part (b) of the proof of Theorem 5.6. The arcs between H_i and H_{i+2} can be used to make Q_2 strong instead of arcs e_p and e_q used in Case 2 of Part (b) of the proof of Theorem 5.6. Thus, Q has a good decomposition, a contradiction. Hence, $t = 3$ and, without loss of generality, $n_1 = n_2 = 2$ and $n_3 \geq 2$.

Suppose that T has opposite arcs. One of these arcs will not be on the Hamiltonian cycle C of T and will correspond to four or more arcs in Q. Now recall Case 2 of Part (b) of the proof of Theorem 5.6. Two of the above-mentioned arcs can be used to make Q_2 strong instead of arcs e_p and e_q used in Case 2 of Part (b) of the proof of Theorem 5.6. Thus, Q has a good decomposition, a contradiction. Hence, $T = \overrightarrow{C}_3$.

Suppose that $n_3 \geq 4$. To get a contradiction, by Lemma 5.5 it suffices to show that $Q = \overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_4}]$ has a decomposition into two strong spanning subdigraphs Q_1, Q_2, where Q_1 consists of a cycle $u_{1,1}u_{2,1}u_{3,1}u_{1,2}u_{2,2}u_{3,2}u_{1,1}$ and two paths $u_{2,1}u_{3,4}u_{1,1}$ and $u_{2,2}u_{3,3}u_{1,2}$ and Q_2 consists of two cycles $u_{1,1}u_{2,2}u_{3,1}u_{1,1}$ and $u_{1,2}u_{2,1}u_{3,2}u_{1,2}$ and two paths $u_{2,1}u_{3,3}u_{1,1}$ and $u_{2,2}u_{3,4}u_{1,2}$. Thus, $n_3 \leq 3$.

Now consider the case of $n_1 = n_2 = 2$ and $n_3 = 3$. Since Q is not isomorphic to $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$, it has an arc in either H_1 or H_2 or H_3, and by Conditions (b) of Theorem 5.6, only one of H_1, H_2, H_3 has an arc a. Without loss of generality, assume that if H_1 has an arc then $a = u_{1,2}u_{1,1}$, if H_2 has an arc then $a = u_{2,1}u_{2,2}$ and if H_3 has an arc then $a = u_{3,2}u_{3,1}$. Then Q has a decomposition into two spanning subdigraphs Q_1, Q_2, where Q_1 consists of a cycle $u_{1,1}u_{2,1}u_{3,1}u_{1,2}u_{2,2}u_{3,2}u_{1,1}$ and a path $u_{2,1}u_{3,3}u_{1,1}$ and Q_2 consists of two cycles $u_{1,1}u_{2,2}u_{3,1}u_{1,1}$ and $u_{1,2}u_{2,1}u_{3,2}u_{1,2}$, a path $u_{2,2}u_{3,3}u_{1,2}$ and arc a. Observe that both Q_1 and Q_2 are strong, a contradiction.

It remains to consider the case of $n_1 = n_2 = n_3 = 2$. Since Q is not isomorphic to $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_2}]$, at least one of H_1, H_2 and H_3 has an arc. By Conditions (b) of Theorem 5.6, only one of H_1, H_2 and H_3 has an arc. Without loss of generality, assume that H_1 has an arc. Suppose that H_1 has two arcs. Then $H_1 = \overrightarrow{C}_2$. Then we can use the arcs of H_1 to make Q_2 strong instead of arcs e_p and e_q used in Case 2 of Part (b) of the proof of Theorem 5.6. Thus, Q has a good decomposition, a contradiction. Hence, if H_1 has an arc, it must have just one arc. This concludes our proof. □

5.3 Products of digraphs

The *Cartesian product* $G\Box H$ of two digraphs G and H is a digraph with vertex set $V(G\Box H) = V(G) \times V(H) = \{(x, x')\colon\ x \in V(G), x' \in V(H)\}$ and arc set $A(G\Box H) = \{(x, x')(y, y')\colon\ xy \in A(G), x' = y',\ or\ x = y, x'y' \in A(H)\}$. By definition, the Cartesian product is associative and commutative (up to isomorphism), and $G\Box H$ is strongly connected if and only if both G and H are strongly connected [42]. We define the kth powers with respect to Cartesian product as $D^{\Box k} = \underbrace{D\Box D\Box \cdots \Box D}_{k \text{ times}}$.

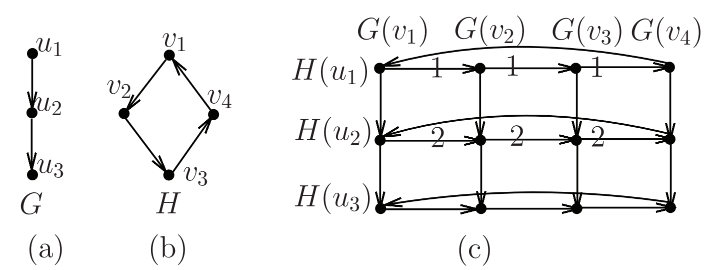


Figure 5.2 Two digraphs G, H and their Cartesian product.

In the arguments of this section, we will use the following terminology and notation. Let G and H be two digraphs with $V(G) = \{u_i\colon\ 1 \le i \le n\}$ and $V(H) = \{v_j\colon\ 1 \le j \le m\}$. For simplicity, we let $u_{i,j} = (u_i, v_j)$ for $1 \le i \le n, 1 \le j \le m$. We use $G(v_j)$ to denote the subdigraph of $G\Box H$ induced by vertex set $\{u_{i,j}\colon\ 1 \le i \le n\}$, where $1 \le j \le m$, and use $H(u_i)$ to denote the subdigraph of $G\Box H$ induced by vertex set $\{u_{i,j}\colon\ 1 \le j \le m\}$, where $1 \le i \le n$.

Clearly, we have $G(v_j) \cong G$ and $H(u_i) \cong H$. (For example, as shown in Figure 5.2, $G(v_j) \cong G$ for $1 \leq j \leq 4$ and $H(u_i) \cong H$ for $1 \leq i \leq 3$.) For $1 \leq j_1 \neq j_2 \leq m$, u_{i,j_1} and u_{i,j_2} belong to the same digraph $H(u_i)$, where $u_i \in V(G)$; we call u_{i,j_2} the *vertex corresponding to* u_{i,j_1} in $G(v_{j_2})$; for $1 \leq i_1 \neq i_2 \leq n$, we call $u_{i_2,j}$ the vertex corresponding to $u_{i_1,j}$ in $H(u_{i_2})$. Similarly, we can define the subdigraph *corresponding* to some other subdigraph. For example, in Fig. 5.2(c), let P_1 (P_2) be the path labelled 1 (2) in $H(u_1)$ ($H(u_2)$), then P_2 is called the path *corresponding* to P_1 in $H(u_2)$.

Lemma 5.8. *For any integer $n \geq 2$, the product digraph $D = \overrightarrow{C}_n \Box \overrightarrow{C}_n$ can be decomposed into two arc-disjoint Hamiltonian cycles.*

Proof. Let $G = H \cong \overrightarrow{C}_n$; moreover $G = u_1u_2 \dots u_nu_1$ and $H = v_1v_2 \dots v_nv_1$. Let $P_i = G(v_i) - u_{n-i,i}u_{n+1-i,i}$ for $1 \leq i \leq n-1$ and $P_n = G(v_n) - u_{n,n}u_{1,n}$. Let $Q_i = H(u_i) - u_{i,n-i}u_{i,n+1-i}$ for $1 \leq i \leq n-1$ and $Q_n = H(u_n) - u_{n,n}u_{n,1}$. Furthermore, let

$$D' = \left(\bigcup_{i=1}^{n-1} (P_i \cup \{u_{n-i,i}u_{n-i,i+1}\}) \right) \cup (P_n \cup \{u_{n,n}u_{n,1}\})$$

and

$$D'' = \left(\bigcup_{i=1}^{n-1} (Q_i \cup \{u_{i,n-i}u_{i+1,n-i}\}) \right) \cup (Q_n \cup \{u_{n,n}u_{1,n}\})$$

By the construction, the subdigraphs D' and D'' are Hamiltonian cycles of D. For example, see Figure 5.3 for the case that $n = 5$ (the Hamiltonian cycle D' consists of five "vertical" paths P_i of order five and five "horizontal" arcs, D'' consists of five "horizontal" paths Q_i of order five and five "vertical" arcs, furthermore, these two cycles are symmetric about the diagonal.) □

Note that deciding whether a digraph D has a collection of arc-disjoint cycles covering all vertices of D can be done in polynomial time using network flows. Indeed, assign lower bound 1 and upper bound $\min\{d^-(x), d^+(x)\}$ to every vertex x in D and lower bound 0 and upper bound 1 to every arc of D. Observe that

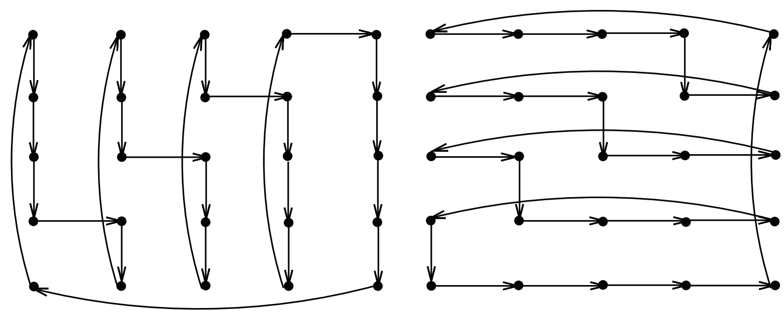


Figure 5.3 Two arc-disjoint Hamiltonian cycles for the case $n = 5$.

the resulting network has a feasible flow if and only if D has a collection of arc-disjoint cycle covering all vertices of D. Observe that the existence of a flow in a network with lower and upper bounds on vertices and arcs can be decided in polynomial time, see e.g. Chapter 4 in [21]. Moreover, we can compute such a flow in polynomial time (if it exists) and obtain the corresponding collection of cycles in D. The following lemma may be of independent interest.

Lemma 5.9. *Let G be a strong digraph with at least two vertices which has a collection of arc-disjoint cycles covering all its vertices. Then the product digraph $D = G\Box G$ has a good decomposition. Moreover, such a good decomposition can be found in polynomial time.*

Proof. By the arguments in the paragraph before this lemma, we may assume that we are given a collection $(P_0, P_1, P_2, \cdots, P_p)$ of arc-disjoint cycles covering all vertices of G. For each $h \in \{0, 1, 2, \cdots, p\}$, let G_h denote the digraph with vertices $\bigcup_{i=0}^{h} V(P_i)$ and arcs $\bigcup_{i=0}^{h} A(P_i)$. Now we will prove the lemma by induction on the number of cycles in the collection.

For the base step, by Lemma 5.8, we have that $G_0\Box G_0 = P_0\Box P_0$ can be decomposed into two arc-disjoint strong spanning subdigraphs.

For the inductive step, we assume that $G_h\Box G_h$ $(0 \leq h \leq p-1)$ can be decomposed into two arc-disjoint strong spanning subdigraphs D'_h and D''_h. We will construct two arc-disjoint strong spanning subdigraphs in $G_{h+1}\Box G_{h+1}$.

If $V(G_h) \subseteq V(P_{h+1})$, then P_{h+1} is a Hamiltonian cycle of G_{h+1}, and we are done by Lemma 5.8. If $V(P_{h+1}) \subseteq V(G_h)$, then G_h is a strong spanning subdigraph of G_{h+1}, and we are also done by the induction hypothesis.

In the following argument, we assume that $V(G_h)\setminus V(P_{h+1}) \neq \emptyset$ and $V(P_{h+1})\setminus V(G_h) \neq \emptyset$. Without loss of generality, for the first copies of G_h and P_{h+1} in $G_h\Box G_h$ and $P_{h+1}\Box P_{h+1}$, let $V(G_h) = \{u_i\colon 1 \leq i \leq t\}$, $V(P_{h+1}) = \{u_i\colon s \leq i \leq \ell\}$. We have $1 < s \leq t < \ell$. For the second copies of G_h and P_{h+1} in $G_h\Box G_h$ and $P_{h+1}\Box P_{h+1}$, we will use v_i's rather than u_i's.

By Lemma 5.8, in $G_{h+1}\Box G_{h+1}$, the subdigraph $P_{h+1}\Box P_{h+1}$ can be decomposed into two arc-disjoint strong spanning subdigraphs $\overline{D}'_h$ and $\overline{D}''_h$. Observe that

$$V(G_h\Box G_h) \cap V(P_{h+1}\Box P_{h+1}) \supseteq \{u_{t,t}\} \text{ and } A(G_h\Box G_h) \cap A(P_{h+1}\Box P_{h+1}) = \emptyset.$$

For $1 \leq j \leq s-1$, let $G_{h,j}$ be the subdigraph of $G(v_j)$ corresponding to P_{h+1}. For $t+1 \leq j \leq \ell$, let $G_{h,j}$ be the subdigraph of $G(v_j)$ corresponding to G_h. For $1 \leq i \leq s-1$, let $H_{h,i}$ be the subdigraph of $H(u_i)$ corresponding to P_{h+1}. For $t+1 \leq i \leq \ell$, let $H_{h,i}$ be the subdigraph of $H(u_i)$ corresponding to G_h.

Now let D'_{h+1} be a union of the following strong digraphs: D'_h, $\overline{D}'_h$, $H_{h,i}$ and $G_{h,j}$ for all $t+1 \leq i,j \leq \ell$. Observe that D'_{h+1} is a strong spanning subdigraph of $G_{h+1}\Box G_{h+1}$ since $\overline{D}'_h$ has at least one common vertex with each of D'_h, $H_{h,i}$ and $G_{h,j}$ for all $t+1 \leq i,j \leq \ell$. Let D''_{h+1} be a spanning subdigraph of $G_{h+1}\Box G_{h+1}$ with $A(D''_{h+1}) = A(G_{h+1}\Box G_{h+1}) \setminus A(D'_{h+1})$. Observe that D''_{h+1} is the union of D''_h, $\overline{D}''_h$, $H_{h,i}$ and $G_{h,j}$ for all $1 \leq i,j \leq s-1$. And D''_h has at least one common vertex with each of $\overline{D}''_h$, $H_{h,i}$ and $G_{h,j}$ for all $1 \leq i,j \leq s-1$, thus D''_{h+1} is strong.

Hence, we complete the inductive step and conclude that $D = G\Box G$ can be decomposed into two arc-disjoint strong spanning subdigraphs. Moreover, by the above argument, these subdigraphs can be found in polynomial time. □

Lemma 5.10. *For any two strong digraphs G and H, if G has a good decomposition, then the product digraph $D = G\Box H$ has a good decomposition.*

Proof. Let $V(G) = \{u_i\colon 1 \le i \le n\}, V(H) = \{v_j\colon 1 \le j \le m\}$, and G contain two arc-disjoint strong spanning subdigraphs G_1 and G_2. For $1 \le i \le 2$ and $1 \le j \le m$, let $G_{i,j}$ be the subdigraph of $G(v_j)$ corresponding to G_i. As shown in Figure 5.4, let D' be the union of $H(u_1)$ and $G_{1,j}$ for all $1 \le j \le m$, and let D'' be a subdigraph of D with $V(D'') = V(D)$ and $A(D'') = A(D) \setminus A(D')$. Note that D'' is the union of $H(u_i)$ and $G_{2,j}$ for all $2 \le i \le n, 1 \le j \le m$. Since $H(u_i)$, $G_{1,j}$ and $G_{2,j}$ $(1 \le i \le n, 1 \le j \le m)$ are strong, both D' and D'' are strong spanning subdigraphs of D. This completes the proof. □

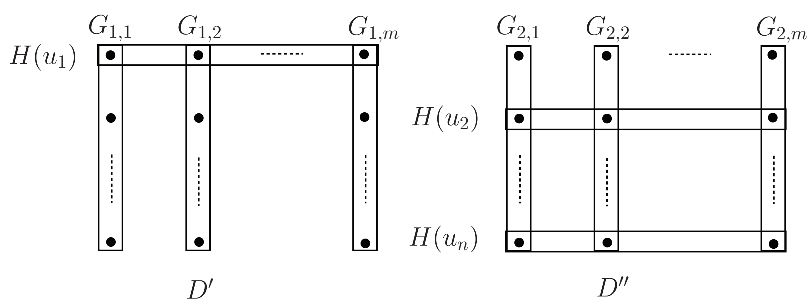


Figure 5.4 Two arc-disjoint strong spanning subdigraphs in Lemma 5.10.

By the definition of $D^{\Box k}$, associativity of the Cartesian product (up to isomorphism), and Lemmas 5.9 and 5.10, we can obtain the following result on $G^{\Box k}$ for any integer $k \ge 2$.

Theorem 5.11. *Let G be a strong digraph of order at least two which has a collection of arc-disjoint cycles covering all its vertices and let $k \ge 2$ be an integer. Then the product digraph $D = G^{\Box k}$ has a good decomposition. Moreover, for any fixed integer k, such a good decomposition can be found in polynomial time.*

The *strong product* $G \boxtimes H$ of two digraphs G and H is a digraph with vertex set $V(G \boxtimes H) = V(G) \times V(H) = \{(x, x')\colon x \in V(G), x' \in V(H)\}$ and arc set

$A(G \boxtimes H) = \{(x, x')(y, y')\colon\ xy \in A(G), x' = y',\ or\ x = y, x'y' \in A(H),\ or\ xy \in A(G),\ x'y' \in A(H)\}$. By definition, $G\Box H$ is a spanning subdigraph of $G \boxtimes H$, and $G \boxtimes H$ is strongly connected if and only if both G and H are strongly connected [42]. In the following argument, we will still use the terminology and notation introduced earlier in this section, since $G\Box H$ is a spanning subdigraph of $G \boxtimes H$.

Lemma 5.12. *For any two integers $n, m \geq 2$, the product digraph $D = \overrightarrow{C}_n \boxtimes \overrightarrow{C}_m$ has a good decomposition.*

Proof. Let $\overrightarrow{C}_n = u_1u_2 \ldots u_nu_1$ and $\overrightarrow{C}_m = v_1v_2 \ldots v_m$. Let D' be the spanning subdigraph of D which is the union of $G(v_j)$ for $1 \leq j \leq m$ and the following additional m arcs: $\{u_{n,j}u_{1,j+1}\colon\ 1 \leq j \leq m-1\} \cup \{u_{1,m}u_{2,1}\}$. Observe that D' is strong. Let D'' be a spanning subdigraph of D with $A(D'') = A(D) \setminus A(D')$. To see that D'' is strong, observe that it contains $H(u_i)$ for $1 \leq i \leq n$ and arcs $\{u_{i,1}u_{i+1,2}\colon\ 1 \leq i \leq n-1\} \cup \{u_{n,m}u_{1,1}\}$. □

We will use the following decomposition of strong digraphs.

An *ear decomposition* of a digraph D is a sequence $\mathcal{P} = (P_0, P_1, P_2, \cdots, P_t)$, where P_0 is a cycle or a vertex and each P_i is a path, or a cycle with the following properties:

(a) P_i and P_j are arc-disjoint when $i \neq j$.

(b) For each $i = 0, 1, 2, \cdots, t$: let D_i denote the digraph with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$. If P_i is a cycle, then it has precisely one vertex in common with $V(D_{i-1})$. Otherwise the end vertices of P_i are distinct vertices of $V(D_{i-1})$ and no other vertex of P_i belongs to $V(D_{i-1})$.

(c) $\bigcup_{j=0}^{t} A(P_j) = A(D)$.

The following result is well-known, see e.g. [21].

Theorem 5.13. *Let D be a digraph with at least two vertices. Then D is strong if and only if it has an ear decomposition. Furthermore, if D is strong, every* *cycle can be used as a starting cycle P_0 for an ear decomposition of D, and there is a linear-time algorithm to find such an ear decomposition.*

Theorem 5.14. *For any strong digraphs G and H with at least two vertices, the product digraph $D = G \boxtimes H$ has a good decomposition. Moreover, such a decomposition can be found in polynomial time.*

Proof. By Theorem 5.14 G has an ear decomposition $\mathcal{P} = (P_0, P_1, P_2, \cdots, P_p)$ and H has an ear decomposition $\mathcal{Q} = (Q_0, Q_1, Q_2, \cdots, Q_q)$, such that P_0 is a cycle of G and Q_0 is a cycle of H by Theorem 5.13. Let G_i denote the subdigraph of G with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$ and let H_i denote the subdigraph of H with vertices $\bigcup_{j=0}^{i} V(Q_j)$ and arcs $\bigcup_{j=0}^{i} A(Q_j)$.

We will prove the theorem by induction on $r \in \{0, 1, \ldots, p+q\}$. For the base step, by Lemma 5.12, we have that $P_0 \boxtimes Q_0$ can be decomposed into two arc-disjoint strong spanning subdigraphs. For the inductive step, we assume that $r = h + g < p + q$ $(h \leq p, g \leq q)$ and $G_h \boxtimes H_g$ can be decomposed into two arc-disjoint strong spanning subdigraphs D' and D''.

Since strong product is a commutative operation, without loss of generality it suffices to prove that $G_{h+1} \boxtimes H_g$ $(h < p)$ can be decomposed into two arc-disjoint strong spanning subdigraphs. Let $V(G_h) = \{u_1, u_2, \ldots, u_\ell\}$, $V(H_g) = \{v_1, v_2, \ldots, v_m\}$ and $v_1 v_s \in A(H_g)$. Let $P_{h+1,j}$ be the subdigraph of $G(v_j)$ corresponding to P_{h+1} for $1 \leq j \leq m$. We will consider two cases.

Case 1: P_{h+1} is a cycle. Let $P_{h+1} = u_\ell u_{\ell+1} \ldots u_n u_\ell$. Observe that every $P_{h+1,j}$ for $1 \leq j \leq m$ shares vertex $u_{\ell,j}$ with D'. Thus, the union U_1 of D' and $P_{h+1,j}$ for $1 \leq j \leq m$ is a strong spanning subdigraph of $G_{h+1} \boxtimes H_g$. Let $V(U_2) = V(G_{h+1} \boxtimes H_g)$ and $A(U_2) = A(G_{h+1} \boxtimes H_g) \setminus A(U_1)$.

Observe that $A(U_2)$ contains $A(D'')$, $A(H(u_i))$ for $\ell + 1 \leq i \leq n$ and $\{u_{i,1}u_{i+1,s} \colon \ell \leq i \leq n-1\} \cup \{u_{n,1}u_{\ell,s}\}$. Thus, U_2 is strong.

Case 2: P_{h+1} is a path. Let $P_{h+1} = u_\ell u_{\ell+1} \dots u_{n-1} u_t$, where $t < \ell$. Let U_1 be the union of D' and $P_{h+1,j}$ for $1 \le j \le m$. Observe that U_1 is a spanning subdigraph of $G_{h+1} \boxtimes H_g$ and strong since every $P_{h+1,j}$ for $1 \le j \le m$ shares its end-vertices with D'. Let $V(U_2) = V(G_{h+1} \boxtimes H_g)$ and $A(U_2) = A(G_{h+1} \boxtimes H_g) \setminus A(U_1)$. Observe that $A(U_2)$ contains $A(D'')$, $A(H(u_i))$ for $\ell + 1 \le i \le n-1$ and $\{u_{i,1}u_{i+1,s} \colon \ell \le i \le n-2\} \cup \{u_{n-1,1}u_{t,s}\}$. Thus, U_2 is strong.

Hence, we complete the inductive step and conclude that $D = G \boxtimes H$ can be decomposed into two arc-disjoint strong spanning subdigraphs. Furthermore, by Theorem 5.13, the proof of Lemma 5.12, and the argument of this theorem, we can conclude that these two strong spanning subdigraphs can be found in polynomial time. □

The *lexicographic product* $G \circ H$ of two digraphs G and H is a digraph with vertex set $V(G \circ H) = V(G) \times V(H) = \{(x, x') \colon x \in V(G), x' \in V(H)\}$ and arc set $A(G \circ H) = \{(x, x')(y, y') \colon xy \in A(G),\ or\ x = y\ and\ x'y' \in A(H)\}$ [42]. By definition, $G \boxtimes H$ is a spanning subdigraph of $G \circ H$, so the following result holds by Theorem 5.14: For any strong connected digraphs G and H with orders at least 2, the product digraph $D = G \circ H$ can be decomposed into two arc-disjoint strong spanning subdigraphs. Moreover, these two arc-disjoint strong spanning subdigraphs can be found in polynomial time. In fact, we can get a more general result.

A digraph is *Hamiltonian decomposable* if it has a family of Hamiltonian dicycles such that every arc of the digraph belongs to exactly one of the dicycles. Ng [53] gives the most complete result among digraph products.

Theorem 5.15. *[53] If G and H are Hamiltonian decomposable digraphs, and $|V(G)|$ is odd, then $G \circ H$ is Hamiltonian decomposable.*

Theorem 5.15 implies that if G and H are Hamiltonian decomposable digraphs, and $|V(G)|$ is odd, then $G \circ H$ can be decomposed into two arc-disjoint strong spanning subdigraphs. It is not hard to extend this result as follows: for any strong digraphs G and H of orders at least 2, if H contains $\ell \geq 1$ arc-disjoint strong spanning subdigraphs, then the product digraph $D = G \circ H$ can be decomposed into $\ell + 1$ arc-disjoint strong spanning subdigraphs.

5.4 Open Problems

We have characterized digraphs $T[H_1, \dots, H_t]$, where T is strong semicomplete and every H_i is arbitrary with at least two vertices, which have a good decomposition. It is a natural open problem to extend the characterization to all such digraphs, where some H_i's can have just one vertex. Of course, the extended characterization would generalize also Theorem 5.3.

A digraph Q is *quasi-transitive*, if for any triple x, y, z of distinct vertices of Q, if xy and yz are arcs of Q then either xz or zx or both are arcs of Q. For a recent survey on quasi-transitive digraphs and their generalizations, see a chapter [36] by Galeana-Sánchez and Hernández-Cruz. Bang-Jensen and Huang [18] proved that a quasi-transitive digraph is strong if and only if $Q = T[H_1, \dots, H_t]$, where T is a strong semicomplete digraph and each H_i is a non-strong quasi-transitive digraph or has just one vertex. Thus, a special case of the above problem is to characterize strong quasi-transitive digraphs with a good decomposition. This would generalize Theorem 5.3 as well.

We believe that these characterizations will confirm Conjecture 5.1 for the classes of quasi-transitive digraphs and digraphs $T[H_1, \dots, H_t]$, where T is strong semicomplete. In the absence of the characterizations, it would still be interesting to confirm the conjecture at least for quasi-transitive digraphs.

In Lemma 5.9, we show that $G \Box H$ contains a pair of arc-disjoint strong spanning subdigraphs when $G \cong H$. However, the following result implies Lemma 5.9 cannot be extended to the case that $G \ncong H$, since it is not hard to show that the Cartesian product digraph of any two cycles has a pair of arc-disjoint strong spanning subdigraphs if and only if it has a pair of arc-disjoint Hamiltonian cycles.

Theorem 5.16. *[62] The Cartesian product $\overrightarrow{C_p} \Box \overrightarrow{C_q}$ is Hamiltonian if and only if there are non-negative integers d_1, d_2 for which $d_1 + d_2 = \gcd(p, q) \geq 2$ and $\gcd(p, d_1) = \gcd(q, d_2) = 1$.*

However, Lemma 5.9 could hold for the case that $G \ncong H$ if we add other conditions. As shown in Lemma 5.10, we know $G \Box H$ contains a pair of arc-disjoint strong spanning subdigraphs when one of G and H contains a pair of arc-disjoint strong spanning subdigraphs. So the following open question is interesting: for any two strong digraphs G and H, neither of which contain a pair of arc-disjoint strong spanning subdigraphs, under what condition the product digraph $G \Box H$ contains a pair of arc-disjoint strong spanning subdigraphs?

Furthermore, we may also consider the following more challenging question: under what conditions the product digraph $G \Box H$ ($G \boxtimes H$) has more (than two) arc-disjoint strong spanning subdigraphs?

Chapter 6

k-Ary spanning trees contained in tournaments

6.1 Introduction

Recall that a *tournament* $T = (V, E)$ is a directed graph (digraph) obtained by assigning a direction for each edge in an undirected complete graph. In this chapter, a tournament of order n is called *n-tournament*. We also use $x \to y$ or (x, y) to denote an arc $xy \in E$, say x *beats* y. Let $A \Rightarrow B$ denote that every vertex in A beats every vertex in B. We call a tournament *transitive* if $x \to y$ and $y \to z$ imply that $x \to z$, in other words, its vertices can be linearly ordered such that each vertex beats all later vertices. We denote by $[X_i]$ the vertex set $\{x_1, \ldots, x_i\}$ for $i \geq 1$.

If $x \in V$ and $X \subseteq V$, we denote by $N^+_X(x)$ ($N^-_X(x)$) the set of out(in)-neighbours of x in X, that is, $N^+_X(x) = N^+(x) \cap X$ ($N^-_X(x) = N^-(x) \cap X$) (here, x may or may not belong to X). A tournament is *k-regular* if all vertices have in-degree and out-degree k. For a subset $X \subseteq V$, we denote by $T[X]$ the subtournament of T induced by X.

A *rooted tree* is a directed tree with a special vertex, called the *root*, such that there exists a unique (directed) path from the root to any other vertex. A rooted tree is called a *k-ary tree*, if all non-leaf vertices have exactly k children, except possibly one non-leaf vertex has at most $k - 1$ children. If all non-leaf vertices have exactly k children, then we call it a *full k-ary tree.* A *k-star* is full $(k-1)$-ary tree with k vertices.

An oriented graph H on n vertices is *unavoidable* if every n-tournament contains H as a subgraph, otherwise, we say that H is *avoidable*. The concept of unavoidable was introduced by Linial et al. [46], in which they studied the maximum number of edges that an unavoidable subgraph on n vertices can have. In particular, if H contains a directed cycle then H must be avoidable, since a transitive tournament contains no directed cycles and hence no copy of H. It is therefore natural to ask which oriented trees are unavoidable.

Rédei [56] showed that every tournament contains a Hamiltonian path. Thomason [60] proved that all orientations of sufficiently long cycles are unavoidable except for those which yield directed cycles. Erdős [57] proved that for any fixed positive integer m, there exists a number $f(m)$ such that every n-tournament contains $\lfloor \frac{n}{m} \rfloor$ vertex-disjoint transitive sub-tournaments of order m if $n \geq f(m)$. Häggkvist and Thomason [41] showed that every oriented tree of order m is contained in every tournament of order $12m$ and El Sahili [31] improved the bound to $3(m-1)$. Lu et al. [47, 49] investigated the avoidable claws. For more results on unavoidable digraphs, we refer to [35, 48, 58].

Actually, Rédei's result [56] can be restated as that a 1-ary spanning tree is unavoidable. It is therefore natural to study the general problem of whether a tournament contains a k-ary spanning tree. Lu et al. [50] proved the following fundamental theorem for the existence of a k-ary spanning tree of a tournament.

Theorem 6.1 ([50])**.** *For any fixed positive integer k, there exists a number $h'(k)$ such that every n-tournament contains a k-ary spanning tree if $n \geq h'(k)$.*

Define $h(k)$ as the minimum number such that every tournament of order at least $h(k)$ contains a k-ary spanning tree. The existence of a Hamiltonian path for any tournament is the same as $h(1) = 1$. Lu et al. [50] determined that $h(2) = 4$ and $h(3) = 8$. The exact values of $h(k)$ remain unknown for $k \geq 4$. In this Chapter, we prove that $h(k) = \Omega(k \log k)$, especially, $h(4) = 10$ and $h(5) \geq 13$.

Theorem 6.2. *For any $k \geq 4$, $h(k) = \Omega(k \log k)$.*

Theorem 6.3. *$h(4) = 10$ and $h(5) \geq 13$.*

6.2 Proof of Theorem 6.2

For any $X, Y \subseteq V(T)$, we say that X *dominates* Y if for every $v \in Y \backslash X$ there exists a $u \in X$ which beats v. The *domination number* of T, denoted $\mu(T)$, is the smallest cardinality of a set that dominates $V(T)$.

Erdős [32] used the probabilistic method to prove the following fact.

Lemma 6.4 ([32]). *For every $\varepsilon > 0$ there is a number K such that for every $k \geq K$ there exists a tournament T_k with no more than $2^k k^2 \log(2+\varepsilon)$ vertices such that $\mu(T_k) > k$.*

By the proof of Lemma 6.4, we can get the following Corollary 6.5 directly which is stated in [51].

Corollary 6.5 ([51]). *There exists a constant $c > 0$ such that for every n there exists a tournament T with n vertices such that $\mu(T) > c \log n$.*

Now we present the proof of Theorem 6.2.

Proof of Theorem 6.2.
Let T be a tournament with n vertices and $\mu(T) > c \log n$. Suppose T contains a k-ary spanning tree R. Since the number of non-leaf vertices of R is $\lceil \frac{n-1}{k} \rceil$ and all non-leaf vertices of R dominates $V(T)$, we have $\lceil \frac{n-1}{k} \rceil \geq \mu(T)$. Then $n > (\mu(T) - 1)k + 1$. By Corollary 6.5, we have $h(k) = \Omega(k \log k)$. □

6.3 Proof of Theorem 6.3

We need the following two useful lemmas proved in [50].

Lemma 6.6 ([50]). *Let R be a k-ary tree of tournament T with the root v and S a k-star of T with the root u, where R and S are vertex disjoint. If $d^+_{V(R)}(u) \geq 1$, then T contains a k-ary tree R' with $V(R') = V(R) \cup V(S)$. Furthermore, if $u \in N^+(v)$, then R' can be chosen to have the root v, which is the same root as R.*

Proof. If $u \to v$, then $E(R) \cup E(S) \cup uv$ induces a k-ary tree. Thus, we may assume that $v \to u$. Since $d^+_{V(R)}(u) \geq 1$, there exists a vertex x such that $u \to x$. Let P be the path of R from v to x, and y the first vertex in P such that $u \to y$. Then $p(y) \to u$, thus, $(E(R) - p(y)y) \cup E(S) \cup \{p(y)u, uy\}$ induces a desired k-ary tree. The 'furthermore' part is evident by the argument above. □

Lemma 6.7 ([50]). *If every $(km+1)$-tournament has a k-ary spanning tree, then so does every km-tournament.*

According to the structure of k-ary spanning trees, we can directly obtain the following result.

Observation 6.8. *For any n-tournament $T = (V, E)$ with $n \geq 2k+1$, let $T_{\geq k} = \{v \in V \mid d^+(v) \geq k\}$. If for every two different vertices $u, v \in T_{\geq k}$, $|(N^+(u) \cup N^+(v)) \backslash \{u, v\}| \leq 2k-2$, then T contains no k-ary spanning tree.*

Proof of Theorem 6.3.
First, we consider the case of $k = 4$. Let T_9 be the 9-tournament with $V(T_9) = \{0, 1, \ldots, 8\}$ and $E(T_9) = \{ij : i - j \equiv 1, 2, 3, 5 \pmod 9\}$. By Observation 6.8, it is straightforward to check that T_9 does not contain a 4-ary spanning tree, since $d^+_{V(T_9)}(i) = 4$ for any $i \in V(T_9)$, and $N^+_{V(T_9)}(j) \cap N^+_{V(T_9)}(i) \neq \emptyset$ for any $j \in N^+_{V(T_9)}(i)$. So $h(4) \geq 10$. In the following, by induction, we will prove that every tournament T of order $n \geq 10$ contains a 4-ary spanning tree.

Let $T = (V, E)$ be a tournament of order n. Note that for any $X \subseteq V$, we have $d^+_X \geq \left\lceil \frac{|X|-1}{2} \right\rceil$. Suppose $n \geq 14$ and the theorem is true for all $n' < n$. Since $n \geq 14$, we can choose $v \in V$ with $d^+(v) \geq 4$, say $\{a, b, c, d\} \subset N^+(v)$. Let $T' = T[V \backslash \{v, a, b, c\}]$. By the induction hypothesis, T' contains a 4-ary spanning tree. By Lemma 6.6, T contains a 4-ary spanning tree. Therefore, by Lemma 6.7, it suffices to prove that every tournament T of order n contains a 4-ary spanning tree, where $n \in \{10, 11, 13\}$. Let u be a vertex of T with the maximum out-degree and $V = \{u\} \cup [X_{n-1}]$.

Claim 1. For $1 \leq d^-(u) \leq 4$, if there exists a vertex $v \in N^-(u)$ such that $d^+_{N^-(u)}(v) = d^-(u) - 1$ and $d^+_{N^+(u)}(v) \geq 4 - d^-(u)$, then T contains a 4-ary spanning tree.

Proof. Let $N^-(u) = [X_{d^-(u)}]$. Suppose $d^+_{N^-(u)}(x_1) = d^-(u) - 1$ and $d^+_{N^+(u)}(x_1) \geq 4 - d^-(u)$, say $x_1 \Rightarrow \{x_2, x_3, x_4\}$. Since $d^+_{\{x_5,\dots,x_{n-1}\}} \geq \lceil \frac{n-6}{2} \rceil \geq n - 9$, we may assume $x_8 \Rightarrow \{x_9, \dots, x_{n-1}\}$. Then we obtain a 4-ary spanning tree of T induced by $\{x_1x_2, x_1x_3, x_1x_4, x_1u, ux_5, \dots, ux_8, x_8x_9, \dots, x_8x_{n-1}\}$. □

Claim 2. Every 10-tournament T contains a 4-ary spanning tree.

Proof. We consider the following five cases.

Case 1: $d^+(u) = 9$.

Since $d^+_{[X_9]} \geq 4$, we assume $x_9 \Rightarrow [X_4]$ and $x_6 \to x_5$. Then we obtain a 4-ary spanning tree induced by $\{ux_6, \dots, ux_9, x_9x_1, \dots, x_9x_4, x_6x_5\}$.

Case 2: $d^+(u) = 8$, say $N^+(u) = [X_8]$.

Since $d^+_{[X_8]} \geq 4$, we assume $x_8 \Rightarrow [X_4]$. Then we obtain a 4-ary spanning tree induced by $\{ux_5, \dots, ux_8, x_8x_1, \dots, x_8x_4, x_9u\}$.

Case 3: $d^+(u) = 7$, say $N^+(u) = [X_7]$ and $x_9 \to x_8$.

By Claim 1, we may assume $d^+_{[X_7]}(x_9) \leq 1$. If $d^+_{[X_7]}(x_8) \geq 3$, assume $x_8 \Rightarrow \{x_7, x_6, x_5\}$, and then $\{x_9x_8, x_8x_5, x_8x_6, x_8x_7, x_8u, ux_1, \dots, ux_4\}$ induces a desired 4-ary spanning tree. So we may assume $d^+_{[X_7]}(x_8) \leq 2$. Then $\left|N^-_{[X_7]}(x_9) \cap N^-_{[X_7]}(x_8)\right| \geq 4$, say $[X_4] \Rightarrow \{x_8, x_9\}$. Without loss of generality, we may assume that $d^+_{[X_4]}(x_3) \geq 2$ with $x_3 \Rightarrow [X_2]$ and $x_6 \to x_7$. Then we obtain a 4-ary spanning tree induced by $\{ux_3, \dots, ux_6, x_3x_1, x_3x_2, x_3x_8, x_3x_9, x_6x_7\}$.

Case 4: $d^+(u) = 6$, say $N^+(u) = [X_6]$ and $x_9 \to x_8, x_8 \to x_7$.

If $d^+_{[X_6]}(x_9) \geq 2$ or $d^+_{[X_6]}(x_8) \geq 2$, say $x_9 \Rightarrow \{x_5, x_6\}$ or $x_8 \Rightarrow \{x_5, x_6\}$, then we obtain a 4-ary spanning tree induced by $\{x_9x_5, x_9x_6, x_9x_8, x_9u, ux_1, \dots, ux_4, x_8x_7\}$ or $\{x_9x_8, x_8x_5, x_8x_6, x_8x_7, x_8u, ux_1, \dots, ux_4\}$. So we assume $\left|N^-_{[X_6]}(x_9) \cap N^-_{[X_6]}(x_8)\right| \geq 4$, say $[X_4] \Rightarrow \{x_9, x_8\}$. Without loss of generality, we may assume that $d^+_{[X_4]}(x_3) \geq 2$ with $x_3 \Rightarrow [X_2]$. Then we obtain a desired 4-ary spanning tree induced by $\{ux_3, \dots, ux_6, x_3x_1, x_3x_2, x_3x_8, x_3x_9, x_8x_7\}$.

Case 5: $d^+(u) = 5$, say $N^+(u) = [X_5]$.

By Claim 1, we may assume $d^+_{N^-(u)} \leq 2$. Without loss of generality, we may assume that $x_9 \Rightarrow \{x_7, x_8\}$, $x_8 \Rightarrow \{x_7, x_6\}$ and $x_6 \rightarrow x_9$. If $d^+_{[X_5]}(x_9) \geq 1$ or $d^+_{[X_5]}(x_8) \geq 1$, say $x_9 \rightarrow x_5$ or $x_8 \rightarrow x_5$, then one can find a desired tree induced by $\{x_6x_9, x_9x_5, x_9x_7, x_9x_8, x_9u, ux_1, \ldots, ux_4\}$ or $\{x_9x_8, x_8x_5, x_8x_6, x_8x_7, x_8u, ux_1, \ldots, ux_4\}$. So we may further assume $[X_5] \Rightarrow \{x_9, x_8\}$. Without loss of generality, assume that $x_7 \rightarrow x_6$. Since $d^+(x_7) \leq 5$, we have $\left|N^-_{[X_5]}(x_7)\right| \geq 2$, say $\{x_4, x_5\} \Rightarrow x_7$ and $x_4 \rightarrow x_5$. Then the set $\{ux_1, \ldots, ux_4, x_4x_5, x_4x_9, x_4x_8, x_4x_7, x_7x_6\}$ induces a desired 4-ary spanning tree. □

Suppose $n \in \{11, 13\}$ and $d^+(u) = n - 1$. By Claim 2, let R be a 4-ary spanning tree of $T[[X_{10}]]$. Without loss of generality, we assume that $R' \subseteq R$ is a full 4-ary tree rooted at x_9 with $V(R') = [X_9]$. Then $R' \cup \{ux_9, \ldots, ux_{n-1}\}$ induces a desired 4-ary spanning tree. So we may further assume $n \in \{11, 13\}$ and $N^+(u) = [X_{d^+(u)}]$ with $d^+(u) \leq n - 2$ in the following.

Claim 3. Every 11-tournament T contains a 4-ary spanning tree.

Proof. We consider the following five cases.

Case 1: $d^+(u) = 9$.

By Claim 1, we may assume $[X_7] \Rightarrow x_{10}$. Without loss of generality, we assume $x_4 \Rightarrow [X_3]$ since $d^+_{[X_7]} \geq 3$, and $x_7 \Rightarrow \{x_8, x_9\}$ since $d^+_{\{x_5,\ldots,x_9\}} \geq 2$. Then we obtain a 4-ary spanning tree of T induced by $\{ux_4, \ldots, ux_7, x_4x_1, x_4x_2, x_4x_3, x_4x_{10}, x_7x_8, x_7x_9\}$.

Case 2: $d^+(u) = 8$.

Without loss of generality, we assume that $x_{10} \rightarrow x_9$ and $x_4 \Rightarrow \{x_5, x_6, x_7, x_8\}$ because $d^+_{[X_8]} \geq 4$. Then we find a desired 4-ary spanning tree induced by $\{x_{10}x_9, x_{10}u, ux_1, \ldots, ux_4, x_4x_5, \ldots, x_4x_8\}$.

Case 3: $d^+(u) = 7$.

Suppose $x_{10} \Rightarrow \{x_8, x_9\}$. By Claim 1, we may assume $[X_7] \Rightarrow x_{10}$ and $x_4 \Rightarrow \{x_5, x_6, x_7\}$ since $d^+_{[X_7]} \geq 3$. We obtain a 4-ary spanning tree of T induced by $\{ux_1, \ldots, ux_4, x_4x_5, x_4x_6, x_4x_7, x_4x_{10}, x_{10}x_9, x_{10}x_8\}$. Suppose $x_{10} \rightarrow x_9$, $x_9 \rightarrow x_8$ and $x_8 \rightarrow x_{10}$. If $d^+_{[X_7]}(x_9) \geq 3$, say $x_9 \Rightarrow \{x_5, x_6, x_7\}$, then we obtain a 4-ary spanning tree of T induced by $\{x_{10}x_9, x_{10}u, ux_1, \ldots, ux_4, x_9x_5, \ldots, x_9x_8\}$. If

$x_9 \Rightarrow \{x_6, x_7\}$ and $x_8 \rightarrow x_5$, then we obtain a desired 4-ary spanning tree induced by $\{x_9x_6, x_9x_7, x_9x_8, x_9u, ux_1, \ldots, ux_4, x_8x_{10}, x_8x_5\}$. By the symmetry of x_8 and x_9, we may assume $[X_4] \Rightarrow \{x_8, x_9\}$ and $x_1 \Rightarrow \{x_2, x_3\}$ since $d^+_{[X_4]} \geq 2$. Then $\{x_1x_2, x_1x_3, x_1x_8, x_1x_9, x_8x_{10}, x_8u, ux_4, \ldots, ux_7\}$ induces a desired 4-ary spanning tree.

Case 4: $d^+(u) = 6$.

By Claim 1, we may assume $d^+_{N^-(u)} \leq 2$. Let $x_{10} \Rightarrow \{x_9, x_8\}$, $x_9 \Rightarrow \{x_8, x_7\}$ and $x_7 \rightarrow x_{10}$. If $d^+_{[X_6]}(x_{10}) \geq 2$ or $d^+_{[X_6]}(x_9) \geq 2$, say $x_{10} \Rightarrow \{x_5, x_6\}$ or $x_9 \Rightarrow \{x_5, x_6\}$, then we obtain a desired tree induced by $\{x_7x_{10}, x_7u, x_{10}x_9, x_{10}x_8, x_{10}x_6, x_{10}x_5, ux_1, \ldots, ux_4\}$ or $\{x_{10}x_9, x_{10}u, x_9x_5, \ldots, x_9x_8, ux_1, \ldots, ux_4\}$. So we may further assume $[X_4] \Rightarrow \{x_{10}, x_9\}$ and $x_3 \Rightarrow [X_2]$ because $d^+_{[X_4]} \geq 2$. Then we obtain a 4-ary spanning tree induced by $\{ux_3, \ldots, ux_6, x_3x_1, x_3x_2, x_3x_{10}, x_3x_9, x_9x_8, x_9x_7\}$.

Case 5: $d^+(u) = 5$.

In this case, T is a 5-regular tournament. Let $1 \leq d^+_{[X_5]}(x_4) \leq 2$ with $x_4 \rightarrow x_5$. We may assume $x_4 \Rightarrow \{x_6, x_7, x_8\}$ because $d^+(x_4) = 5$, and let $x_{10} \rightarrow x_9$. Then we obtain a 4-ary spanning tree induced by $\{x_{10}x_9, x_{10}u, ux_1, \ldots, ux_4, x_4x_5, \ldots, x_4x_8\}$.
□

Claim 4. Every 13-tournament T contains a 4-ary spanning tree.

Proof. We consider the following six cases.

Case 1: $d^+(u) = 11$.

By Claim 1, we may assume $[X_9] \Rightarrow x_{12}$ and $x_4 \Rightarrow [X_3]$ because $d^+_{[X_9]} \geq 4$. First we suppose $d^+_{\{x_5,\ldots,x_{11}\}} \geq 4$, say $x_7 \Rightarrow \{x_8, \ldots, x_{11}\}$. Then we obtain a 4-ary spanning tree induced by $\{ux_4, \ldots, ux_7, x_7x_8, \ldots, x_7x_{11}, x_4x_1, x_4x_2, x_4x_3, x_4x_{12}\}$. Next we consider $d^+_{\{x_5,\ldots,x_{11}\}} = 3$. If $x_4 \Rightarrow \{x_5, \ldots, x_{11}\}$, then $d^+(x_4) = 11$. Since $d^+_{N^+(x_4)}(u) \geq 3$, we obtain a 4-ary spanning tree of T by Claim 1. Otherwise there exists a vertex $v \in \{x_5, \ldots, x_{11}\}$ such that $v \rightarrow x_4$, without loss of generality, say $x_8 \Rightarrow \{x_4, \ldots, x_7\}$. Then we obtain a 4-ary spanning tree induced by $\{ux_8, \ldots, ux_{11}, x_8x_4, \ldots, x_8x_7, x_4x_1, x_4x_2, x_4x_3, x_4x_{12}\}$.

Case 2: $d^+(u) = 10$, say $x_{12} \rightarrow x_{11}$.

By Claim 1, we may assume $[X_9] \Rightarrow x_{12}$. If $d^+_{[X_9]}(x_{11}) \geq 3$, say $x_{11} \Rightarrow \{x_7, x_8, x_9\}$ and $x_3 \Rightarrow \{x_4, x_5, x_6\}$ because $d^+_{[X_6]} \geq 3$, then we obtain a 4-ary spanning tree induced by $\{x_{11}x_7, x_{11}x_8, x_{11}x_9, x_{11}u, ux_1, ux_2, ux_3, ux_{10}, x_3x_4, x_3x_5, x_3x_6, x_3x_{12}\}$. So we may assume $[X_7] \Rightarrow \{x_{11}, x_{12}\}$ and $x_3 \Rightarrow [X_2]$ because $d^+_{[X_7]} \geq 3$. Suppose $d^+_{\{x_4,\ldots,x_{10}\}} \geq 4$, say $x_6 \Rightarrow \{x_7, \ldots, x_{10}\}$. Then $\{ux_3, \ldots, ux_6, x_3x_1, x_3x_2, x_3x_{11}, x_3x_{12}, x_6x_7, \ldots, x_6x_{10}\}$ induces a desired 4-ary spanning tree. Next we consider the case when $d^+_{\{x_4,\ldots,x_{10}\}} = 3$. Since $d^+(x_3) \leq 10$, there exists $v \in \{x_4, \ldots, x_{10}\}$ such that $v \to x_3$, without loss of generality, say $x_7 \Rightarrow \{x_3, \ldots, x_6\}$. Then we get a 4-ary spanning tree induced by $\{ux_7, ux_8, ux_9, ux_{10}, x_7x_3, \ldots, x_7x_6, x_3x_1, x_3x_2, x_3x_{11}, x_3x_{12}\}$.

Case 3: $d^+(u) = 9$.

Let $T[\{x_{12}, x_{11}, x_{10}\}]$ be a transitive 3-tournament with $x_{12} \Rightarrow \{x_{11}, x_{10}\}$ and $x_{11} \to x_{10}$. By Claim 1, we may assume $[X_9] \Rightarrow x_{12}$. If $d^+_{[X_9]}(x_{11}) \geq 2$, say $x_{11} \Rightarrow \{x_8, x_9\}$ and $x_4 \Rightarrow \{x_5, x_6, x_7\}$ since $d^+_{[X_7]}(x_4) \geq 3$. Then we obtain a 4-ary spanning tree induced by $\{x_{11}x_8, x_{11}x_9, x_{11}x_{10}, x_{11}u, ux_1, \ldots, ux_4, x_4x_5, x_4x_6, x_4x_7, x_4x_{12}\}$. So we assume $[X_8] \Rightarrow \{x_{12}, x_{11}\}$. If $d^+_{[X_8]}(x_{10}) \geq 3$, say $x_{10} \Rightarrow \{x_6, x_7, x_8\}$ and $x_3 \Rightarrow \{x_1, x_2\}$ because $d^+_{[X_5]} \geq 2$. Then $\{x_{10}x_6, x_{10}x_7, x_{10}x_8, x_{10}u, ux_3, ux_4, ux_5, ux_9, x_3x_1, x_3x_2, x_3x_{11}, x_3x_{12}\}$ induces a desired 4-ary spanning tree. So we may further assume $[X_6] \Rightarrow \{x_{12}, x_{11}, x_{10}\}$ and $x_1 \to x_2$. Finally, we obtain a 4-ary spanning tree of T by a similar discussion for $d^+_{\{x_3,\ldots,x_9\}}$ as Case 2.

Let $x_{12} \to x_{11} \to x_{10} \to x_{12}$. Suppose $d^+_{[X_9]}(x_{12}) \geq 3$, say $x_{12} \Rightarrow \{x_7, x_8, x_9\}$. If $d^+_{[X_6]}(x_{11}) \geq 2$ or $d^+_{[X_6]}(x_{10}) \geq 2$, say $x_{11} \Rightarrow \{x_5, x_6\}$ or $x_{10} \Rightarrow \{x_5, x_6\}$, then we obtain a 4-ary spanning tree induced by $\{x_{12}x_7, x_{12}x_8, x_{12}x_9, x_{12}x_{11}, x_{11}x_5, x_{11}x_6, x_{11}x_{10}, x_{11}u, ux_1, \ldots, ux_4\}$ or $\{x_{10}x_5, x_{10}x_6, x_{10}x_{12}, x_{10}u, ux_1, \ldots, ux_4, x_{12}x_7, x_{12}x_8, x_{12}x_9, x_{12}x_{11}\}$. So we assume $[X_2] \Rightarrow \{x_{12}, x_{11}, x_{10}\}$ when $d^+_{[X_9]}(x_{12}) \leq 5$. When $d^+_{[X_9]}(x_{12}) \geq 6$, we assume $x_{12} \Rightarrow \{x_4, \ldots, x_9\}$ and $x_7 \Rightarrow \{x_8, x_9\}$ because $d^+_{\{x_4,\ldots,x_9\}} \geq 2$. If $x_7 \Rightarrow \{x_{11}, x_{10}\}$, then we obtain a 4-ary spanning tree induced by $\{x_{12}x_5, x_{12}x_6, x_{12}x_7, x_{12}u, x_7x_8, \ldots, x_7x_{11}, ux_1, \ldots, ux_4\}$. Since there are at least two vertices with out-degree more than one in $\{x_4, \ldots, x_9\}$, say x_6 and x_7. So we assume $x_{11} \to x_7$, $x_{10} \to x_6$ and $[X_2] \Rightarrow \{x_{12}, x_{11}, x_{10}\}$. By the symmetry of x_{12}, x_{11} and x_{10}, we get $[X_2] \Rightarrow \{x_{12}, x_{11}, x_{10}\}$ and $x_1 \to x_2$ in each case. Finally, we obtain a 4-ary spanning tree of T by a similar discussion for $d^+_{\{x_3,\ldots,x_9\}}$ as Case 2.

Case 4: $d^+(u) = 8$.

By Claim 1, we may assume $d^+_{N^-(u)} \leq 2$. Let $x_{12} \Rightarrow \{x_{11}, x_{10}\}$, $x_{11} \Rightarrow \{x_{10}, x_9\}$ and $x_9 \rightarrow x_{12}$. First we suppose $d^+_{[X_8]}(x_{12}) \geq 2$, say $x_{12} \Rightarrow \{x_7, x_8\}$. If $d^+_{[X_6]}(x_{11}) \geq 2$ or $d^+_{[X_6]}(x_9) \geq 2$, say $x_{11} \Rightarrow \{x_5, x_6\}$ or $x_9 \Rightarrow \{x_5, x_6\}$, then we get a desired set $\{x_{12}x_{11}, x_{12}x_{10}, x_{12}x_8, x_{12}x_7, x_{11}x_9, x_{11}x_5, x_{11}x_6, x_{11}u, ux_1, \ldots, ux_4\}$ or $\{x_9x_{12}, x_9x_5, x_9x_6, x_9u, x_{12}x_{11}, x_{12}x_{10}, x_{12}x_8, x_{12}x_7, ux_1, \ldots, ux_4\}$. In particular, if $d^+_{[X_8]}(x_{12}) \geq 3$, say $x_{12} \Rightarrow \{x_6, x_7, x_8\}$, and $d^+_{[X_5]}(x_{11}) \geq 1$, then we get a 4-ary spanning tree induced by $\{x_{12}x_{11}, x_{12}x_8, x_{12}x_7, x_{12}x_6, x_{11}x_{10}, x_{11}x_9, x_{11}x_5, x_{11}u, ux_1, \ldots, ux_4\}$. Since $d^+_{[X_8]}(x_{12}) \leq 5$, we may assume $[X_2] \Rightarrow \{x_{12}, x_{11}, x_9\}$ and $x_1 \rightarrow x_2$. And it follows that, when $d^+_{[X_8]}(x_{12}) \geq 2$, $\{x_1x_2, x_1x_{12}, x_1x_{11}, x_1x_9, x_{12}x_{10}, x_{12}x_8, x_{12}x_7, x_{12}u, ux_3, \ldots, ux_6\}$ induces a desired spanning tree.

We next consider the case when $d^+_{[X_8]}(x_{12}) \leq 1$, say $N^+_{[X_8]}(x_{12}) \subseteq \{x_8\}$. If $x_9 \rightarrow x_{10}$, then we assume $[X_5] \Rightarrow \{x_{12}, x_{11}, x_9\}$ by the symmetry of x_{12}, x_{11} and x_9. If $x_{10} \Rightarrow [X_5]$, say $x_2 \rightarrow x_1$, then $\{x_{10}x_2, x_{10}x_3, x_{10}x_4, x_{10}u, ux_5, \ldots, ux_8, x_2x_1, x_2x_{12}, x_2x_{11}, x_2x_9\}$ induces a desired 4-ary spanning tree. So we may further suppose $x_{10} \rightarrow x_9$. If $d^+_{[X_7]}(x_{11}) \geq 1$, say $x_{11} \rightarrow x_7$ and assume $x_4 \Rightarrow [X_3]$ because $d^+_{[X_6]} \geq 3$, then we obtain a 4-ary spanning tree induced by $\{x_{11}x_{10}, x_{11}x_9, x_{11}x_7, x_{11}u, ux_4, ux_5, ux_6, ux_8, x_4x_1, x_4x_2, x_4x_3, x_4x_{12}\}$. So we may assume $[X_7] \Rightarrow x_{11}$. If $d^+_{[X_7]}(x_{10}) \geq 2$, say $x_{10} \Rightarrow \{x_6, x_7\}$, assume $x_3 \Rightarrow [X_2]$ because $d^+_{[X_5]} \geq 2$, then we obtain a desired set $\{x_{10}x_9, x_{10}x_6, x_{10}x_7, x_{10}u, ux_3, ux_4, ux_5, ux_8, x_3x_1, x_3x_2, x_3x_{11}, x_3x_{12}\}$. So we may assume $[X_6] \Rightarrow \{x_{12}, x_{11}, x_{10}\}$. If $x_9 \Rightarrow [X_6]$, say $x_2 \rightarrow x_1$, then we obtain a 4-ary spanning tree induced by $\{x_9x_2, x_9x_3, x_9x_4, x_9u, ux_5, \ldots, ux_8, x_2x_1, x_2x_{12}, x_2x_{11}, x_2x_{10}\}$. Consequently, there exists a vertex $v \in [X_6]$ such that $v \Rightarrow \{x_9, \ldots, x_{12}\}$, say $v = x_1$. Then we obtain a 4-ary spanning tree of T by a similar discussion for $d^+_{\{x_2,\ldots,x_8\}}$ as Case 2.

Case 5: $d^+(u) = 7$.

Suppose $d^+_{N^-(u)} = d^+_{N^-(u)}(x_{12})$.

Firstly, suppose $d^+_{N^-(u)}(x_{12}) = 4$, say $x_{12} \Rightarrow \{x_8, \ldots, x_{11}\}$. If there exists some vertex, say x_4, such that $d^+_{[X_7]}(x_4) \geq 3$ and $x_4 \rightarrow x_{12}$, then we assume $x_4 \Rightarrow \{x_5, x_6, x_7\}$ and obtain a 4-ary spanning tree induced by $\{ux_1, \ldots, ux_4, x_4x_5, x_4x_6, x_4x_7, x_4x_{12}, x_{12}x_8, \ldots, x_{12}x_{11}\}$. Since $d^+(x_{12}) \leq 7$, we may assume $x_{12} \Rightarrow \{x_6, x_7\}$ and $T[[X_5]]$ is 2-regular with $x_1 \Rightarrow \{x_2, x_3\}$. Let $d^+_{N^-(u)}(x_{11}) \geq 2$. Since $d^+(x_{11}) \leq 7$, there exists some vertex $v \in [X_5]$ such that $v \rightarrow x_{11}$, say $v = x_1$. Then $\{x_1x_2, x_1x_3, x_1x_{11}, x_1x_{12}, x_{12}x_8, x_{12}x_9, x_{12}x_{10}, x_{12}u, ux_4, \ldots, ux_7\}$ induces a desired

4-ary spanning tree.

Next, suppose $d^+_{N^-(u)}(x_{12}) = 3$, say $x_{12} \Rightarrow \{x_{11}, x_{10}, x_9\}$. If there exists some vertex, say x_4, such that $d^+_{[X_7]}(x_4) \geq 3$ and $x_4 \to x_8$, then we assume $x_4 \Rightarrow \{x_5, x_6, x_7\}$ and obtain a 4-ary spanning tree induced by $\{x_{12}x_{11}, x_{12}x_{10}, x_{12}x_9, x_{12}u, ux_1, \ldots, ux_4, x_4x_5, x_4x_6, x_4x_7, x_4x_8\}$. If $d^+_{[X_7]}(x_8) \geq 3$, say $x_8 \Rightarrow \{x_5, x_6, x_7\}$, then we get a 4-ary spanning tree induced by $\{x_8x_5, x_8x_6, x_8x_7, x_8x_{12}, x_{12}x_{11}, x_{12}x_{10}, x_{12}x_9, x_{12}u, ux_1, \ldots, ux_4\}$. So we may assume $x_8 \Rightarrow \{x_6, x_7\}$, $T[[X_5]]$ is 2-regular with $x_1 \Rightarrow \{x_2, x_3\}$ and $\{x_6, x_7\} \Rightarrow [X_5]$. If $d^+_{[X_5]}(x_{12}) \geq 1$, say $x_{12} \to x_5$, then we obtain a 4-ary spanning tree induced by $\{x_8u, x_8x_6, x_8x_7, x_8x_{12}, , x_6x_1, \ldots, x_6x_4, x_{12}x_{11}, x_{12}x_{10}, x_{12}x_9, x_{12}x_5\}$. So we may assume $[X_5] \Rightarrow \{x_8, x_{12}\}$. Then we obtain a 4-ary spanning tree induced by $\{x_1x_2, x_1x_3, x_1x_8, x_1x_{12}, x_{12}x_9, x_{12}x_{10}, x_{12}x_{11}, x_{12}u, ux_4, \ldots, ux_7\}$.

Finally, we consider the case when $T[N^-(u)]$ is 2-regular, say $x_{12} \Rightarrow \{x_{11}, x_{10}\}$ and $x_{11} \to x_{10}$. If $d^+(v) \leq 6$ for any $v \in V(T)\backslash\{u\}$, then the out-degree sequence of T is $\{5, 6, \ldots, 6, 7\}$. So there exist three vertices, say x_5, x_6 and x_7, such that $x_{10} \Rightarrow \{x_6, x_7\}$ and $x_{12} \to x_5$. Then we obtain a desired tree induced by $\{x_{12}x_{11}, x_{12}x_{10}, x_{12}x_5, x_{12}u, x_{10}x_6, \ldots, x_{10}x_9, ux_1, \ldots, ux_4\}$. We next consider the remaining two cases. If $d^+(x_{12}) = 7$, say $x_{12} \Rightarrow [X_4]$, we may assume $T[\{x_5, \ldots, x_9\}]$ is 2-regular with $x_9 \Rightarrow \{x_8, x_7\}$ by the symmetry of x_{12} and u. Then we obtain a desired tree induced by $\{x_9x_8, x_9x_7, x_9x_{12}, x_9u, x_{12}x_{11}, x_{12}x_{10}, x_{12}x_1, x_{12}x_2, ux_3, \ldots, ux_6\}$. Without loss of generality, if $d^+(x_1) = 7$, we may assume $x_1 \Rightarrow \{x_4, \ldots, x_{10}\}$ because $T[N^-(x_1)]$ is 2-regular by the symmetry of x_1 and u, then we obtain a 4-ary spanning tree induced by $\{x_{12}x_{11}, x_{12}x_{10}, x_{12}x_1, x_{12}u, x_1x_6, \ldots, x_1x_9, ux_2, \ldots, ux_5\}$.

Case 6: $d^+(u) = 6$.

In this case, T is 6-regular. Firstly, suppose $d^+_{N^-(u)} = d^+_{N^-(u)}(x_{12}) = 5$, say $x_{12} \Rightarrow \{x_7, \ldots, x_{11}\}$. Let $1 \leq d^+_{N^-(u)}(x_8) \leq 3$ and assume $x_8 \Rightarrow \{x_5, x_6, x_7\}$. Then we obtain a 4-ary spanning tree induced by $\{x_{12}x_{11}, \ldots, x_{12}x_8, x_8x_7, x_8x_6, x_8x_5, x_8u, ux_1, \ldots, ux_4\}$. Then, suppose $d^+_{N^-(u)} = 4$, say $x_{12} \Rightarrow \{x_8, \ldots, x_{11}\}$. If $d^+_{[X_6]}(x_7) \geq 2$, say $x_7 \Rightarrow \{x_5, x_6\}$, then we obtain a desired set $\{x_7x_5, x_7x_6, x_7x_{12}, x_7u, x_{12}x_{11}, \ldots, x_{12}x_8, ux_1, \ldots, ux_4\}$. Notice that T is 6-regular, so we may assume $x_7 \Rightarrow \{x_6, x_8, x_9, x_{10}\}$. If $d^+_{[X_5]}(x_{12}) \geq 1$, say $x_{12} \to x_5$, then we obtain a desired tree induced by $\{x_7x_6, x_7x_8, x_7x_{12}, x_7u, x_{12}x_{11}, x_{12}x_{10}, x_{12}x_9, x_{12}x_5, ux_1, \ldots, ux_4\}$. So we may assume $[X_5] \Rightarrow x_{12}$ and $x_3 \Rightarrow \{x_1, x_2\}$ because $d^+_{[X_5]} \geq 2$, and then we obtain a 4-ary spanning tree induced by

$\{ux_3, \ldots, ux_6, x_3x_1, x_3x_2, x_3x_7, x_3x_{12}, x_{12}x_{11}, \ldots, x_{12}x_8\}$. Finally, suppose $d^+_{N^-(u)} = 3$, say $x_{12} \Rightarrow \{x_9, x_{10}, x_{11}\}$ and $x_7 \rightarrow x_8$. Since $d^+_{[X_6]}(x_7) \geq 2$, we assume $x_7 \Rightarrow \{x_5, x_6\}$. Then $\{x_7x_5, x_7x_6, x_7x_8, x_7x_{12}, x_{12}x_{11}, x_{12}x_{10}, x_{12}x_9, x_{12}u, ux_1, \ldots, ux_4\}$ induces a desired 4-ary spanning tree. □

Now suppose $k = 5$. Let T_{12} be a 12-tournament with $V(T_{12}) = \{0, 1, \ldots, 11\}$ and $E(T_{12}) = \{(0,3), (0,5), (0,9), (0,10), (0,11), (1,0), (1,4), (1,6), (1,8), (1,9), (1,11), (2,0), (2,1), (2,7), (2,8), (2,10), (2,11), (3,1), (3,2), (3,6), (3,9), (3,10), (4,0), (4,2), (4,3), (4,7), (4,9), (5,1), (5,2), (5,3), (5,4), (5,8), (5,11), (6,0), (6,2), (6,4), (6,5), (6,10), (7,0), (7,1), (7,3), (7,5), (7,6), (8,0), (8,3), (8,4), (8,6), (8,7), (9,2), (9,5), (9,6), (9,7), (9,8), (9,11), (10,1), (10,4), (10,5), (10,7), (10,8), (10,9), (11,3), (11,4), (11,6), (11,7), (11,8), (11,10)\}$. It is easy to check that T_{12} satisfies the condition of Observation 6.8. Therefore, T_{12} contains no 5-ary spanning tree, which implies that $h(5) \geq 13$.

Remark 6.9. *Using the similar method as $h(4)$, we can prove that $h(5) = 13$. However, the proof is too long to include here. Some new methods are needed to determine the exact values of $h(k)$ for $k \geq 5$.*

Chapter 7

Kings in Multipartite Hypertournaments

7.1 Introduction

Given two integers n and k, $n \geq k > 1$, a *k-hypertournament* T on n vertices is a pair (V, A), where V is a set of vertices, $|V| = n$ and A is a set of k-tuples of vertices, called arcs, so that for any k-subset S of V, A contains exactly one of the $k!$ tuples whose entries belong to S. For an arc $x_1x_2 \dots x_k$, we say that x_i *precedes* x_j if $i < j$. A 2-hypertournament is merely an (ordinary) tournament. Hypertournaments have been studied in a large number of papers, see e.g. [12, 13, 15, 16, 34, 39, 44, 55, 63].

Recently, Petrovic [54] introduced multipartite hypertournaments in a similar way. Let n and k be integers such that $n > k \geq 2$. Let V be a set of n vertices and $V = V_1 \uplus V_2 \uplus \cdots \uplus V_p$ be a partition of V into $p \geq 2$ non-empty subsets. A *p-partite k-tournament* (or, *multipartite hypertournament*) H can be obtained from a k-hypertournament T on vertex set V by deleting all arcs $x_1x_2 \dots x_k$ such that $\{x_1, x_2, \dots, x_k\} \subseteq V_i$ for some $i \in [p]$. We call V_i's *partite sets* of H. The set of arcs of $H = (V, A)$ will be denoted by $A(H)$, i.e., $A(H) = A$. A p-partite 2-tournament is a *p-partite tournament.*

For $u \in V_i, w \in V_j$ with $i \neq j$, $A_H(u, w)$ is the set of arcs of H which contain u and w and where u precedes w. We will write xey if $e \in A_H(x, y)$. We let

$A_H(x,y) = \emptyset$ if x and y belong to the same partite set of H. A *path* in H is an alternating sequence $P = x_1a_1x_2a_2 \ldots x_{q-1}a_{q-1}x_q$ of distinct vertices x_i and arcs a_j such that $x_ja_jx_{j+1}$ for every $j \in [q-1]$. We will call P an *(x_1, x_q)-path* of *length* $q-1$.

Let $q \geq 1$ be a natural number. A vertex x of H is a *q-king* if for every $y \in V$, H has an (x,y)-path of length at most q. Generalizing a well-known theorem of Landau that every tournament has a 2-king (see e.g. [17]), Brcanov et al. [15] showed that every hypertournament has a 2-king. A vertex v of H is a *transmitter* if for every vertex u from a different partite set than v, $A_H(u,v) = \emptyset$.

Note that for every $u \in V_i, w \in V_j$ $(i \neq j)$, we have $|A_H(u,w)| + |A_H(w,u)| = \binom{n-2}{k-2}$. A *majority multipartite tournament* $M(H)$ of H has the same partite sets as H and for every $u \in V_i$ and $w \in V_j$ with $i \neq j$, $uw \in M(H)$ if $|A_H(u,w)| > \frac{1}{2}\binom{n-2}{k-2}$. If $|A_H(u,w)| = \frac{1}{2}\binom{n-2}{k-2}$ then we can choose either uw or wu for $M(H)$.

For a graph $G = (V,E)$ and $U \subseteq V$, let $N_G(U) = \{v \in V \setminus U : \ uv \in E, u \in U\}$.

Gutin [38] and independently Petrovic and Thomassen [55] proved the following:

Theorem 7.1. *[38, 55] Every multipartite tournament with at most one transmitter contains a 4-king.*

Petrovic [54] proved that the same result holds for bipartite k-tournaments:

Theorem 7.2. *[54] Every bipartite k-tournament $(k \geq 2)$ with at most one transmitter contains a 4-king.*

In the same paper he conjectured the following:

Conjecture 7.3. *[54] Every multipartite k-tournament $(k \geq 2)$ with at most one transmitter contains a 4-king.*

In this short Chapter, we will solve this conjecture in the affirmative.

The next conjecture of Petrovic [54] is motivated by the fact that Petrovic and Thomassen [55] proved that the assertion of the conjecture holds for bipartite tournaments.

Theorem 7.4. *[55] Every bipartite tournament B without transmitters has at least two 4-kings in each partite set of B.*

Conjecture 7.5. *[54] Every bipartite k-tournament B ($k \geq 2$) without transmitters has at least two 4-kings in each partite set of B.*

In this Chapter, we will first show a counterexample to Conjecture 7.5 and then exhibit a wide family of bipartite hypertournaments for which the conclusion of the conjecture holds.

The Chapter is organized as follows. In the next section, we prove a lemma (Lemma 7.7) which we call *the Majority Lemma*, and which is used to show the positive above-mentioned results. In Section 7.3, we prove the counterexample and positive results.

7.2 The Majority Lemma

The Majority Lemma, Lemma 7.7, is the main technical result of this Chapter. To prove Lemma 7.7, we will use the following simple lemma.

Lemma 7.6. *Let G be a bipartite graph with partite sets U and W and let every vertex in U have degree at least $p \geq 1$ and every vertex in W have degree at most p, except one vertex has degree at most $2p-1$. Then G has a matching saturating U.*

Proof. By Hall's theorem, if for every $S \subseteq U$, $|S| \leq |N_G(S)|$ then G has a matching saturating U. Suppose that there is a subset S of U such that $|S| \geq |N_G(S)| + 1$. Let e be the number of edges in the subgraph of G induced by $S \cup N_G(S)$ and observe that

$$p|S| \leq e \leq (|N(S)| - 1)p + (2p - 1) \leq (|S| - 2)p + (2p - 1) = |S|p - 1,$$

a contradiction. □

Lemma 7.7. *Let H be a p-partite k-tournament with $p \geq 2$. Let $n \geq 5$ and $n > k \geq 3$. If a majority p-partite tournament $M(H)$ has an (x, y)-path P of length at most 4, then H has such a path of length at most 4.*

Proof. It suffices to prove this lemma for the case when P is of length 4 as the other cases are simpler and similar. Thus, assume that $P = x_1x_2x_3x_4x_5$. By definition of a path, for every $i \in [4]$, x_i and x_{i+1} belong to different partite sets of H. Now consider the following cases covering all possibilities.

Case 1: $n \geq 9$ and $3 \leq k < n$ or $n \geq 7$ and $4 \leq k < n-1$. Observe that if for every $i \in \{1, 2, 3, 4\}$,

$$|A_H(x_i, x_{i+1})| > 3 \tag{7.2.1}$$

then we can choose distinct arcs $a_i \in A_H(x_i, x_{i+1})$ such that $x_1a_1x_2a_2x_3a_3x_4a_4x_5$ is the required path in H. In particular, inequalities (7.2.1) will hold if $\frac{1}{2}\binom{n-2}{k-2} > 3$.

If $n \geq 9$ and $3 \leq k < n$, we have

$$\frac{1}{2}\binom{n-2}{k-2} \geq \frac{n-2}{2} > 3$$

and hence inequalities (7.2.1) hold. If $n \geq 7$ and $4 \leq k < n-1$, we have

$$\frac{1}{2}\binom{n-2}{k-2} \geq \frac{(n-2)(n-3)}{4} > 3.$$

Case 2: $k = 3$ and $5 \leq n \leq 8$. Then

$$|A_H(x_i, x_{i+1})| \geq \frac{1}{2}\binom{n-2}{k-2} \geq \frac{1}{2}\binom{3}{1} = \frac{3}{2} \tag{7.2.2}$$

for $i = 1, 2, 3, 4$. Consider a bipartite graph G with partite sets $Z = \{z_1, z_2, z_3, z_4\}$ and $A(H)$. We have an edge z_ia_j if $a_j \in A_H(x_i, x_{i+1})$. By (7.2.2), each vertex in Z has degree at least two. Since $k = 3$, vertices z_i and z_j in G have no common neighbor unless $|i - j| = 1$. Thus, every vertex of G in $A(H)$ has degree at most 2. Thus, by Lemma 7.6, G has a matching saturating Z. In other words, there are distinct $a_1, a_2, a_3, a_4 \in A(H)$ such that $x_1a_1x_2a_2x_3a_3x_4a_4x_5$ is a path in H.

Case 3: $k = 4$ and $5 \leq n \leq 6$. Consider the bipartite graph G constructed as in the previous case. Using the computations analogous to those in (7.2.2), we see that the minimum degree of a vertex in Z is at least 3 when $n = 6$ and at least 2 when $n = 5$. Since $k = 4$, there is no common neighbor of all vertices in Z. Thus, every vertex of G in $A(H)$ has degree at most 3. Now consider two subcases.

Subcase 1: $n = 6$. Since every vertex of G in $A(H)$ has degree at most 3 and every vertex of G in Z has degree at least 3, by Lemma 7.6, G has a matching saturating Z and we are done as in Case 2.

Subcase 2: $n = 5$**.** Recall that the minimum degree of a vertex in Z is at least 2. Suppose that there are two vertices of G in $A(H)$ of degree 3. This means that

$$N_G(z_i) \cap N_G(z_{i+1}) \cap N_G(z_{i+2}) \neq \emptyset \tag{7.2.3}$$

for $i = 1$ or 2. Indeed, since $k = 4$, $N_G(z_1) \cap N_G(z_j) \cap N_G(z_4) = \emptyset$ when either $j = 2$ or 3. Without loss of generality, we assume that (7.2.3) holds when $i = 1$ and let $e_1 \in N_G(z_1) \cap N_G(z_2) \cap N_G(z_3)$. Thus, $e_1 = x_1x_2x_3x_4$.

If x_1 and x_4 are in different partite sets of H, then $x_1e_1x_4$. Since e_1 does not contain x_5, we can choose an arc e_2 of H which is different from e_1 such that $x_4e_2x_5$. Then $x_1e_1x_4e_2x_5$ is a path in H. Now we assume that x_1 and x_4 are in the same partite set of H. Then $x_1e_1x_3$. Since the degree of z_3 in G is at least 2, we can choose an arc e_2 of H which is different from e_1 such that $x_3e_2x_4$. We can also choose an arc e_3 of H which is different from e_1 and e_2 such that $x_4e_3x_5$. Indeed, $e_3 \neq e_1$ since e_1 does not contain x_5 and $e_3 \neq e_2$ since the degree of z_4 in G is at least 2. Then $x_1e_1x_3e_2x_4e_3x_5$ is a path in H. Thus, we may assume that every vertex of G in $A(H)$ has degree at most 2, except for one vertex which has degree at most 3. Then we can use Lemma 7.6 and thus we are done as above.

Case 4: $k \in \{5, 6, 7\}$ **and** $n = k+1$**.** Consider the bipartite graph G constructed as in Case 2.

Subcase 1: $k \in \{6, 7\}$**.** Using the computations analogous to those in (7.2.2), we see that the minimum degree of a vertex in Z is at least 3. If there is a vertex with degree 4 in $A(H)$, then it means $\{x_1, x_2, x_3, x_4, x_5\}$ is a subset of a vertex set of an arc e_1 and the relative order is x_1, x_2, x_3, x_4, x_5. If x_1 and x_5 are in different partite sets, then $x_1e_1x_5$ is a path in H. Otherwise x_1 and x_4 are in different partite sets, so $x_1e_1x_4$. There is an arc e_2 different from e_1 such that $x_4e_2x_5$ (since the degree of z_4 is at least 3). Now $x_1e_1x_4e_2x_5$ is a path in H. Thus, we assume each vertex in $A(H)$ has degree at most 3, and we are done by Lemma 7.6.

Subcase 2: $k = 5$**.** Suppose that the lemma does not hold in this case. Using the computations analogous to those in (7.2.2), we see that the minimum degree of a vertex in Z is at least 2. To obtain a contradiction, it suffices to show that G has at most one vertex of degree at least 3 in $A(H)$. Suppose that G has at least two vertices of degree at least 3 in $A(H)$. This means that (7.2.3) holds for $i = 1$

or 2. Since H can have only one arc with vertex set $\{x_1, x_2, x_3, x_4, x_5\}$, we have

$$\sum_{j=2}^{3} |N_G(z_1) \cap N_G(z_j) \cap N_G(z_4)| \leq 1 \tag{7.2.4}$$

Without loss of generality, we assume that (7.2.3) holds when $i = 1$ and let $e_1 \in N_G(z_1) \cap N_G(z_2) \cap N_G(z_3)$. If we restrict e_1 to the vertices $\{x_1, x_2, x_3, x_4\}$, then $\{x_1, x_2, x_3, x_4\}$ is a subset of a vertex set of e_1 and the relative order is x_1, x_2, x_3, x_4.

If x_1 and x_4 are in the different partite sets, then $x_1e_1x_4$. Since the degree of z_4 in G is at least 2, we can choose an arc e_2 of H which is different from e_1 such that $x_4e_2x_5$. Then $x_1e_1x_4e_2x_5$ is a path in H, a contradiction. Now we assume x_1 and x_4 are in the same partite set. Then $x_1e_1x_3$. Since the degree of z_3 in G is at least 2, we can choose an arc e_2 of H which is different from e_1 such that $x_3e_2x_4$. Since the degree of z_4 in G is at least 2, we can choose an arc e_3 of H such that $x_4e_3x_5$ and $e_3 \neq e_2$. Suppose $e_3 = e_1$. Then $e_1 = x_1x_2x_3x_4x_5$ and $x_1e_1x_5$, a contradiction. Thus, $e_3 \neq e_1$ and $x_1e_1x_3e_2x_4e_3x_5$ is a path in H, a contradiction. □

Proposition 7.14 proved in the next section shows that Lemma 7.7 cannot be extended to $n = 4$ and $p = 2$.

7.3 Main Results

In Section 7.3.1, using the Majority Lemma and other results, we solve Conjecture 7.3 in affirmative. In Section 7.3.2, we describe a family of couterexamples to Conjecture 7.5 and prove a sufficient condition of when the conclusion of Conjecture 7.5 holds.

7.3.1 Results on Conjecture 7.3

Lemma 7.8. *Let $H = (V, A)$ be a multipartite k-tournament with at most one transmitter and let $M(H)$ be a majority multipartite tournament of H. Let $n \geq 5$ and $n > k \geq 3$. If $M(H)$ has at least one transmitter, then H has a 2-king.*

Proof. Let V_1 be the partite vertex set containing all transmitters of $M(H)$. Let v be the transmitter of H, if H has a transmitter, and an arbitrary transmitter of $M(H)$, otherwise. Clearly, $v \in V_1$. Observe that for every $u \in V \setminus V_1$, there is an arc $a \in A_H(v, u)$ implying that vau. Note that for every $w \in V_1 \setminus \{v\}$, there are a vertex $u \in V \setminus V_1$ and an arc e of H such that uew. As in Lemma 7.7, it is easy to see that $|A_H(v, u)| \geq 2$. Thus, there is an arc $a \in A_H(v, u)$ distinct from e implying that $vauew$ is a path. □

Lemma 7.9. *Let $H = (V, A)$ be a multipartite k-tournament and let $n \geq 5$ and $n > k \geq 3$. If H has at most one transmitter then H has a 4-king.*

Proof. Let $M(H)$ be a majority multipartite tournament of H. If $M(H)$ has no transmitters, then by Theorem 7.1, $M(H)$ has a 4-king x. By Lemma 7.7, x is a 4-king of H. If $M(H)$ has transmitters, then we apply Lemma 7.8. □

Lemma 7.10. *Let $H = (V, A)$ be a p-partite k-hypertournament with $k = 3$, $n = 4$ and $p \geq 2$. If H has at most one transmitter then H has a 4-king.*

Proof. By Theorem 7.2, this lemma holds for $p = 2$ and so we may assume that $p \geq 3$. It is well known that every k-hypertournament with more than k vertices has a Hamilton path [39]. Observe that for $p = 4$ the first vertex of a Hamilton path in H is a 3-king. Now we may assume that $p = 3$. Let $V = V_1 \cup V_2 \cup V_3$ be a partition of vertices of H. Without loss of generality, we may assume that $V_1 = \{x_1, x_2\}$, $V_2 = \{x_3\}$ and $V_3 = \{x_4\}$.

First assume that H has the unique transmitter v. If $v = x_3$ or $v = x_4$, then v is a 1-king of H. Thus, we assume without loss of generality that $v = x_1$. Since v is a transmitter, va_1x_3 and va_2x_4 for some arcs a_1 and a_2 of H. Since x_2 is not a transmitter, there is an arc e_1 such that ye_1x_2, where $y \in V_2 \cup V_3$. By the definition of a transmitter, v precedes y in every arc containing v and y. Consequently, there is an arc e_2 different from e_1 such that ve_2y. Hence $ve_2ye_1x_2$ is a path from v to x_2. So v is a 2-king.

Now assume that H has no transmitter. Consider the arc e_1 containing x_1, x_3, and x_4. If x_1 is in the first position of e_1, since x_2 is not a transmitter, there is an arc e_2 different from e_1 such that $x_3e_2x_2$ or $x_4e_2x_2$. Hence $x_1e_1x_3e_2x_2$ or

$x_1e_1x_4e_2x_2$ is a path from x_1 to x_2, implying that x_1 is a 2-king. Without loss of generality, we now assume that x_3 is in the first position of e_1. Since x_2 is not a transmitter, there is an arc e_2, where x_3 or x_4 precedes x_2. Hence x_3 is a 2-king. □

Lemmas 7.9 and 7.10 imply the following result solving Conjecture 7.3 in affirmative.

Theorem 7.11. *Every multipartite hypertournament with at most one transmitter has a 4-king.*

7.3.2 Results on Conjecture 7.5

The next result describes a family of counterexamples to Conjecture 7.5.

Proposition 7.12. *For every $k \geq 3$, there is a bipartite k-tournament B without transmitters which has at most one 4-king in both U and W, where U and W are partite sets of B.*

Proof. Let U and W be partite sets of B. Choose a vertex u in U and a vertex w in W. Let every arc of B with both u and w have both of them in the first and second position such that in at least one such arc u is the first and in at least one such arc w is the first. Let every arc of B containing u but not w have u in the first position and let every arc of B containing w but not u have w in the first position. Clearly, B has no transmitters, but no vertex v in $(U \cup W) \setminus \{u, w\}$ can be a 4-king as there is no path from v to either u or w. □

The next result is a sufficient condition of when the conclusion of Conjecture 7.5 holds. It follows directly from Theorem 7.4 and the Majority Lemma.

Theorem 7.13. *Let B be a bipartite hypertournament with partite sets U and W and with at least 5 vertices. If a majority bipartite tournament $M(B)$ has no transmitters, then B has at least two 4-kings in each U and W.*

Our final result shows that the Majority Lemma cannot be extended to $n = 4$ and $p = 2$. The proof provides another counterexample to Conjecture 7.5.

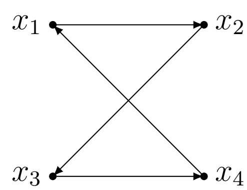


Figure 7.1 $M(H)$

Proposition 7.14. *For $k = 3$ and $n = 4$, there is a bipartite hypertournament H with partite sets U and W such that (i) $|U| = |W| = 2$, (ii) a majority bipartite tournament $M(H)$ has no transmitters, (iii) $M(H)$ has an (x, y)-path of length 3, but H has no (x, y)-path, (iv) H has only one 4-king in U.*

Proof. Let H be a bipartite hypertournament with partite sets $U = \{x_1, x_3\}$ and $W = \{x_2, x_4\}$, arc set $\{a_1, a_2, a_3, a_4\}$ where

$$a_1 = x_4x_1x_2, a_2 = x_2x_3x_4, a_3 = x_3x_2x_1, a_4 = x_4x_3x_1.$$

Let the arcs of $M(H)$ be $x_4x_1, x_1x_2, x_2x_3, x_3x_4$ (see Fig. 7.1). Clearly, (i) and (ii) hold and $x_1x_2x_3x_4$ is an (x_1, x_4)-path in $M(H)$.

Now consider H. Suppose that H has an (x_1, x_4)-path P. Since $A_B(x_1, x_4) = \emptyset$, $P = x_1b_1x_2b_2x_3b_3x_4$ for some distinct arcs b_1, b_2, b_3 of H. By inspection of the arcs of H, we conclude that $b_1 = a_1, b_2 = a_2, b_3 = a_2$, which is impossible since b_1, b_2, b_3 must be distinct. So H has no (x_1, x_4)-path and (iii) holds. Observe that x_3 is a 4-king of H since $x_3a_3x_2$, $x_3a_2x_4$ and $x_3a_2x_4a_1x_1$ is an $(x_{3,}x_1)$-path of length 2. Moreover, x_1 cannot be a 4-king by the discussion in (iii), so (iv) holds. $\square$

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