A Joint Particle Filter for Quaternion-Valued α-Stable Signals via the Characteristic Function

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***Abstract*—The filtering paradigm is revisited through the perspective of characteristic functions. This results in the derivation of a novel particle filtering technique for sequential estimation/tracking of quaternion-valued α-stable random signals. Importantly, the derived particle filter incorporates an efficient information fusion format and collaborative/distributed estimation framework to accommodate the push toward use of sensor networks. The distributed setting provides for the distribution of computational complexity among agents of a sensor network, while allowing each agent to retain an estimate of the state. Furthermore, the quaternion-valued structure allows for the derivation of a rigorous algorithm that is advantageous when dealing with signals of a multidimensional nature commonly encountered in sensor arrays.**

***Index Terms*—α-stable random signals, quaternion-valued signal processing, particle filtering, distributed estimation.**

I. INTRODUCTION

Recent observations in an increasing number of applications [1]–[5] have come to indicate that the underlying signal and/or noise exhibits sharp spikes resulting in distributions that do not decay as fast as the Gaussian case [1]–[3]. In these applications, outliers are not mistakes, but an integral part of the signal. In these settings, the "*α-stable*" class of distributions has proven to be a useful tool in modelling the behaviour of signals, mainly due to the fact that they admit the generalized central limit theorem [1]–[3,5]. However, a closed-form expression for the probability distribution function (pdf) of the generality of α-stable random processes does not exist, making deriving signal processing techniques cumbersome.

For the case of real-valued α-stable random variables with elliptically contoured distributions, it is shown that the characteristic function (CF) will take the form [1]

$$\mathbf{\Phi}_{\boldsymbol{Z}}(\boldsymbol{\vartheta}) = \mathsf{E}\left\{e^{\eta\boldsymbol{\vartheta}^T\mathbf{z}}\right\} = e^{-\left(\frac{1}{2}\boldsymbol{\vartheta}^T\boldsymbol{\mathcal{C}}_{\mathbf{z}}\boldsymbol{\vartheta}\right)^{\frac{\alpha}{2}}} \quad (1)$$

where $\eta^2 = -1$, $\mathsf{E}\{\cdot\}$, is the statistical expectation, and $\mathbf{\Phi}_{\boldsymbol{Z}}(\cdot)$ denotes the CF of real-valued random variable $\boldsymbol{Z}$, with the semi-positive definite matrix $\boldsymbol{\mathcal{C}}_{\mathbf{z}}$ determining the elliptical distribution of $\boldsymbol{Z}$ and referred to as the covariation matrix, while $0 < \alpha \leq 2$ is referred to as the characteristic exponent.

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In recent years, the introduction of the $\mathbb{HR}$-calculus [13] and the augmented quaternion statistics [14]–[16], have kindled a great deal of interest in quaternion-valued signal processing as quaternions provide a natural representation for three-dimensional signals encountered in many applications including, smart grid [17], target tracking [18], machine intelligence [19], and attitude estimation [20]. The use of quaternion algebra has particularly proven advantageous, when dealing with multidimensional signals encountered in sensor arrays [17]–[19,23,24]. However, these works rely on second-order statistics and are not generalizable for quaternion-valued α-stable random signals.

This work focuses on deriving a truly distributed quaternion-valued particle filter for sequential tracking of quaternion-valued α-stable signals that include Gaussian signals as a particular case. This is achieved through exploiting the one-to-one relation between a distribution and its CF, where it is shown that the operations of a quaternion-valued particle filter can be approximated in a distributed fashion. Furthermore, the distributed format accommodates use of the derived filter in modern sensor networks, while the quaternion setting accommodates for the rigorous modelling of multidimensional signals from sensor arrays.

Mathematical notations: Scalars, column vectors, and matrices are denoted by lowercase, bold lowercase, and bold uppercase letters respectively, while uppercase bold italic letters denote multivariate random processes, with uppercase italic letters denoting univariate random processes. The transpose and Hermitian transpose operators are denoted by $(\cdot)^{\mathsf{T}}$ and $(\cdot)^{\mathsf{H}}$, whereas $\mathsf{E}\{\cdot\}$ denotes the statistical expectation operator. The pdf of the random process $\boldsymbol{X}$ is denoted by $P_{\boldsymbol{X}}(\cdot)$ and ln$(\cdot)$ denotes the natural logarithm. Finally, the real and quaternion domains are denoted by $\mathbb{R}$ and $\mathbb{H}$.

II. PRELIMINARIES

The skew-field of quaternions is a four-dimensional, non-commutative, associative, division algebra. A quaternion variable $q \in \mathbb{H}$ consists of a real part, $\Re\{q\}$, and a three-dimensional imaginary part $\Im\{q\}$, comprised of the three imaginary components, $\Im_i\{q\}$, $\Im_j\{q\}$, and $\Im_k\{q\}$. Hence, q

can be expressed as

$$\begin{aligned} q =& \Re\{q\} + \Im\{q\} = \Re\{q\} + \Im_i\{q\} + \Im_j\{q\} + \Im_k\{q\} \\ =& q_r + iq_i + jq_j + kq_k \end{aligned}$$

where $q_r, q_i, q_j, q_k \in \mathbb{R}$, while i, j, and k are imaginary units obeying the following product rules

$$ij = k, jk = i, ki = j, i^2 = j^2 = k^2 = ijk = -1$$

whereas the conjugate and norm of $q \in \mathbb{H}$ are respectively given by $q^* = \Re\{q\} - \Im\{q\}$ and $|q| = \sqrt{qq^*}$. A quaternion $q \in \mathbb{H}$ can alternatively be expressed in its polar presentation, given by [21]

$$q = |q|e^{\xi\theta} = |q|\big(\cos(\theta) + \xi\sin(\theta)\big)$$

where

$$\xi = \frac{\Im\{q\}}{|\Im\{q\}|}, \quad \theta = \text{atan}\left(\frac{|\Im\{q\}|}{\Re\{q\}}\right).$$

The involution of $q \in \mathbb{H}$ around $\kappa \in \mathbb{H}$ is defined as $q^\kappa = \kappa q \kappa^{-1}$ [22] and can be seen as the quaternion counterpart of the complex conjugate, as the four real-valued components of a quaternion variable, $q \in \mathbb{H}$, can be expressed using involutions as [13]- [18]

$$\begin{aligned} q_r =& \frac{1}{4}\left(q + q^i + q^j + q^k\right) \quad q_i = \frac{1}{4i}\left(q + q^i - q^j - q^k\right) \\ q_j =& \frac{1}{4j}\left(q - q^i + q^j - q^k\right) \quad q_k = \frac{1}{4k}\left(q - q^i - q^j + q^k\right). \end{aligned} \tag{2}$$

The quaternion involution has seen extensive use in modelling three-dimensional rotations. In this setting, the Cartesian coordinates $\rho = (x, y, z)$ are modelled as the quaternion $q_\rho = ix + jy + ky$; then, the rotation of q_ρ around the unit vector η by the angle φ is expressed in terms of quaternion involutions by $q_{\rho'} = e^{\eta\varphi/2} q_\rho e^{-\eta\varphi/2}$, where $q_{\rho'}$ denotes the post-rotation coordinates [20]. Advantageously, quaternions do not suffer form gimbal lock [20].

***Remark* 1**: The expressions in (2) establish a relation between the augmented quaternion vector, $\mathbf{q}^a = [q, q^i, q^j, q^k]^T \in \mathbb{H}^4$ and the real-valued vector $[q_r, q_i, q_j, q_k]^T \in \mathbb{R}^4$. This duality forms the basis of the $\mathbb{HR}$-calculus [13] and the augmented quaternion statistics [14]- [16].

Consider the quaternion-valued random variable $\boldsymbol{Q}$, the joint statistical information of its real-valued components are fully described by their joint CF given by

$$\boldsymbol{\Phi}_{\boldsymbol{Q}}(\mathbf{s}_r, \mathbf{s}_i, \mathbf{s}_j, \mathbf{s}_k) = \mathsf{E}\left\{e^{\eta(\mathbf{s}_r^T\mathbf{q}_r + \mathbf{s}_i^T\mathbf{q}_i + \mathbf{s}_j^T\mathbf{q}_j + \mathbf{s}_k^T\mathbf{q}_k)}\right\} \tag{3}$$

where following the same approach as the $\mathbb{HR}$-calculus and through exploiting the relation between $\mathbb{H}$ and $\mathbb{R}$, the characteristic function in (3) can be expressed directly in the quaternion domain as

$$\boldsymbol{\Phi}_{\boldsymbol{Q}^a}(\mathbf{s}^a) = \mathsf{E}\left\{e^{\eta\Re\{\mathbf{s}^T\mathbf{q}\}}\right\} = \mathsf{E}\left\{e^{\left(\frac{\eta}{4}\mathbf{s}^{a\mathsf{H}}\mathbf{q}^a\right)}\right\} \tag{4}$$

with $\mathbf{s} = \mathbf{s}_r + i\mathbf{s}_i + j\mathbf{s}_j + k\mathbf{s}_k$. Note that in (4) the quaternion random vector must be used in its augmented form and therefore the expression in (4) is referred to as the augmented quaternion characteristic function (AQCF).

The random vector $\boldsymbol{Q}_{\boldsymbol{\mu}}$ with mean vector $\boldsymbol{\mu}$ can be decomposed into $\boldsymbol{Q}_{\boldsymbol{\mu}} = \boldsymbol{Q} + \boldsymbol{\mu}$ where $\boldsymbol{Q}$ is a zero-mean random vector; therefore, given the definition in (4), the AQCF of $\boldsymbol{Q}_{\boldsymbol{\mu}}$ can be expressed as

$$\boldsymbol{\Phi}_{\boldsymbol{Q}^a_{\boldsymbol{\mu}}}(\mathbf{s}^a) = e^{\left(\frac{\eta}{4}\mathbf{s}^{a\mathsf{H}}\boldsymbol{\mu}^a\right)}\boldsymbol{\Phi}_{\boldsymbol{Q}^a}(\mathbf{s}^a)$$

which in essence is the AQCF of $\boldsymbol{Q}$ modulated by

$$e^{\left(\frac{\eta}{4}\mathbf{s}^{a\mathsf{H}}\boldsymbol{\mu}^a\right)} = \cos\left(\frac{1}{4}\mathbf{s}^{a\mathsf{H}}\boldsymbol{\mu}^a\right) + \eta\sin\left(\frac{1}{4}\mathbf{s}^{a\mathsf{H}}\boldsymbol{\mu}^a\right).$$

A quaternion-valued random vector is referred to as α-stable if its real-valued components are jointly α-stable and has the following important properties:

1) For an elliptically contoured zero-mean quaternion-valued α-stable random variable with $1 < \alpha \leq 2$, given the expression (1) and the duality between $\mathbb{H}$ and $\mathbb{R}$ established in (2), the AQCF takes the form

$$\boldsymbol{\Phi}_{\boldsymbol{Q}^a}(\mathbf{s}) = \mathsf{E}\left\{e^{\frac{1}{4}\eta\mathbf{s}^{a\mathsf{H}}\mathbf{q}^a}\right\} = e^{-\left(\frac{1}{32}\mathbf{s}^{a\mathsf{H}}\boldsymbol{\mathcal{C}}_{\mathbf{q}^a}\mathbf{s}^a\right)^{\frac{\alpha}{2}}}$$

where $\boldsymbol{\mathcal{C}}_{\mathbf{q}^a}$, referred to as the augmented covariation matrix, determines the elliptical shape of the distribution, and is the augmented covariance matrix if $\alpha = 2$.

2) Consider a real-valued univariate random variable, Z with $\boldsymbol{\Phi}_Z(\vartheta) = e^{-|\vartheta|^{\alpha/2}}$ and a quaternion-valued Gaussian random vector $\boldsymbol{G}$ with covariance matrix $\mathcal{C}_{\mathbf{g}^a}$ that is independent of Z; then, the quaternion-valued random vector $\boldsymbol{Q} = \sqrt{Z}\boldsymbol{G}$ is an α-stable random variable with covariation matrix $\mathcal{C}_{\mathbf{g}^a}$.[1]

The proofs of these characteristics have been omitted for brevity, as they closely follow those of the real-valued vector cases that are presented in [1].

Remark 2: Hereafter, the focus of this work is on elliptically distributed α-stable signals with $1 < \alpha \leq 2$, where first-order moments are finite, allowing to build a framework for establishing conditional statistical expectations and statistical inference.

III. The Quaternion Distributed Particle Filter

Consider the evolution of the quaternion-valued augmented state vector sequence $\{\mathbf{x}^a_n, n = 0, 1, 2, \ldots\}$, given by

$$\mathbf{x}^a_n = f_n\left(\mathbf{x}^a_{n-1}, \boldsymbol{\nu}^a_n\right)$$

where $f_n(\cdot,\cdot)$ is the state evolution function at time instant n and $\{\boldsymbol{\nu}^a_n, n = 0, 1, 2, \ldots\}$ is the augmented state evolution noise sequence. The objective is to track $\mathbf{x}^a_n$ in real-time from the observations made by a set of sensors, denoted by $\mathcal{N}$, that are interconnected in a network. These observations are given by

$$\mathbf{y}^a_{l,n} = h_{l,n}\left(\mathbf{x}^a_n\right) + \boldsymbol{\omega}^a_{l,n}$$

where $\mathbf{y}^a_{l,n}$ and $h_{l,n}(\cdot)$ are respectively the augmented observation vector and observation function at time instant

[1]For details on generating samples of a real-valued random variable with $\boldsymbol{\Phi}_Z(\vartheta) = e^{-|\vartheta|^{\alpha/2}}$ the keen reader is referred to [1].

n at sensor l, while $\{\boldsymbol{\omega}^a_{l,n}, n = 0, 1, 2, \ldots\}$ is the augmented measurement noise sequence at sensor l.

The observation sequence from all the sensors in the network that can be expressed as $\mathbf{y}^a_{col,1:n} = \{\mathbf{y}^a_{col,1}, \ldots, \mathbf{y}^a_{col,n}\}$ where

$$\mathbf{y}^a_{col,n} = \left[\mathbf{y}^{aT}_{1,n}, \ldots, \mathbf{y}^{aT}_{|\mathcal{N}|,n}\right]^T$$

and $|\mathcal{N}|$ is number of nodes in the network. Taking the conventional particle filtering approach [9], [25]- [26], the AQCF of $\mathbf{x}^a_{0:n} = \{\mathbf{x}^a_0, \ldots, \mathbf{x}^a_n\}$ conditional on the observation sequence $\mathbf{y}^a_{col,1:n}$ can now be expressed as

$$\begin{aligned}\mathbf{\Phi}_{\boldsymbol{X}^a_{0:n}|\boldsymbol{Y}^a_{col,1:n}}(\mathbf{s}^a) =& \int_{\mathcal{D}_{\boldsymbol{X}^a_{0:n}}} e^{\frac{\eta}{4}\mathbf{s}^{a\mathsf{H}}\mathbf{x}^a_{0:n}} P_{\boldsymbol{X}^a_{0:n}}(\mathbf{x}^a_{0:n}|\mathbf{y}^a_{1:n}) \mathrm{d}\mathbf{x}^a_{0:n} \\ \approx& \frac{1}{\sum_{m=1}^M w_n^{\{m\}}} \sum_{m=1}^M w_n^{\{m\}} e^{\frac{\eta}{4}\mathbf{s}^{a\mathsf{H}}\mathbf{x}^{a\{m\}}_{0:n}}\end{aligned}$$

where $P_{\boldsymbol{X}^a_{0:n}}(\mathbf{x}^a_{0:n}|\mathbf{y}^a_{col,1:n})$ denotes the probability of the augmented state sequence $\mathbf{x}^a_{0:n}$ given the augmented observation sequence $\mathbf{y}^a_{col,1:n}$, whereas $\mathcal{D}_{\boldsymbol{X}^a_{0:n}}$ denotes the domain of $\boldsymbol{X}_{0:n}$, while $\mathbf{x}^{a\{m\}}_{0:n}$ and $w_n^{\{m\}}$ are independent particles drawn from the distribution of $\boldsymbol{X}_{0:n}$, or its importance function $\mathcal{P}(\mathbf{x}^a_{0:n}|\mathbf{y}^a_{1:n})$, and their associated weights given by

$$w_n^{\{m\}} \propto \frac{P_{\boldsymbol{X}^a_{0:n}}(\mathbf{x}^a_{0:n}) P_{\boldsymbol{Y}^a_{col,1:n}|\boldsymbol{X}^a_{0:n}}(\mathbf{y}^a_{col,1:n}|\mathbf{x}^a_{0:n})}{\mathcal{P}(\mathbf{x}^a_{0:n}|\mathbf{y}^a_{1:n})}. \tag{5}$$

Now, assuming that the current state is independent of future observations and that the importance function is selected to be factorisable so that

$$\mathcal{P}(\mathbf{x}^a_{0:n+1}|\mathbf{y}^a_{1:n+1}) = \mathcal{P}(\mathbf{x}^a_{0:n}|\mathbf{y}^a_{1:n}) \mathcal{P}(\mathbf{x}^a_{n+1}|\mathbf{x}^a_{0:n}, \mathbf{y}^a_{col,1:n+1})$$

allows the weights to be updated in a sequential manner as

$$\begin{aligned}&w_{n+1}^{\{m\}} \propto \\ &\frac{P_{\boldsymbol{Y}^a_{col,n+1}|\boldsymbol{X}^a_{n+1}}(\mathbf{y}^a_{col,n+1}|\mathbf{x}^a_{n+1}) P_{\boldsymbol{X}^a_{n+1}|\boldsymbol{X}^a_n}(\mathbf{x}^a_{n+1}|\mathbf{x}^a_n)}{\mathcal{P}(\mathbf{x}^a_{n+1}|\mathbf{x}^a_{0:n}, \mathbf{y}^a_{col,1:n+1})} w_n^{\{m\}}\end{aligned} \tag{6}$$

where the probability function $P_{\boldsymbol{X}^a_{n+1}|\boldsymbol{X}^a_n}(\mathbf{x}^a_{n+1}|\mathbf{x}^a_n)$ is determined by the state evolution function.

In order to construct a likelihood distribution we consider the averaged observation, i.e. diffused observation, given by

$$\boldsymbol{\psi}_n = \frac{1}{|\mathcal{N}|} \sum_{l=1}^{|\mathcal{N}|} \mathbf{y}^a_{l,n} \tag{7}$$

where the AQCF of $\boldsymbol{\psi}^a_n$ can be expressed as

$$\mathbf{\Phi}_{\boldsymbol{\psi}_n|\boldsymbol{X}^a_n}(\mathbf{s}^a) = e^{\frac{\eta}{4}\mathbf{s}^{a\mathsf{H}}\boldsymbol{\psi}_n} e^{-\left(\frac{1}{32}\mathbf{s}^{a\mathsf{H}}\mathbf{\Gamma}_n\mathbf{s}^a\right)^{\frac{\alpha}{2}}} \tag{8}$$

where $\mathbf{\Gamma}_n$ represents the covariation matrix of $\boldsymbol{\psi}_n$.

Making the assumption that the observational noise of one node is independent from that of other nodes in the network and replacing (7) into (8) yields

$$\mathbf{\Phi}_{\boldsymbol{\psi}_n|\boldsymbol{X}^a_n}(\mathbf{s}^a) = \prod_{l=1}^{|\mathcal{N}|} \mathbf{\Phi}_{\frac{\boldsymbol{Y}^a_{l,n}}{|\mathcal{N}|}\Big|\boldsymbol{X}^a_n}(\mathbf{s}^a) \tag{9}$$

where after some mathematical manipulations we have

$$\begin{aligned}\mathbf{\Phi}_{\frac{\boldsymbol{Y}^a_{l,n}}{|\mathcal{N}|}\Big|\boldsymbol{X}^a_n}(\mathbf{s}^a) =& \mathbf{\Phi}_{\boldsymbol{Y}^a_{l,n}|\boldsymbol{X}^a_n}\left(\frac{1}{|\mathcal{N}|}\mathbf{s}^a\right) \\ =& e^{\frac{\eta}{4|\mathcal{N}|}\mathbf{s}^{a\mathsf{H}}\mathbf{y}^a_{l,n}} e^{-\left(\frac{1}{32|\mathcal{N}|^2}\mathbf{s}^{a\mathsf{H}}\boldsymbol{\mathcal{C}}_{\boldsymbol{\omega}^a_{l,n}}\mathbf{s}^a\right)^{\frac{\alpha}{2}}}.\end{aligned} \tag{10}$$

Moreover, replacing (10) into (9) gives

$$\begin{aligned}&\mathbf{\Phi}_{\boldsymbol{\psi}_n|\boldsymbol{X}^a_n}(\mathbf{s}^a) = \\ &\left(\prod_{l=1}^{|\mathcal{N}|} e^{\frac{\eta}{4|\mathcal{N}|}\mathbf{s}^{a\mathsf{H}}\mathbf{y}^a_{l,n}}\right) \left(\prod_{l=1}^{|\mathcal{N}|} e^{-\left(\frac{1}{32|\mathcal{N}|^2}\mathbf{s}^{a\mathsf{H}}\boldsymbol{\mathcal{C}}_{\boldsymbol{\omega}^a_{l,n}}\mathbf{s}^a\right)^{\frac{\alpha}{2}}}\right)\end{aligned} \tag{11}$$

therefore, equating the expressions in (11) and (8) allows the covariation matrix $\mathbf{\Gamma}_n$ to be calculated through the widely-linear quaternion-valued regression

$$\mathbf{s}^{aH}\mathbf{\Gamma}_n\mathbf{s}^a = \left(\sum_{l=1}^{|\mathcal{N}|} \left(\frac{1}{|\mathcal{N}|^2}\mathbf{s}^{a\mathsf{H}}\boldsymbol{\mathcal{C}}_{\boldsymbol{\omega}^a_{l,n}}\mathbf{s}^a\right)^{\frac{\alpha}{2}}\right)^{\frac{2}{\alpha}}. \tag{12}$$

Assuming that the network is connected, that is, there exists a path between any two nodes in the network, allows each node to calculate $\mathbf{\Phi}_{\boldsymbol{Y}^a_{col,n}|\boldsymbol{X}^a_n}(\mathbf{s}^a)$ in the formulation given in (11) through the diffusion of observations $\mathbf{y}^a_{l,n}$ and the quaternion-valued regression in (12). This, in turn, permits each node to reconstruct the likelihood distribution at each node due to the one-to-one relation between the pdf and the AQCF. Furthermore, approximating the distribution of $\boldsymbol{X}^a_{0:n}$ to that of an elliptically countered one[2] allows the distribution of $\boldsymbol{X}^a_{0:n}$ to be fully obtained at each node through the diffusion of the local mean estimates given by

$$\boldsymbol{\mu}_{l,n} = \frac{1}{\sum_{m=1}^M w_n^{\{m\}}} \sum_{m=1}^M w_n^{\{m\}} \mathbf{x}^{a\{m\}}_{0:n} \tag{13}$$

and local covariation matrix estimates calculable from the widely-linear regression

$$\mathbf{s}^{a\mathsf{H}}\boldsymbol{\mathcal{C}}_{l,n}\mathbf{s}^a = \left(-(32)^{\frac{\alpha}{2}} \ln\left(\hat{\mathbf{\Phi}}_{l,\boldsymbol{X}^a_{0:n}}(\mathbf{s}^a)\right)\right)^{\frac{2}{\alpha}} \tag{14}$$

where

$$\hat{\mathbf{\Phi}}_{l,\boldsymbol{X}^a_{0:n}}(\mathbf{s}^a) = \frac{1}{\sum_{m=1}^M w_n^{\{m\}}} \sum_{m=1}^M w_n^{\{m\}} e^{\left(\mathbf{x}^{a\{m\}}_{0:n} - \boldsymbol{\mu}_{l,n}\right)}.$$

Thus, the operations of the particle filter can be performed in a distributed fashion as summarized in Algorithm 1, where $\mathcal{N}_i$ denotes the set of nodes in the neighbourhood of node i, that is the set of nodes that can communicate with node i including self-communication.

[2]This is equivalent to stating that the state evolution function at each time instant can be approximated with the quaternion-valued widely-linear function $\mathbf{x}^a_n = \mathbf{A}^a_n\mathbf{x}^a_{n-1} + \mathbf{B}^a_n\boldsymbol{\nu}^a_{n-1}$ where $\mathbf{A}^a_n$ and $\mathbf{B}^a_n$ are the Jacobian matrices of $f_n(\cdot,\cdot)$ with regards to $\mathbf{x}^a_{n-1}$ and $\boldsymbol{\nu}^a_{n-1}$.

Algorithm 1. Quaternion Distributed Particle Filter (QDPF)

For node $l = \{1, \cdots, |\mathcal{N}|\}$:

Initialize:

Draw samples $\mathbf{x}_0^{a\{m\}}$ and assign weights $w_0^{\{m\}}$ using $\mathcal{P}(\mathbf{x}_0^a)$.

At each time instant:

1) Sample from the importance density $\mathcal{P}(\mathbf{x}_{0:n-1}^a | \mathbf{y}_{col,1:n-1}^a)$ and assign weights through (5).
2) Track samples through the state evaluation function using the distribution $P_{\boldsymbol{X}_n^a | \boldsymbol{X}_{n-1}^a}(\boldsymbol{x}_n^a | \boldsymbol{x}_{n-1}^a)$.
3) Share $\mathbf{y}_{l,n}^a$ and $\mathcal{C}_{\boldsymbol{\omega}_{l,n}^a}$ with neighbouring nodes.
4) Approximate $\boldsymbol{\Phi}_{\boldsymbol{Y}_{col,n}^a | \boldsymbol{X}_{0:n}^a}(\mathbf{s}^a)$ through (11)-(12) where $|\mathcal{N}|$ is replaced with $|\mathcal{N}_i|$.
5) Reassign weights through (6) using $\boldsymbol{\Phi}_{\mathbf{Y}_{col,n}^a | \boldsymbol{X}_{0:n}^a}(\mathbf{s}^a)$ to reconstruct the likelihood function.
6) Share $\boldsymbol{\mu}_{l,n}$ and $\boldsymbol{\mathcal{C}}_{l,n}$ with neighbouring nodes.
7) Approximate the distribution of $\boldsymbol{X}_{0:n}^a$ with that of an elliptically contoured α-stable distribution with mean and covariation matrix

$$\boldsymbol{\mu}_n = \frac{1}{|\mathcal{N}_i|} \sum_{\forall l \in \mathcal{N}_i} \boldsymbol{\mu}_{l,n} \text{ and } \boldsymbol{\mathcal{C}}_{\mathbf{x}_{0:n}^a} = \frac{1}{|\mathcal{N}_i|} \sum_{\forall l \in \mathcal{N}_i} \boldsymbol{\mathcal{C}}_{l,n}.$$

8) Draw particles from a quaternion-valued elliptically contoured α-stable distribution with mean $\boldsymbol{\mu}_n$ and covariation matrix $\boldsymbol{\mathcal{C}}_{\mathbf{x}_{0:n}^a}$ to be propagated to the next stage.

IV. Numerical Example

One of the most important applications of quaternion algebra is to track three dimensional rotations using data from multiple redundant gyroscopes measuring the three Euler angles θ, β, and γ, that respectively represent roll, pitch, and yaw of the rotating body and are within the range $[-\pi, \pi]$. The total rotation of the body (e.g., aircraft) is now fully characterized by the quaternion $\phi = \ln\left(e^{i\theta} e^{j\beta} e^{k\gamma}\right)$ where $\phi/|\phi|$ gives the rotation axis and $|\phi|$ gives the rotation angle [28].

In order to track three-dimensional rotations in real-time, the state vector $\mathbf{x}_n = [\phi_n, \dot{\phi}_n]^T$ with the state space model

$$\mathbf{x}_n = \begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix} \mathbf{x}_{n-1} + \begin{bmatrix} \frac{1}{2}(\Delta T)^2 \\ \Delta T \end{bmatrix} \nu_n$$
$$y_{l,n} = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}_n + \boldsymbol{\omega}_{l,n}$$

was used, where $\dot{\phi}_n$ indicates the first-order rate of change of ϕ_n at time instant n, with its second-order rate of change modeled as the state evolution noise ν_n, whereas ΔT denotes the sampling interval. The observational noise was considered to be a quaternion-valued 1.93-stable random process to best reflect the observational noise of laser gyroscopes [29] and the state evolution noise was selected to be a quaternion-valued 1.98-stable random process in order to to be able to account for sharp terns with higher rates of angular change than can be modelled using Gaussian random processes. In addition, for simulations the sampling interval was $\Delta T = 0.04\,$s.

The sensor network shown in Fig. 1 was used to track synthetically generated three-dimensional rotations through implementing the developed quaternion distributed particle filter (QDPF). The estimates of the rotation parameters are shown in Fig. 2. Observe that all the sensors in the network accurately tracked the three-dimensional rotations.

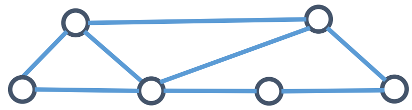

Fig. 1. The sensor network used in simulations.

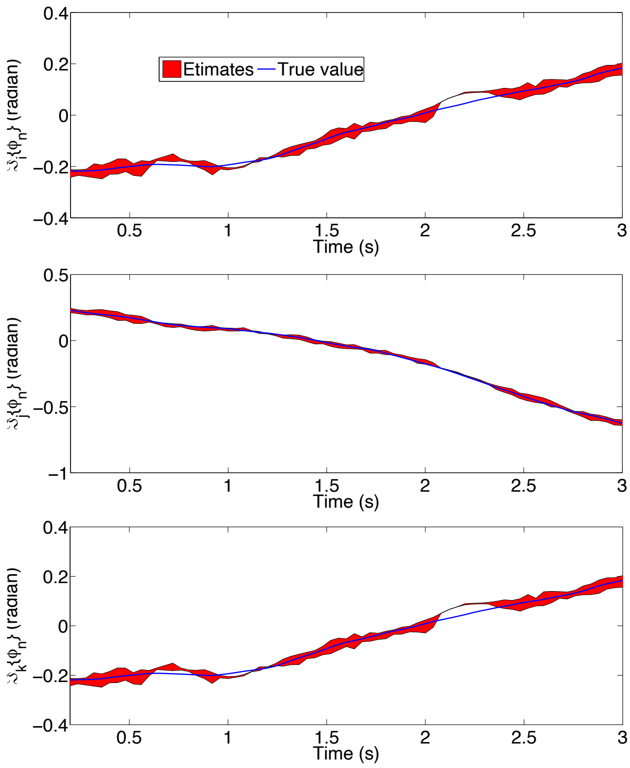

Fig. 2. The real-valued components of the quaternion rotation parameter ϕ_n. Estimates obtained at different nodes of the sensor network lie within the red region.

V. Conclusion

The dilemma of tracking multidimensional non-Gaussian signals was visited. Stemming from the natural ability of quaternion division algebra for modelling multidimensional signals and the capacity of α-stable distributions to characterise the behaviour of non-Gaussian phenomena, a quaternion-valued particle filter was derived based on the characteristic functions of α-stable random signals with elliptically contoured distributions. In addition, the framework was derived to be suitable for collaborative estimation tasks over sensor networks. The performance of the derived particle filter was demonstrated in a numerical example.

References

[1] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian random Processes*, Chapman & Hall, 1994.

[2] V. V. Uchaikin V. M. Zolotarev, *Chance and stability: Stable distributions and their applications*, VSP, 1999.

[3] M. Shao and C. L. Nikias, "Signal processing with fractional lower order moments: Stable processes and their applications," *In Proceedings of the IEEE*, vol. 81, no. 7, pp. 986-1010, July 1993.

[4] J. P. Nolan and D. Ojeda-Revah, "Linear and nonlinear regression with stable errors," *Journal of Econometrics*, vol. 172, no. 2, pp. 186-194, February 2013.

[5] N. Azzaoui and L. Clavier, "Statistical channel model based on α-stable random processes and application to the 60 GHz ultra wide band channel," *IEEE Transactions on Communications*, vol. 58, no. 5, pp. 1457-1467, May 2010.

[6] O. Hlinka, F. Hlawatsch, and P. M. Djuric, "Distributed particle filtering in agent networks: A survey, classification, and comparison," *IEEE Signal Processing Magazine*, vol. 30, no. 1, pp. 61-81, January 2013.

[7] C. J. Bordin and M. G. S. Bruno, "Consensus-based distributed particle filtering algorithms for cooperative blind equalization in receiver networks," *In Proceedings of IEEE International Conference on Acoustics, Speech, and Signal Processing*, pp. 3968-3971, 2011.

[8] A. Mohammadi and A. Asif, "Distributed consensus + innovation particle filtering for bearing/range tracking with communication constraints," *IEEE Transactions on Signal Processing*, vol. 63, no. 3, pp. 620-635, February 2015.

[9] B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman filter: Particle filters for tracking applications*, Artech House, 2004.

[10] O. Hlinka, O. Sluciak, F. Hlawatsch, P. M. Djuric, and M. Rupp, "Distributed Gaussian particle filtering using likelihood consensus," *In Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing*, pp. 3756-3759, 2011.

[11] W. Xia, M. Sun and Q. Wang, "Direct target tracking by distributed Gaussian particle filtering for heterogeneous networks," *IEEE Transactions on Signal Processing*, vol. 68, pp. 1361-1373, 2020.

[12] W. Song, Z. Wang, J. Wang, F. E. Alsaadi and J. Shan, "Particle filtering for nonlinear/non-Gaussian systems with energy harvesting sensors subject to randomly occurring sensor saturations," *IEEE Transactions on Signal Processing*, vol. 69, pp. 15-27, 2021.

[13] D. P. Mandic, C. Jahanchahi and C. C. Took, "A quaternion gradient operator and its applications," *IEEE Signal Processing Letters*, vol. 18, no. 1, pp. 47-50, January 2011.

[14] C. C. Took and D. P. Mandic, "Augmented second-order statistics of quaternion random signals," *Signal Processing*, vol. 91, no. 2, pp. 214-224, February 2011.

[15] J. Via, D. P. Palomar, L. Vielva and I. Santamaria, "Quaternion ICA from second-order statistics," *IEEE Transactions on Signal Processing*, vol. 59, no. 4, pp. 1586-1600, April 2011.

[16] J. Via, D. Ramirez, and I. Santamaria, "Properness and widely linear processing of quaternion random vectors," *IEEE Transactions on Information Theory*, vol. 56, no. 7, pp. 3502-3515, July 2010.

[17] S. P. Talebi and D. P. Mandic, "A quaternion frequency estimator for three-phase power systems," *In Proceedings of IEEE International Conference on Acoustics, Speech, and Signal Processing*, pp. 3956-3960. 2015.

[18] S. P. Talebi, S. Werner and D. P. Mandic, "Quaternion-valued distributed filtering and control," *IEEE Transactions on Automatic Control*, vol. 65, no. 10, pp. 4246-4257, October 2020.

[19] L. Xiaodong, L. Aijun, Y. Changjun and S. Fulin, "Widely linear quaternion unscented Kalman filter for quaternion-valued feedforward neural network," *IEEE Signal Processing Letters*, vol. 24, no. 9, pp. 1418-1422, September 2017.

[20] J. L. Crassidis, F. L. Markley, and Y. Cheng, "Survey of nonlinear attitude estimation methods," *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 1, pp. 12-28, January 2007.

[21] S. Said, N. Le Bihan, and S. J. Sangwine, "Fast complexified quaternion Fourier transform," *IEEE Transactions on Signal Processing*, vol. 56, no. 4, pp. 1522-1531, April 2008.

[22] T. A. Ell and S. J. Sangwine, "Quaternion involutions and anti-involutions," *Computers & Mathematics with Applications*, vol. 53, no. 1, pp. 137-143, January 2007.

[23] T. K. Paul and T. Ogunfunmi, "A Kernel adaptive algorithm for quaternion-valued inputs," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, no. 10, pp. 2422-2439, October 2015.

[24] F. A. Tobar and D. P. Mandic, "Quaternion reproducing kernel Hilbert spaces: Existence and uniqueness conditions," *IEEE Transactions on Information Theory*, vol. 60, no. 9, pp. 5736-5749, September 2014.

[25] D. H. Dini, P. M. Djuric, and D. P. Mandic, "The augmented complex particle filter," *IEEE Transactions on Signal Processing*, vol. 61, no. 17, pp. 4341-4346, September 2013.

[26] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking," *IEEE Transactions on Signal Processing*, vol. 50, no. 2, pp. 174-188, February 2002.

[27] R. P. G. Collinson, *Introduction to avionics systems*. Springer, 2011.

[28] J. B. Kuipers, "Quaternions and rotation sequences: A primer with applications to orbits, aerospace and virtual reality," *Princeton University Press*, August 2002.

[29] X. Shen, H. Zhang, Y. Xu, and S. Meng, "Observation of alpha-stable noise in the laser gyroscope data," *IEEE Sensors Journal*, vol. 16, no. 7, pp. 1998-2003, April 2016.