Representations of the general linear group with multilinear constructions

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Declaration of authorship

I, Eoghan McDowell, hereby declare that this thesis and the work presented in it is entirely my own. Where I have consulted the work of others, this is always clearly stated.

Signed: 

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Abstract

This thesis presents a collection of results in the representation theory of the general linear group in defining characteristic, with a focus on multilinear constructions, explicit maps and combinatorial techniques.

We use tableaux to describe concrete models for the the Schur and Weyl endofunctors, and hence in particular the Weyl modules, their duals, and the Specht modules.

We establish a number of modular plethystic isomorphisms – isomorphisms between modules obtained by iterated Schur and Weyl endofunctors – for $\mathrm{GL}_2(K)$, where K is an arbitrary field. Our isomorphisms are generalisations of classical results, and require dualities that were not present in characteristic 0. An example is Hermite reciprocity $\mathrm{Sym}_m \mathrm{Sym}^l E \cong \mathrm{Sym}_l \mathrm{Sym}^m E$, where E is the natural representation. We exhibit explicit maps for our isomorphisms.

We study the image under the inverse Schur functor of the Specht module for the symmetric group, proving a necessary and sufficient condition on the indexing partition for this image to be isomorphic to the dual Weyl module in characteristic 2, and giving an elementary proof that this isomorphism holds in all cases in all other characteristics. We use this result to identify some indecomposable Specht modules. When the isomorphism does not hold, we describe some particular examples and prove some additional results, including a bound on the dimension of the kernel of the quotient map onto the dual Weyl module.

Finally we investigate a family of Markov chains on the set of simple representations of the finite group $\mathrm{SL}_2(\mathbb{F}_p)$, defined by tensoring with a fixed simple module and choosing an indecomposable non-projective summand. We draw connections between the properties of the chain and the representation theory of $\mathrm{SL}_2(\mathbb{F}_p)$, emphasising symmetries of the tensor product. We also give a novel elementary proof of the decomposition of tensor products of simple modular $\mathrm{SL}_2(\mathbb{F}_p)$-representations.

Contents

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Introduction

0.1. Overview

This thesis collects a number of results on the representation theory of the general linear group in defining characteristic, with a focus on multilinear constructions, explicit maps and combinatorial techniques. The multilinear constructions are known as *Schur* and *Weyl endofunctors*, which we denote ∇^λ and Δ^λ respectively. When applied to the natural representation E of the general linear group $\mathrm{GL}_n(K)$ over a field K, these functors yield the *dual Weyl* and *Weyl modules*. These modules are of great importance in the study of the representation theory of the general linear group: the socles of the dual Weyl modules (or equivalently, the heads of the Weyl modules) form a complete irredundant set of simple modules in the category of polynomial representations. When these endofunctors are applied to the natural permutation module of the symmetric group S_r, after restriction to a certain subspace we obtain the *Specht modules* S^λ and their duals, which likewise give rise to a complete irredundant set of simple modules (the heads of the Specht modules indexed by p-regular partitions, or the socles of the dual Specht modules indexed by p-restricted partitions).

The first goal of this thesis is to provide concrete descriptions of the Schur and Weyl endofunctors – and hence the Weyl, dual Weyl and Specht modules – viewing both functors both as explicit submodules and as explicit quotients of symmetric or exterior powers. This is achieved in Chapter I. Tableaux and tabloids are used to model the elements of our modules in a visual way which is amenable to combinatorial techniques, and which holds for representations of any group over any field. The majority of this construction is well-known, at least in the context of the general linear or symmetric groups; however, it has not appeared all in one place in our generality, and some existing sources rely on excess machinery such as the theory of algebraic groups and the Schur algebra. Additionally, our description of the row Garnir relations satisfied by the Weyl endofunctors is a new generalisation of the row relations for the dual Specht module. A review of existing constructions is given below.

We aim to provide a unified, elementary treatment; the author hopes that Chapter I will serve as a helpful and thorough reference for the Schur and Weyl endofunctors which brings out both their simplicity and utility.

A crucial property of the Weyl and dual Weyl modules is that they are polynomial. In Chapter II we introduce this property and its connection with the representation theory of the symmetric group: the Schur functor $\mathcal{F}$ and its one-sided inverses $\mathcal{G}_{\otimes}$ and $\mathcal{G}_{\mathrm{Hom}}$ (not to be confused with the Schur endofunctors above). Once again we give elementary definitions, bypassing the need for the Schur algebra and thereby presenting constructions of the Schur functor and its inverses which generalise to finite fields (see Remark 6.22 for a comparison with the Schur algebra approach).

The remaining three chapters collect the author's results on representations of the general linear group in defining characteristic. Each of these chapters is logically independent from the other two. Chapter III draws on [McDW21], investigating isomorphisms between $K\mathrm{GL}_2(K)$-modules obtained by iterated Schur and Weyl endofunctors, generalising some well-known classical results to the modular case; Chapter IV draws on [McD21a], investigating the image of the Specht modules under the inverse Schur functor in arbitrary characteristic; Chapter V draws on [McD21b], investigating tensor products of representations of the finite group $\mathrm{SL}_2(\mathbb{F}_p)$ and a random walk on their indecomposable summands. An overview of each of these chapters can be found below.

0.2. Existing constructions of the Schur and Weyl endofunctors

There are many treatments of the Schur and Weyl endofunctors, especially in the case of their application to representations of the general linear group $\mathrm{GL}_n(K)$ and the symmetric group S_r.

Our submodule construction of the Schur endofunctors, including the use of the Garnir relations, is a generalisation to arbitrary groups of the construction of the dual Weyl modules given by de Boeck, Paget and Wildon [dBPW21]. Their GL-polytabloids we call simply polytabloids, and their snake relations are a particular case of our Garnir relations. This is also the construction used by the author in [McD21a]. When used to obtain the Specht module, this construction becomes that due to James [Jam78].

Green [EGS08, §4] constructs the dual Weyl module (which he denotes $D_{\lambda,K}$) as a module for the Schur algebra spanned by bideterminants, and shows that it can also be obtained as the quotient of the tensor power $E^{\otimes|\lambda|}$ by relations combining our J_{Alt} and Garnir relations ([EGS08, §4.6]), or by inducing a certain character from a Borel subgroup to $\mathrm{GL}_n(K)$ ([EGS08, §4.8]). The Weyl module is defined by Green [EGS08, Section 5] to be its dual (which he denotes $V_{\lambda,K}$), and is shown to be isomorphic to the Carter–Lusztig Weyl module [CL74, Section 3.2] (denoted $\bar{V}^{\lambda}$).

Green also shows that the Weyl module is generated by its highest weight vector [EGS08, (5.3b)]. Wildon [Wil20] uses this as a characterising property of the module as a submodule of the exterior power in order to identify, in the case that K is infinite, a basis consisting of the elements we call copolytabloids.

In the case $K = \mathbb{C}$, Fulton constructs the dual Weyl module (which he denotes E^{λ} and calls the Schur module) as the quotient of an exterior power by his *quadratic relations* [Ful97]. This is equivalent to our construction of the Schur endofunctor as the quotient by the Garnir relations.

James constructs the dual Weyl module (which he denotes W^{λ} and, unfortunately in the light of modern terminology, calls the Weyl module) by summing the images of the space of polytabloids of symmetric type under maps which induce each possible weight [Jam78, Definitions 17.2, 17.4 and 26.4, pp. 65,127,129]. The correspondence between this and our construction is noted in [dBPW21, Remark 2.16].

Constructions of the Schur and Weyl endofunctors themselves appear in [Kou91] and [ABW82] (the latter using the name "coSchur" in place of "Weyl"). Kouwenhoven's construction is through the letter place algebra. Akin, Buchsbaum and Weyman's definition, given for skew partitions, is as the image of a recursively defined map; this presents the endofunctors as quotients, but in contrast to our construction does not give an explicit model.

The row Garnir relations – described in §3.4 to present the Weyl endofunctor as a quotient of a symmetric power – do not appear in the literature. These relations are a non-trivial generalisation of the row relations for the dual Specht module.

0.3. Chapter III: Modular plethystic isomorphisms

In the context of representation theory, *plethysm* refers to the composition of Schur and Weyl endofunctors, and the goal is to describe modules of the form $\nabla^\mu \nabla^\lambda E$, and those with a Δ in place of a ∇, where E is the natural $\mathrm{GL}_n(K)$-module. In this chapter we describe or rule out the existence of a number of isomorphisms between representations of $\mathrm{GL}_2(K)$ of this form, in arbitrary characteristic. We call such isomorphisms *plethystic isomorphisms*. This chapter draws on the author's joint work with Mark Wildon [McDW21].

The characteristic-free isomorphisms we prove, stated for the special linear group $\mathrm{SL}_2(K)$ where K is an arbitrary field, are: Hermite reciprocity

$$\mathrm{Sym}_m \mathrm{Sym}^l E \cong \mathrm{Sym}_l \mathrm{Sym}^m E;$$

the Wronskian isomorphism

$$\mathrm{Sym}_m \mathrm{Sym}^l E \cong \textstyle\bigwedge^m \mathrm{Sym}^{l+m-1} E;$$

and the complementary partition isomorphism

$$\nabla^\lambda \mathrm{Sym}^l E \cong \nabla^{\lambda^\circ} \mathrm{Sym}_l E,$$

where λ° denotes the complement of λ in a rectangle with $l+1$ rows. In each case we exhibit an explicit map. Here Sym^l denotes the usual symmetric power, a quotient of the tensor power; Sym_l denotes its dual, the *lower symmetric power*, defined in §3.2.

On the other hand, we show that the conjugate partition isomorphism

$$\nabla^\lambda \mathrm{Sym}^{m+\lambda'_1-1} E \cong \nabla^{\lambda'} \mathrm{Sym}^{m+\lambda_1-1} E,$$

known to hold over $\mathbb{C}$ under certain conditions on λ by [Kin85, §4], does not have a modular analogue under any combination of swapping Schur and Weyl endofunctors and upper and lower symmetric powers. We prove this by considering hook partitions of prime power arm and leg length, and introducing a new invariant called the *defect set* to distinguish the modules. The author credits Mark Wildon for the introduction of this invariant.

Many classical (characteristic 0) plethystic isomorphisms are known, including the classical versions of the above isomorphisms. Some have been known for a long time – Hermite reciprocity, for example, dates back to the 19th century – while more recent works on the subject include King's [Kin85] and Paget and Wildon's [PW21]. In the modular case, only compositions of symmetric and exterior powers have previously been studied: [Kou90b] shows the failure of the naïve generalisation of Hermite reciprocity; [AFP$^+$19] gives correct generalisations of Hermite reciprocity and the Wronskian isomorphism, but lacks our explicit description of the maps. Moreover, our approach to the Wronskian isomorphism is more general than, and uses different methods to, both [AFP$^+$19] and [McDW21]: here we prove that there is an injection $\mathrm{Sym}_{\overline{m}} \mathrm{Sym}^l E \hookrightarrow \bigwedge^{\overline{m}} \mathrm{Sym}^{l+m-1} E$ where $\overline{m} = \binom{m+n-1}{m}$ and E is the natural representation of $\mathrm{SL}_n(K)$ for any n. Further comparison of our work with existing results, both classical and modular, is given in the introduction to Chapter III.

Though this thesis does not study it, we remark that *plethysm* also has a combinatorial definition: it is a certain product of symmetric functions, denoted $\circ$. This is the original meaning of the term, defined for example in [Mac98, Section 1.8]. The connection with representation theoretic plethysm is characters: the formal character of $\nabla^\mu E$ is the *Schur polynomial* s_λ, and the formal character of $\nabla^\mu \nabla^\lambda E$ is the plethysm product $s_\mu \circ s_\lambda$. Here, the *formal character* of a representation V of $\mathrm{GL}_n(K)$ is the polynomial whose coefficient of $\prod_i x_{\alpha_i}$ is the dimension of the α-weight space of V (see Definition 6.1 for the definition of weight spaces). In characteristic zero, a representation is determined by its formal character, and so, in the classical setting, decomposing $\nabla^\mu \nabla^\lambda E$ into irreducibles is equivalent to writing $s_\mu \circ s_\lambda$ as a linear combination of Schur polynomials. Finding a combinatorial interpretation of these coefficients is the task set by Stanley's Problem 9 ([Sta99]).

0.4. Chapter IV: The Specht module under the inverse Schur functor

The inverse Schur functor $\mathcal{G}_\otimes$ is a map from the category of representations of the symmetric group to the category of (polynomial) representations of the general linear group (see §6 for the definition of $\mathcal{G}_\otimes$). This chapter studies, in all characteristics, the image of the Specht module S^λ under the inverse Schur functor, drawing on the author's [McD21a].

It has been known for some time that in characteristics other than 2 and 3, the image $\mathcal{G}_\otimes(S^\lambda)$ is isomorphic to the dual Weyl module $\nabla^\lambda E$ [KN01, 3.2]. This fact is also known in the more general context of q-Schur algebras and Hecke algebras of quantum characteristic at least 4 [HN04, Theorem 3.4.2]. A related result identifying the image of the twisted Young module in characteristics other than 2 is given in [CPS96, Theorem 5.2.4].

Here we give a necessary and sufficient condition on the indexing partition for the isomorphism $\mathcal{G}_\otimes(S^\lambda) \cong \nabla^\lambda E$ to hold in characteristic 2, and give an elementary proof that this isomorphism holds in all cases in all other characteristics. The condition is this: provided $\dim E \geqslant |\lambda| - 2$, there is an isomorphism $\mathcal{G}_\otimes(S^\lambda) \cong \nabla^\lambda E$ if and only if λ is 2-regular, or if $\lambda_1 = \lambda_2 \geqslant \lambda_3+2$ and λ minus its first part is 2-regular. The novelty of these results is in characteristics 2 and 3. Additionally, the approach here establishes the isomorphisms in characteristics other than 2 without the level of homological algebra used in the accounts cited above.

We deduce from this result some new examples of indecomposable Specht modules: whenever λ meets the condition above, S^λ is indecomposable. Determining the decomposability Specht modules in characteristic 2 is an open problem; the first family of decomposable Specht modules was identified by Murphy [Mur80], after which there was little progress until the recent results of Dodge and Fayers [DF12] and Donkin amd Geranios [DG20]. Our result adds to the list of Specht modules whose decomposability is known.

When the isomorphism $\mathcal{G}_\otimes(S^\lambda) \cong \nabla^\lambda E$ does not hold, the dual Weyl module is still a quotient of the image $\mathcal{G}_\otimes(S^\lambda)$. We prove some additional results in this case: we demonstrate that the image need not have a filtration by dual Weyl modules; we bound the dimension of the kernel of the quotient map; and we give some explicit descriptions for particular cases.

0.5. Chapter V: Tensor products of representations of $\mathrm{SL}_2(\mathbb{F}_p)$

Our final chapter presents decompositions of tensor products of certain representations of the finite group $\mathrm{SL}_2(\mathbb{F}_p)$ in defining characteristic, and studies a random walk on the representations of $\mathrm{SL}_2(\mathbb{F}_p)$ driven by taking tensor products. These results are drawn from the author's [McD21b].

We offer a novel elementary proof of the so-called Clebsch–Gordan rule, which describes the decomposition of tensor products of simple modules. Our approach is to use a family of $\mathrm{GL}_2(K)$-homomorphisms which exhibit explicit submodules and quotients of the tensor products, and use self-duality to inductively show that these maps split over $\mathrm{SL}_2(\mathbb{F}_p)$. This yields a proof of the rule which finds the projective summands more efficiently than inductively tensoring by the natural representation (as in [Glo78, (5.5) and (6.3)] or [Kou90a, Corollary 1.2(a) and Proposition 1.3(c)]), and that does not require the machinery of tilting theory (as in [EH02, Lemma 4]). Adding to the work of [McD21b], we then apply the Clebsch–Gordan rule to decompose tensor products involving projective indecomposable modules and to decompose symmetric squares.

The random walk we study is defined by tensoring by a fixed simple module and choosing a non-projective indecomposable summand of the result (with probability depending on a weighting given to each simple module). This is inspired by [BDLT20], which considers Markov chains defined by choosing composition factors (rather than indecomposable summands) of tensor products; the Benkart–Diaconis–Liebeck–Tiep chains for $\mathrm{SL}_2(\mathbb{F}_p)$ are examined in §3.2 of [BDLT20], in the cases of tensoring with the natural module and the Steinberg module. We show our new family of random walks are reversible and find their connected components and their stationary distributions (for any choice of simple module to tensor with). We draw connections between these properties of the chain and the representation theory of $\mathrm{SL}_2(\mathbb{F}_p)$, emphasising symmetries of the tensor product.

0.6. Conventions and notation

Let K denote a field, which may be of characteristic 0 or of prime characteristic p. Let G denote a (not necessarily finite) group. Most commonly G will be either a general linear group $\mathrm{GL}_n(K)$ of $n \times n$ invertible matrices; its subgroup the special linear group $\mathrm{SL}_n(K)$ of $n \times n$ matrices with determinant 1; or the symmetric group S_r of permutations of r symbols (where n and r are some positive integers). We write KG for the group algebra of formal K-linear sums of elements of G.

We still study finite dimensional representations of G over K. Such a representation can be thought of either as a module for the group algebra KG, or a group homomorphism $\rho\colon G \to \mathrm{GL}_d(K)$ for some integer d which is the dimension of the representation as a K-vector space. Given a KG-module V, we let ρ_V denote the corresponding group homomorphism $G \to \mathrm{GL}_{\dim V}(K)$; note that constructing ρ_V from V requires a choice of basis for V. We will use the terms "representation of G over K" and "KG-module" interchangeably.

We generally work with left KG-modules, but have permutation groups acting on the right. This allows us to use the action of permutation groups to construct interesting modules, without confusing the permutation group with the group G whose representation theory is our primary interest.

Given a K-vector space V, we let $V^* = \mathrm{Hom}_K(V, K)$ denote the dual K-vector space. Given a K-basis $\{v_1, \ldots, v_d\}$ for V, the dual basis for V^* is $\{v_1^*, \ldots, v_d^*\}$ where $v_i^*(v_i) = 1$ and $v_i^*(v_j) = 0$ for all $j \neq i$. In §3.1 we will describe two ways in which the dual vector space also becomes a representation.

A representation of particular importance is the natural representation E of a group $G \leqslant \mathrm{GL}_n(K)$ of $n \times n$ matrices. This representation has dimension n and with respect to a given basis the matrices of G act by matrix multiplication; that is, with respect to this basis, the corresponding homomorphism is the embedding $\rho_E\colon G \hookrightarrow \mathrm{GL}_n(K)$. Explicitly we choose a basis $X_1, \ldots, X_n$ such that for all $g \in G \leqslant \mathrm{GL}_n(K)$ we have

$$gX_i = \sum_{j=1}^{n} g_{j,i} X_j.$$

Given any set X we let S_X denote the symmetric group on X (that is, the group of permutations of X). We write elements of S_X on the right of their arguments, and hence multiply permutations left to right (for example, we have $(x\ y\ z)(x\ y) = (y\ z)$ for elements $x, y, z \in X$).

For an integer $r \geqslant 1$, we write $[r] = \{1, \ldots, r\}$ and $[r]_0 = \{0, 1, \ldots, r\}$.

We write $\mathbb{1}$ for the indicator function for propositions (so that, for example, $\mathbb{1}[p > 2]$ evaluates to 1 if $p > 2$ and to 0 otherwise).

Given a group G and a subgroup H, we write G/H for the set of left cosets gH of H in G, and $H\backslash G$ for the set of right cosets Hg of H in G. Given also a subgroup F, we write $F\backslash G/H$ for the set of double cosets FgH of F on the left and H on the right in G. Abusing notation, we denote sets of coset representatives in the same way.

For an integer $r \geqslant 1$, we write Sym^r and $\bigwedge^r$ for the rth symmetric and exterior powers, viewed as *quotients* of the tensor power. Explicitly,

$$\mathrm{Sym}^r V \cong {}^{V^{\otimes r}}\!\big/_{\langle\, w \cdot \sigma - w \;\mid\; w \in V^{\otimes r},\, \sigma \in S_r \,\rangle_K},$$

$$\textstyle\bigwedge^r V \cong {}^{V^{\otimes r}}\!\big/_{\langle\, w \in V^{\otimes r} \;\mid\; w \cdot \tau = w \text{ for some transposition } \tau \in S_r \,\rangle_K}.$$

The dual notion of the symmetric power is introduced in §3.2; this is defined as the submodule of the tensor power consisting of symmetric tensors.

A glossary of symbols is provided on page 208.

CHAPTER I

Multilinear constructions

This chapter presents elementary constructions of the Schur and Weyl endofunctors, which are endofunctors on the category of representations of a group G over a field K.

Our approach relies on *tableaux* and *tabloids*, which are introduced in §1. The Schur endofunctor is constructed, as both a submodule of a symmetric power and a quotient of an exterior power, in §2. The Weyl endofunctor is defined by pre- and post-composing the Schur endofunctor with the duality functor; in §3, we describe it as both a submodule of an exterior power and a quotient of (the dual of) a symmetric power. The row Garnir relations we describe in §3.4 are new.

The work of the first three sections of this chapter is valid for any group. Nevertheless our primary interests are the general linear group $\mathrm{GL}_n(K)$ and the symmetric group S_r. In §4 we apply the Schur and Weyl endofunctors to the natural representation of $\mathrm{GL}_n(K)$, obtaining the dual Weyl and Weyl modules, and to the natural permutation representation of S_r, obtaining (after restriction to a certain subspace) the Specht modules and their duals.

1. Tableaux, tabloids and Garnir relations

In this section we define the combinatorial objects which will be used to construct our representations. These objects are tableaux and their equivalence classes, tabloids.

The *entries* of our tableaux will be basis vectors for a left KG-module V for a group G. We denote the basis $\mathcal{B}$ and choose a total order on it. For convenience, we will often think of $\mathcal{B}$ as being the set $[d]$, where d is the dimension of V, writing i for the basis vector labelled by i.

The G-action on V induces a "diagonal" left G-action on the space of tableaux (and their equivalence classes) by entrywise action and multilinear expansion, as illustrated in Example 2.2. The K-vector spaces defined in this chapter thus become left KG-modules. We denote these group actions by concatenation.

1.1. Partitions and tableaux

A *composition* of r is a sequence of strictly positive integers whose sum is r. A *partition* of r is a weakly decreasing composition of r; we use λ to denote a partition throughout. The sum of the parts of a partition λ is called its *size*, denoted $|\lambda|$. The number of parts of a partition λ is called its *length*, denoted $\ell(\lambda)$; by convention we interpret $\lambda_i = 0$ for $i > \ell(\lambda)$.

The *Young diagram* of a partition λ is the set $[\lambda] = \{\, (i,j) \mid 1 \leqslant i \leqslant \ell(\lambda), 1 \leqslant j \leqslant \lambda_i \,\}$, which we picture lying in the plane using the "English" notation: the x-direction downward and the y-direction rightward. An element of a Young diagram is called a *box*. Let $\mathrm{row}_i[\lambda]$ and $\mathrm{col}_j[\lambda]$ denote the sets of boxes in row i and column j of $[\lambda]$ respectively.

The *conjugate* (or *transpose*) of a partition λ, denoted λ', is the partition defined by $\lambda'_i = |\{\, j \geqslant 1 \mid \lambda_j \geqslant i \,\}|$ for $1 \leqslant i \leqslant \lambda_1$. This is the partition obtained by reflecting the Young diagram of λ over the main diagonal.

A *tableau* of shape λ with entries in $\mathcal{B}$ is a function $[\lambda] \to \mathcal{B}$. The image of a box $b \in [\lambda]$ under a tableau t is the *entry* of t in b. The *weight* of t is the multiset of entries of a tableau t, expressed as a composition of n via the total ordering on $\mathcal{B}$. We depict a tableau t by filling the boxes in the Young diagram of λ with their entries in t.

Example 1.1. Suppose $\lambda = (3, 2)$. The size of λ is $|\lambda| = 5$; that is, λ is a partition of 5. The length of λ is $\ell(\lambda) = 2$. The Young diagram of λ is

$$[\lambda] = \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \end{array}.$$

Three tableaux of shape λ with entries in $[3]$ are depicted below.

$$\begin{array}{|c|c|c|}\hline 2 & 1 & 3 \\ \hline 1 & 3 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline 1 & 1 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 3 & 2 \\ \cline{1-2} \end{array}$$

Each has weight $(2, 1, 2)$ (indicating precisely two entries of 1, one of 2 and two of 3). Each has entry 3 in the box $(1, 3)$.

Let $\mathrm{YT}_{\mathcal{B}}(\lambda)$ denote the set of tableaux of shape λ with entries in $\mathcal{B}$. Let $\mathsf{Tbx}^{\lambda}(V)$ be the KG-module with basis $\mathrm{YT}_{\mathcal{B}}(\lambda)$. There is a (non-unique) isomorphism $\mathsf{Tbx}^{\lambda}(V) \cong V^{\otimes|\lambda|}$.

If the entries of a tableau strictly increase along the rows or down the columns, we say it is *row standard* or *column standard* respectively. The set of column standard tableaux of shape λ with entries in $\mathcal{B}$ is denoted $\mathrm{CSYT}_{\mathcal{B}}(\lambda)$. If the entries of a tableau weakly increase along the rows or down the columns, we say it is *row semistandard* or *column semistandard* respectively. The set of row semistandard tableaux of shape λ with entries in $\mathcal{B}$ is denoted $\mathrm{RSSYT}_{\mathcal{B}}(\lambda)$.

If a tableau is both row semistandard and column semistandard, we abbreviate this description to *row-and-column semistandard*.

If a tableau is both row standard and column standard, we say it is *standard*. If a tableau is both row semistandard and column standard, we say it is *semistandard*. The set of semistandard tableaux of shape λ with entries in $\mathcal{B}$ is denoted $\mathrm{SSYT}_{\mathcal{B}}(\lambda)$. For this and other sets just defined, when the set $\mathcal{B}$ is clear from context it is suppressed in the notation.

Example 1.2. None of the tableaux depicted in Example 1.1 are semistandard: the first is neither row semistandard nor column standard; the second is row semistandard but not column standard; the third is column standard but not row semistandard. A semistandard tableaux of shape λ with entries in $[3]$ and with weight $(2,1,2)$ is depicted below.

1	1	3
2	3	

1.2. Tableaux of symmetric type

We say a tableau t is *of symmetric type* if all entries of t are distinct. Let $\mathsf{Tbx}^{\lambda}_{\mathrm{sym}}(V)$ be the K-subspace of $\mathsf{Tbx}^{\lambda}(V)$ spanned by tableaux of symmetric type. Likewise, for all constructions of spaces in this chapter, let $-_{\mathrm{sym}}$ denote the construction restricted to tableaux of symmetric type.

Note that these restricted constructions yield K-subspaces which in general may not be KG-submodules. However, if V is a permutation KG-module and $\mathcal{B}$ is a permutation basis (as is the case in our specialisation to the symmetric group and the natural permutation module to yield the Specht module in §4.1), then indeed they are KG-submodules.

1.3. Place permutation action on tableaux

We write elements of $S_{[\lambda]}$ on the right of their arguments. The group $S_{[\lambda]}$ then acts on tableaux on the right by permuting the boxes via

$$(t \cdot \sigma)(b) = t(b\sigma^{-1})$$

for $\sigma \in S_{[\lambda]}$ and $b \in [\lambda]$. This action is essential notation for defining more complicated structures, but $S_{[\lambda]}$ should not be considered the group whose representation theory we are interested in.

Remark 1.3. We are writing elements of $S_{[\lambda]}$ on the right of their arguments, and a simple calculation demonstrates that the inverse is necessary in the above definition. If we were to instead write elements of $S_{[\lambda]}$ on the left of their arguments, then we would define the right action of $S_{[\lambda]}$ on tableaux by $(t \cdot \sigma)(b) = t(\sigma(b))$.

It is also possible to define a *left* place permutation action of $S_{[\lambda]}$ (and there are two ways to denote this depending on the choice of how to write elements of $S_{[\lambda]}$). We have made the choice to use a right action, in order to distinguish the place permutation action from the left group action of G.

Example 1.4. Suppose $\lambda = (3,2)$ with Young diagram $\ydiagram{3,2}$, and let t be the semistandard tableau depicted in Example 1.2. We illustrate the action of the permutation $\sigma = \big((1,1)\ (2,1)\ (2,2) \big)$ below; colour is used to indicate the boxes being moved.

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \cdot \big((1,1) \quad (2,1) \quad (2,2) \big) = \begin{array}{|c|c|c|} \hline 3 & 1 & 3 \\ \hline 1 & 2 \\ \cline{1-2} \end{array}$$

Observe that the tableau $t \cdot \sigma$ can be found from t by moving each box $b \in [\lambda]$ to the place previously occupied by $b\sigma$ (carrying its entry with it).

Define the sets of *row-preserving* and *column-preserving place permutations*, subgroups of $S_{[\lambda]}$, by

$$\mathrm{RPP}(\lambda) = \prod_{i=1}^{\lambda'_1} S_{\mathrm{row}_i[\lambda]} \qquad \text{and} \qquad \mathrm{CPP}(\lambda) = \prod_{j=1}^{\lambda_1} S_{\mathrm{col}_j[\lambda]}.$$

Given a tableau t, let $\mathrm{rstab}(t) = \mathrm{stab}(t) \cap \mathrm{RPP}(\lambda)$ and $\mathrm{cstab}(t) = \mathrm{stab}(t) \cap \mathrm{CPP}(\lambda)$ denote the *row stabiliser* and *column stabiliser* of t respectively.

1.4. Row tabloids

A *row tabloid* is an equivalence class of tableaux under row equivalence. Concretely, we quotient the space of tableaux $\mathsf{Tbx}^\lambda(V)$ by the subspace

$$J_{\mathsf{Sym}} = \langle\, x \cdot \sigma - x \mid x \in \mathsf{Tbx}^\lambda(V),\, \sigma \in \mathrm{RPP}(\lambda) \,\rangle_K,$$

and say the row tabloid corresponding to a tableau t is the element $t + J_{\mathsf{Sym}}$ in the quotient $\mathsf{Tbx}^\lambda(V)/J_{\mathsf{Sym}}$. We write the row tabloid corresponding to t as $[t]$, and draw a row tabloid $[t]$ by deleting the vertical lines from a drawing of t, as depicted below in the case $\lambda = (3,2)$.

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} \qquad \Longrightarrow \qquad [t] = \begin{array}{ccc} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}$$

By definition, $[t \cdot \sigma] = [t]$ for any $\sigma \in \mathrm{RPP}(\lambda)$. Thus the space of row tabloids is naturally isomorphic as a KG-module to the symmetric power

$\operatorname{Sym}^\lambda V = \bigotimes_{i=1}^{\lambda'_1} \operatorname{Sym}^{\lambda_i} V$, and we therefore use $\operatorname{Sym}^\lambda V$ to denote the space of row tabloids. This space has K-basis $\{\, [t] \mid t \in \mathrm{RSSYT}(\lambda) \,\}$.

1.5. Column tabloids

When defining column tabloids, we wish to also associate signs to the equivalence classes. This is achieved by quotienting the space of tableaux $\mathsf{Tbx}^\lambda(V)$ by the subspace

$$J_{\mathsf{Alt}} = \langle\, x \in \mathsf{Tbx}^\lambda(V) \mid x \cdot \tau = x \text{ for some transposition } \tau \in \mathrm{CPP}(\lambda) \,\rangle_K.$$

The *(alternating) column tabloid* corresponding to a tableau t is the element $t + J_{\mathsf{Alt}}$ in the quotient $\mathsf{Tbx}^\lambda(V)/J_{\mathsf{Alt}}$. We write this tabloid as $|t|$, and draw an alternating column tabloid by deleting the horizontal lines from a drawing of the corresponding tableau, as depicted below in the case $\lambda = (3, 2)$.

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} \quad \Longrightarrow \quad |t| = \begin{array}{|c|c|c|} 1 & 2 & 4 \\ 3 & 5 & \end{array}$$

Observe that $|t \cdot \sigma| = |t| \operatorname{sgn}(\sigma)$ for any $\sigma \in \mathrm{CPP}(\lambda)$, and furthermore $|t| = 0$ if t has a repeated entry in a column. For example, with $\lambda = (1, 1)$, the elements $\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}$ and $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ of $\mathsf{Tbx}^\lambda V$ are fixed by the transposition swapping the only two boxes, so these element lies in J_{Alt} and hence $\begin{vmatrix} 1 \\ 1 \end{vmatrix} = 0$ and $\begin{vmatrix} 1 \\ 2 \end{vmatrix} = -\begin{vmatrix} 2 \\ 1 \end{vmatrix}$. (To see that $|t \cdot \sigma| = |t| \operatorname{sgn}(\sigma)$ when $\sigma \in \mathrm{CPP}(\lambda)$ is a product of several transpositions, consider the collection of elements of the form $t \cdot \tau_1 \cdots \tau_i + t \cdot \tau_1 \cdots \tau_{i-1} \in J_{\mathsf{Alt}}$ where $\tau_1, \tau_2, \ldots$ is a sequence of transpositions whose product is σ.)

The space of alternating column tabloids is therefore naturally isomorphic as a KG-module to the exterior power $\bigwedge^{\lambda'} V = \bigotimes_{i=1}^{\lambda_1} \bigwedge^{\lambda'_i} V$, and we use $\bigwedge^{\lambda'} V$ to denote the space of alternating column tabloids. This space has K-basis $\{\, |t| \mid t \in \mathrm{CSYT}(\lambda) \,\}$.

In Chapter IV we introduce a different form of column tabloid, called a *skew column tabloid* (Definition 12.1).

1.6. Column and row ordering on tableaux

There exist many ways to order tableaux and tabloids. Here we define two orders on tableaux which we call the *column ordering* and the *row ordering*; a comparison with two well-known orderings is made in Remark 1.7.

Before giving the complete definition, we note that the order is easier to interpret in the case of tableaux of symmetric type: to compare two distinct tableaux of symmetric type, identify the largest entry which does not appear in the same column in both tableaux, and declare the $<_{\mathrm{c}}$-greater tableau to be the one for which this element is further left. We illustrate the $<_{\mathrm{c}}$-least and $<_{\mathrm{c}}$-greatest standard tableaux of symmetric type in the case $\lambda = (4^3, 2, 1)$ and $\mathcal{B} = [15]$ in Figure 1.1.

For the general definition, we require the symmetric difference operation on multisets, which we denote $\triangle$, and the multiset difference operation, which we denote $\setminus$. For example, using double braces to denote multisets, we have $\{\{1,3,3,4\}\} \triangle \{\{1,2,2,3\}\} = \{\{2,2,3,4\}\}$ and $\{\{1,3,3,4\}\} \setminus \{\{1,2,2,3\}\} = \{\{3,4\}\}$.

Definition 1.5. The *column ordering* is defined on the set of tableaux of a fixed shape as follows. Consider tableaux t and u of shape λ.

- If there is equality $\mathrm{col}_j(t) = \mathrm{col}_j(u)$ (as multisets) for all $1 \leqslant j \leqslant \lambda_1$, then we say t and u are *column equivalent* and write $t \sim_{\mathrm{c}} u$.
- Otherwise, let $m \in \mathcal{B}$ be maximal such that there exists j such that $m \in \mathrm{col}_j(t) \triangle \mathrm{col}_j(u)$, and let j be minimal such that $m \in \mathrm{col}_j(t) \triangle \mathrm{col}_j(u)$. If $m \in \mathrm{col}_j(u) \setminus \mathrm{col}_j(t)$, then we say $t <_{\mathrm{c}} u$.

The tableaux t and u are $<_{\mathrm{c}}$-incomparable if and only if $t \sim_{\mathrm{c}} u$. We write $t \lesssim_{\mathrm{c}} u$ to mean $t <_{\mathrm{c}} u$ or $t \sim_{\mathrm{c}} u$.

The relation $<_{\mathrm{c}}$ is a strict partial order. The relation $\sim_{\mathrm{c}}$ is an equivalence relation. The relation $\lesssim_{\mathrm{c}}$ is a total preorder, also known as a weak order (that is, $\lesssim_{\mathrm{c}}$ is a partial order with the antisymmetry requirement relaxed – permitting $t \lesssim_{\mathrm{c}} u$ and $u \lesssim_{\mathrm{c}} t$ to hold simultaneously for distinct t and u – and with the property that at least one of $t \lesssim_{\mathrm{c}} u$ and $u \lesssim_{\mathrm{c}} t$ holds for any pair of tableaux t and u).

1	6	10	13
2	7	11	14
3	8	12	15
4	9		
5			

(a) $<_{\mathrm{c}}$-least.

1	2	3	4
5	6	7	8
9	10	11	12
13	14		
15			

(b) $<_{\mathrm{c}}$-greatest.

FIGURE 1.1. Extremal standard tableaux of symmetric type for $\lambda = (4^3, 2, 1)$ and $\mathcal{B} = [15]$.

We make the identical definitions for the *row ordering* and the symbols $<_{\mathrm{r}}$, $\sim_{\mathrm{r}}$ and $\lessapprox_{\mathrm{r}}$, replacing all instances of "column" with "row". For two tableaux of symmetric type, the $<_{\mathrm{r}}$-greater tableau is the one in which the largest entry which does not appear in the same row in both tableaux appears further up the tableau (that is, in the numerically smaller row). Equivalently, for tableaux t and u of shape λ, we say $t <_{\mathrm{r}} u$ if and only if $t' <_{\mathrm{c}} u'$ (and $t \sim_{\mathrm{r}} u$ if and only if $t' \sim_{\mathrm{c}} u'$), where t' and u' are the tableaux of shape λ' obtained from t and u by conjugation

It is clear that $t \sim_{\mathrm{r}} u$ if and only if $[t] = [u]$, and that $t \sim_{\mathrm{c}} u$ implies $|t| = \pm|u|$ (the converse also holds, provided that $|t|, |u| \neq 0$).

Example 1.6. The following inequalities between (semistandard) tableaux of shape $(4^3, 2, 1)$ with entries in $[9]$ hold in the row ordering.

$$\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 6 \\ 3 & 5 & 7 & 8 \\ 5 & 7 & & \\ 9 & & & \end{array} \quad <_{\mathrm{r}} \quad \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 7 & 8 \\ 5 & 6 & & \\ 9 & & & \end{array} \quad <_{\mathrm{r}} \quad \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 7 \\ 3 & 5 & 8 & 8 \\ 4 & 6 & & \\ 9 & & & \end{array}$$

To see the first inequality, note that the tableaux differ only by a single transposition, swapping a pair of boxes which contain a 6 and a 7; the $<_{\mathrm{r}}$-greater tableau is the one in which the larger of these entries, 7, appears higher up. For the second inequality, the critical difference between the tableaux is that the largest entry which appears in the multiset symmetric difference of any row is 8, due to the contribution from the third row of the right-hand tableau; the differences in other rows are then irrelevant, and the right-hand tableau is $<_{\mathrm{r}}$-greater.

Remark 1.7. In the case of row semistandard tableaux of symmetric type, our row ordering $<_{\mathrm{r}}$ is the reverse of the ordering $<$ defined by James in [Jam78, Definition 3.1] (though ours is defined on tableaux rather than row equivalence classes). Our column ordering $<_{\mathrm{c}}$ is the column analogue of this.

On arbitrary row semistandard tableaux, the reverse of our row ordering $<_{\mathrm{r}}$ is an extension of the dominance order $\triangleleft$ defined by de Boeck, Paget and Wildon in [dBPW21, Definition 2.7] (which is a generalisation of [Jam78, Definition 3.11] on tabloids of symmetric type), in the sense that $t \triangleright u$ implies $t <_{\mathrm{r}} u$ (but not conversely). The column analogue of the dominance order, which our column ordering $<_{\mathrm{c}}$ extends, is defined by James in [Jam78, Definition 13.8] (for column equivalence classes of symmetric type).

1.7. Garnir relations

We here define Garnir relations as certain linear combinations of alternating column tabloids (that is, as certain elements of $\bigwedge^{\lambda'} V$). The motivation for considering these relations is that they are relations obeyed (in the sense of being sent to 0 by the appropriate quotient map) by the images of Schur endofunctors, defined in §2.

In the context of tableaux of symmetric type, James [Jam78, Section 7] encapsulated the same concept with *Garnir elements*: elements of the group algebra KS_n that annihilate the Specht modules. In that context, James's Garnir elements yield our notion of a Garnir relation when they act on suitable column tabloids of symmetric type. The relations used by de Boeck, Paget and Wildon [dBPW21, Lemma 2.4 and Equation 2.5] are images of our Garnir relations under the map e defined in §2.2. Fulton [Ful97, Section 8] describes a similar collection of linear combinations of alternating column tabloids which he calls quadratic relations; these generate the same K-subspace of $\bigwedge^{\lambda'} V$ as our Garnir relations.

Definition 1.8 (Garnir relations)**.** Let t be a tableau of shape λ with entries in $\mathcal{B}$. Let $1 \leqslant j < j' \leqslant \lambda_1$, and let $A \subseteq \mathrm{col}_j(\lambda)$ and $B \subseteq \mathrm{col}_{j'}(\lambda)$ be such that $|A| + |B| > \lambda'_j$. Choose $\mathcal{S} = S_{A \sqcup B}/S_A \times S_B$ a set of left coset representatives for $S_A \times S_B$ in $S_{A\sqcup B}$. The *Garnir relation* labelled by (t, A, B) is

$$\mathsf{R}_{(t,A,B)} = \sum_{\tau \in \mathcal{S}} |t \cdot \tau| \operatorname{sgn} \tau.$$

Let $\mathsf{GR}^\lambda(V)$ denote the subspace of $\bigwedge^{\lambda'} V$ which is spanned by the Garnir relations.

A Garnir relation does not depend on the choice of coset representatives: in the notation of the definition, if $\tau \in \mathcal{S}$ and $\sigma \in S_A \times S_B$, then because $S_A \times S_B \subseteq \mathrm{CPP}(\lambda)$ we have $|t \cdot \tau\sigma| \operatorname{sgn}(\tau\sigma) = |t \cdot \tau| \operatorname{sgn}(\tau)$, and so replacing τ with $\tau\sigma$ does not change the sum.

The K-subspace $\mathsf{GR}^\lambda(V)$ is moreover a KG-submodule. Indeed, the group action commutes with the place permutation action, so if $g \in G$ is such that $gt = \sum_{u \in \mathrm{YT}(\lambda)} \alpha_u u$ for some $\alpha_u \in K$, then $g\mathsf{R}_{(t,A,B)} = \sum_{u\in\mathrm{YT}(\lambda)} \alpha_u \mathsf{R}_{(u,A,B)}$.

In the following lemma we make the simple observation that a Garnir relation is zero if it involves boxes containing equal entries.

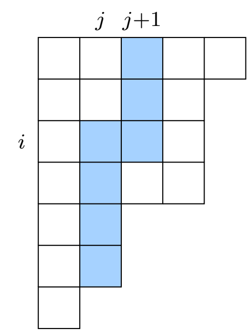
Lemma 1.9. *Let A, B be sets of boxes as in Definition 1.8. Suppose t is a tableau with an entry occurring with multiplicity greater than 1 in $A \sqcup B$. Then $\mathsf{R}_{(t,A,B)} = 0$.*

Proof. Let $b_1, b_2 \in A \sqcup B$ be boxes such that $t(b_1) = t(b_2)$; let $\tau = (b_1\ b_2) \in S_{A\sqcup B}$. Then τ acts on the left cosets of $S_A \times S_B$ in $S_{A\sqcup B}$ by left multiplication. For the orbits of size 1, choose coset representatives arbitrarily. For the orbits of size 2, choose a representative of one of the cosets in each orbit arbitrarily, and let the representative of the other coset be obtained from the first by left multiplication by τ. Let $\mathcal{S}$ be the set of representatives chosen in this way.

If $\sigma \in S_{[\lambda]}$ is any permutation, then $t \cdot \tau\sigma = t \cdot \sigma$. In particular if $\{\sigma, \tau\sigma\} \subseteq \mathcal{S}$ are the representatives of cosets in an orbit of size 2, then $|t \cdot \tau\sigma| = |t \cdot \sigma|$ and $\operatorname{sgn}(\tau\sigma) = -\operatorname{sgn}(\sigma)$, and hence the contribution to the Garnir relation $\mathsf{R}_{(t,A,B)}$ from this orbit is zero.

If $\sigma \in \mathcal{S}$ is the representative of a coset in an orbit of size 1, then $\sigma^{-1}\tau\sigma \in S_A \times S_B \subseteq \mathrm{CPP}(\lambda)$, and so the boxes $b_1\sigma$ and $b_1\tau\sigma = b_2\sigma$ lie in the same column. But $(t \cdot \sigma)(b_1\sigma) = t(b_1) = t(b_2) = (t \cdot \sigma)(b_2\sigma)$, so $t \cdot \sigma$ has a repeated entry in a column. Thus $|t \cdot \sigma| = 0$ and the contribution to the Garnir relation $\mathsf{R}_{(t,A,B)}$ from this orbit is zero. $\square$

It is often the case that we need only consider Garnir relations in which the chosen columns are adjacent and boxes are taken from the bottom of the left-hand column and the top of the right-hand column, with a single row containing chosen boxes from both columns. Following the terminology introduced in [dBPW21, Equation 2.5] (but requiring also that the columns in question be adjacent), we call such relations *snake relations* due to the shape of the outline of the chosen boxes (depicted in the margin). Formally they are as defined as follows.



Definition 1.10 (Snake relations)**.** A Garnir relation $\mathsf{R}_{(t,A,B)}$ is called a *snake relation* when, in the notation of Definition 1.8, $j' = j + 1$ and there exists i such that $A = \{\, (x, j) \mid i \leqslant x \leqslant \lambda'_j \,\}$ and $B = \{\, (x, j') \mid 1 \leqslant x \leqslant i \,\}$. In this case, we may also label the Garnir relation by (t, i, j).

We define sets of relations dual to the Garnir relations in Definition 3.16.

2. Two constructions of the Schur endofunctors

In this section we present two ways to construct the Schur endofunctors, one as a submodule of a symmetric power and one as quotient of an exterior power. We take the former as our definition and show that it is equivalent to the latter, establishing a well-known basis along the way.

2.1. A submodule of a symmetric power

Our submodule will consist of the following elements of the symmetric power.

Definition 2.1. The *polytabloid* corresponding to a tableau t is the element of $\operatorname{Sym}^\lambda V$ given by

$$\mathrm{e}(t) = \sum_{\sigma \in \mathrm{CPP}(\lambda)} [t \cdot \sigma] \operatorname{sgn} \sigma.$$

Since the action of G commutes with the place permutation action, we can compute the action of an element $g \in G$ on a polytabloid $\mathrm{e}(t)$ by applying g to each entry of t, expanding multilinearly, and taking the polytabloids corresponding to the resulting tableaux. That is, if $g \in G$ is such that $gt = \sum_{u \in \mathrm{YT}(\lambda)} \alpha_u u$ for some $\alpha_u \in K$, then $\mathrm{e}(t) = \sum_{u \in \mathrm{YT}(\lambda)} \alpha_u \,\mathrm{e}(u)$. We illustrate this convenient way to compute the action of G on a polytabloid with the following example.

Example 2.2. Suppose that $\mathcal{B} = \{v_1, v_2, v_3\}$, and as usual write i for v_i in diagrams for convenience. Suppose that $g \in G$ has action on V defined by $gv_1 = v_1 + \alpha v_3$, $gv_2 = v_2$, $gv_3 = \beta v_1 + v_3$. Then

$$\begin{aligned} g\,\mathrm{e}\left(\begin{ytableau} 1 & 2 & 2 \\ 3 & 3 \end{ytableau}\right) &= \mathrm{e}\left(\begin{ytableau} v_1{+}\alpha v_3 & v_2 & v_2 \\ \beta v_1{+}v_3 & \beta v_1{+}v_3 \end{ytableau}\right) \\ &= \mathrm{e}\left(\begin{ytableau} 1 & 2 & 2 \\ 3 & 3 \end{ytableau}\right) - \beta\,\mathrm{e}\left(\begin{ytableau} 1 & 1 & 2 \\ 3 & 2 \end{ytableau}\right) - \alpha\beta\,\mathrm{e}\left(\begin{ytableau} 1 & 2 & 2 \\ 3 & 3 \end{ytableau}\right) + \alpha\beta^2\,\mathrm{e}\left(\begin{ytableau} 1 & 1 & 2 \\ 3 & 2 \end{ytableau}\right) \end{aligned}$$

where the first line is interpreted purely formally.

Note that in particular the subspace spanned by the polytabloids is a KG-submodule of $\operatorname{Sym}^\lambda V$. This allows us to make the following definition of a Schur endofunctor as a submodule of a symmetric power.

Definition 2.3. The *Schur endofunctor* labelled by λ, denoted ∇^λ, is the endofunctor on the category of KG-modules defined on a KG-module V by

$$\nabla^\lambda V = \langle\, \mathrm{e}(t) \;\mid\; t \in \mathrm{YT}(\lambda)\,\rangle_K \subseteq \mathrm{Sym}^\lambda V,$$

and for a map f, $\nabla^\lambda f$ is defined by applying f to each entry in a polytabloid and expanding multilinearly.

Remark 2.4. The Schur endofunctors ∇^λ are more commonly referred to simply as the Schur functors. However, they are not related to the Schur functor $\mathcal{F}$ defined in §6, and so we refer to them as the Schur endofunctors to distinguish them from $\mathcal{F}$.

Example 2.5 (Symmetric powers and exterior powers)**.**

(i) Suppose $\lambda = (r)$ consists of a single row. Then $\mathrm{e}(t) = [t]$ and $\nabla^{(r)}V = \mathrm{Sym}^r V$.

(ii) Suppose $\lambda = (1^r)$ consists of a single column. Then

$$\begin{aligned}\mathrm{e}(t) &= \sum_{\sigma \in S_{[\lambda]}} t \cdot \sigma \\ &= \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma)\, t(1\sigma^{-1},1) \otimes t(2\sigma^{-1},1) \otimes \cdots \otimes t(r\sigma^{-1},1),\end{aligned}$$

and we claim that $\nabla^{(1^r)}V \cong \bigwedge^r V$. Indeed, the map

$$\begin{aligned}\nabla^{(1^r)}V &\to \textstyle\bigwedge^r V \\ \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma)\, t(1\sigma^{-1},1) \otimes \cdots \otimes t(r\sigma^{-1},1) &\mapsto t(1,1) \wedge \cdots \wedge t(r,1)\end{aligned}$$

is an isomorphism. Alternatively, in §2.2 we identify $\nabla^\lambda V$ as the quotient $\bigwedge^{\lambda'} V/\mathsf{GR}^\lambda(V)$ for all partitions, and plainly $\mathsf{GR}^{(1^r)}(V) = 0$.

An interesting property of the Schur endofunctors is that they are determined by the images of the natural representations of the general linear groups, in the sense of the following proposition. Recall we write ρ_V for the group homomorphism $G \to \mathrm{GL}_{\dim V}(K)$ representing V (given some choice of basis for V).

Proposition 2.6. *Suppose V is d-dimensional and let E be the natural representation of* $\mathrm{GL}_d(K)$. *Then* $\rho_{\nabla^\lambda V} = \rho_{\nabla^\lambda E}\rho_V$.

PROOF. Consider the action of $g \in G$ on $\mathrm{e}(t) \in \nabla^\lambda V$. As illustrated in Example 2.2, the action is given by acting by g on each entry of t – which by definition is represented by the matrix $\rho(g) \in \mathrm{GL}_d(K)$ – and expanding multilinearly. This is precisely the action of the matrix $\rho(g)$ on $\nabla^\lambda E$, which is what the proposition claims. $\square$

It it well-known that $\nabla^\lambda V$ has basis the set of polytabloids for semistandard tableaux. We see that this set is spanning in the next subsection §2.2; we see that it is linearly independent immediately, using the row ordering $<_\mathrm{r}$ defined in §1.6.

Lemma 2.7. *Let t be a column standard tableau. Then*

$$\mathrm{e}(t) = [t] + \sum_{u <_\mathrm{r} t} m_u [u]$$

for some elements m_u in the subring of K generated by 1. *In particular, the set* $\{\, \mathrm{e}(s) \mid s \in \mathrm{SSYT}(\lambda) \,\}$ *is K-linearly independent.*

PROOF. Since t is column standard, we have $t \cdot \sigma \lneq_\mathrm{r} t$ for all $\sigma \in \mathrm{CPP}(\lambda)$ with row equivalence if and only if $\sigma = \mathrm{id}$; the claimed expression for $\mathrm{e}(t)$ is then clear. The linear independence of $\{\, \mathrm{e}(s) \mid s \in \mathrm{SSYT}(\lambda) \,\}$ follows: the $<_\mathrm{r}$-greatest row semistandard tableau whose row tabloid appears in $\mathrm{e}(s)$ is, being s itself, distinct for each $s \in \mathrm{SSYT}(\lambda)$. $\square$

2.2. A quotient of an exterior power and the semistandard basis

An immediate consequence of the definition of a polytabloid is that $\mathrm{e}(t \cdot \sigma) = \mathrm{e}(t) \operatorname{sgn} \sigma$ for $\sigma \in \mathrm{CPP}(\lambda)$, and that $\mathrm{e}(t) = 0$ if t has a repeated entry in a column. It follows that the map $e\colon \bigwedge^{\lambda'} V \to \nabla^\lambda(V)$ defined by K-linear extension of

$$e\colon |t| \mapsto \mathrm{e}(t)$$

is well-defined and surjective. It is also G-equivariant. We thus see that $\nabla^\lambda V$ is the quotient of $\bigwedge^{\lambda'} V$ by the kernel of e. To make this into an explicit model for $\nabla^\lambda V$, we must identify the kernel of e.

The two main aims of this subsection are to show that $\ker e$ is the space of Garnir relations (defined in Definition 1.8) and to show that the set of semistandard polytabloids forms a basis of $\nabla^\lambda V$. These aims are intertwined: we require the Garnir relations to rewrite polytabloids in terms of semistandard ones.

We begin by showing that $\mathsf{GR}^\lambda(V) \subseteq \ker e$.

Proposition 2.8. *If* $\mathsf{R}_{(t,A,B)}$ *is any Garnir relation, then* $e(\mathsf{R}_{(t,A,B)}) = 0$.

Proof. The element we are required to show is zero is

$$\begin{aligned} e(\mathsf{R}_{(t,A,B)}) &= \sum_{\sigma \in \mathrm{CPP}(\lambda)} \sum_{\tau \in S_{A \sqcup B}/S_A \times S_B} [t \cdot \tau\sigma] \operatorname{sgn}(\tau\sigma) \\ &= \sum_{\sigma \in S_A \times S_B \backslash \mathrm{CPP}(\lambda)} \sum_{\pi \in S_A \times S_B} \sum_{\tau \in S_{A \sqcup B}/S_A \times S_B} [t \cdot \tau\pi\sigma] \operatorname{sgn}(\tau\pi\sigma) \\ &= \sum_{\sigma \in S_A \times S_B \backslash \mathrm{CPP}(\lambda)} \sum_{\tau \in S_{A \sqcup B}} [t \cdot \tau\sigma] \operatorname{sgn}(\tau\sigma) \end{aligned}$$

where we have first broken up the sum over $\mathrm{CPP}(\lambda)$ into sums over right cosets, and then collected up the sums over left cosets in $S_{A \sqcup B}$. From this expression we see that it suffices to fix $\sigma \in \mathrm{CPP}(\lambda)$ and show $\sum_{\tau \in S_{A \sqcup B}} [t \cdot \tau\sigma] \operatorname{sgn}(\tau) = 0$.

Recall from the definition of a Garnir relation that $A \subseteq \mathrm{col}_j[\lambda]$ and $B \subseteq \mathrm{col}_{j'}[\lambda]$ for some $1 \leqslant j < j' \leqslant \lambda_1$, and that $|A| + |B| > \lambda'_j$. Thus by the pigeonhole principle there exists a row containing both a box in A and a box in B. Moreover the same claim holds if we act by σ first; that is, there exist $a \in A$, $b \in B$ and $1 \leqslant i \leqslant \lambda'_1$ such that $a\sigma = (i, j)$ and $b\sigma = (i, j')$. Let $\omega = (a\ b) \in S_{A \sqcup B}$, and note that $\sigma^{-1}\omega\sigma = (a\sigma\ b\sigma) \in \mathrm{RPP}(\lambda)$.

Let $\mathcal{A} \subseteq S_{A \sqcup B}$ be the subgroup of even permutations in $S_{A \sqcup B}$; then $S_{A \sqcup B} = \mathcal{A} \sqcup \mathcal{A}\omega$. Because $\sigma^{-1}\omega\sigma \in \mathrm{RPP}(\lambda)$, we have $[t \cdot \tau\omega\sigma] = [t \cdot \tau\sigma(\sigma^{-1}\omega\sigma)] = [t \cdot \tau\sigma]$. Thus

$$\begin{aligned} \sum_{\tau \in S_{A \sqcup B}} [t \cdot \tau\sigma] \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) &= \sum_{\tau \in \mathcal{A}} ([t \cdot \tau\sigma] - [t \cdot \tau\omega\sigma]) \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \\ &= 0 \end{aligned}$$

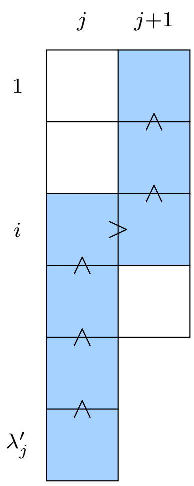
as required. □

Remark 2.9. The manipulation of the sums in the proof of Proposition 2.8 avoids the detour into integral forms and reduction modulo p taken by the usual proof of this fact (see for example [Jam78, Lemma 8.4] or [dBPW21, Lemma 2.4 and Equation 2.5]).

Lemma 2.10. *Let t be a column standard tableau, and suppose (i, j) is such that $t(i, j) > t(i, j+1)$. Then*

$$\mathsf{R}_{(t,i,j)} = |t| + \sum_{u <_{\mathrm{c}} t} m_u |u|$$

for some elements m_u in the subring of K generated by 1*.*



Proof. By assumption, the sets $A = \{ (r, j) \mid i \leqslant r \leqslant \lambda'_j \}$ and $B = \{ (r, j+1) \mid 1 \leqslant r \leqslant i \}$ defining the Garnir relation satisfy

$$t(1, j+1) < \ldots < t(i, j+1) < t(i, j) < t(i+1, j) < \ldots < t(\lambda'_j, j).$$

(These boxes and the inequalities between their entries are illustrated in the margin.) Thus for any $\sigma \in S_{A \sqcup B}$, we have $t \cdot \sigma \lesssim_{\mathrm{c}} t$, with $t \cdot \sigma \sim_{\mathrm{c}} t$ if and only if $\sigma \in S_A \times S_B$. $\square$

Lemma 2.11. *Let t be any tableau. Then there exists some K-linear combination γ of snake relations (with coefficients in the subring of K generated by* 1*) such that*

$$|t| + \gamma = \sum_{s \in \mathrm{SSYT}(\lambda)} a_s |s|$$

for some elements a_s in the subring of K generated by 1*. Consequently, the set $\{ \mathrm{e}(s) \mid s \in \mathrm{SSYT}(\lambda) \}$ spans $\nabla^\lambda V$.*

Proof. Without loss of generality, we may assume t is column standard. If t is also row semistandard, we are done. Otherwise, choose a box (i, j) such that $t(i, j) > t(i, j+1)$. By Lemma 2.10, $\mathsf{R}_{(t,i,j)} = |t| + \sum_{u <_{\mathrm{c}} t} m_u |u|$ for some elements m_u in the subring of K generated by 1. Then $|t| - \mathsf{R}_{(t,i,j)}$ is a linear combination of column tabloids whose tableaux precede t in the column ordering. The first part of the lemma then follows by induction.

Applying the map e to the given expression and using that $\mathsf{GR}^\lambda(V) \subseteq \ker e$ shows that any polytabloid can be written as a linear combination of semistandard polytabloids. $\square$

We can now meet the two aims of this subsection.

Proposition 2.12. *The set* $\{\, \mathrm{e}(s) \mid s \in \mathrm{SSYT}(\lambda) \,\}$ *is a* K*-basis for* $\nabla^\lambda(V)$*.*

PROOF. The set is linearly independent by Lemma 2.7 and spanning by Lemma 2.11. $\square$

Proposition 2.13. *There is equality* $\ker e = \mathsf{GR}^\lambda(V)$ *and hence a* KG*-isomorphism*

$$\nabla^\lambda V \cong \bigwedge\nolimits^{\lambda'} V \big/ \mathsf{GR}^\lambda(V) \, .$$

PROOF. From Proposition 2.8, we have that $\mathsf{GR}^\lambda(V) \subseteq \ker e$. It therefore suffices to show that the snake relations span $\ker e$.

Let $\kappa \in \ker e$. By Lemma 2.11 there exists a K-linear combination γ of snake relations such that

$$\kappa + \gamma = \sum_{s \in \mathrm{SSYT}(\lambda)} \alpha_s |s|$$

for some elements $\alpha_s \in K$. Applying e to this equation and using that $\mathsf{GR}^\lambda(V) \subseteq \ker e$, we find

$$0 = \sum_{s \in \mathrm{SSYT}(\lambda)} \alpha_s \, \mathrm{e}(s).$$

The semistandard polytabloids are K-linearly independent by Lemma 2.7, so this implies that $\alpha_s = 0$ for all s. Hence $\kappa = -\gamma$ is in the span of the snake relations, as required. $\square$

2.3. Basis for the Garnir relations

The work of the previous subsection is easily modified to identify a basis for the space of Garnir relations (which will be essential knowledge in Chapter IV). Our basis is the following subset of the snake relations.

Definition 2.14 (Basic snake relations)**.** Let Φ be a function on column standard tableaux which are not row semistandard whose output on such a tableau t is a box (i, j) such that $t(i, j) > t(i, j + 1)$. A snake relation $\mathsf{R}_{(t,i,j)}$ is called Φ-*basic* if t is column standard but not row semistandard and $(i, j) = \Phi(t)$.

The purpose of Φ is to associate a unique snake relation to each column standard tableau which is not row semistandard. Any such function suffices: except in the proofs of Propositions 13.3 and 14.12, the choice of Φ is irrelevant (that is, all the claims, including the statements of those propositions, hold for any choice of Φ). Accordingly, Φ is suppressed in the notation. An example of a suitable function Φ is to let $\Phi(t) = (i, j)$ where j is least (primarily) and i is greatest (secondarily) such that $t(i, j) > t(i, j + 1)$; outside the specified proofs, we may consider this to be the function in the definition of basic snake relations.

Proposition 2.15. *The set of basic snake relations is a basis for the space* $\mathsf{GR}^\lambda(V)$.

Proof. By Lemma 2.10, the basic snake relations have distinct leading tableaux with the respect to the column ordering, and hence are linearly independent.

It was shown in Proposition 2.13 that the snake relations span $\mathsf{GR}^\lambda(V)$. The proof relied on Lemma 2.11, in which a choice of box (i, j) such that $t(i, j) > t(i, j + 1)$ was made. By letting this choice be $\Phi(t)$, all the snake relations referred to in these proofs are basic, and so they in fact show that the basic snake relations span $\mathsf{GR}^\lambda(V)$. $\square$

2.4. Schur endofunctors on submodules of symmetric type

Our construction of the Specht module in §4.1 requires restriction to the subspace of symmetric type; we record here that all the results of this section hold upon this restriction.

Recall from §1.2 that we say a tableau is *of symmetric type* if all its entries are distinct, and that for the constructions in this chapter we write $-_{\mathrm{sym}}$

for the restriction of that construction to tableaux of symmetric type. Thus $\nabla^\lambda_{\mathrm{sym}} V$ is defined as the subspace of $\mathrm{Sym}^\lambda_{\mathrm{sym}} V$ spanned by the polytabloids of symmetric type. As noted in §1.2, however, the K-vector space $\nabla^\lambda_{\mathrm{sym}} V$ may not be a KG-submodule unless V is a permutation module and $\mathcal{B}$ is a permutation basis.

Proposition 2.16. *There is equality* $\ker(e|_{\bigwedge^{\lambda'}_{\mathrm{sym}} V}) = \mathsf{GR}^\lambda_{\mathrm{sym}}(V)$ *and hence a K-linear isomorphism*

$$\nabla^\lambda_{\mathrm{sym}} V \cong {\textstyle\bigwedge^{\lambda'}_{\mathrm{sym}}} V \big/ \mathsf{GR}^\lambda_{\mathrm{sym}}(V) \, .$$

Moreover, if V is a permutation KG-module and $\mathcal{B}$ a permutation basis, then the above map is a KG-isomorphism.

Proposition 2.17. *The basic snake relations of symmetric type form a basis of* $\mathsf{GR}^\lambda_{\mathrm{sym}}(V)$.

Remark 2.18. Note that $\nabla^\lambda_{\mathrm{sym}}$ is not a functor (even on the category of K-vector spaces): given K-vector spaces V and W and a linear map $f\colon V \to W$, the map $\nabla^\lambda f\colon \nabla^\lambda V \to \nabla^\lambda W$ does not in general restrict to a map $\nabla^\lambda_{\mathrm{sym}} V \to \nabla^\lambda_{\mathrm{sym}} W$. That is, restriction to subspaces of symmetric type is a well-defined operation on objects but not on morphisms.

3. Duality and the Weyl endofunctors

In this section we dualise the constructions of the previous section to obtain explicit models for the Weyl endofunctors. We first examine precisely what we mean by duality in §3.1, and then consider the special case of the symmetric power in §3.2. In §3.3 we define the Weyl endofunctor as the dual of the Schur endofunctor, and hence describe it as a submodule of an exterior power and a quotient of a (dual) symmetric power. In §3.4 we present a new set of relations obeyed by the Weyl endofunctor.

3.1. Two notions of duality

Recall that the *dual space* of V is the K-vector space $V^* = \mathrm{Hom}_K(V, K)$. This space becomes a representation via inversion in the group: the module structure is given by $(gf)(v) = f(g^{-1}v)$ for all $v \in V$ and all $f \in \mathrm{Hom}_K(V, k)$. With respect to the dual basis, which we denote $\mathcal{B}^*$, the representing group homomorphism is $\rho_{V^*}(g) = \rho_V(g^{-1})^\top$.

A different module structure is possible when G is a matrix group closed under transposition. Following the notation and terminology of Green [EGS08, p. 20], the *contravariant dual* of V, denoted V°, has the same underlying vector space V^* but group action given by $(gf)(v) = f(g^\top v)$ for all $v \in V$ and all $f \in \mathrm{Hom}_K(V, K)$. With respect to the dual basis, the representing group homomorphism is $\rho_{V^\circ}(g) = \rho_V(g^\top)^\top$.

Remark 3.1. We discuss the concept of polynomial representations in §5. We will see (Proposition 5.6(iv)) that contravariant duality preserves the property of being polynomial (and furthermore preserves the degree), but that (usual) duality does not.

When $G = \mathrm{SL}_2(K)$, the two notions of duality coincide, as the following proposition shows.

Proposition 3.2. *Suppose $G = \mathrm{SL}_2(K)$. Then $V^* \cong V^\circ$.*

PROOF. Let $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(K)$. It is simple to verify that for any matrix $g \in \mathrm{SL}_2(K)$, we have $Jg^{-1}J^{-1} = g^\top$. Then $R = \rho_V(J^{-1})^\top$ satisfies

$$\begin{aligned} R\rho_{V^*}(g)R^{-1} &= R\rho_V(g^{-1})^\top R^{-1} \\ &= \big(\rho_V(J)\rho_V(g^{-1})\rho_V(J^{-1})\big)^\top \\ &= \rho_V(g^\top)^\top \\ &= \rho_{V^\circ}(g), \end{aligned}$$

and the proposition follows. $\square$

The interactions between these dualities and symmetric powers is explored further in §3.2. The case of exterior powers is more straight-forward: exterior powers commute with duals, via the obvious map.

Proposition 3.3. *Suppose $\mathcal{B} = \{v_1, \ldots, v_d\}$ is the chosen basis for V and $\mathcal{B}^* = \{v_1^*, \ldots, v_d^*\}$ is the dual basis for V^*. Let $\{\,(v_{i_1} \wedge \cdots \wedge v_{i_r})^* \mid 1 \leqslant i_1 < \ldots < i_r \leqslant d\,\}$ be the basis for $(\bigwedge^r V)^*$ dual to the basis $\{\,v_{i_1} \wedge \cdots \wedge v_{i_r} \mid 1 \leqslant i_1 < \ldots < i_r \leqslant d\,\}$ for $\bigwedge^r V$. Then there is a KG-isomorphism*

$$\begin{gathered} (\textstyle\bigwedge^r V)^* \cong \textstyle\bigwedge^r V^* \\ (v_{i_1} \wedge \cdots \wedge v_{i_r})^* \mapsto v_{i_1}^* \wedge \cdots \wedge v_{i_r}^*. \end{gathered}$$

If G is a matrix group closed under transposition, the same map defines a KG-isomorphism when $-^*$ is replaced with the contravariant dual $-^\circ$.

PROOF. Clearly the map is a K-linear bijection. Let ρ_V be the homomorphism representing the action on V with respect to the given basis, and likewise for the other relevant modules. Let $g \in G$. Observe that

$$\begin{aligned} g(v_{i_1} \wedge \cdots \wedge v_{i_r}) &= \Big(\sum_{j=1}^{d} \rho_V(g)_{j,i_1} v_j\Big) \wedge \cdots \wedge \Big(\sum_{j=1}^{d} \rho_V(g)_{j,i_r} v_j\Big) \\ &= \sum_{(j_1,\ldots,j_r)\in[d]^r} \rho_V(g)_{j_1,i_1} \cdots \rho_V(g)_{j_r,i_r}\; v_{j_1} \wedge \cdots \wedge v_{j_r} \\ &= \sum_{\sigma\in S_r} \mathrm{sgn}(\sigma) \sum_{1\leqslant j_1<\ldots<j_r\leqslant d} \rho_V(g)_{j_{1\sigma},i_1} \cdots \rho_V(g)_{j_{r\sigma},i_r}\, v_{j_1} \wedge \cdots \wedge v_{j_r}. \end{aligned}$$

Thus for $1 \leqslant i_1 < \ldots < i_r \leqslant d$ and $1 \leqslant j_1 < \ldots < j_r \leqslant d$ we have

$$\rho_{\bigwedge^r V}(g)_{(j_1,\ldots,j_r),(i_1,\ldots,i_r)} = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\rho_V(g)_{j_{1\sigma},i_1} \cdots \rho_V(g)_{j_{r\sigma},i_r}.$$

This allows us to deduce the action of g on the two modules of interest: on $(\bigwedge^r V)^*$ it is given by

$$\begin{aligned}\rho_{(\bigwedge^r V)^*}(g)_{(j_1,\ldots,j_r),(i_1,\ldots,i_r)} &= \rho_{\bigwedge^r V}(g^{-1})_{(i_1,\ldots,i_r),(j_1,\ldots,j_r)} \\ &= \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\rho_V(g^{-1})_{i_{1\sigma},j_1} \cdots \rho_V(g^{-1})_{i_{r\sigma},j_r}\end{aligned}$$

while on $\bigwedge^r V^*$ it is given by

$$\begin{aligned}\rho_{\bigwedge^r V^*}(g)_{(j_1,\ldots,j_r),(i_1,\ldots,i_r)} &= \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\rho_{V^*}(g)_{j_{1\sigma},i_1} \cdots \rho_{V^*}(g)_{j_{r\sigma},i_r} \\ &= \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\rho_V(g^{-1})_{i_1,j_{1\sigma}} \cdots \rho_V(g^{-1})_{i_r,j_{r\sigma}}.\end{aligned}$$

Using that $M_{i_1,j_{1\sigma}} \ldots M_{i_r,j_{r\sigma}} = M_{i_{1\sigma^{-1}},j_1} \ldots M_{i_{r\sigma^{-1}},j_r}$ for a matrix M and reindexing the sum by replacing σ with σ^{-1} shows that the two matrices $\rho_{(\bigwedge^r V)^*}(g)$ and $\rho_{\bigwedge^r V^*}(g)$ are equal, as required.

To show that $\rho_{(\bigwedge^r V)^\circ}(g)$ and $\rho_{\bigwedge^r V^\circ}(g)$ are equal, we use the same argument with $g^\top$ occurring in place of g^{-1}. $\square$

3.2. Duality and symmetric powers

In this subsection, we define functors dual to the symmetric powers, which we call *lower symmetric powers* (calling the true symmetric powers *upper symmetric powers*). (The lower symmetric powers are sometimes known as *divided powers*, being grades of the divided power algebra.) These functors are important in their own right, featuring prominently in Chapter III where the duality is necessary to generalise certain classical results. The lower symmetric powers are also useful for constructing and studying the Weyl endofunctors. Indeed, whereas in §2 we showed that the Schur endofunctors can be viewed both as *submodules of upper symmetric powers* and as *quotients of exterior powers*, we will show later that the Weyl endofunctors can be viewed both as *submodules of exterior powers* and as *quotients of lower symmetric powers*. We will see also that the lower symmetric powers are in fact a particular case of the Weyl endofunctors (those labelled by rows); nevertheless it is helpful to define the lower symmetric powers explicitly before studying the Weyl endofunctors.

Recall that the upper symmetric power is defined as a quotient of the tensor power. More concretely, letting the symmetric groups act on pure tensors on the right by place permutation, we have

$$\operatorname{Sym}^r V = {V^{\otimes r}}\big/{\langle\, x - x\cdot\sigma \;\mid\; x \in V^{\otimes r} \text{ is a pure tensor}, \sigma \in S_r \,\rangle_K}.$$

By contrast, the lower symmetric power is defined as a submodule of the tensor power.

Definition 3.4 (Lower symmetric powers)**.** The *rth lower symmetric power* of V is $\operatorname{Sym}_r V = (V^{\otimes r})^{S_r}$ the space of invariants of the place permutation action of S_r on $V^{\otimes r}$. That is, it is the subspace of symmetric tensors

$$\operatorname{Sym}_r V = \left\langle \sum_{\sigma \in \operatorname{stab} x \backslash S_r} x \cdot \sigma \;\middle|\; x \in V^{\otimes r} \text{ is a pure tensor} \right\rangle_K \subseteq V^{\otimes r}.$$

where $\operatorname{stab} x \leqslant S_r$ denotes the stabiliser of the place permutation action on a pure tensor x.

It is clear that $\operatorname{Sym}_r V$ is a KG-module: since the group action and the place permutation action commute, if $x \in V^{\otimes r}$ is such that $x \cdot \sigma = x$ for all $\sigma \in S_r$, then this is also true for gx.

Writing $\mathcal{B} = \{v_1, \ldots, v_d\}$ for the basis of V, a basis for $\operatorname{Sym}_r V$ is

$$\left\{ \sum_{\sigma \in \operatorname{stab}(i_1,\ldots,i_r)\backslash S_r} v_{i_{1\sigma^{-1}}} \otimes \cdots \otimes v_{i_{r\sigma^{-1}}} \;\middle|\; 1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant d \right\}.$$

Proposition 3.5. *If* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > r$*, then* $\operatorname{Sym}^r V$ *and* $\operatorname{Sym}_r V$ *are isomorphic.*

Proof. This is easily verified using the restriction to $\operatorname{Sym}_r V$ of the canonical surjection $V^{\otimes r} \twoheadrightarrow \operatorname{Sym}^r V$: this map sends

$$\sum_{\tau \in \operatorname{stab}(i_1,\ldots,i_r)\backslash S_r} v_{i_{1\tau^{-1}}} \otimes \cdots \otimes v_{i_{r\tau^{-1}}} \mapsto |S_r : \operatorname{stab}(i_1,\ldots,i_r)|\, v_{i_1} \cdot \ldots \cdot v_{i_r},$$

and so is an isomorphism when these coefficients are nonzero. $\square$

Further to Proposition 3.5, we show in Proposition 11.12 (using the invariant introduced in that section) that if K has characteristic p, then a necessary and sufficient condition for the two powers to be isomorphic is that $r < p$ or $r = p^\varepsilon - 1$ for some integer ε. Proposition 3.5 can also be seen as a special case of the isomorphism noted in Remark 3.14.

Remark 3.6. We could analogously construct the *lower exterior power* as the submodule $\bigwedge_r V \subseteq V^{\otimes r}$ consisting of antisymmetrisations of tensors. However, this is isomorphic to the usual exterior power: the antisymmetrisation of a pure tensor is equal to a signed sum over *all* permutations (if its stabiliser is nontrivial the antisymmetrisation is 0), and thus the map

$$\begin{aligned} \textstyle\bigwedge_r V &\to \textstyle\bigwedge^r V \\ \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) v_{i_{1\sigma^{-1}}} \otimes \cdots \otimes v_{i_{r\sigma^{-1}}} &\mapsto v_{i_1} \wedge \cdots \wedge v_{i_r} \end{aligned}$$

is an isomorphism (regardless of the characteristic).

Although not always isomorphic, the lower symmetric powers and the upper symmetric powers are always dual, in the following sense.

Proposition 3.7 (cf. Proposition 3.3)**.** *Suppose $\mathcal{B} = \{v_1, \ldots, v_d\}$ is the chosen basis for V and $\mathcal{B}^* = \{v_1^*, \ldots, v_d^*\}$ is the dual basis for V^*. Let $\{\, (v_{i_1} \cdot \ldots \cdot v_{i_r})^* \mid 1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant d \,\}$ be the basis for $(\operatorname{Sym}^r V)^*$ dual to the basis $\{\, v_{i_1} \cdot \ldots \cdot v_{i_r} \mid 1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant d \,\}$ for $\operatorname{Sym}^r V$. There is a KG-isomorphism*

$$\begin{aligned} (\operatorname{Sym}^r V)^* &\cong \operatorname{Sym}_r V^* \\ (v_{i_1} \cdot \ldots \cdot v_{i_r})^* &\mapsto \sum_{\sigma \in \operatorname{stab}(i_1, \ldots, i_r) \backslash S_r} v^*_{i_{1\sigma^{-1}}} \otimes \cdots \otimes v^*_{i_{r\sigma^{-1}}}. \end{aligned}$$

If G is a matrix group closed under transposition, the same map defines a KG-isomorphism when $-^*$ is replaced with the contravariant dual $-^\circ$.

Proof. Clearly the map is a K-linear bijection. Let ρ_V be the homomorphism representing the action on V with respect to the given basis, and likewise for the other relevant modules. Let $g \in G$. Observe that the action of g on the tensor power is given by

$$\begin{aligned}
&g(v_{i_1} \otimes \cdots \otimes v_{i_r}) \\
&\quad = \Big(\sum_{j=1}^{d} \rho_V(g)_{j,i_1} v_j\Big) \otimes \cdots \otimes \Big(\sum_{j=1}^{d} \rho_V(g)_{j,i_r} v_j\Big) \\
&\quad = \sum_{(j_1,\ldots,j_r)\in[d]^r} \rho_V(g)_{j_1,i_1} \cdots \rho_V(g)_{j_r,i_r}\, v_{j_1} \otimes \cdots \otimes v_{j_r} \\
&\quad = \sum_{1 \leqslant j_1 \leqslant \ldots \leqslant j_r \leqslant d} \ \sum_{\sigma \in \mathcal{S}_{j_1,\ldots,j_r}} \rho_V(g)_{j_{1\sigma^{-1}},i_1} \cdots \rho_V(g)_{j_{r\sigma^{-1}},i_r} v_{j_{1\sigma^{-1}}} \otimes \cdots \otimes v_{j_{r\sigma^{-1}}}
\end{aligned}$$

where $\mathcal{S}_{j_1,\ldots,j_r} = \operatorname{stab}(j_1,\ldots,j_r)\backslash S_r$. Then the action of g on the symmetric powers is given by

$$g(v_{i_1} \cdot \ldots \cdot v_{i_r}) = \sum_{1 \leqslant j_1 \leqslant \ldots \leqslant j_r \leqslant d} \ \sum_{\sigma \in \mathcal{S}_{j_1,\ldots,j_r}} \rho_V(g)_{j_{1\sigma^{-1}},i_1} \cdots \rho_V(g)_{j_{r\sigma^{-1}},i_r}\, v_{j_1} \cdot \ldots \cdot v_{j_r}$$

and

$$\begin{aligned}
&g\Big(\sum_{\tau \in \mathcal{S}_{i_1,\ldots,i_r}} v_{i_{1\tau^{-1}}} \otimes \cdots \otimes v_{i_{r\tau^{-1}}}\Big) \\
&\ = \sum_{\tau \in \mathcal{S}_{i_1,\ldots,i_r}} \ \sum_{1 \leqslant j_1 \leqslant \ldots \leqslant j_r \leqslant d} \rho_V(g)_{j_1,i_{1\tau^{-1}}} \cdots \rho_V(g)_{j_r,i_{r\tau^{-1}}} \sum_{\sigma \in \mathcal{S}_{j_1,\ldots,j_r}} v_{j_{1\sigma^{-1}}} \otimes \cdots \otimes v_{j_{r\sigma^{-1}}},
\end{aligned}$$

where we have used that since $\operatorname{Sym}_r V$ is a submodule, the coefficient of $v_{j_{1\sigma^{-1}}} \otimes \cdots \otimes v_{j_{r\sigma^{-1}}}$ does not depend on the permutation σ. Thus for $1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant d$ and $1 \leqslant j_1 \leqslant \ldots \leqslant j_r \leqslant d$ we have

$$\begin{aligned}
\rho_{\operatorname{Sym}^r V}(g)_{(j_1,\ldots,j_r),(i_1,\ldots,i_r)} &= \sum_{\sigma \in \mathcal{S}_{j_1,\ldots,j_r}} \rho_V(g)_{j_{1\sigma^{-1}},i_1} \cdots \rho_V(g)_{j_{r\sigma^{-1}},i_r}, \\
\rho_{\operatorname{Sym}_r V}(g)_{(j_1,\ldots,j_r),(i_1,\ldots,i_r)} &= \sum_{\sigma \in \mathcal{S}_{i_1,\ldots,i_r}} \rho_V(g)_{j_1,i_{1\sigma^{-1}}} \cdots \rho_V(g)_{j_r,i_{r\sigma^{-1}}}.
\end{aligned}$$

This allows us to deduce the action of g on the two modules of interest: on $(\operatorname{Sym}^r V)^*$ it is given by

$$\begin{aligned}
\rho_{(\operatorname{Sym}^r V)^*}(g)_{(j_1,\ldots,j_r),(i_1,\ldots,i_r)} &= \rho_{\operatorname{Sym}^r V}(g^{-1})_{(i_1,\ldots,i_r),(j_1,\ldots,j_r)} \\
&= \sum_{\sigma \in \mathcal{S}_{i_1,\ldots,i_r}} \rho_V(g^{-1})_{i_{1\sigma^{-1}},j_1} \cdots \rho_V(g^{-1})_{i_{r\sigma^{-1}},j_r}
\end{aligned}$$

while on $\mathrm{Sym}_r V^*$ it is given by

$$\begin{aligned}\rho_{\mathrm{Sym}_r V^*}(g)_{(j_1,\dots,j_r),(i_1,\dots,i_r)} &= \sum_{\sigma\in\mathcal{S}_{i_1,\dots,i_r}} \rho_{V^*}(g)_{j_1,i_{1\sigma^{-1}}}\cdots\rho_{V^*}(g)_{j_r,i_{r\sigma^{-1}}} \\ &= \sum_{\sigma\in\mathcal{S}_{i_1,\dots,i_r}} \rho_V(g^{-1})_{i_{1\sigma^{-1}},j_1}\cdots\rho_V(g^{-1})_{i_{r\sigma^{-1}},j_r}.\end{aligned}$$

Thus these two matrices $\rho_{(\mathrm{Sym}^r V)^*}(g)$ and $\rho_{\mathrm{Sym}_r V^*}(g)$ are equal, as required. To show that $\rho_{(\mathrm{Sym}^r V)^\circ}(g)$ and $\rho_{\mathrm{Sym}_r V^\circ}(g)$ are equal, we use the same argument with $g^\top$ occurring in place of g^{-1}. $\square$

As with the upper symmetric powers, we write $\mathrm{Sym}_\lambda V = \bigotimes_{i=1}^{\lambda'_1} \mathrm{Sym}_{\lambda_i} V$, and can model this space using a tabular construction.

Definition 3.8. Let t be a tableau. Define the *row symmetrisation* of t to be the element of $\mathsf{Tbx}^\lambda(V)$ given by

$$\mathrm{rsym}(t) = \sum_{\tau\in\mathrm{rstab}(t)\backslash\mathrm{RPP}(\lambda)} t\cdot\tau \ .$$

The subspace of $\mathsf{Tbx}^\lambda(V)$ spanned by the row symmetrisations is isomorphic as a KG-module to $\mathrm{Sym}_\lambda V$. Moreover, the set $\{\,\mathrm{rsym}(t) \mid t\in \mathrm{RSSYT}(\lambda)\,\}$ is a basis for $\mathrm{Sym}_\lambda V$.

3.3. Weyl endofunctors

We define the Weyl endofunctors Δ^λ as the duals of the Schur endofunctors ∇^λ, in the sense given in the definition below. As a consequence, we will see that the Weyl endofunctors can be viewed as either submodules of exterior powers or quotients of lower symmetric powers. Taking the submodule viewpoint, we give an explicit basis consisting of *copolytabloids*, generalising the result given by [Wil20] in the case when K is infinite and $G = \mathrm{GL}(V)$. Taking the quotient viewpoint, in §3.4 we describe the kernel as consisting of *row Garnir relations*, a new description of this module.

Definition 3.9. The *Weyl endofunctor* labelled by λ, denoted Δ^λ, is the endofunctor on the category of KG-modules obtained by pre- and post-composing the Schur endofunctor with the duality functor $-^*$. That is, it is defined on a KG-module V by

$$\Delta^\lambda V = (\nabla^\lambda V^*)^*$$

and on a map f, by $\Delta^\lambda f = (\nabla^\lambda f^*)^*$.

The Weyl endofunctors are determined by the natural representations, in the same sense as for the Schur endofunctors (Proposition 2.6). Examining the representing homomorphism also demonstrates that, if G is a matrix group closed under transposition, the definition of the Weyl endofunctor Δ^λ can use the contravariant dual $-^\circ$ in place of $-^*$.

Proposition 3.10.

(i) Suppose V is d-dimensional and let E be the natural representation of $\mathrm{GL}_d(K)$. Then $\rho_{\Delta^\lambda V} = \rho_{\Delta^\lambda E}\rho_V$.

(ii) Suppose G is a matrix group closed under transposition. Then $\Delta^\lambda V \cong (\nabla^\lambda V^\circ)^\circ$.

Proof. By definition of $\Delta^\lambda V$, the representing homomorphism satisfies

$$\rho_{\Delta^\lambda V}(g) = \rho_{(\nabla^\lambda V^*)^*}(g) = \rho_{\nabla^\lambda V^*}(g^{-1})^\top.$$

But $\rho_{\nabla^\lambda V^*} = \rho_{\nabla^\lambda E}\rho_{V^*}$ by Proposition 2.6, so this becomes

$$\rho_{\Delta^\lambda V}(g) = \big(\rho_{\nabla^\lambda E}\rho_{V^*}(g^{-1})\big)^\top = \big(\rho_{\nabla^\lambda E}(\rho_V(g)^\top)\big)^\top. \tag{3.10.1}$$

In particular, by setting $V = E$ we find that $\rho_{\Delta^\lambda E}(M) = \big(\rho_{\nabla^\lambda E}(M^\top)\big)^\top$ for all $M \in \mathrm{GL}_d(K)$. Choosing $M = \rho_V(g)$ we obtain $\rho_{\Delta^\lambda V}(g) = \rho_{\Delta^\lambda E}\rho_V(g)$, demonstrating (i).

Calculations analogous to the above, but with $g^\top$ appearing in place of g^{-1}, show that $\rho_{(\nabla^\lambda V^\circ)^\circ}(g) = \big(\rho_{\nabla^\lambda E}(\rho_V(g)^\top)\big)^\top$. This is precisely the expression found for $\rho_{\Delta^\lambda V}(g)$ in (3.10.1), proving (ii). $\square$

Since $\nabla^\lambda V^*$ is a submodule of the upper symmetric power $\mathrm{Sym}^\lambda V^*$, using Proposition 3.7 we see that $\Delta^\lambda V$ is a quotient of the lower symmetric power $\mathrm{Sym}_\lambda V$. Likewise since $\nabla^\lambda V^*$ is a quotient of the exterior power $\bigwedge^{\lambda'} V^*$, using Proposition 3.3 we see that $\Delta^\lambda V$ is a submodule of the exterior power $\bigwedge^{\lambda'} V$. In the remainder of this section we find an explicit basis for this submodule of $\bigwedge^{\lambda'} V$.

Definition 3.11. Let t be a tableau. The *copolytabloid* of t is the element of $\bigwedge^{\lambda'} V$ given by[1]

$$ə(t) = \sum_{\tau \in \mathrm{rstab}(t)\backslash \mathrm{RPP}(\lambda)} |t \cdot \tau|.$$

In Proposition 3.13 we show that $\Delta^\lambda V$ is precisely the subspace of $\bigwedge^{\lambda'} V$ spanned by the copolytabloids. Unlike the case of polytabloids, it is not immediately obvious from direct computation that the subspace of copolytabloids is a KG-submodule: the row stabiliser of a tableau in the image gt may differ from that of t, so it is not sufficient to observe that the group action commutes with the place permutation action. However, the copolytabloid of t is the image of $\mathrm{rsym}(t)$ under the canonical map $\Lambda\colon \mathsf{Tbx}^\lambda(V) \to \bigwedge^{\lambda'} V$, so the space of copolytabloids is the image of $\mathrm{Sym}_\lambda V$ under a G-equivariant map and hence a KG-module.

We first make the obvious dualisation of Lemma 2.7. We use the column ordering $<_{\mathrm{c}}$ defined in §1.6.

Lemma 3.12. *Let t be a row semistandard tableau. Then*

$$ə(t) = |t| + \sum_{u <_{\mathrm{c}} t} m_u |u|$$

for some elements m_u in the subring of K generated by 1. In particular, the set $\{\, ə(s) \mid s \in \mathrm{SSYT}(\lambda)\,\}$ is K-linearly independent.

Proof. By definition, $ə(t) = \sum_{\tau \in \mathrm{rstab}(t)\backslash \mathrm{RPP}(\lambda)} |t \cdot \tau|$. Since t is row semistandard, $t \cdot \tau \lesssim_{\mathrm{c}} t$ for all $\tau \in \mathrm{RPP}(\lambda)$, with equality if and only if $\tau \in \mathrm{rstab}(t)$; the claimed expression for $ə(t)$ is then clear. The linear independence of $\{\, \mathrm{e}(s) \mid s \in \mathrm{SSYT}(\lambda)\,\}$ follows: the $<_{\mathrm{c}}$-greatest column standard tableau whose column tabloid appears in $ə(s)$ is, being s itself, distinct for each $s \in \mathrm{SSYT}(\lambda)$. $\square$

The key step of the next proposition is showing that copolytabloids are contained in $\Delta^\lambda V$. A special case of this argument, for the tableau t defined by $t(i,j) = i$, is given by Wildon [Wil20, §3.2].

[1]The symbol ə is a *schwa*; it is a rotation of the Roman e.

Proposition 3.13. *The submodule of $\bigwedge^{\lambda'} V$ spanned by the copolytabloids is isomorphic to $\Delta^\lambda V$, and has K-basis $\{\, \text{ə}(s) \mid s \in \mathrm{SSYT}(\lambda) \,\}$.*

Proof. It suffices to show that $\Delta^\lambda V$ is isomorphic to a submodule of $\bigwedge^{\lambda'} V$ containing the copolytabloids, for then the set $\{\, \text{ə}(s) \mid s \in \mathrm{SSYT}(\lambda) \,\}$ is a linearly independent set contained in a module of dimension $\dim \Delta^\lambda V = \dim \nabla^\lambda V = |\mathrm{SSYT}(\lambda)|$, and hence the set is a basis.

We view $(\nabla^\lambda V^*)^*$ as a submodule of $(\bigwedge^{\lambda'} V^*)^*$ via the dual to the surjective map $e\colon \bigwedge^{\lambda'} V^* \to \nabla^\lambda V^*$; that is, by the injective map $e^*\colon (\nabla^\lambda V^*)^* \to (\bigwedge^{\lambda'} V^*)^*$ defined by $e^*(\theta)(x) = \theta(e(x))$ for $\theta \in (\nabla^\lambda V^*)^*$ and $x \in \bigwedge^{\lambda'} V^*$. Meanwhile we view $(\bigwedge^{\lambda'} V^*)^*$ as isomorphic to $\bigwedge^{\lambda'} V$ via the isomorphism from Proposition 3.3 (defined on each tensor factor, with V^* in place of V), which we denote ψ. Since by definition $\Delta^\lambda V \cong (\nabla^\lambda V^*)^*$, our objective, then, is to show that the copolytabloids lie in the image of ψe^*. Since $\text{ə}(t) = \text{ə}(u)$ if t and u are row-equivalent tableaux, it suffices to show that copolytabloids of row semistandard tableaux are contained in this image.

Fix $t \in \mathrm{RSSYT}_{\mathcal{B}}(\lambda)$. Let t^* denote the tableau obtained from t by replacing each entry from $\mathcal{B}$ with its dual from $\mathcal{B}^*$. Let $[t^*]^*$ denote the element dual to $[t^*]$ in the basis of $(\mathrm{Sym}^\lambda V^*)^*$ dual to $\{\, [s] \mid s \in \mathrm{RSSYT}_{\mathcal{B}^*}(\lambda) \,\}$. Since $\nabla^\lambda V^* \subseteq \mathrm{Sym}^\lambda V^*$, we can restrict the function $[t^*]^*$ to the subspace $\nabla^\lambda V^*$, thus obtaining an element of $(\nabla^\lambda V^*)^*$.

To view $[t^*]^*$ as an element of $(\bigwedge^{\lambda'} V^*)^*$, we compute $e^*([t^*]^*)$. For any $u \in \mathrm{CSYT}_{\mathcal{B}^*}(\lambda)$ we have

$$\begin{aligned}
e^*([t^*]^*)(|u|) &= [t^*]^*(e(u)) \\
&= [t^*]^* \sum_{\sigma \in \mathrm{CPP}(\lambda)} \mathrm{sgn}(\sigma)[u \cdot \sigma] \\
&= \sum_{\sigma \in \mathrm{CPP}(\lambda)} \mathrm{sgn}(\sigma) \mathbb{1}[[u \cdot \sigma] = [t^*]] \\
&= \sum_{\tau \in \mathrm{rstab}(t^*) \backslash \mathrm{RPP}(\lambda)} \sum_{\sigma \in \mathrm{CPP}(\lambda)} \mathrm{sgn}(\sigma) \mathbb{1}[t^* \cdot \tau = u \cdot \sigma]
\end{aligned}$$

where the last equality holds because there is at most one element $\tau \in \mathrm{rstab}(t^*) \backslash \mathrm{RPP}(\lambda)$ such that $t^* \cdot \tau = u \cdot \sigma$, and such an element exists if and only if t^* and $u \cdot \sigma$ have the same multisets of entries in each row (that is, if and only if $[t^*] = [u \cdot \sigma]$).

We employ a similar argument to collapse the sum over $\mathrm{CPP}(\lambda)$. Since u is column standard and hence has distinct entries within a column, there is at most one element $\sigma \in \mathrm{CPP}(\lambda)$ such that $t^* \cdot \tau = u \cdot \sigma$, and such an element exists if and only if $t^* \cdot \tau$ and u have the same multisets of entries in each column. Write $s^{\downarrow}$ for the unique column semistandard tableau obtained from a tableau s by sorting all the columns into ascending order; thus $t^* \cdot \tau$ and u have the same multisets of entries if and only if $(t^* \cdot \tau)^{\downarrow} = u$. Defining $\mathrm{sgn}(s \mapsto s^{\downarrow})$ to be the sign of the unique column-preserving permutation which makes s into a column standard tableau (if it exists; defining $\mathrm{sgn}(s \mapsto s^{\downarrow}) = 0$ if s does not have distinct column entries), the expression above becomes

$$e^*([t^*]^*)(|u|) = \sum_{\tau \in \mathrm{rstab}(t^*) \backslash \mathrm{RPP}(\lambda)} \mathbb{1}[(t^* \cdot \tau)^{\downarrow} = u] \, \mathrm{sgn}(t^* \cdot \tau \mapsto (t^* \cdot \tau)^{\downarrow}).$$

Thus we have

$$e^*([t^*]^*) = \sum_{\tau \in \mathrm{rstab}(t^*) \backslash \mathrm{RPP}(\lambda)} |(t^* \cdot \tau)^{\downarrow}|^* \, \mathrm{sgn}(t^* \cdot \tau \mapsto (t^* \cdot \tau)^{\downarrow}),$$

where for a column standard tableau u, we denote by $|u|^*$ the element dual to $|u|$ in the basis of $(\bigwedge^{\lambda'} V^*)^*$ dual to $\{\, |s| \mid s \in \mathrm{CSYT}_{\mathcal{B}^*}(\lambda) \,\}$. Applying ψ we find that

$$\begin{aligned} \psi(e^*([t^*]^*)) &= \sum_{\tau \in \mathrm{rstab}(t) \backslash \mathrm{RPP}(\lambda)} |(t \cdot \tau)^{\downarrow}| \, \mathrm{sgn}(t \cdot \tau \mapsto (t \cdot \tau)^{\downarrow}) \\ &= \sum_{\tau \in \mathrm{rstab}(t) \backslash \mathrm{RPP}(\lambda)} |t \cdot \tau| \\ &= \text{ə}(t), \end{aligned}$$

so $\text{ə}(t)$ is in the image $\psi((\nabla^{\lambda} V^*)^*)$ as required. □

Note that the map in Proposition 3.13 does *not* send $\mathrm{e}(t^*)^*$, the element dual to a polytabloid, to the copolytabloid $\text{ə}(t)$. Furthermore, it is *not* the case that the basis $\{\, \text{ə}(s) \mid s \in \mathrm{SSYT}_{\mathcal{B}}(\lambda) \,\}$ is dual to the basis $\{\, \mathrm{e}(s) \mid s \in \mathrm{SSYT}_{\mathcal{B}^*}(\lambda) \,\}$. The change of basis matrix between these is given by the Désarménien matrix [EGS08, §5.3].

Remark 3.14. Let t be the tableau defined by $t(i,j) = i$, and note that the row symmetrisation of t is $\mathrm{rsym}(t) = t$, and so $\text{ə}(t) = |t|$. Suppose K is infinite.

(i) Suppose $V = E$ is the natural representation of $\mathrm{GL}_n(K)$. It can be shown that the copolytabloid $\text{ə}(t)$ generates $\Delta^\lambda E$ [EGS08, (5.3b)]. The element $\text{ə}(t)$ is a weight vector in the sense described in §6.1 (and in the sense described for $\mathrm{SL}_2(K)$ in §11.1), and moreover is a unique highest weight vector in a suitable sense. That is, $\Delta^\lambda E$ is generated by its unique highest weight vector $\text{ə}(t)$.

(ii) There is a natural map $\Delta^\lambda V \to \nabla^\lambda V$ obtained by viewing $\Delta^\lambda V$ as a submodule of $\bigwedge^{\lambda'} V$ and applying the map $e\colon \bigwedge^{\lambda'} V \to \nabla^\lambda V$. This map is nonzero: on the copolytabloid $\text{ə}(t) = |t|$ it has image $\mathrm{e}(t) \neq 0$ in $\nabla^\lambda V$. If either $\operatorname{char} K = 0$ or $\operatorname{char} K > |\lambda|$, this map is an isomorphism. As the Schur and Weyl endofunctors are determined by the natural representations (Propositions 2.6 and 3.10), it suffices to show this in the case $V = E$ the natural representation of $\mathrm{GL}_n(K)$. By Schur's Lemma, it suffices to know that $\Delta^\lambda E$ and $\nabla^\lambda E$ are both simple under the given conditions on K; this follows from the semisimplicity of the Schur algebra [EGS08, (2.6e)].

In general this map $\Delta^\lambda V \to \nabla^\lambda V$ is not an isomorphism. In the case $V = E$ the natural representation of $\mathrm{GL}_n(K)$, we have that $\text{ə}(t)$ generates $\Delta^\lambda E$ as remarked above, and furthermore $\mathrm{e}(t)$ generates the unique nonzero minimal submodule of $\nabla^\lambda E$ which is isomorphic to the simple head of $\Delta^\lambda E$ [EGS08, (5.4c), (5.4d)]. It follows that the map has image the unique nonzero minimal submodule of $\nabla^\lambda E$ and kernel the unique proper maximal submodule of $\Delta^\lambda E$.

Example 3.15 (Lower symmetric powers; exterior powers)**.**

(i) Suppose $\lambda = (n)$ consists of a single row. Then $\Delta^{(n)} V \cong \mathrm{Sym}_n V$ because $\nabla^{(n)} V = \mathrm{Sym}^n V$ and using Proposition 3.7. Also, $\text{ə}(t) = \mathrm{rsym}(t)$.

(ii) Suppose $\lambda = (1^n)$ consists of a single column. Then $\Delta^{(1^n)} V \cong \bigwedge^n V$ because $\nabla^{(1^n)} V \cong \bigwedge^n V$ and using Proposition 3.3. Also, $\text{ə}(t) = |t|$.

3.4. Row Garnir relations

In this subsection we identify the kernel of the map $\mathsf{\Lambda}\colon \operatorname{Sym}_\lambda V \twoheadrightarrow \Delta^\lambda V$, thus obtaining a concrete model of $\Delta^\lambda V$ as a quotient of the lower symmetric power. The method is analogous to the treatment of the Schur endofunctor in §2.2, but is complicated by the consideration of stabiliser sizes. Our description of the kernel as consisting of the *row Garnir relations* below is new.

Definition 3.16 (Row Garnir relations)**.** Let t be a tableau of shape λ with entries in $\mathcal{B}$. Let $1 \leqslant i < i' \leqslant \lambda'_1$, and let $A \subseteq \operatorname{row}_i(\lambda)$ and $B \subseteq \operatorname{row}_{i'}(\lambda)$ be such that $|A| + |B| > \lambda_i$. Let $\mathcal{T} = \{\, t \cdot \tau \mid \tau \in S_{A \sqcup B} \,\}$ be the set (*not* multiset) of tableaux which can be obtained from t by permuting boxes in $A \sqcup B$. Let $\mathcal{T}/{\sim_{\mathrm{r}}}$ denote the set of equivalence classes of tableaux in $\mathcal{T}$ modulo row equivalence. The *row Garnir relation* labelled by (t, A, B) is the element of $\operatorname{Sym}_\lambda V$ given by

$$\text{ꓤ}_{(t,A,B)} = \sum_{u \in \mathcal{T}/\sim_{\mathrm{r}}} \bigl|\operatorname{rstab}(u) : \operatorname{rstab}(u) \cap (S_{A\sqcup B} \times S_{[\lambda]\setminus A\sqcup B})\bigr| \operatorname{rsym}(u).$$

Let $\mathsf{G}\text{ꓤ}^\lambda(V)$ denote the subspace of $\operatorname{Sym}_\lambda V$ which is spanned by the row Garnir relations.

A row Garnir relation does not depend on the choice of equivalence class representatives: in the notation of the definition, if $u, u' \in \mathcal{T}$ are such that $u \sim_{\mathrm{r}} u'$, then clearly $\operatorname{rsym}(u) = \operatorname{rsym}(u')$; furthermore there exists $\sigma \in \operatorname{RPP}(\lambda) \cap (S_{A\sqcup B} \times S_{[\lambda]\setminus A\sqcup B})$ such that $u\sigma = u'$ and hence $\operatorname{rstab}(u) = \sigma \operatorname{rstab}(u')\sigma^{-1}$, and so the index of $\operatorname{rstab}(u) \cap (S_{A\sqcup B} \times S_{[\lambda]\setminus A\sqcup B})$ in $\operatorname{rstab}(u)$ is unchanged if u is replaced with u'. The representatives must indeed be chosen from $\mathcal{T}$, however: if $s \notin \mathcal{T}$, then $u \sim_{\mathrm{r}} s$ does not imply that the relevant indices are equal.

It is not immediately clear that the K-subspace $\mathsf{G}\text{ꓤ}^\lambda(V)$ is a KG-submodule. We do not show this fact directly, but instead deduce it after we have shown $\mathsf{G}\text{ꓤ}^\lambda(V) = \ker \mathsf{\Lambda}$.

A row Garnir relation can also be expressed as a sum over double cosets. This expression, given in Lemma 3.17 below, is helpful for proving that the row Garnir relations lie in the kernel of Λ (Proposition 3.21), but may be more cumbersome for explicit calculations.

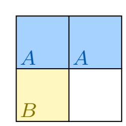
Lemma 3.17. *Let* $\text{ꓤ}_{(t,A,B)}$ *be any row Garnir relation. Let* $\mathcal{S} = \operatorname{stab}(t) \cap S_{A\sqcup B}\backslash S_{A\sqcup B}/S_A \times S_B$ *be a set of double coset representatives for* $\operatorname{stab}(t) \cap S_{A\sqcup B}$ *on the left and* $S_A \times S_B$ *on the right in* $S_{A\sqcup B}$*. Then*

$$\text{ꓤ}_{(t,A,B)} = \sum_{\tau\in\mathcal{S}} \left|\operatorname{rstab}(t\cdot\tau) : \operatorname{rstab}(t\cdot\tau) \cap (S_{A\sqcup B} \times S_{[\lambda]\backslash A\sqcup B})\right| \operatorname{rsym}(t\cdot\tau).$$

PROOF. Both the definition of $\text{ꓤ}_{(t,A,B)}$ and the expression in the statement above can be viewed as sums over $S_{A\sqcup B}$ modulo certain equivalence relations: in the definition, by equality in $\mathcal{T}$ and row equivalence; in the claim, by left multiplication by $\operatorname{stab}(t) \cap S_{A\sqcup B}$ and right multiplication by $S_A \times S_B$. Reducing modulo left multiplication by $\operatorname{stab}(t) \cap S_{A\sqcup B}$ precisely corresponds to reduction modulo equality in $\mathcal{T}$: given $\tau, \tau' \in S_{A\sqcup B}$, we have that $t\cdot\tau = t\cdot\tau'$ if and only if $\tau^{-1}\tau' \in \operatorname{stab}(t) \cap S_{A\sqcup B}$. Reduction modulo right multiplication by $S_A \times S_B$ precisely corresponds to the reduction modulo row equivalence in $\mathcal{T}$: given $u, u' \in \mathcal{T}$, we have $u \sim_{\mathrm{r}} u'$ if and only if there exists $\sigma \in S_A \times S_B$ such that $u \cdot \sigma = u'$ (where we have used that $\operatorname{RPP}(\lambda) \cap S_{A\sqcup B} = S_A \times S_B$). $\square$

We illustrate the definition of a row Garnir relation with an example. This example also demonstrates why it is the right definition to make.

Example 3.18. Suppose $\lambda = (2,2)$, $\mathcal{B} = [2]$, and $t = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}$. Let $A = \{(1,1),(1,2)\}$ and $B = \{(2,1)\}$ (these sets of boxes are indicated in the margin).

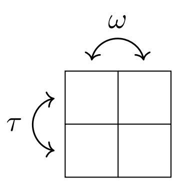


There are three distinct tableaux obtained by the action of $S_{A\sqcup B}$ on t:

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\,, \qquad \begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array}\,, \qquad \begin{array}{|c|c|}\hline 1&2\\\hline 1&2\\\hline\end{array}\,.$$

The latter two are row equivalent, so a set of representatives is $\mathcal{T}/{\sim_{\mathrm{r}}} = \left\{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\,, \begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array}\right\}$. The second tableau has trivial row stabiliser. The first tableau, t itself, has row stabiliser $S_{\mathrm{row}_1[\lambda]} \times S_{\mathrm{row}_2[\lambda]}$, of size 4, whose intersection with $S_{A\sqcup B} \times S_{[\lambda]\backslash A\sqcup B}$ is $S_{\mathrm{row}_1[\lambda]}$, of size 2. Thus the row Garnir relation is

$$\begin{aligned}\text{ꓤ}_{(t,A,B)} &= 2\,\mathrm{rsym}\Big(\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\Big) + \mathrm{rsym}\Big(\begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array}\Big)\\ &= 2\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&1\\\hline 2&1\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 2&1\\\hline\end{array}.\end{aligned}$$



Alternatively we can use the expression from Lemma 3.17. The group $S_{A\sqcup B}$ is isomorphic as an abstract group to S_3, and is generated by the transpositions $\tau = \big((1,1)\ (2,1)\big)$ and $\omega = \big((1,1)\ (1,2)\big)$ (these permutations are depicted in the margin). Note that $\mathrm{stab}(t) \cap S_{A\sqcup B} = S_A \times S_B$, and ω is the unique nontrivial element of this subgroup. Thus there are only two double cosets, $\{\mathrm{id}, \omega\}$ and $\{\tau, \tau\omega, \omega\tau, \omega\tau\omega\}$, and a choice of double coset representatives is

$$\mathrm{stab}(t) \cap S_{A\sqcup B}\backslash S_{A\sqcup B}/S_A \times S_B = \{\mathrm{id}, \tau\}.$$

The set $\{t, t\cdot\tau\}$ obtained from the action of these double coset representatives is precisely the above choice of representatives for $\mathcal{T}/{\sim_{\mathrm{r}}}$, and thus we obtain $\text{ꓤ}_{(t,A,B)}$ as above.

Observe that the image of $\text{ꓤ}_{(t,A,B)}$ under the map $\mathsf{\Lambda}\colon \mathsf{Tbx}^{\lambda}(V) \twoheadrightarrow \bigwedge^{\lambda'} V$ is

$$\begin{aligned}\mathsf{\Lambda}(\text{ꓤ}_{(t,A,B)}) &= 2\,\text{ə}(t) + \text{ə}(t\cdot\tau)\\ &= 2\begin{array}{|c|c|}1&1\\2&2\end{array} + \begin{array}{|c|c|}2&1\\1&2\end{array} + \begin{array}{|c|c|}1&2\\2&1\end{array}\\ &= 0,\end{aligned}$$

as the upcoming Proposition 3.21 claims. This allows us to rewrite the non-semistandard copolytabloid $\text{ə}\Big(\begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array}\Big)$ in terms of semistandard copolytabloids (as described in general in the upcoming Lemma 3.24).

We now use this example to demonstrate why our definition of the row Garnir relations is the correct definite to make.

The obvious candidate for a simpler definition is in direct analogy with the usual Garnir relations: define $\text{ꓤ}^*_{(t,A,B)}$ as a sum over left coset representatives of $S_A \times S_B$ in $S_{A\sqcup B}$, without any coefficients appearing in the sum. A choice of coset representatives is $S_{A\sqcup B}/S_A \times S_B = \{\mathrm{id}, \tau, \omega\tau\}$. In our example we have

$$\begin{aligned}
\text{ꓤ}^*_{(t,A,B)} &= \operatorname{rsym}\left(\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\right) + 2\operatorname{rsym}\left(\begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array}\right)\\
&= \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} + 2\begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array} + 2\begin{array}{|c|c|}\hline 1&2\\\hline 1&2\\\hline\end{array} + 2\begin{array}{|c|c|}\hline 2&1\\\hline 2&1\\\hline\end{array} + 2\begin{array}{|c|c|}\hline 1&2\\\hline 2&1\\\hline\end{array}
\end{aligned}$$

whose image under Λ is

$$\begin{aligned}
\Lambda(\text{ꓤ}^*_{(t,A,B)}) &= \text{ə}\left(\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\right) + 2\,\text{ə}\left(\begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array}\right)\\
&= \begin{array}{|c|c|} 1&1\\ 2&2 \end{array} + 2\begin{array}{|c|c|} 2&1\\ 1&2 \end{array} + 2\begin{array}{|c|c|} 1&2\\ 2&1 \end{array}\\
&= -3\begin{array}{|c|c|} 1&1\\ 2&2 \end{array}
\end{aligned}$$

which is nonzero (in characteristics other than 3). Similarly it can be shown that summing over the entire group $S_{A\sqcup B}$ also fails to yield an element of the kernel (in this example we would obtain twice the quantity above).

An alternative definition that does yield elements of the kernel is to, as above, sum over left coset representatives of $S_A \times S_B$ in $S_{A\sqcup B}$ (without any additional coefficients), but replace the row symmetrisation with a sum over the entire group of row preserving permutations. Define

$$\text{ꓤ}^{**}_{(t,A,B)} = \sum_{\tau\in S_{A\sqcup B}/S_A\times S_B}\ \sum_{\sigma\in\mathrm{RPP}(\lambda)} t\cdot\tau\sigma.$$

It can be shown, analogously to the proof of Proposition 2.8, that these elements lie in the kernel of Λ; this proof is much simpler than that of Proposition 3.21 because the sums over the row permutations do not depend on the tableaux to which they are being applied. However, in general these elements have scalar factors, and hence cannot be used in a straightening algorithm (as described in Lemma 3.24) to express a copolytabloid in terms of semistandard copolytabloids. In our example, we have

$$\begin{aligned}
\text{ꓤ}^{**}_{(t,A,B)} &= \sum_{\sigma\in\mathrm{RPP}(\lambda)}\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\cdot\sigma \quad + \quad 2\sum_{\sigma\in\mathrm{RPP}(\lambda)}\begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array}\cdot\sigma\\
&= 4\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} + 2\begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array} + 2\begin{array}{|c|c|}\hline 1&2\\\hline 1&2\\\hline\end{array} + 2\begin{array}{|c|c|}\hline 2&1\\\hline 2&1\\\hline\end{array} + 2\begin{array}{|c|c|}\hline 1&2\\\hline 2&1\\\hline\end{array}\\
&= 2\,\text{ꓤ}_{(t,A,B)}\,.
\end{aligned}$$

Remark 3.19. If t is of symmetric type, then a row Garnir relation labelled by t is simpler to write down: the row stabiliser of any place permutation of t is trivial, so all the coefficients in the expression are 1, and the row symmetrisations are sums over the entire group of row preserving permutations. Additionally, the proof of the upcoming Proposition 3.21 is much easier (as noted in Example 3.18). In this case, when $G = S_r$ and $V = W$ the natural permutation representation, the row Garnir relations become well-known relations for the dual Specht module $(S^\lambda)^*$ [Ful97, Exercise 14, p. 101]. Similar relations hold working in the cellular basis of the dual Specht module for the Hecke algebra [Mat99, §3.2].

We now show that the row Garnir relations lie in the kernel of $\mathsf{\Lambda}$. The strategy is to rewrite the double coset expression for $\text{ꓤ}_{(t,A,B)}$ to remove the dependence between the sums, and then argue as in the proof of Proposition 2.8. To this end, we first record some expressions for sets of coset representatives.

Lemma 3.20. *Let Γ be any group, and let I, J and L be subgroups of Γ. Denote compontentwise multiplication of sets by concatenation. The following equalities hold, interpreted as statements about choices of coset representatives.*

(i) $I \setminus \Gamma = \bigsqcup_{g \in I\setminus\Gamma/J} \bigsqcup_{h \in (I^g \cap J)\setminus J} \{gh\}$.
(ii) $(L \cap I{\times}J) \setminus \Gamma = \big(L \cap I{\times}J \setminus I{\times}J\big)\big(I{\times}J \setminus \Gamma\big)$.
(iii) $(L \cap I{\times}J) \setminus \Gamma = \big(L \cap I{\times}J \setminus L \cap \Gamma\big)\big(L \cap \Gamma \setminus \Gamma\big)$.
(iv) $(L \cap I{\times}J) \setminus I{\times}J = \big(L \cap I \setminus I\big)\big(L \cap J \setminus J\big)$ *if I and J commute and are disjoint.*

Proof. All the statements are routine exercises in bookkeeping. □

Proposition 3.21 (cf. Proposition 2.8)**.** *If $\text{ꓤ}_{(t,A,B)}$ is any row Garnir relation, then $\mathsf{\Lambda}(\text{ꓤ}_{(t,A,B)}) = 0$.*

Proof. For convenience, we introduce some abbreviations: $H = \operatorname{stab} t$, $C = A \sqcup B$ and $Z = [\lambda] \setminus A \sqcup B$, and for $D \subseteq [\lambda]$ we write $\mathrm{R}_D = \mathrm{RPP}(\lambda) \cap S_D$ (and hence $\mathrm{R}_{[\lambda]} = \mathrm{RPP}(\lambda)$). We then have, for $\tau \in S_{[\lambda]}$, that $\operatorname{stab}(t \cdot \tau) =$

$H^\tau = \tau^{-1}H\tau$, that $\mathrm{rstab}(t\cdot\tau) = H^\tau \cap \mathrm{R}_{[\lambda]}$, that $\mathrm{rstab}(t\cdot\tau)\cap(S_C \times S_{[\lambda]\backslash C}) = H^\tau \cap (\mathrm{R}_C \times \mathrm{R}_Z)$, and that $\mathrm{R}_C = S_A \times S_B$.

For each $\tau \in S_{[\lambda]}$, we use Lemma 3.20(iii) with $\Gamma = \mathrm{R}_{[\lambda]}$, $L = H^\tau$, $I = \mathrm{R}_C$, $J = \mathrm{R}_Z$ to find that

$$\begin{aligned}\sum_{\pi\in H^\tau\cap(\mathrm{R}_C\times\mathrm{R}_Z)\backslash \mathrm{R}_{[\lambda]}} |t\cdot\tau\pi| &= \sum_{\varphi\in H^\tau\cap(\mathrm{R}_C\times\mathrm{R}_Z)\backslash H^\tau\cap\mathrm{R}_{[\lambda]}} \;\sum_{\sigma\in H^\tau\cap\mathrm{R}_{[\lambda]}\backslash\mathrm{R}_{[\lambda]}} |t\cdot\tau\varphi\sigma| \\ &= \left|H^\tau\cap\mathrm{R}_{[\lambda]} : H^\tau\cap\mathrm{R}_C{\times}\mathrm{R}_Z\right| \sum_{\sigma\in H^\tau\cap\mathrm{R}_{[\lambda]}\backslash\mathrm{R}_{[\lambda]}} |t\cdot\tau\sigma| \\ &= \left|H^\tau\cap\mathrm{R}_{[\lambda]} : H^\tau\cap\mathrm{R}_C{\times}\mathrm{R}_Z\right| \mathsf{\Lambda}(\mathrm{rsym}(t\cdot\tau))\end{aligned}$$

where we have used that elements of $H^\tau\cap\mathrm{R}_{[\lambda]}$ fix $t\cdot\tau$. Note the index is precisely the index occurring in the definition of the row Garnir relations. Therefore the element we are required to show is zero is

$$\begin{aligned}\mathsf{\Lambda}(\text{ꓤ}_{(t,A,B)}) &= \sum_{\tau\in H\cap S_C\backslash S_C/\mathrm{R}_C} \;\sum_{\pi\in H^\tau\cap(\mathrm{R}_C\times\mathrm{R}_Z)\backslash\mathrm{R}_{[\lambda]}} |t\cdot\tau\pi| \\ &= \sum_{\tau\in H\cap S_C\backslash S_C/\mathrm{R}_C} \;\sum_{\varphi\in H^\tau\cap(\mathrm{R}_C\times\mathrm{R}_Z)\backslash\mathrm{R}_C\times\mathrm{R}_Z} \;\sum_{\sigma\in\mathrm{R}_C\times\mathrm{R}_Z\backslash\mathrm{R}_{[\lambda]}} |t\cdot\tau\varphi\sigma| \\ &= \sum_{\tau\in H\cap S_C\backslash S_C/\mathrm{R}_C} \;\sum_{\chi\in H^\tau\cap\mathrm{R}_C\backslash\mathrm{R}_C} \;\sum_{\psi\in H^\tau\cap\mathrm{R}_Z\backslash\mathrm{R}_Z} \;\sum_{\sigma\in\mathrm{R}_C\times\mathrm{R}_Z\backslash\mathrm{R}_{[\lambda]}} |t\cdot\tau\chi\psi\sigma|\end{aligned}$$

where the last two equalities hold by parts (ii) and (iv) of Lemma 3.20 respectively, each with $R = \mathrm{R}_{[\lambda]}$, $L = H^\tau$, $I = \mathrm{R}_C$, $J = \mathrm{R}_Z$.

Now we combine terms using Lemma 3.20(i) with $\Gamma = S_C$, $I = H\cap S_C$ and $J = \mathrm{R}_C$. We see that the final line above becomes

$$\mathsf{\Lambda}(\text{ꓤ}_{(t,A,B)}) = \sum_{\tau\in H\cap S_C\backslash S_C} \;\sum_{\psi\in H^\tau\cap\mathrm{R}_Z\backslash\mathrm{R}_Z} \;\sum_{\sigma\in\mathrm{R}_C\cap\mathrm{R}_Z\backslash\mathrm{R}_{[\lambda]}} |t\cdot\tau\psi\sigma|.$$

Notice that, because the boxes of Z are fixed by $\tau \in S_{A\sqcup B}$, we have that $H^\tau\cap\mathrm{R}_Z = H\cap\mathrm{R}_Z$ is independent of τ. The rightmost sum above also has indexing set independent of τ, and both of these indexing sets are subsets of $\mathrm{R}_{[\lambda]}$, so it suffices to show that $\sum_{\tau\in H\cap S_C\backslash S_C} |t\cdot\tau\sigma| = 0$ for all $\sigma\in\mathrm{R}_{[\lambda]}$.

Fix $\sigma\in\mathrm{R}_{[\lambda]}$. Recall from the definition of a row Garnir relation that $A \subseteq \mathrm{row}_i[\lambda]$ and $B \subseteq \mathrm{row}_{i'}[\lambda]$ for some $1 \leqslant i < i' \leqslant \lambda_1$, and that $|A| + |B| > \lambda_i$. Thus by the pigeonhole principle there exists a column containing both a box in A and a box in B. Moreover the same claim holds if we act by σ first; that is, there exist $a \in A$, $b \in B$ and $1 \leqslant j \leqslant \lambda_1$ such that $a\sigma = (i, j)$ and $b\sigma = (i', j)$. Let $\omega = (a\ b) \in S_{A \sqcup B}$, and note that $\omega^\sigma = \sigma^{-1}\omega\sigma = (a\sigma\ b\sigma) \in \mathrm{CPP}(\lambda)$.

Define an action of ω on the set of cosets $H \cap S_{A\sqcup B} \backslash S_{A\sqcup B}$ by right multiplication. If $\tau \in H \cap S_{A\sqcup B} \backslash S_{A\sqcup B}$ is in an orbit of size 1, then $t \cdot \tau = t \cdot \tau\omega$ and hence $t \cdot \tau\sigma = t \cdot \tau\sigma\omega^\sigma$. But then $t \cdot \tau\sigma$ has the same entries in $a\sigma$ and $b\sigma$; since these are in the same column, this implies $|t \cdot \tau\sigma| = 0$. If $\{\tau, \tau\omega\}$ is an orbit of size 2, then since $\omega^\sigma \in \mathrm{CPP}(\lambda)$ we have $|t \cdot \tau\omega\sigma| = |t \cdot \tau\sigma\omega^\sigma| = -|t \cdot \tau\sigma|$, and so the contributions to the sum of these orbits cancel out. Thus the entire sum is zero, as required. $\square$

Just as for the original Garnir relations, it suffices to consider a certain subset of the row Garnir relations: those in which the chosen rows are adjacent, and boxes are taken from the right of the upper row and the left of the lower row. We call these relations *row snake relations*; they are defined formally below. Unlike the snake relation, which permitted the chosen boxes to overlap only in a single row, here we permit them to overlap in multiple columns. This is due to our straightening algorithm (Lemmas 3.23 and 3.24) requiring the chosen boxes to contain all or none of the instances of a particular entry in a row.

Definition 3.22. A row Garnir relation $\rotatebox{180}{R}_{(t,A,B)}$ is called a *row snake relation* when, in the notation of Definition 3.16, $i' = i + 1$ and there exist $j \leqslant j'$ such that $A = \{\, (i, r) \mid j \leqslant r \leqslant \lambda_i \,\}$ and $B = \{\, (i', r) \mid 1 \leqslant r \leqslant j' \,\}$. In this case, we may also label the row Garnir relation by $(t, i, (j, j'))$.

Lemma 3.23 (cf. Lemma 2.10)**.** *Let t be a row semistandard tableau, and suppose i and $j \leqslant j'$ are such that there exists $j \leqslant j_0 \leqslant j'$ such that:*

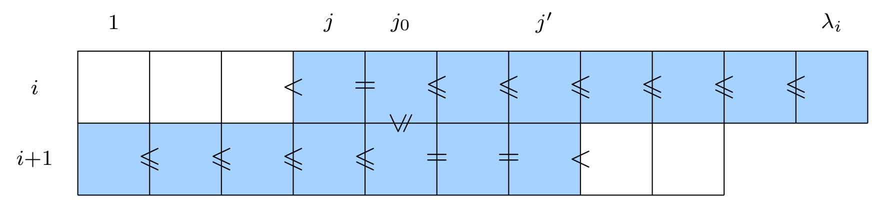
- *$t(i, j_0) \geqslant t(i{+}1, j_0)$;*
- *$t(i, j) = t(i, j_0)$ and $t(i + 1, j') = t(i + 1, j_0)$;*
- *$t(i, j{-}1) < t(i, j)$ (or $j = 1$) and $t(i{+}1, j') < t(i{+}1, j'{+}1)$ (or $j' = \lambda_{i+1}$).*

Then

$$\text{ꓤ}_{(t,i,(j,j'))} = \operatorname{rsym}(t) + \sum_{u <_{\mathrm{r}} t} m_u \operatorname{rsym}(u)$$

for some elements m_u in the subring of K generated by 1.

Proof. The diagram below illustrates the hypotheses satisfied by the sets $A = \{ (i,r) \mid j \leqslant r \leqslant \lambda_i \}$ and $B = \{ (i+1,r) \mid 1 \leqslant r \leqslant j' \}$ defining the Garnir relation (these sets are shaded in the diagram).



In particular all the boxes in B contain entries less than or equal to all the entries in boxes in A. Therefore for any $\tau \in S_{A \sqcup B}$, we have $t \cdot \tau \lesssim_{\mathrm{r}} t$, and furthermore row equivalence $t \cdot \tau \sim_{\mathrm{r}} t$ holds if and only if $\tau \in (\operatorname{stab}(t) \cap S_{A \sqcup B})(S_A \times S_B)$. Thus the row symmetrisation $\operatorname{rsym}(t)$ appears precisely once in $\text{ꓤ}_{(t,i,(j,j'))}$, with coefficient $\left|\operatorname{rstab}(t) : \operatorname{rstab}(t) \cap (S_{A \sqcup B} \times S_{[\lambda] \setminus A \sqcup B})\right|$. But by the assumptions, as displayed above, every entry in row i of t occurs either only in A or only in $\operatorname{row}_i[\lambda] \setminus A$, and likewise every entry in row $i+1$ of t occurs either only in B or only in $\operatorname{row}_{i+1}[\lambda] \setminus B$. Then $\operatorname{rstab}(t) \subseteq S_{A \sqcup B} \times S_{[\lambda] \setminus A \sqcup B}$, and so this coefficient is 1. □

Lemma 3.24 (cf. Lemma 2.11)**.** *Let t be any tableau. Then there exists some K-linear combination γ of row snake relations (with coefficients in the subring of K generated by* 1*) such that*

$$\operatorname{rsym}(t) + \gamma = \sum_{s \in \mathrm{SSYT}(\lambda)} a_s \operatorname{rsym}(s)$$

for some elements a_s in the subring of K generated by 1 *(which may be all zero).*

Proof. Without loss of generality, we may assume t is row semistandard. If t is also column standard, we are done. Otherwise, choose a box (i, j_0) such that $t(i{+}1, j_0) \leqslant t(i, j_0)$. Then pick $j \leqslant j_0 \leqslant j'$ such that $t(i, j{-}1) < t(i, j) =$

$t(i, j_0)$ (or $j = 1$) and $t(i{+}1, j_0) = t(i{+}1, j') < t(i{+}1, j'{+}1)$ (or $j' = \lambda_{i+1}$). By Lemma 3.23 we have that $\text{ᴚ}_{(t,i,(j,j'))} = \operatorname{rsym}(t) + \sum_{u <_\mathrm{r} t} m_u \operatorname{rsym}(u)$ for some elements m_u in the subring of K generated by 1. Then $\operatorname{rsym}(t) - \text{ᴚ}_{(t,i,j)}$ is a linear combination of row symmetrisations of tableaux which precede t in the row ordering. The lemma then follows by induction. □

Analogously to Lemma 2.11, the above lemma gives a direct proof that the semistandard copolytabloids span $\Delta^\lambda V$ (and hence form a basis by Lemma 3.12), a fact which we deduced by dimension counting in Proposition 3.13.

Proposition 3.25 (cf. Proposition 2.13)**.** *There is equality* $\ker(\mathsf{\Lambda}|_{\mathrm{Sym}_\lambda V}) = \mathsf{G}\text{ᴚ}^\lambda(V)$ *(and consequently* $\mathsf{G}\text{ᴚ}^\lambda(V)$ *is a* KG*-module), and hence there is a* KG*-isomorphism*

$$\Delta^\lambda V \cong {}^{\mathrm{Sym}_\lambda V}\!\big/_{\mathsf{G}\text{ᴚ}^\lambda(V)}.$$

PROOF. From Proposition 3.21, we have that $\mathsf{G}\text{ᴚ}^\lambda(V) \subseteq \ker \mathsf{\Lambda}$. It therefore suffices to show that the row snake relations span $\ker \mathsf{\Lambda}$.

Let $\kappa \in \ker \mathsf{\Lambda}$. By Lemma 3.24 there exists a K-linear combination γ of row snake relations such that

$$\kappa + \gamma = \sum_{s \in \mathrm{SSYT}(\lambda)} \alpha_s \operatorname{rsym}(s)$$

for some elements $\alpha_s \in K$. Applying $\mathsf{\Lambda}$ to this equation and using that $\mathsf{G}\text{ᴚ}^\lambda(V) \subseteq \ker \mathsf{\Lambda}$, we find

$$0 = \sum_{s \in \mathrm{SSYT}(\lambda)} \alpha_s \, \text{ə}(s).$$

By Lemma 3.12 the semistandard copolytabloids are K-linearly independent, so this implies that $\alpha_s = 0$ for all s. Hence $\kappa = -\gamma$ is in the span of the row snake relations, as required. □

4. Specht modules and Weyl modules

We now specialise the constructions of this chapter to representations of the general linear and symmetric groups. We obtain some important families of modules: the Specht modules for the symmetric group, and the Weyl and dual Weyl modules for the general linear group.

4.1. Specht modules and their duals

Let W be the natural permutation representation of the symmetric group S_r. That is, W has a basis $w_1, \ldots, w_r$ such that $\sigma w_i = w_{i\sigma^{-1}}$ for $\sigma \in S_r$. With respect to this basis, W is a permutation module, and so the multilinear constructions of the previous chapter yield KS_r-modules when restricted to tableaux of symmetric type (recall tableaux of symmetric type are those with all entries distinct).

Let $S^\lambda = \nabla^\lambda_{\mathrm{sym}} W$. We call S^λ the *Specht module*. The set of standard tableaux of symmetric type label a basis, and moreover a permutation basis, for S^λ.

Due to the submodule construction of the Schur endofunctor, S^λ can be viewed as a submodule of $\mathrm{Sym}^\lambda_{\mathrm{sym}} W$ (called the *Young permutation module*). Due to the quotient construction of the Schur endofunctor, S^λ is known to obey the Garnir relations.

This construction of the Specht modules is due to James [Jam78]. James also gives the following well-known classification of the simple modules of the symmetric group S_r:

- when K has characteristic 0, the Specht modules form a complete irredundant set of simple KS_r-modules;
- when K has characteristic p, the Specht modules labelled by p-regular partitions have simple heads, and these heads form a complete irredundant set of simple KS_r-modules (a partition is said to be p-regular if no part is repeated p or more times).

The dual $(S^\lambda)^*$ of the Specht module can be obtained from the Weyl endofunctor (restricted to tableaux of symmetric type). Indeed, as W is a permutation module, we have $W \cong W^*$, and so $\Delta^\lambda W \cong (\nabla^\lambda W)^*$. Thus $\Delta^\lambda_{\mathrm{sym}} W \cong (S^\lambda)^*$.

Remark 4.1. Some authors (such as Mathas [Mat99]) instead call $\Delta^\lambda_{\mathrm{sym}} W$ the Specht module, the dual of our definition.

4.2. Weyl and dual Weyl modules

Suppose K is infinite. Recall E denotes the natural representation of $\mathrm{GL}_n(K)$; that is, E has basis $X_1, \dots, X_n$ with respect to which $\rho_E(g) = g$ for all $g \in \mathrm{GL}_n(K)$.

We call $\Delta^\lambda E$ the *Weyl module* and we call $\nabla^\lambda E$ the *dual Weyl module*. Since $E^\circ \cong E$, we have that $(\Delta^\lambda E)^\circ \cong \nabla^\lambda E$, as the name suggests.

We will see in §5 in the following chapter that both modules $\Delta^\lambda E$ and $\nabla^\lambda E$ are polynomial of degree $|\lambda|$ (Proposition 5.6).

As shown for example in [EGS08], the Weyl and dual Weyl modules are indecomposable, and moreover the Weyl modules have simple heads and (hence) the dual Weyl modules have simple socles. We write

$$L^\lambda(E) = {\Delta^\lambda E}\big/{\mathrm{rad}\, \Delta^\lambda E} = \mathrm{soc}\, \nabla^\lambda E.$$

These simple modules form a complete irredundant set of simple modules in a certain subcategory of representations of $\mathrm{GL}_n(K)$ (the category of polynomial representations of $\mathrm{GL}_n(K)$ of degree $|\lambda|$).

Many alternative constructions of the Weyl and dual Weyl modules are available; comparisons with our construction were made in §0.2.

CHAPTER II

Polynomial representations of matrix groups

In this chapter we give a brief introduction to the category of *polynomial* representations of matrix groups. This category includes the Weyl and dual Weyl modules constructed at the end of the previous chapter.

Definitions, examples and first results are given in §5; notably we show that a polynomial representation of $\mathrm{GL}_n(K)$ is determined up to a power of the determinant by the action of $\mathrm{SL}_n(K)$. An important connection between polynomial representations and representations of the symmetric group is given by the Schur functor and its inverse (not to be confused with the Schur endofunctor constructed in Chapter I); these functors are introduced in §6. In the short §7 we define the dimension reduction functor, which gives a connection between polynomial representations of $\mathrm{GL}_n(K)$ for different n.

Polynomial representations, and the Schur functor and its inverses, are usually described in the language of the Schur algebra (see [EGS08, §2 and §6]). Here we give the explicit, intuitive interpretation of the property of being polynomial (see [Wil14]), and an elementary construction of the Schur functor and its inverses which requires only the notions of weight spaces, tensor products and hom spaces. While we are primarily concerned with the case where K is an infinite field, our interpretation has the advantage that it permits extension to finite fields (see Remark 6.22 for a comparison with the Schur algebra approach).

5. Elementary results on polynomial representations

In this section we define polynomial representations and see that various common operations on modules preserve the property of being polynomial, including the multilinear constructions of the previous chapter. It follows that the Weyl and dual Weyl modules are polynomial. We also show that the action of the special linear group is sufficient to determine a polynomial representation of the general linear group, up to a power of the determinant.

5.1. Definitions and examples

Definition 5.1 (Polynomial representations)**.** Suppose K is an infinite field and $G \leqslant \mathrm{GL}_n(K)$ is an infinite matrix group. A representation ρ of G is called *polynomial* if there exist polynomials $\rho^{(i,j)} \in K[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$ over K in n^2 variables such that for all $g \in G$ we have

$$\rho(g)_{i,j} = \rho^{(i,j)}(g_{1,1}, g_{1,2}, \ldots, g_{n,n}).$$

We call the polynomials $\rho^{(i,j)}$ the *representing polynomials*. A polynomial representation of $\mathrm{GL}_n(K)$ has *degree* r if all the representing polynomials are homogeneous of degree r.

That is, a representation of a matrix group G is polynomial if, with respect to some basis, the entries of the matrix representing the action of $g \in G$ are (fixed) polynomials in the entries of g.

Note that if ρ has representing polynomials $\rho^{(i,j)}$ and M is a fixed invertible matrix, then the polynomials $\hat{\rho}^{(i,j)} = \sum_{a,b} M_{i,a}\rho^{(a,b)}M^{-1}_{b,j}$ are representing polynomials for $M\rho M^{-1}$. Thus if representing polynomials exist for one choice of basis then they exist for all choices of basis.

We illustrate the definition with some examples and non-examples.

Example 5.2 (Examples of polynomial representations)**.**

(i) The natural representation E of $\mathrm{GL}_n(K)$ is polynomial of degree 1, with representing polynomials $\rho_E^{(i,j)} = x_{i,j}$.

(ii) The determinant representation $\det E$ is polynomial of degree n.

(iii) Let E be the natural representation of $\mathrm{GL}_2(K)$ with standard basis $\{X, Y\}$, and consider the representation $\mathrm{Sym}^2 E$. Denoting elements of the symmetric power by concatenating their factors, we use the basis $\{X^2, XY, Y^2\}$ for $\operatorname{Sym}^2 E$. The matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ is represented by

$$\rho_{\operatorname{Sym}^2 E}\left(\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\right) = \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta+\beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix}.$$

Representing polynomials are $\rho^{(1,1)}_{\operatorname{Sym}^2 E} = x_{1,1}^2$, $\rho^{(2,1)}_{\operatorname{Sym}^2 E} = 2x_{1,1}x_{2,1}$, etc, and thus $\operatorname{Sym}^2 E$ is polynomial of degree 2. (We show in Proposition 5.6 that all images of polynomial representations under Schur and Weyl endofunctors are polynomial.)

Example 5.3 (Non-examples of polynomial representations)**.**

(i) The 1-dimensional representation of $\mathrm{GL}_n(K)$ on which $g \in \mathrm{GL}_n(K)$ acts by $(\det g)^{-1}$ is not polynomial, as the reciprocal of the determinant cannot be written as a polynomial in the entries of the matrix.

(ii) Consider the dual E^* of the natural representation of $\mathrm{GL}_2(K)$. The matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ is represented by

$$\rho_{E^*}\left(\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)^{-\top} = \frac{1}{\alpha\delta-\beta\gamma}\begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}$$

whose entries cannot be written as polynomials in $\alpha, \beta, \gamma, \delta$, again due to the reciprocal of the determinant. This illustrates that the dual of a polynomial representation need not be polynomial.

(iii) [cf. Example 1.2, Wil14] Suppose $K = \mathbb{R}$ and $G = \mathrm{GL}_n(\mathbb{R})$, and consider the 2-dimensional representation $\rho\colon G \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\rho(g) = \begin{pmatrix} 1 & \log\lvert\det g\rvert \\ 0 & 1 \end{pmatrix}.$$

This is not polynomial as the logarithm cannot be written as a polynomial.

Proposition 5.4. *Suppose K is an infinite field. There is a unique choice of representing polynomials for a polynomial representation of $\mathrm{GL}_n(K)$. In particular, the degree of a polynomial representation is well-defined.*

PROOF. This follows from two facts: the set of invertible matrices is Zariski-dense in the set of all matrices, and over an infinite field a polynomial in N variables which vanishes on all inputs is the zero polynomial (this latter fact can be shown by induction on N, by fixing $N-1$ variables and using that a nonzero polynomial in one variable has only finitely many roots).

Explicitly, suppose polynomials $f, h \in K[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$ agree on all invertible matrices. Let $\det \in K[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$ be the determinant polynomial. The polynomial $(f-h)\det$ vanishes on all matrices, and so is the zero polynomial; since $\det \neq 0$, we conclude $f = h$. $\square$

Remark 5.5. Proposition 5.4 may fail for infinite matrix groups other than $\mathrm{GL}_n(K)$. For example, if $G = \mathrm{SL}_n(K)$, then representing polynomials may be multiplied by powers of the determinant without changing their evaluation on elements of G.

Proposition 5.6.

(i) Submodules and quotients of a polynomial representation (of degree r) are polynomial (of degree r).

(ii) Direct sums of polynomial representations (of degree r) are polynomial (of degree r).

(iii) Tensor products of polynomial representations (of degrees r_i) are polynomial (of degree the sum of the r_i).

(iv) The contravariant dual of a polynomial representation (of degree r) is polynomial (of degree r).

(v) Images under Schur and Weyl endofunctors ∇^λ and Δ^λ of a polynomial representation (of degree r) are polynomial (of degree $|\lambda|r$).

In particular, the Weyl and dual Weyl modules $\Delta^\lambda E$ and $\nabla^\lambda E$ are polynomial, of degree $|\lambda|$.

PROOF. Parts (i)–(iv) are clear from considering the forms of the representing matrices. Part (v) follows from the previous parts. $\square$

Remark 5.7. The definition of a polynomial representation still makes sense if the field K is finite, or if the matrix group G is finite. However, it is not a useful definition to make, for the following reasons.

(i) All representations of a finite matrix group satisfy the definition of being polynomial. Indeed, let $F \subseteq K$ be the set of field elements which appear as entries of matrices in G, and for each $\alpha \in F$ define a polynomial $\mathbb{1}_\alpha(x) = \prod_{\beta \in F \setminus \{\alpha\}} (x - \beta)(\alpha - \beta)^{-1}$ so that $\mathbb{1}_\alpha(\alpha) = 1$ and $\mathbb{1}_\alpha(\beta) = 0$ if $\beta \neq \alpha$. Then define for each $g \in G$ a polynomial $\mathbb{1}_g(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) = \prod_{i,j} \mathbb{1}_{g_{i,j}}(x_{i,j})$ so that $\mathbb{1}_g(g_{1,1}, g_{1,2}, \ldots, g_{n,n}) = 1$ and $\mathbb{1}_g$ vanishes on all other inputs. Then given any representation ρ of G we can choose representing polynomials $\rho^{(i,j)} = \prod_{g \in G} \rho(g)_{i,j} \mathbb{1}_g$.

(ii) The degree of a polynomial representation of the finite $\mathrm{GL}_n(K)$ (or any finite matrix group) would not be well-defined. This can be seen by modifying the construction in (i) above: the representing polynomials constructed there are homogeneous of degree $(|F| - 1)|G|$, but for example we can replace the polynomials $\mathbb{1}_\alpha$ with their squares and obtain representing polynomials which are homogeneous of twice that degree. Alternatively, in the case of a finite field, the degrees of polynomials being ill-defined is clear from the fact that $\alpha^q = \alpha$ for any $\alpha \in K$ when K is of order q.

Remark 5.8. We note that the action of $\mathrm{GL}_n(K)$ on a polynomial representation can be extended to an action of $\mathrm{Mat}_n(K)$, the semigroup of all (not necessarily invertible) $n \times n$ matrices. This is achieved simply by evaluating the representing polynomials on arbitrary matrices. We need to check, however, that the semigroup multiplication is compatible with this action. Writing $\rho^{(i,j)}$ for the representing polynomials, the compatibility requirement is precisely the identity

$$\begin{aligned}\sum_{k=1}^{n} &\rho^{(i,k)}(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) \rho^{(k,j)}(y_{1,1}, y_{1,2}, \ldots, y_{n,n}) \\ &= \rho^{(i,j)}\Big(\sum_{k=1}^{n} x_{1,k} y_{k,1}, \sum_{k=1}^{n} x_{1,k} y_{k,2}, \ldots, \sum_{k=1}^{n} x_{n,k} y_{k,n} \Big)\end{aligned}$$

between polynomials in $2n^2$ variables, for all $1 \leqslant i, j \leqslant n$. Indeed, since this identity holds for all choices of variables corresponding to pairs of invertible matrices, then (as in the proof of Proposition 5.4) it holds for all choices of variables.

5.2. Restriction to special linear groups

It is often simpler to work over the special linear group $\mathrm{SL}_n(K)$ than the full general linear group. In this subsection we record the results required to pass between these groups.

We first identify generating sets for the general and special linear groups. We say a matrix is an *elementary transvection* if it has 1s on the diagonal and a unique nonzero off-diagonal entry; we say a matrix is a *scalar matrix* if it is a (nonzero) scalar multiple of the identity matrix.

Lemma 5.9 (Generation of $\mathrm{SL}_n(K)$ and $\mathrm{GL}_n(K)$).

(i) The group $\mathrm{SL}_n(K)$ is generated by elementary transvections.

(ii) The group $\mathrm{GL}_n(K)$ is generated by elementary transvections and diagonal matrices.

(iii) If K is algebraically closed, then $\mathrm{GL}_n(K)$ is generated by elementary transvections and scalar matrices.

PROOF. [Lan02, Chapter XIII, Proposition 9.1] proves (i) and (ii). When K is algebraically closed, we can choose an nth root of the determinant, and a simple modification to the proof yields (iii). □

The key result for passing from representations of $\mathrm{SL}_n(K)$ to $\mathrm{GL}_n(K)$ is the following proposition.

Proposition 5.10. *Suppose K is infinite. Let V and W be polynomial representation of $\mathrm{GL}_n(K)$ of degrees r and s, where $r \geqslant s$. If $V|_{\mathrm{SL}_n(K)} \cong W_{\mathrm{SL}_n(K)}$, then there exists $m \geqslant 0$ such that $r - s = mn$ and $V \cong W \otimes (\det E)^{\otimes m}$.*

PROOF. Suppose $V|_{\mathrm{SL}_n(K)} \cong W|_{\mathrm{SL}_n(K)}$; then there exist bases for V and W such that $\rho_V(g) = \rho_W(g)$ for all $g \in \mathrm{SL}_n(K)$. Then for each pair of coordinates (i,j), the difference of the representing polynomials $\rho_V^{(i,j)} - \rho_W^{(i,j)}$ vanishes on $\mathrm{SL}_n(K)$.

Suppose first that K is algebraically closed. Then $\mathrm{SL}_n(K)$ is the algebraic set of zeroes of the polynomial $\det -1$, and so by Hilbert's Nullstellensatz (as formulated for example in [AM69, Chapter 7, Exercise 14]), there exists

$l \in \mathbb{N}$ such that $(\rho_V^{(i,j)} - \rho_W^{(i,j)})^l$ lies in the ideal generated by $\det -1$. Since the polynomial $\det -1$ is irreducible and hence this ideal is prime, in fact $\rho_V^{(i,j)} - \rho_W^{(i,j)}$ lies in the ideal. Thus we can write $\rho_V^{(i,j)} - \rho_W^{(i,j)} = (\det -1)f$ for some polynomial $f \in K[x_{1,1}, x_{1,2}, \dots, x_{n,n}]$, and decompose into a system of equations of homogeneous polynomials. Solving this system leads routinely to the requirement $r = s + mn$ for some $m \geqslant 0$ and that $\rho_V^{(i,j)} = (\det)^m \rho_W^{(i,j)}$.

If K is not algebraically closed, let $\overline{K}$ denote its algebraic closure, and construct polynomial representations $\overline{V}$ and $\overline{W}$ of $\mathrm{GL}_n(\overline{K})$ over $\overline{K}$ by using the same representing polynomials as V and W. That is, we set $\rho_{\overline{V}}^{(i,j)} = \rho_V^{(i,j)}$ and $\rho_{\overline{W}}^{(i,j)} = \rho_W^{(i,j)}$. (This indeed defines a representation, as can be seen similarly to the argument in Remark 5.8: the polynomial equalities which imply compatibility with the group multiplication hold when evaluated on any invertible matrix over K, and so are genuine equalities of polynomials.)

It suffices to show that $\rho_{\overline{V}}$ and $\rho_{\overline{W}}$ agree on $\mathrm{SL}_n(\overline{K})$, for then we can apply the algebraically closed case above to find that $r - s = mn$ and $\rho_V^{(i,j)} = \rho_{\overline{V}}^{(i,j)} = (\det)^m \rho_{\overline{W}}^{(i,j)} = (\det)^m \rho_W^{(i,j)}$. By Lemma 5.9(i), it suffices to show that $\rho_{\overline{V}}(g)_{i,j} = \rho_{\overline{W}}(g)_{i,j}$ for all elementary transvections g, for all $1 \leqslant i, j \leqslant d$. Fix $1 \leqslant a \neq b \leqslant n$ and consider elementary transvections whose unique off-diagonal entries are in position (a, b). Define univariate polynomials $f_V, f_W \in K[y]$ by specialising the representing polynomials $\rho_V^{(i,j)}$ and $\rho_W^{(i,j)}$ at $x_{a,b} = y$, $x_{c,c} = 1$ for $1 \leqslant c \leqslant n$, and all other variables equal to 0. Writing $g^{(\alpha)}$ for the elementary transvection having $g_{a,b} = \alpha$ for $\alpha \in \overline{K}$, we therefore have

$$f_V(\alpha) = \rho_{\overline{V}}(g^{(\alpha)})_{i,j}$$

and likewise for W. But $\rho_{\overline{V}}$ and $\rho_{\overline{W}}$ agree on elements of $\mathrm{SL}_n(K)$, so $f_V(\alpha) = f_W(\alpha)$ for all $\alpha \in K$. Since K is infinite, this implies $f_V = f_W$, and so $\rho_{\overline{V}}(g^{(\alpha)})_{i,j} = \rho_{\overline{W}}(g^{(\alpha)})_{i,j}$ for all $\alpha \in \overline{K}$ as required. □

Remark 5.11. A proof of Proposition 5.10 which avoids the Nullstellensatz is possible when it is known that $r - s$ is an integer multiple of n. Writing m for this integer, we are required to show $\rho_V = \rho_W \otimes \rho_{\det}^{\otimes m}$. When K is algebraically closed, Lemma 5.9 tells us that $\mathrm{GL}_n(K)$ is generated by $\mathrm{SL}_n(K)$

and the scalar matrices, and since the homomorphisms agree on $\mathrm{SL}_n(K)$ by hypothesis, it suffices to show that $\rho_V(\alpha I_n) = \alpha^{mn}\rho_W(\alpha I_n)$ for all $\alpha \in K^\times$.

Consider specialising the representing polynomials $\rho_V^{(i,j)}$ at $x_{i,i} = y$ for all $1 \leqslant i \leqslant n$ and all other variables at 0, yielding univariate polynomials in $K[y]$. Since each $\rho_V^{(i,j)}$ is homogeneous of degree r, the result is either the zero polynomial or a multiple of y^r, and considering the case $y = 1$ determines that the diagonal representing polynomials specialise to y^r and all others vanish. Thus $\rho_V(\alpha I_n) = \alpha^r I_d$ where d is the common dimension of V and W, and likewise $\rho_W(\alpha I_n) = \alpha^s I_d$. The requirement follows.

Separately, we can completely classify the 1-dimensional polynomial representations using the following group-theoretic fact.

Lemma 5.12. *Unless $n = 2$ and $K = \mathbb{F}_2$, the general linear group $\mathrm{GL}_n(K)$ has derived subgroup $\mathrm{SL}_n(K)$ and abelianisation $K^\times$.*

Proof. The quotient ${\mathrm{GL}_n(K)}/{\mathrm{SL}_n(K)} \cong K^\times$ is abelian, and so the derived subgroup of $\mathrm{GL}_n(K)$ is contained in $\mathrm{SL}_n(K)$. On the other hand, $\mathrm{SL}_n(K)$ is its own derived subgroup except when $n = 2$ and $K = \mathbb{F}_2$ ([Lan02, Chapter XIII, Theorems 8.3 and 9.2]), and hence the derived subgroup of $\mathrm{GL}_n(K)$ contains $\mathrm{SL}_n(K)$. □

Remark 5.13. When $n = 2$ and $K = \mathbb{F}_2$, we have $\mathrm{GL}_2(\mathbb{F}_2) \cong \mathrm{SL}_2(\mathbb{F}_2) \cong S_3$. Thus the derived subgroup is isomorphic to $A_3 \ncong \mathrm{SL}_2(\mathbb{F}_2)$, and the abelianisation is isomorphic to $C_2 \ncong \mathbb{F}_2^\times$.

Proposition 5.14. *Let V be a 1-dimensional representation of $\mathrm{GL}_n(K)$, and suppose that either V is polynomial or K is finite. Then V is isomorphic to a non-negative power of the determinant representation (and in particular $V|_{\mathrm{SL}_n(K)}$ is the trivial representation).*

Proof. Let $\rho\colon \mathrm{GL}_n(K) \to K^\times$ be a one-dimensional representation of $\mathrm{GL}_n(K)$. Suppose first that it is not the case that $n = 2$ and $K = \mathbb{F}_2$; then the derived subgroup of $\mathrm{GL}_n(K)$ is $\mathrm{SL}_n(K)$. By the universal property of the abelianisation, ρ factors through the surjection $\det\colon \mathrm{GL}_n(K) \twoheadrightarrow K^\times$; let $\varphi\colon K^\times \to K^\times$ be the map such that $\varphi \det = \rho$. It then suffices to show that

$\varphi\colon K^\times \to K^\times$ is a non-negative integer power map. If K is finite, then $K^\times$ is a finite cyclic group and this is clear. If V is polynomial, then this follows from the requirement that $\rho^{(1,1)}$ is a polynomial.

For the case $n = 2$ and $K = \mathbb{F}_2$, observe that the determinant representation is trivial, and that $\mathrm{GL}_2(\mathbb{F}_2) \cong \mathrm{SL}_2(\mathbb{F}_2) \cong S_3$ so in characteristic 2 the trivial representation is the unique 1-dimensional representation. $\square$

Remark 5.15. Proposition 5.14 also holds, by the same proof, for representations of finite groups in non-defining characteristic, with the exception of the case $n = 2$ and $K = \mathbb{F}_2$. In this case we have $\mathrm{GL}_2(\mathbb{F}_2) \cong \mathrm{SL}_2(\mathbb{F}_2) \cong S_3$ and the determinant representation is the trivial representation, but the 1-dimensional sign representation exists in characteristics other than 2.

6. The Schur functor and its inverses

The Schur functor and its one-sided inverses provide an important connection between the representation theory of the general linear groups and of the symmetric groups. In this section we define the Schur functor $\mathcal{F}$ as a weight space; show it is isomorphic to a hom functor and hence define its left-adjoint $\mathcal{G}_\otimes$; and finally use duality to deduce $\mathcal{F}$ has a right-adjoint $\mathcal{G}_{\mathrm{Hom}}$ and hence is isomorphic to a tensor product.

The approach here differs from the usual description of the Schur functor in the language of the Schur algebra (see [EGS08, Section 6]), where the definition as a weight space can be seen to be equivalent to multiplication by a certain idempotent. Our approach avoids the machinery of coalgebras, and opens the possibility of considering these functors over finite fields.

6.1. Weight spaces and the Schur functor

Definition 6.1 (Weight space)**.** Suppose K is infinite. Let V be a representation of $\mathrm{GL}_n(K)$. Let ν be a composition with at most n nonzero parts. The *ν-weight space* of V is the subspace

$$V_\nu = \left\{ v \in V \;\middle|\; \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} v = \alpha_1^{\nu_1} \cdots \alpha_n^{\nu_n} v \text{ for all } \alpha_1, \ldots, \alpha_n \in K^\times \right\}.$$

A composition ν such that $V_\nu \neq 0$ is called a *weight* of V. A nonzero element $v \in V_\nu$ is called a *weight vector* with weight ν.

The assumption that K is infinite is made so that there exist field elements of arbitrarily large order, and hence so that weights are well-defined (that is, so that weight vectors have a unique weight).

Example 6.2. The canonical basis of the natural representation E of $\mathrm{GL}_n(K)$ consists of weight vectors: the basis element X_i has weight $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 occurs in the ith position.

The polytabloid basis for the dual Weyl module $\nabla^\lambda E$ consists of weight vectors: the polytabloid $\mathrm{e}(t)$ has weight the weight of t (that is, the multiset of entries of t expressed as a composition, as defined in §1.1). Likewise $\mathsf{Tbx}^\lambda E$, $\mathrm{Sym}^\lambda E$, $\mathrm{Sym}_\lambda E$, $\bigwedge^\lambda E$ and $\Delta^\lambda E$ have weight vector bases labelled by tableaux, with the weights of the vectors being the weight of the tableaux. This explains the terminology for the multiset of entries of t.

For $n \geqslant r$, we write the permutation matrix corresponding to $\sigma \in S_r$ as $g_\sigma \in \mathrm{GL}_n(K)$, defined by

$$(g_\sigma)_{i,j} = \mathbb{1}[i\sigma = j] \tag{6.3}$$

(where we view $i\sigma = i$ for $i > r$). A simple calculation shows this definition satisfies $g_{\sigma\tau} = g_\sigma g_\tau$ for $\sigma, \tau \in S_r$; that is, $\sigma \mapsto g_\sigma$ is a group homomorphism. Let $\tilde{S}_r \leqslant \mathrm{GL}_n(K)$ denote this subgroup of permutation matrices (those which fix $r+1, \ldots, n$). Of course, $\tilde{S}_r \cong S_r$.

Lemma 6.4. *Suppose K is infinite and $n \geqslant r$. Let V be a left $K\mathrm{GL}_n(K)$-module. The weight space $V_{(1^r,0^{n-r})}$ is invariant under the action of $\tilde{S}_r$, and therefore becomes a left KS_r-module.*

Proof. Let $v \in V_{(1^r,0^{n-r})}$ and let $\sigma \in S_r$; we are required to show that $g_\sigma v \in V_{(1^r,0^{n-r})}$. Let

$$h = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$$

and observe that

$$\begin{aligned}(g_\sigma^{-1} h g_\sigma)_{i,j} &= \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \mathbb{1}[i\sigma^{-1} = k]\mathbb{1}[k = l]\mathbb{1}[l\sigma = j] \\ &= \alpha_{i\sigma^{-1}} \mathbb{1}[i = j].\end{aligned}$$

Thus

$$h g_\sigma v = g_\sigma \begin{pmatrix} \alpha_{1\sigma^{-1}} & & \\ & \ddots & \\ & & \alpha_{n\sigma^{-1}} \end{pmatrix} v = \alpha_1 \cdots \alpha_r g_\sigma v$$

so $g_\sigma v \in V_{(1^r,0^{n-r})}$ as required. □

Definition 6.5 (Schur functor)**.** Suppose K is infinite and $n \geqslant r$. The *Schur functor* $\mathcal{F}$ is the functor from the category of polynomial left $K\mathrm{GL}_n(K)$-modules of degree r to the category of left KS_r-modules defined by

$$\mathcal{F}(V) = V_{(1^r,\, 0^{n-r})}$$

on modules, and by restriction on maps.

Note that the action of $\mathcal{F}$ on maps is indeed well-defined: if $\varphi\colon V \to V'$ is a $\mathrm{GL}_n(K)$-equivariant map and diagonal matrices act by certain scalars on $v \in V$, then diagonal matrices act by the same scalars on $\varphi(v) \in V'$; thus restriction to a weight space in the domain permits restriction to the same weight space in the codomain.

We remark that the definition of the Schur functor makes sense on any polynomial representation of $\mathrm{GL}_n(K)$ (or indeed any representation of $\mathrm{GL}_n(K)$ when K is infinite). However, if V is a polynomial representation of $\mathrm{GL}_n(K)$ of degree $r' \neq r$, then $\mathcal{F}(V) = 0$. This is because if there exists a nonzero $v \in V_{(1^r,0^{n-r})}$, then choosing a basis of V whose first element is v we have that the representing polynomial $\rho^{(1,1)}$ has the monomial $x_{1,1} \cdots x_{r,r}$ of degree r as a summand.

The following identification of the image of the dual Weyl module under the Schur functor $\mathcal{F}$ is well-known. (On the other hand, the image of the Specht module under the one-sided inverse Schur functors defined in the following subsections is harder to determine, and is the subject of Chapter IV.)

Proposition 6.6 (cf. [EGS08, §6.3])**.** *Suppose K is infinite and $n \geqslant r$. Let λ be a partition of r. There is an isomorphism $\mathcal{F}(\nabla^\lambda E) \cong S^\lambda$.*

Proof. As noted in Example 6.2, the polytabloid basis for $\nabla^\lambda E$ is a basis of weight vectors, with $\mathrm{e}(t)$ having weight the weight of t. Thus $\mathcal{F}(\nabla^\lambda E)$ is spanned by polytabloids labelled by tableaux of weight $(1^r, 0^{n-r})$, which are the tableaux of symmetric type with entries in $\{X_1, \ldots, X_r\}$ (an r-subset of the basis for E). The isomorphism to S^λ is given by the obvious map sending such a polytabloid to the polytabloid of symmetric type with entries in the basis for the natural permutation representation W (also an r-set). Indeed this map respects the S_r-action, as permutation matrices act on the basis elements $X_1, \ldots, X_r$ of E precisely as permutations act on the basis of the natural permutation representation W. $\square$

6.2. The tensor-hom adjunction and the left-adjoint inverse Schur functor

We show that the Schur functor can be viewed as a hom functor. This demonstrates that $\mathcal{F}$ is left exact, and furthermore allows us to identify via the tensor-hom adjunction a left-adjoint to $\mathcal{F}$, which turns out to also be a right-inverse.

We first recall how a bimodule gives rise to hom and tensor functors. Let G_1 and G_2 be groups, and suppose V is a left KG_1-module, U is a left KG_2-module, and M is a (KG_1, KG_2)-bimodule. Then the hom space $\mathrm{Hom}_{KG_1}(M, V)$ is a left KG_2-module, with action on $f \in \mathrm{Hom}_{KG_1}(M, V)$ defined by $(g_2 f)(m) = f(mg_2)$ for all $g_2 \in G_2$, $m \in M$. Meanwhile the tensor product $M \otimes_{KG_2} U$ is a left KG_1-module, with action on $m \otimes u \in M \otimes_{KG_2} U$ defined by $g_1(m \otimes u) = (g_1 m) \otimes u$ for all $g_1 \in G_1$. Moreover, the functor $\mathrm{Hom}_{KG_1}(M, -)\colon KG_1\text{-}\mathsf{mod} \to KG_2\text{-}\mathsf{mod}$ is right adjoint to the functor $M \otimes -\colon KG_2\text{-}\mathsf{mod} \to KG_1\text{-}\mathsf{mod}$ (where $\mathcal{A}\text{-}\mathsf{mod}$ denotes the category of left modules of an algebra $\mathcal{A}$); that is, there are isomorphisms of abelian groups

$$\mathrm{Hom}_{KG_1}(M \otimes_{KG_2} U, V) \cong \mathrm{Hom}_{KG_2}(U, \mathrm{Hom}_{KG_1}(M, V))$$

which are natural in V and U.

In our setting the bimodule we use is the $(K\mathrm{GL}_n(K), KS_r)$-bimodule $E^{\otimes r}$, the rth tensor power of the natural representation of $\mathrm{GL}_n(K)$. The left $\mathrm{GL}_n(K)$-action is the usual diagonal action on tensor products of representations of groups. The right S_r-action is the place permutation action which permutes the tensor factors: given $X_{i_1} \otimes \cdots \otimes X_{i_r} \in E^{\otimes r}$ and $\sigma \in S_r$, we have

$$(X_{i_1} \otimes \cdots \otimes X_{i_r}) \cdot \sigma = X_{i_{1\sigma^{-1}}} \otimes \cdots \otimes X_{i_{r\sigma^{-1}}}.$$

When $n \geqslant r$, the element $X_1 \otimes \cdots \otimes X_r \in E^{\otimes r}$ generates $E^{\otimes r}$ as a $K\mathrm{GL}_n(K)$-module (for any field K); write $X_{[r]}$ for this element. Note that the action of S_r on $X_{[r]}$ can be written in terms of the action of $\mathrm{GL}_n(K)$ via the subgroup of permutations matrices:

$$X_{[r]} \cdot \sigma = g_\sigma X_{[r]} \tag{6.7}$$

for any $\sigma \in S_r$ (beware this behaviour applies only to the generator $X_{[r]}$, and in particular not to $X_{[r]} \cdot \tau$ for $\tau \in S_r$, so there is no expectation for $X_{[r]} \cdot \tau\sigma$ to be equal to $g_\sigma g_\tau X_{[r]}$; rather we have $X_{[r]} \cdot \tau\sigma = g_{\tau\sigma} X_{[r]} = g_\tau g_\sigma X_{[r]}$).

Lemma 6.8. *Suppose K is infinite and $n \geqslant r$. Let V be a polynomial representation of $\mathrm{GL}_n(K)$ of degree r, and let $v \in V_{(1^r,0^{n-r})}$. Then there exists a unique $\mathrm{GL}_n(K)$-equivariant map $E^{\otimes r} \to V$ sending $X_{[r]} \mapsto v$.*

Proof. The element $X_{[r]} \in E^{\otimes r}$ generates $E^{\otimes r}$ as a $K\mathrm{GL}_n(K)$-module, so the image of $X_{[r]}$ determines a $\mathrm{GL}_n(K)$-equivariant map. Uniqueness is therefore clear.

For existence, we are required to show that if $\gamma \in K\mathrm{GL}_n(K)$ is such that $\gamma X_{[r]} = 0$, then $\gamma v = 0$. Suppose $\gamma = \sum_{l \in L} \gamma_l g^{(l)} \in K\mathrm{GL}_n(K)$ is such that $\gamma X_{[r]} = 0$, where L is some finite indexing set, $\gamma_l \in K$ and $g^{(l)} \in \mathrm{GL}_n(K)$. Observe that the coefficient of $X_{i_1} \otimes \cdots \otimes X_{i_r}$ in $\gamma X_{[r]}$ is $\sum_{l \in L} \gamma_l g^{(l)}_{i_1,1} \cdots g^{(l)}_{i_r,r}$. Since this coefficient is zero, we have that $\sum_{l \in L} \gamma_l g^{(l)}_{i_1,1} \cdots g^{(l)}_{i_r,r} = 0$ for every choice of $1 \leqslant i_1, \ldots, i_r \leqslant n$.

Let $d = \dim V$, write $v = v_1$, and extend to a basis $\{v_1, \ldots, v_d\}$ of V. We consider the representing polynomials $\rho^{(i,j)}$ of V with respect to this basis. Recall we write representing polynomials in the n^2 variables $x_{i,j}$ corresponding to the (i,j)th coordinate of a matrix. Fixing $i \in [n]$, we aim to show that the polynomials $\rho^{(i,1)}$ (those corresponding to the action on v) are sums of monomials of the form $x_{i_1,1} \cdots x_{i_r,r}$; by the previous paragraph, each of these monomials vanishes when applied to γ, giving the requirement that $\gamma v = 0$. This approach is illustrated in Example 6.9 below.

Given that $\rho^{(i,1)}$ is of degree r, any monomial not of the required form has some $m \in [r]$ which does not appear as the second label in any of its factors. Thus to prove the lemma it suffices to show, given a set of r variables $\{x_{i_1,j_1}, \ldots, x_{i_r,j_r}\}$ such that there exists $m \in [r]$ with $m \notin \{j_1, \ldots, j_r\}$, that any monomial in these r variables has zero coefficient in $\rho^{(i,1)}$ (when written with respect to the monomial basis). Let $\hat\rho^{(i,1)}$ be the linear combination in $\rho^{(i,1)}$ of the monomials in the r variables in consideration; we show that $\hat{\rho}^{(i,1)} = 0$.

Pick elements $\alpha_1, \ldots, \alpha_r \in K$, and let g be the (singular, having at most r nonzero entries) matrix defined by

$$g_{i,j} = \begin{cases} \alpha_a & \text{if } (i,j) = (i_a, j_a) \text{ for } a \in [r], \\ 0 & \text{if } (i,j) \notin \{(i_1, j_1), \ldots, (i_r, j_r)\}. \end{cases}$$

Any monomial including a variable not in our chosen set $\{x_{i_1,j_1}, \ldots, x_{i_r,j_r}\}$ vanishes on g, and hence

$$\rho(g)_{i,1} = \hat{\rho}^{(i,1)}(g_{1,1}, g_{1,2}, \ldots, g_{n,n}).$$

Moreover, $\hat{\rho}^{(i,1)}$ can be viewed as a polynomial in just the r chosen variables $x_{i_1,j_1}, \ldots, x_{i_r,j_r}$; suppressing the other variables, we obtain

$$\rho(g)_{i,1} = \hat{\rho}^{(i,1)}(\alpha_1, \ldots, \alpha_r). \tag{6.8.1}$$

Let h be the (singular) diagonal matrix h whose diagonal entries are all 1 except for $h_{m,m} = 0$. Observe that

$$\begin{aligned} (gh)_{i,j} &= \sum_{k=1}^{n} g_{i,k} h_{k,j} \\ &= \sum_{a=1}^{r} \alpha_a \mathbb{1}[(i,j) = (i_a, j_a)] \mathbb{1}[j \neq m] \\ &= g_{i,j} \end{aligned}$$

where the last equality holds by the property that $m \notin \{j_1, \ldots, j_r\}$. Thus $gh = g$.

Recall that the action of $\mathrm{GL}_n(K)$ on V extends to an action of $\mathrm{Mat}_n(K)$ (see Remark 5.8). Since $v \in V_{(1^r, 0^{n-r})}$ and h is a diagonal matrix with a 0 in one of the first r diagonal entries, we have $hv = 0$. Thus $gv = ghv = 0$, and hence $\rho(g)_{i,1} = 0$. By (6.8.1), this says $\hat{\rho}^{(i,1)}(\alpha_1, \ldots, \alpha_r) = 0$. Since this holds for any choice of $\alpha_1, \ldots, \alpha_r \in K$, we have $\hat{\rho}^{(i,1)} = 0$ as required. $\square$

Example 6.9. Suppose $n = r = 2$, and let $V = \mathrm{Sym}_2 E$, a polynomial representation of $\mathrm{GL}_2(K)$ of degree 2. Write $\{X, Y\}$ for the canonical basis of E, so $X_{[r]} = X \otimes Y$. The weight space $(\mathrm{Sym}_2 E)_{(1^2)}$ is spanned by the element $X \otimes Y + Y \otimes X$. We verify that the K-linear map sending $X \otimes Y \mapsto X \otimes Y + Y \otimes X$ is well-defined by considering the representing polynomials, as in the proof of Lemma 6.8.

Extending the basis of the weight space, let $\{X \otimes Y + Y \otimes X, X \otimes X, Y \otimes Y\}$ be our basis of $\mathrm{Sym}_2 E$. We can explicitly compute the representing polynomials with respect to this basis:

$$\rho_{\mathrm{Sym}_2 E}\left(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}\right) = \begin{pmatrix} x_{1,1}x_{2,2} + x_{1,2}x_{2,1} & x_{1,1}x_{2,1} & x_{1,2}x_{2,2} \\ 2x_{1,1}x_{1,2} & x_{1,1}^2 & x_{1,2}^2 \\ 2x_{2,1}x_{1,2} & x_{2,1}^2 & x_{2,2}^2 \end{pmatrix}.$$

The polynomials in the first column (those representing the action on $X \otimes Y + Y \otimes X$) are linear combinations of monomials of the form $x_{i_1,1}x_{i_2,2}$ for some i_1, i_2. To illustrate how this fact was proved in Lemma 6.8, suppose, for example, that the monomial $x_{1,1}x_{2,1}$ occurred in the first column. Then given $\alpha_1, \alpha_2 \in K^\times$ the matrix $\left(\begin{smallmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{smallmatrix}\right)$ would have nonzero action on $X \otimes Y + Y \otimes X$, contradicting that $X \otimes Y + Y \otimes X$ lies in the (1^2)-weight space.

In order for the map $X \otimes Y \mapsto X \otimes Y + Y \otimes X$ to be well-defined, we require that if $\gamma \in K\mathrm{GL}_2(K)$ is such that $\gamma(X \otimes Y) = 0$, then $\gamma(X \otimes Y + Y \otimes X) = 0$. For concreteness, consider the element

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in K\mathrm{GL}_2(K).$$

The coefficient of, for example, $X \otimes X$ in $\gamma(X \otimes Y)$ is given by evaluating the monomial $x_{1,1}x_{1,2}$ on γ: the coefficient is $1 + 0 + 0 - 1 = 0$, as this monomial vanishes on γ. Moreover, $\gamma(X \otimes Y) = 0$, precisely because every monomial of the form $x_{i_1,1}x_{i_2,2}$ vanishes on γ. Since the polynomials representing the action on $X \otimes Y + Y \otimes X$ are linear combinations of monomials of this form, we have $\gamma(X \otimes Y + Y \otimes X) = 0$ as required.

Proposition 6.10. *Suppose K is infinite and $n \geqslant r$. There is an isomorphism of functors*

$$\mathcal{F}(-) \cong \mathrm{Hom}_{K\mathrm{GL}_n(K)}(E^{\otimes r}, -)$$

on the category of polynomial representations of $K\mathrm{GL}_n(K)$ of degree r.

PROOF. Given a $K\mathrm{GL}_n(K)$-module V and an element $v \in V_{(1^r,0^{n-r})}$, let $\eta_V(v)\colon E^{\otimes r} \to V$ be the unique $\mathrm{GL}_n(K)$-equivariant map sending $X_{[r]} \mapsto v$, whose existence is verified in Lemma 6.8. This makes $\eta_V\colon \mathcal{F}(V) \to \mathrm{Hom}_{K\mathrm{GL}_n(K)}(E^{\otimes r}, V)$ a K-linear map. This map is clearly injective; it is surjective because given any $\mathrm{GL}_n(K)$-equivariant map $\chi\colon E^{\otimes r} \to V$, diagonal matrices act on the element $\chi(X_{[r]})$ as they do on $X_{[r]}$, so $\chi(X_{[r]}) \in V_{(1^r,0^{n-r})}$ and $\chi = \eta_V(\chi(X_{[r]}))$.

We claim that the map η_V respects the action of S_r. We have

$$\eta_V(\sigma v)(X_{[r]}) = \sigma v = g_\sigma v$$

by the definition of the action of S_r on the weight space $\mathcal{F}(V)$. Meanwhile by definition of the action of S_r on the hom space, we have

$$(\sigma\eta_V(v))(X_{[r]}) = \eta_V(v)(X_{[r]}\cdot\sigma) = \eta_V(v)(g_\sigma X_{[r]}) = g_\sigma v$$

as required.

It remains only to observe that the following diagram commutes, where V, V' are $K\mathrm{GL}_n(K)$-modules and $\varphi\colon V \to V'$ is a $\mathrm{GL}_n(K)$-equivariant map.

$$\begin{array}{ccc}
\mathcal{F}(V) & \xrightarrow{\eta_V} & \mathrm{Hom}_{K\mathrm{GL}_n(K)}(E^{\otimes r}, V) \\
\Big\downarrow{\scriptstyle \mathcal{F}(\varphi)} & & \Big\downarrow{\scriptstyle \varphi\circ -} \\
\mathcal{F}(V') & \xrightarrow{\eta_{V'}} & \mathrm{Hom}_{K\mathrm{GL}_n(K)}(E^{\otimes r}, V')
\end{array}$$

Indeed the image of an element $v \in \mathcal{F}(V)$ under either composition in the diagram is the map sending $X_{[r]} \mapsto \varphi(v)$. □

Remark 6.11. Proposition 6.10 claims an isomorphism between functors on the category of polynomial representations of degree r. As observed below Definition 6.5, the definition of $\mathcal{F}$ extends to all polynomial representations, but $\mathcal{F}$ vanishes on polynomial representations of degree $r' \neq r$. Likewise the functor $\mathrm{Hom}_{K\mathrm{GL}_n(K)}(E^{\otimes r}, -)$ vanishes on polynomial representations of degree $r' \neq r$: the module $E^{\otimes r}$ is polynomial of degree r, so if there exists a nonzero $\mathrm{GL}_n(K)$-equivariant map $f\colon E^{\otimes r} \to V$, then $\operatorname{im} f \leqslant V$ is also polynomial of degree r.

Definition 6.12 (Left-adjoint inverse Schur functor)**.** The *left-adjoint inverse Schur functor* $\mathcal{G}_\otimes^n$ is the functor from the category of left KS_r-modules to the category of left $K\mathrm{GL}_n(K)$-modules defined by

$$\mathcal{G}_\otimes^n(-) = E^{\otimes r} \otimes_{KS_r} -.$$

We suppress the dependence of $\mathcal{G}_\otimes^n$ on n except where there is need to emphasise it. Note that K infinite and $n \geqslant r$ are *not* required here.

The functor $\mathcal{G}_\otimes$ is left-adjoint to $\mathcal{F}$ by the tensor-hom adjunction. We see next that the functor $\mathcal{G}_\otimes$ is right-inverse to $\mathcal{F}$ (though it is not in general a left-inverse).

Proposition 6.13. *Suppose K is infinite. The image $\mathcal{G}_\otimes(U)$ of a KS_r-module U is polynomial of degree r.*

Proof. Let U be a KS_r module, suppose $d = \dim U$, and choose a basis $u_1, \ldots, u_d$ of U. Define a map by

$$\begin{aligned}(E^{\otimes r})^{\oplus d} &\to \mathcal{G}_\otimes(U)\\ (x_1, \ldots, x_d) &\mapsto x_1 \otimes u_1 + \ldots + x_d \otimes u_d\end{aligned}$$

for $x_i \in E^{\otimes r}$. This is a surjective map of $K\mathrm{GL}_n(K)$-modules, so $\mathcal{G}_\otimes(U)$ is a quotient of $(E^{\otimes r})^{\oplus d}$. Since $(E^{\otimes r})^{\oplus d}$ is polynomial of degree r (by Proposition 5.6(ii),(iii)), so too is $\mathcal{G}_\otimes(U)$ (by Proposition 5.6(i)). $\square$

Proposition 6.14 (cf. [EGS08, (6.2d)])**.** *Suppose K is infinite and $n \geqslant r$. The functor $\mathcal{G}_\otimes$ is right-inverse to the functor $\mathcal{F}$ (that is, there is a natural isomorphism $\mathcal{F}\mathcal{G}_\otimes \cong \mathrm{id}$ of functors on the category of KS_r-modules).*

Proof. Let $\varepsilon_U \colon U \to \mathcal{F}\mathcal{G}_\otimes(U)$ be the map defined by

$$\begin{aligned}\varepsilon_U \colon U &\to (E^{\otimes r} \otimes_{KS_r} U)_{(1^r,0^{n-r})}\\ u &\mapsto X_{[r]} \otimes u\end{aligned}$$

for $u \in U$. Indeed the image lies in the required weight space because $X_{[r]} \in (E^{\otimes r})_{(1^r,0^{n-r})}$, and a matrix which acts like a scalar on $X_{[r]}$ has the same action on $X_{[r]} \otimes u$. It is clear that ε_U is K-linear and S_r-equivariant.

To see that ε_U is injective, consider the S_r-balanced K-bilinear form defined by K-linear extension of

$$\begin{aligned} E^{\otimes r} \times U &\to U \\ (X_{i_1} \otimes \cdots \otimes X_{i_r}, u) &\mapsto \begin{cases} \sigma u & \text{if } (i_1, \ldots, i_r) = (1, \ldots, r) \cdot \sigma, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since this form is nonzero on $(X_{[r]}, u)$ for $u \neq 0$, the universal property of the tensor product requires that $X_{[r]} \otimes u \neq 0$ for $u \neq 0$.

We next show that ε_U is also surjective. Let $v \in (E^{\otimes r} \otimes_{KS_r} U)_{(1^r, 0^{n-r})}$, and we are required to show that v can be written in the form $X_{[r]} \otimes u$ for some $u \in U$. Using the action of S_r to order the left-hand tensor factors, it is clear that we can write $v = \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant n} X_{i_1} \otimes \cdots \otimes X_{i_r} \otimes u_{i_1, \ldots, i_r}$ for some elements $u_{i_1, \ldots, i_r} \in U$. We claim that the only nonzero summand is $X_{[r]} \otimes u_{1, \ldots, r}$.

We first use the universal property of the tensor product to show that the nonzero summands of v are K-linearly independent. For each tuple $(i_1, \ldots, i_r)$ such that $X_{i_1} \otimes \cdots \otimes X_{i_r} \otimes u_{i_1, \ldots, i_r} \neq 0$, there exists some abelian group A and some S_r-balanced K-bilinear form $\langle -, - \rangle \colon E^{\otimes r} \times U \to A$ such that $\langle X_{i_1} \otimes \cdots \otimes X_{i_r}, u_{i_1, \ldots, i_r} \rangle \neq 0$; define a new S_r-balanced K-bilinear form $\langle -, - \rangle' \colon E^{\otimes r} \times U \to A$ by

$$\langle X_{j_1} \otimes \cdots \otimes X_{j_r}, u \rangle' = \begin{cases} \langle X_{i_1} \otimes \cdots \otimes X_{i_r}, u \rangle & \text{if } \{i_1, \ldots, i_r\} = \{j_1, \ldots, j_r\}, \\ 0 & \text{otherwise,} \end{cases}$$

extended K-linearly. By evaluating the induced map $E^{\otimes r} \otimes_{KS_r} U \to A$ of abelian groups on any linear combination of the nonzero summands of v, we see that the linear combination is zero only if the coefficient of $X_{i_1} \otimes \cdots \otimes X_{i_r} \otimes u_{i_1, \ldots, i_r}$ is zero.

To recognise the nonzero summands of v we consider the action of diagonal matrices. Pick $m \in [r]$ and $\alpha \in K$ with $\alpha \notin \{0, 1\}$, and let $g \in \mathrm{GL}_n(K)$ be the diagonal matrix with $g_{m,m} = \alpha$ and $g_{i,i} = 1$ for all other $i \in [n]$. Then

$$gv = \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant n} \alpha^{|\{\, a \in [r] \,|\, i_a = m \,\}|} X_{i_1} \otimes \cdots \otimes X_{i_r} \otimes u_{i_1, \ldots, i_r}.$$

Since v is in the $(1^r, 0^{n-r})$-weight space, we have also $gv = \alpha v$. Thus by K-linear independence of the summands, we have that if $X_{i_1} \otimes \cdots \otimes X_{i_r} \otimes u_{i_1,\ldots,i_r} \neq 0$ then $\alpha = \alpha^{|\{\, a \in [r] \mid i_a = m \,\}|}$, and hence $|\{\, a \in [r] \mid i_a = m \,\}| \geqslant 1$. Since this holds for all $m \in [r]$, we have that the only nonzero summand of v is $X_{[r]} \otimes u_{1,\ldots,r}$, as required.

Finally we need only observe that the following diagram commutes, where U, U' are KS_r-modules and $\psi\colon U \to U'$ is a S_r-equivariant map.

$$\begin{tikzcd} U \arrow[r, "\varepsilon_U"] \arrow[d, "\psi"] & \mathcal{F}\mathcal{G}_{\otimes}(U) \arrow[d, "\mathrm{id} \otimes \psi"] \\ U' \arrow[r, "\varepsilon_{U'}"] & \mathcal{F}\mathcal{G}_{\otimes}(U') \end{tikzcd}$$

Indeed the image of an element $u \in U$ under either composition in the diagram is the element $X_{[r]} \otimes \psi(u)$. $\square$

Remark 6.15. By Proposition 6.10, the functor $\mathcal{G}_{\otimes}$ is, of course, also right-inverse to the functor $\mathrm{Hom}_{K\mathrm{GL}_n(K)}(E^{\otimes r}, -)$. Provided $n \geqslant r$, this isomorphism of functors can in fact be shown for any field $K \neq \mathbb{F}_2$, not necessarily infinite (whereas the definition of $\mathcal{F}$ requires K to be infinite). The natural isomorphism of KS_r-modules

$$U \cong \mathrm{Hom}_{K\mathrm{GL}_n(K)}(E^{\otimes r}, E^{\otimes r} \otimes_{KS_r} U)$$

is given by sending an element $u \in U$ to the map $- \otimes u$ (that is, the map determined by $X_{[r]} \mapsto X_{[r]} \otimes u$). This is easily shown to be S_r-equivariant by writing the action of S_r in terms of permutation matrices in $\mathrm{GL}_n(K)$. The proofs of injectivity and surjectivity require essentially the same arguments as the proof of Proposition 6.14.

The assumption that $K \neq \mathbb{F}_2$ is essential here, because otherwise the only diagonal matrix is the identity matrix and hence maps which do not preserve powers of the X_i are permitted. For example, take $n = r = 2$ and U the trivial representation of S_2. Then $\mathcal{G}_{\otimes}(U) = E^{\otimes 2} \otimes_{KS_2} U \cong \mathrm{Sym}^2 E$, and so the image of the composition of our functors is $\mathrm{Hom}_{K\mathrm{GL}_2(K)}(E^{\otimes 2}, \mathrm{Sym}^2 E)$. We claim that that this module is 2-dimensional when $K = \mathbb{F}_2$, and in particular not isomorphic to the 1-dimensional module U. Indeed, writing

$X = X_1$ and $Y = X_2$, consider the linear map defined by

$$\begin{aligned} E^{\otimes 2} &\to \operatorname{Sym}^2 E \\ X \otimes X &\mapsto X^2 \\ X \otimes Y &\mapsto X^2 + Y^2 \\ Y \otimes X &\mapsto X^2 + Y^2 \\ Y \otimes Y &\mapsto Y^2 \end{aligned}$$

extended linearly. This map is easily verified to be $\mathrm{GL}_2(\mathbb{F}_2)$-equivariant, and is not a scalar multiple of the canonical quotient map (and all $\mathrm{GL}_2(\mathbb{F}_2)$-equivariant maps are linear combinations of these two maps).

6.3. Duality and the right-adjoint inverse Schur functor

The functor $\mathcal{F}$ also has a right-adjoint right-inverse. Once again the adjunction is a case of the tensor-hom adjunction, this time with $\mathcal{F}$ playing the role of the tensor product. However, we deduce the adjunction by considering the interaction between duality and the functors $\mathcal{F}$ and $\mathcal{G}_\otimes$; we show that a certain hom-functor is right-adjoint to $\mathcal{F}$, and deduce by the uniqueness of adjunctions that $\mathcal{F}$ is isomorphic to a tensor product.

We first define the functor which we later show is right-adjoint and right-inverse to $\mathcal{F}$. To define this functor requires viewing the $(K\mathrm{GL}_n(K), KS_r)$-bimodule $E^{\otimes r}$ instead as a $(KS_r, K\mathrm{GL}_n(K))$-bimodule. We write $E^\top$ for the *right* natural representation of $\mathrm{GL}_n(K)$, with left S_r-action given by place permutation, both denoted $\diamond$. Given $x \in E$, we denote the corresponding element of $E^\top$ by $x^\top$; the right $\mathrm{GL}_n(K)$-action on the natural basis is

$$X_i^\top \diamond g = \sum_{j=1}^{n} g_{i,j} X_j^\top = (g^\top X_i)^\top$$

where $g \in \mathrm{GL}_n(K)$. The left S_r-action is

$$\sigma \diamond (X_{i_1}^\top \otimes \cdots \otimes X_{i_r}^\top) = X_{i_{1\sigma}}^\top \otimes \cdots \otimes X_{i_{r\sigma}}^\top = ((X_{i_1} \otimes \cdots \otimes X_{i_r}) \cdot \sigma^{-1})^\top$$

where $\sigma \in S_r$. The S_r-action on the generator $X_{[r]}^\top$ can be written in terms of the permutation matrices:

$$\sigma \diamond X_{[r]}^\top = X_{[r]}^\top \diamond g_\sigma = (g_\sigma^\top X_{[r]})^\top.$$

Definition 6.16 (Right-adjoint inverse Schur functor)**.** The *right-adjoint inverse Schur functor* $\mathcal{G}_{\mathrm{Hom}}^n$ is the functor from the category of left KS_r-modules to the category of left $K\mathrm{GL}_n(K)$-modules defined by

$$\mathcal{G}_{\mathrm{Hom}}^n(-) = \mathrm{Hom}_{KS_r}((E^\top)^{\otimes r}, -).$$

We suppress the dependence of $\mathcal{G}_{\mathrm{Hom}}^n$ on n except where there is need to emphasise it. Note that K infinite and $n \geqslant r$ are *not* required here.

Proposition 6.17 (cf. [CHN10, §2.2])**.**

(i) There is a natural isomorphism of functors $\mathcal{G}_\otimes(-)^\circ \cong \mathcal{G}_{\mathrm{Hom}}(-^*)$.

(ii) Suppose K is infinite and $n \geqslant r$. Then there is a natural isomorphism of functors $\mathcal{F}(-^\circ) \cong \mathcal{F}(-)^*$.

Proof. [(i)] Given a KS_r-module U, by the tensor-hom adjunction there is a natural isomorphism of abelian groups

$$\begin{array}{ccc} \mathrm{Hom}_K(E^{\otimes r} \otimes_{KS_r} U, K) & \cong & \mathrm{Hom}_{KS_r}\big((E^\top)^{\otimes r}, \mathrm{Hom}_K(U,K)\big) \\ \| & & \| \\ \mathcal{G}_\otimes(U)^\circ & & \mathcal{G}_{\mathrm{Hom}}(U^*) \end{array}$$

which sends a map $\varphi \in \mathrm{Hom}_K(E^{\otimes r} \otimes_{KS_r} U, K)$ to the map sending $x^\top \in (E^\top)^{\otimes r}$ to $\varphi(x \otimes -)$. We claim that this is also $\mathrm{GL}_n(K)$-equivariant. Indeed, acting by an element $g \in \mathrm{GL}_n(K)$ before or after applying the adjunction to a map φ yields the map sending $x^\top \in (E^\top)^{\otimes r}$ to the map $\varphi((g^\top x) \otimes -)$.

[(ii)] Given a $K\mathrm{GL}_n(K)$-module V, note that as K-vector spaces we have $\mathcal{F}(V^\circ) \subseteq \mathrm{Hom}_K(V, K)$ and that $\mathcal{F}(V)^* = \mathrm{Hom}_K(V_{(1^r,0^{n-r})}, K)$. Let $\theta_V : \mathcal{F}(V^\circ) \to \mathcal{F}(V)^*$ be the restriction map (sending a function $f \in \mathcal{F}(V^\circ)$ to the function $f|_{V_{(1^r,0^{n-r})}}$). This is clearly K-linear and natural in V.

A permutation $\sigma \in S_r$ acts on a function $f \in \mathcal{F}(V^\circ)$ by multiplication by the permutation matrix g_σ, which by definition of the contravariant dual is given by precomposing with the transpose matrix:

$$(\sigma f)(v) = (g_\sigma f)v = f(g_\sigma^\top v).$$

Meanwhile σ acts on the function $\theta_V(f) \in \mathcal{F}(V)^*$ by precomposing with the inverse permutation, which acts by multiplication by the corresponding permutation matrix:

$$(\sigma\theta_V(f))(v) = \theta_V(f)(\sigma^{-1}v) = \theta_V(f)(g_\sigma^{-1}v).$$

The transpose of a permutation matrix is its inverse, so θ_V is S_r-equivariant.

To see that θ_V is bijective, choose a basis $v_1, \ldots, v_c$ for $V_{(1^r,0^{n-r})}$, extend to a basis $v_1, \ldots, v_d$ for V, and let $v_1^*, \ldots, v_d^*$ denote the dual basis. With respect to these bases, a diagonal matrix acts on V° exactly as it does on V, and so $v_1^*, \ldots, v_c^*$ is a basis for $(V^\circ)_{(1^r,0^{n-r})}$. It is then clear that θ_V is both injective and surjective. $\square$

This is sufficient to deduce that $\mathcal{G}_{\mathrm{Hom}}$ is right-inverse to $\mathcal{F}$.

Proposition 6.18. *Suppose K is infinite.*

(i) The image $\mathcal{G}_{\mathrm{Hom}}(U)$ of a KS_r-module U is polynomial of degree r.

(ii) Suppose $n \geqslant r$. The functor $\mathcal{G}_{\mathrm{Hom}}$ is right-inverse to $\mathcal{F}$ (that is, there is a natural isomorphism $\mathcal{F}\mathcal{G}_{\mathrm{Hom}} \cong \mathrm{id}$ of functors on the category of KS_r-modules).

Proof. By Proposition 6.17(i), we have $\mathcal{G}_{\mathrm{Hom}}(U) \cong \mathcal{G}_\otimes(U^*)^\circ$; the module $\mathcal{G}_\otimes(U^*)$ is polynomial of degree r by Proposition 6.13, and hence so is its contravariant dual by Proposition 5.6(iv). This proves (i). Part (ii) follows from $\mathcal{G}_\otimes$ being right-inverse to $\mathcal{F}$ (Proposition 6.14) and using both parts of Proposition 6.17. $\square$

We next show that $\mathcal{G}_{\mathrm{Hom}}$ is right-adjoint to $\mathcal{F}$. We require the following lemma.

Lemma 6.19. *Let G be a group and let U, V be KG-modules. There is a natural isomorphisms of abelian groups*

$$\mathrm{Hom}_{KG}(V, U) \cong \mathrm{Hom}_{KG}(U^*, V^*).$$

If G is a matrix group closed under transposition, the same map defines a natural isomorphism when the dual $-^*$ is replaced by the contravariant dual $-^\circ$.

Proof. Choose bases $u_1, \dots, u_c$ and $v_1, \dots, v_d$ for U and V, and let $u_1^*, \dots, u_c^*$ and $v_1^*, \dots, v_d^*$ denote the dual bases for U^* and V^* respectively. Let R be the $d \times c$ matrix representing a G-equivariant map $U \to V$ with respect to the given bases. We claim that the map $U^* \to V^*$ represented by the transpose $R^\top$ with respect to the given bases is G-equivariant. Indeed, we have by assumption that $R\rho_U(g) = \rho_V(g)R$ for all elements $g \in G$; taking transposes and inverting g, we have $\rho_U(g^{-1})^\top R^\top = R^\top \rho_V(g^{-1})^\top$ for all $g \in G$. Since $\rho_{U^*}(g) = \rho_U(g^{-1})^\top$ and likewise for V, this is the requirement that the map represented by $R^\top$ is G-equivariant.

Matrix transposition thus yields the required isomorphism of abelian groups, and naturality is easily verified. When G is a matrix group, the same argument with $g^\top$ occurring in place of g^{-1} establishes the statement for contravariant duals. $\square$

Proposition 6.20. *Suppose K is infinite and $n \geqslant r$.*

(i) The functor $\mathcal{G}_{\mathrm{Hom}}$ is right-adjoint to $\mathcal{F}$.

(ii) There is a natural isomorphism of functors

$$\mathcal{F}(-) \cong (E^\top)^{\otimes r} \otimes_{K\mathrm{GL}_n(K)} -.$$

Proof. The functor $\mathcal{G}_{\mathrm{Hom}}$ is right-adjoint to $(E^\top)^{\otimes r} \otimes_{K\mathrm{GL}_n(K)} -$ by the tensor-hom adjunction, so part (ii) follows from (i) by uniqueness of adjunctions.

For part (i), let U be a KS_r-module and let V be a $K\mathrm{GL}_n(K)$-module. Consider the following chain of natural isomorphisms of abelian groups:

$$\begin{aligned}
&\mathrm{Hom}_{K\mathrm{GL}_n(K)}(V, \mathcal{G}_{\mathrm{Hom}}(U)) \\
&\qquad \cong \mathrm{Hom}_{K\mathrm{GL}_n(K)}(V, \mathcal{G}_\otimes(U^*)^\circ) && \text{(by Proposition 6.17(i))} \\
&\qquad \cong \mathrm{Hom}_{K\mathrm{GL}_n(K)}(\mathcal{G}_\otimes(U^*), V^\circ) && \text{(by Lemma 6.19)} \\
&\qquad \cong \mathrm{Hom}_{KS_r}(U^*, \mathcal{F}(V^\circ)) && \text{(as } \mathcal{G}_\otimes \text{ is left-adjoint to } \mathcal{F}) \\
&\qquad \cong \mathrm{Hom}_{KS_r}(U^*, \mathcal{F}(V)^*) && \text{(by Proposition 6.17(ii))} \\
&\qquad \cong \mathrm{Hom}_{KS_r}(\mathcal{F}(V), U) && \text{(by Lemma 6.19).}
\end{aligned}$$

This is precisely the requirement that $\mathcal{G}_{\mathrm{Hom}}$ is right-adjoint to $\mathcal{F}$. $\square$

Finally in this section we record exactness properties of the Schur functor and its inverses.

Proposition 6.21. *The functor $\mathcal{G}_\otimes$ is right exact and the functor $\mathcal{G}_{\mathrm{Hom}}$ is left exact. The functor $\mathcal{F}$ is exact (when K is infinite and $n \geqslant r$).*

Proof. These are general properties of tensor and hom functors (or more generally of adjoints), using Proposition 6.10 and Proposition 6.20(ii) to view $\mathcal{F}$ as tensor functor and a hom functor (that is, both a left- and right-adjoint). $\square$

Remark 6.22. The Schur functor is usually described in the language of the Schur algebra S, where the definition as a weight space is easily seen to be equivalent to multiplication by a certain idempotent. Call this idempotent e. Verifying that the Schur functor is isomorphic to a tensor product and to a hom space is straightforward using this characterisation: the roles of $E^{\otimes r}$ and $(E^\top)^{\otimes r}$ are replaced by Se and eS, and the isomorphisms $\mathrm{Hom}_S(Se, -) \cong e(-) \cong eS \otimes_S -$ are clear. The tensor-hom adjunction yields the left-adjoint inverse $Se \otimes_{eSe} -$ and the right-adjoint inverse $\mathrm{Hom}_{eSe}(eS, -)$.

Our treatment, aside from bypassing the need to construct the Schur algebra, gives constructions of the inverse Schur functors which are valid for any field K and any choice of parameters n and r. It gives two candidates for such a construction of the Schur functor itself: $\mathrm{Hom}_{K\mathrm{GL}_n(K)}(E^{\otimes r}, -)$ and $(E^\top)^{\otimes r} \otimes_{K\mathrm{GL}_n(K)} -$. The results of this chapter show that these two functors are isomorphic to each other and to $\mathcal{F}$ when K is infinite and $n \geqslant r$; they are dual to each other, analogously to $\mathcal{G}_{\mathrm{Hom}}$ and $\mathcal{G}_\otimes$ in Proposition 6.17(i); and each is left-inverse to its adjoint inverse Schur functor provided $K \neq \mathbb{F}_2$ and $n \geqslant r$ (see Remark 6.15). It would be interesting to identify whether these functors are isomorphic in all cases.

7. Dimension reduction functor

In this section we consider connections between polynomial representations of $\mathrm{GL}_n(K)$ for different values of n, when K is infinite. We use a functor which was defined in the context of the Schur algebra by Green [EGS08, Section 6.5]; here we simply multiply by the appropriate idempotent in $\mathrm{Mat}_n(K)$.

Recall that the algebra $\mathrm{Mat}_n(K)$ of all $n \times n$ matrices with entries in K acts on any polynomial representation of $\mathrm{GL}_n(K)$ by extending the domain of the defining polynomials (see Remark 5.8).

Let $n' \leqslant n$, and let $\varepsilon = \left(\begin{smallmatrix} I_{n'} & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Mat}_n(K)$, a block matrix, where $I_{n'}$ is the $n' \times n'$ identity matrix. Note that ε is an idempotent in $\mathrm{Mat}_n(K)$ and that the subalgebra $\varepsilon K\mathrm{GL}_n(K)\varepsilon$ is isomorphic to $K\mathrm{GL}_{n'}(K)$. Given V a polynomial representation of $\mathrm{GL}_n(K)$, the image εV of V under the action of ε is a polynomial representation of $\mathrm{GL}_{n'}(K)$ with the same representing polynomials (with the variables with labels greater than n' set to 0).

Definition 7.1 (Dimension reduction functor)**.** The *dimension reduction functor* from n to n' is the functor $\varepsilon(-)$ from the category of polynomial representations of $\mathrm{GL}_n(K)$ of degree r to the category of polynomial representations of $\mathrm{GL}_{n'}(K)$ of degree r defined by left multiplication by ε.

Proposition 7.2.

(i) The dimension reduction functor $\varepsilon(-)$ is exact.

(ii) For any $K\mathrm{GL}_n(K)$-module V, we have $\varepsilon\nabla^\lambda(V) \cong \nabla^\lambda(\varepsilon V)$.

(iii) For any KS_r-module U, we have $\varepsilon\mathcal{G}^n_\otimes(U) \cong \mathcal{G}^{n'}_\otimes(U)$.

Proof. Part (i) is a property of any functor defined by multiplication by an idempotent (see [EGS08, (6.2a)]). Part (ii) is clear from the construction of ∇^λ; the case of $V = E$ is noted in [EGS08, Remark following (6.5f)]. For part (iii), let E' denote the natural $K\mathrm{GL}_{n'}(K)$-module, and observe that $\varepsilon E \cong E'$ and that furthermore $\varepsilon(E^{\otimes r}) \cong (E')^{\otimes r}$; the claim then follows by the definition of $\mathcal{G}_\otimes$. $\square$

This proposition tells us that, informally, the structure of $\mathcal{G}_{\otimes}^{n}(U)$ for a KS_r-module U is independent of n. More precisely, we have the following corollary.

Corollary 7.3. *Suppose K is infinite. Let U be a KS_r-module, and suppose $\underline{\mu} = (\mu^{(1)}, \ldots, \mu^{(l)})$ is the sequence of labels for the simple modules in a composition series for $\mathcal{G}_{\otimes}^{n}(U)$ for some fixed $n \geqslant r$. Then $\underline{\mu}$ is also the sequence of labels for the simple modules in a composition series for $\mathcal{G}_{\otimes}^{n'}(U)$ for any n' (after excluding the labels for zero modules).*

CHAPTER III

Modular plethystic isomorphisms

In this chapter, we establish or rule out the existence of several *plethystic isomorphisms* – isomorphisms between modules of the form $\nabla^\mu \nabla^\lambda E$ and those with a Δ in place of a ∇ – over the two-by-two general linear group $\mathrm{GL}_2(K)$ where K is an arbitrary field. We give explicit maps in the case of existence; these results generalise classical results, but require dualities that were not present in characteristic 0.

The results of this chapter are taken from the author's joint work with Mark Wildon, [McDW21]; in the case of the Wronskian isomorphism we prove a more general result (see below).

Each of the four sections in this chapter is dedicated to one of the four plethystic (non-)isomorphisms described below. The first two sections are logically independent; the latter two make use of the results of the previous sections.

Complementary partition isomorphism. King [Kin85, §4.2] used the character theory of SU_2 to prove that, if λ° is the complement of the partition λ in a rectangle with $l+1$ rows, then

$$\nabla^\lambda \operatorname{Sym}^l E \cong \nabla^{\lambda^\circ} \operatorname{Sym}^l E$$

where E is the natural representation of the special linear group $\mathrm{SL}_2(\mathbb{C})$. In §8 we generalise this to the modular case, and furthermore to arbitrary groups, obtaining the following theorem.

Theorem A (Complementary partition isomorphism)**.** *Let G be a group, and let V be a d-dimensional representation of G. Let $c \in \mathbb{N}$, and let λ be a partition with $0 \leqslant \lambda_1 \leqslant c$ and $0 \leqslant \lambda'_1 \leqslant d$. Let λ° denote the box-complement of λ in the $d \times c$ rectangle. Then there is an isomorphism*

$$\nabla^\lambda V \cong \nabla^{\lambda^\circ} V^* \otimes (\det V)^{\otimes c}$$

where $\det V \cong \bigwedge^d V$.

Our map is explicit, sending a polytabloid $\mathrm{e}(t)$ of a tableau t to (plus or minus) the polytabloid $\mathrm{e}(t^\circ)$ of the 'complementary' tableau t° (see Definition 8.5 for the precise definition including the sign, and an illustrative example after it).

Two interesting special cases of this theorem are that $\bigwedge^l V \cong \bigwedge^{d-l} V^*$ and $\nabla^{(d,d-1,\ldots,1)} V \cong \nabla^{(d,d-1,\ldots,1)} V^*$ whenever $\det V$ is trivial – an assumption which holds, for instance, when V is obtained by restricting a polynomial representation of $\mathrm{GL}_2(K)$ to a subgroup of $\mathrm{SL}_2(K)$. Thus we obtain (Corollary 8.3) an explicit isomorphism $\bigwedge^l \mathrm{Sym}^{l+m-1} E \cong \bigwedge^m \mathrm{Sym}_{l+m-1} E$, where E is the natural representation of $\mathrm{SL}_2(K)$. More generally, we obtain as a corollary of Theorem A the following modular version of King's plethystic isomorphism.

Corollary B. *Let* $l, c \in \mathbb{N}_0$, *and let* λ *be a partition with* $\lambda'_1 \leqslant l+1$ *and* $\lambda_1 \leqslant c$. *Let* λ° *denote the complement of* λ *in the* $(l+1) \times c$ *rectangle. Let* E *be the natural* 2*-dimensional representation of* $\mathrm{SL}_2(K)$. *Then there is an isomorphism*

$$\nabla^\lambda \mathrm{Sym}^l E \cong \nabla^{\lambda^\circ} \mathrm{Sym}_l E.$$

Wronskian isomorphism. The Wronskian isomorphism is the classical result

$$\mathrm{Sym}^m \mathrm{Sym}^l E \cong \bigwedge^m \mathrm{Sym}^{l+m-1} E$$

for $m, l \in \mathbb{N}$, where E is the natural representation of $\mathrm{SL}_2(\mathbb{C})$ (see for instance [AC07, §2.5]). Our explicit modular version is as follows, where $\{X, Y\}$ is the canonical basis for the natural representation E of $\mathrm{GL}_2(K)$.

Theorem C (Characteristic-free Wronskian isomorphism)**.** *Let* m, $l \in \mathbb{N}$. *Let* K *be a field and let* E *be the natural* 2*-dimensional representation of* $\mathrm{GL}_2(K)$. *There is an isomorphism of* $\mathrm{GL}_2(K)$*-representations*

$$\mathrm{Sym}_m \mathrm{Sym}^l E \otimes (\det E)^{\otimes m(m-1)/2} \cong \bigwedge^m \mathrm{Sym}^{l+m-1} E$$

given by restriction of the K-linear map $(\mathrm{Sym}^l E)^{\otimes m} \to \bigwedge^m \mathrm{Sym}^{l+m-1} E$ *defined on the canonical basis of* $(\mathrm{Sym}^l E)^{\otimes m}$ *by*

$$\begin{aligned} &X^{i_1}Y^{l-i_1} \otimes X^{i_2}Y^{l-i_2} \otimes \cdots \otimes X^{i_m}Y^{l-i_m} \\ &\qquad\qquad \mapsto X^{i_1+m-1}Y^{l-i_1} \wedge X^{i_2+m-2}Y^{l-i_2+1} \wedge \cdots \wedge X^{i_m}Y^{l+m-1-i_m}. \end{aligned}$$

Recently, [AFP$^+$19, §3.4] also proved a modular version of this isomorphism, namely $\mathrm{Sym}^m \mathrm{Sym}_l E \cong \bigwedge^l \mathrm{Sym}^{l+m-1} E$ where E is the natural representation of $\mathrm{SL}_2(K)$. This isomorphism is equivalent to the existence of the isomorphism in Theorem C: using Corollary 8.3 (stated also in [AFP$^+$19]), their codomain $\bigwedge^l \mathrm{Sym}^{l+m-1} E$ is isomorphic to $\bigwedge^m (\mathrm{Sym}^{l+m-1} E)^*$ and hence by Proposition 3.3 to $(\bigwedge^m \mathrm{Sym}^{l+m-1} E)^*$, the dual of our right-hand side; meanwhile by the duality of symmetric powers and Proposition 3.2, their domain $\mathrm{Sym}^m \mathrm{Sym}_l E$ is isomorphic to $(\mathrm{Sym}_m \mathrm{Sym}^l E)^*$, the dual of our left-hand side. The isomorphism in [AFP$^+$19] is constructed indirectly using maps into, and out of, the ring of symmetric functions; the proof that it is $\mathrm{SL}_2(K)$-invariant requires Pieri's rule and a somewhat intricate inductive argument. By contrast our isomorphism has a simple one-line definition.

We in fact prove a result which is more general than that of [AFP$^+$19] and [McDW21], using different methods. We show that there is an injective map

$$\mathrm{Sym}_{\overline{m}} \mathrm{Sym}^l E \otimes (\det E)^{m\overline{m}/n} \hookrightarrow \bigwedge^{\overline{m}} \mathrm{Sym}^{l+m} E$$

where $\overline{m} = \binom{m+n-1}{m}$ and E is the natural representation of the $n \times n$ general linear group $\mathrm{GL}_n(K)$ (and that when $n = 2$ the map is a bijection, and is the map described in Theorem C). We prove this in §9.

Hermite reciprocity. Known also as the Cayley–Sylvester formula, Hermite reciprocity was discovered by the eponymous mathematicians in the setting of invariant theory. In our language it states that

$$\mathrm{Sym}^m \mathrm{Sym}^l E \cong \mathrm{Sym}^l \mathrm{Sym}^m E$$

for all m, $l \in \mathbb{N}$, where E is the natural 2-dimensional representation of the general linear group $\mathrm{GL}_2(\mathbb{C})$ ([FH04, Exercise 6.18]). Our modular version, which we obtain in §10 by composing our Wronskian isomorphism with a special case (Corollary 8.3) of the complementary partition isomorphism, is as follows.

Theorem D (Characteristic-free Hermite reciprocity). *Let $m, l \in \mathbb{N}$ and let E be the natural 2-dimensional representation of $\mathrm{GL}_2(K)$. Then*

$$\mathrm{Sym}_m \mathrm{Sym}^l E \cong \mathrm{Sym}^l \mathrm{Sym}_m E.$$

This result is obtained, without an explicit description of the maps, in a similar manner in [AFP$^+$19, Remark 3.2]. We illustrate our explicit composition in Example 10.1.

It is well known (as described in §3.2) that when K has characteristic p and $m \leqslant p-1$, the functors Sym_m and Sym^m are naturally isomorphic. Thus Theorem D implies that $\mathrm{Sym}^m \mathrm{Sym}^l E \cong \mathrm{Sym}^l \mathrm{Sym}^m E$ when $m \leqslant p-1$. This special case of the corollary was first proved by Kouwenhoven [Kou90b, pp. 1699–1700], where it is also shown that $\mathrm{Sym}^p \mathrm{Sym}^l E \ncong \mathrm{Sym}^l \mathrm{Sym}^p E$ if $p < l < p(p-1)$. In Proposition 11.14 we give infinitely many examples of such non-isomorphisms, considering different combinations of duality, thus demonstrating that Theorem D is the unique modular generalisation of Hermite reciprocity.

Conjugate hook partition isomorphism. Another classical result, due to King [Kin85, §4] (reproved as the main theorem in [CP16], and proved in a stronger version in [PW21, Theorem 1.3]) states that under certain conditions on the partition λ, there is an isomorphism

$$\nabla^\lambda \mathrm{Sym}^{m+\lambda'_1-1} E \cong \nabla^{\lambda'} \mathrm{Sym}^{m+\lambda_1-1} E,$$

where E is the natural representation of $\mathrm{SL}_2(\mathbb{C})$ and $m \in \mathbb{N}_0$. Hook partitions satisfy the conditions, and so we have

$$\nabla^{(a+1,1^b)} \mathrm{Sym}^{m+b} E \cong \nabla^{(b+1,1^a)} \mathrm{Sym}^{m+a} E$$

for all $a, b, m \in \mathbb{N}_0$. By the final theorem of this chapter, proved using the new modular invariant introduced in Definition 11.3, this isomorphism has, in general, no modular analogue, even after considering all possible dualities.

Theorem E (Obstructions to the conjugate hook partition isomorphism)**.** *Let α, β, $\varepsilon \in \mathbb{N}$ with $\alpha < \beta < \varepsilon$. If K has characteristic p and $|K| > 1 + 2(p^\varepsilon + p^\beta)(p^\alpha + p^\beta + 1) - p^\alpha(p^\alpha + 1)$, then the eight representations of $\mathrm{SL}_2(K)$ obtained from $\Delta^{(p^\alpha+1, 1^{p^\beta})} \mathrm{Sym}^{p^\varepsilon + p^\beta} E$ by any combination of*

- *replacing Δ with ∇,*
- *replacing Sym^- with Sym_-,*
- *swapping α and β,*

are pairwise non-isomorphic.

8. Complementary partition isomorphism

This section proves the following theorem and its corollaries.

Theorem A (Complementary partition isomorphism)**.** *Let G be a group, and let V be a d-dimensional representation of G. Let $c \in \mathbb{N}$, and let λ be a partition with $0 \leqslant \lambda_1 \leqslant c$ and $0 \leqslant \lambda'_1 \leqslant d$. Let λ° denote the box-complement of λ in the $d \times c$ rectangle. Then there is an isomorphism*

$$\nabla^\lambda V \cong \nabla^{\lambda^\circ} V^* \otimes (\det V)^{\otimes c}$$

where $\det V \cong \bigwedge^d V$.

We adopt the notation of this theorem throughout. Additionally, write $\lambda^{\circ\prime}$ for $(\lambda^\circ)'$, and let $\mathcal{B} = \{v_1, \ldots, v_d\}$ be an ordered K-basis for V and let $\mathcal{B}^* = \{v_1^*, \ldots, v_d^*\}$ be the (ordered) dual basis for V^*.

Our strategy is to define a G-equivariant map $\bigwedge^{\lambda'} V \to \bigwedge^{\lambda^{\circ\prime}} V^* \otimes (\det V)^{\otimes c}$ and show that its image on $\mathsf{GR}^\lambda(V)$ is contained in $\mathsf{GR}^{\lambda^\circ}(V^*) \otimes (\det V)^{\otimes c}$. The map will therefore descend to a G-equivariant map $\nabla^\lambda V \to \nabla^{\lambda^\circ} V^* \otimes (\det V)^{\otimes c}$, which is bijective by counting dimensions.

8.1. Map between exterior powers

We begin by constructing a K-linear isomorphism $\bigwedge^l V \to \bigwedge^{d-l} V^*$ for $0 \leqslant l \leqslant d$; we extend this to a K-linear isomorphism $\bigwedge^{\lambda'} V \to \bigwedge^{\lambda^{\circ\prime}} V^*$ in §8.2. We show that, accounting for powers of determinants, theses maps are also KG-equivariant.

Let $\Pi \subseteq S_d$ be the set of permutations of $[d]$ which preserve the relative orders within each subset $\{1, \ldots, l\}$ and $\{l+1, \ldots, d\}$; that is, $\sigma \in \Pi$ if and only if $1\sigma < \ldots < l\sigma$ and $(l+1)\sigma < \ldots < d\sigma$. Then we can write the standard basis of $\bigwedge^l V$ as $\{\, v_{1\sigma} \wedge \cdots \wedge v_{l\sigma} \mid \sigma \in \Pi \,\}$.

Definition 8.1. Let $\psi\colon \bigwedge^l V \to \bigwedge^{d-l} V^*$ be the K-linear bijection defined by

$$\psi(v_{1\sigma} \wedge \cdots \wedge v_{l\sigma}) = \operatorname{sgn}(\sigma)\, v^*_{(l+1)\sigma} \wedge \cdots \wedge v^*_{d\sigma}$$

for each $\sigma \in \Pi$ (and hence any $\sigma \in S_d$). Furthermore, let $\bar{\psi}\colon \bigwedge^l V \to \bigwedge^{d-l} V^* \otimes \det V$ be the K-linear bijection defined by $\bar{\psi}(x) = \psi(x) \otimes 1$.

Let $\{\,(v_{i_1} \wedge \cdots \wedge v_{i_l})^* \mid 1 \leqslant i_1 < \ldots < i_l \leqslant d\,\}$ be the basis of $(\bigwedge^l V)^*$ dual to the basis $\{\,v_{i_1} \wedge \cdots \wedge v_{i_l} \mid 1 \leqslant i_1 < \ldots < i_l \leqslant d\,\}$ for $\bigwedge^l V$.

Proposition 8.2. *The map $\bar{\psi}$ is G-equivariant.*

Proof. Let $\varepsilon = (v_1 \wedge \cdots \wedge v_d)^*$ be the unique element of the canonical basis of $(\bigwedge^d V)^*$. Our strategy is to show that ψ is the image of ε under a sequence of G-equivariant maps. Assuming this is done, since $(\bigwedge^d V)^* \cong (\det V)^{-1}$, for each $g \in G$ and $x \in \bigwedge^l V$ we have $(g \cdot \psi)(x) = (\det g^{-1})\psi(x)$, as required.

In the following steps we apply to ε the comultiplication map $(\bigwedge^d V)^* \to (\bigwedge^l V \otimes \bigwedge^{d-l} V)^*$ with respect to the standard bases introduced above; compose with the standard isomorphism $(U \otimes W)^* \cong U^* \otimes W^*$; and then apply the isomorphism $(\bigwedge^r V)^* \cong \bigwedge^r V^*$ from Proposition 3.3 on the right-hand factor:

$$\begin{aligned}
\varepsilon &\mapsto \sum_{\sigma \in \Pi} \operatorname{sgn}(\sigma)(v_{1\sigma} \wedge \cdots \wedge v_{l\sigma} \otimes v_{(l+1)\sigma} \wedge \cdots \wedge v_{d\sigma})^* \\
&\mapsto \sum_{\sigma \in \Pi} \operatorname{sgn}(\sigma)(v_{1\sigma} \wedge \cdots \wedge v_{l\sigma})^* \otimes (v_{(l+1)\sigma} \wedge \cdots \wedge v_{d\sigma})^* \\
&\mapsto \sum_{\sigma \in \Pi} \operatorname{sgn}(\sigma)(v_{1\sigma} \wedge \cdots \wedge v_{l\sigma})^* \otimes v^*_{(l+1)\sigma} \wedge \cdots \wedge v^*_{d\sigma}.
\end{aligned}$$

Finally we apply the standard isomorphism $U^* \otimes W \cong \operatorname{Hom}_K(U, W)$ to obtain the K-linear isomorphism

$$v_{1\sigma} \wedge \cdots \wedge v_{l\sigma} \mapsto \operatorname{sgn}(\sigma)\, v^*_{(l+1)\sigma} \wedge \cdots \wedge v^*_{d\sigma}$$

which is precisely the map ψ. □

An alternative proof of Proposition 8.2 is possible; see Remark 8.4 below.

Proposition 8.2 establishes the case of Theorem A when $c = 1$ and λ is a column. From this, we can already obtain the following plethystic isomorphism for $\mathrm{GL}_2(K)$.

Corollary 8.3. *Let $l, m \in \mathbb{N}$. Let E denote the natural 2-dimensional representation of $\mathrm{GL}_2(K)$. Then*

$$\textstyle\bigwedge^l \operatorname{Sym}^{l+m-1} E \cong \bigwedge^m \operatorname{Sym}_{l+m-1} E \otimes (\det E)^{\otimes \frac{1}{2}(l+m-1)(l-m)}.$$

Proof. It suffices to show that $\bigwedge^l \mathrm{Sym}^{l+m-1} E \cong \bigwedge^m \mathrm{Sym}_{l+m-1} E$ as representations of $\mathrm{SL}_2(K)$ (by Proposition 5.10). Indeed, suppose $G = \mathrm{SL}_2(K)$, $d = l + m$ and $V = \mathrm{Sym}^{l+m-1} E$. By Proposition 8.2, $\bar{\psi}$ is an isomorphism $\bigwedge^l V \cong \bigwedge^m V^*$. But $V^* \cong \mathrm{Sym}_{l+m-1} E$ by Propositions 3.2 and 3.7 (and the observation that $E \cong E^\circ$). $\square$

Remark 8.4. We discuss, with a connection to combinatorics, an alternative proof that the map $\bar{\psi}$ is G-equivariant (as seen in Proposition 8.2). The matrices by which an element $g \in G$ acts on $\bigwedge^l V$ and $\bigwedge^{d-l} V^*$ can be computed directly; their entries are *minors* of the matrices $\rho_V(g)$ and $\rho_V(g^{-1})^\top$. Recall a *minor* of a matrix is a determinant of a submatrix: given a $d \times d$ matrix M and subsets $A, B \subseteq [d]$, let $M[A, B]$ be the submatrix obtained by retaining only the rows and columns indexed by elements of A and B respectively; the corresponding minor of M is $\det(M[A, B])$.

Our basis for $\bigwedge^l V$ is labelled by l-subsets of $[d]$; when g acts on the basis element labelled by an l-subset A, the coefficient of the basis element labelled by the l-subset B is the minor

$$\rho_{\bigwedge^l V}(g)_{A,B} = \det\bigl(\rho_V(g)[A, B]\bigr).$$

Our basis for $\bigwedge^{d-l} V^*$ is likewise labelled by $(d - l)$-subsets of $[d]$, which are in correspondence with the l-subsets via complementation $-^{\mathsf{c}}$ in $[d]$. When g acts on the basis element labelled by the complement A^{c} of an l-subset A, the coefficient of the basis element labelled by the complement B^{c} of the l-subset B is the minor

$$\rho_{\bigwedge^{d-l} V^*}(g)_{A^{\mathsf{c}},B^{\mathsf{c}}} = \det\bigl(\rho_V(g^{-1})^\top[A^{\mathsf{c}}, B^{\mathsf{c}}]\bigr).$$

We require that these matrices $\rho_{\bigwedge^l V}(g)$ and $\rho_{\bigwedge^{d-l} V^*}(g)$ are equal, up to a factor of the determinant of $\rho_V(g)$ and the sign in the map $\bar{\psi}$. Indeed this is the case by Jacobi's complementary minor formula [CSS13, Lemma A.1(e), p. 96], which states that for any $d \times d$ matrix M and any subsets $A, B \subseteq [d]$, there is equality

$$\det\bigl(M[A, B]\bigr) = (-1)^{\Sigma A + \Sigma B} \det(M) \det\bigl(M^{-\top}[A^{\mathsf{c}}, B^{\mathsf{c}}]\bigr)$$

where ΣA, ΣB denote the sums of the entries of A and B.

This approach is of particular interest in the case $G = \mathrm{GL}_2(K)$ and $V = \mathrm{Sym}^{d-1} E$ of Corollary 8.3. When g is an elementary transvection (that is, has 1s on the diagonal and a unique nonzero off-diagonal entry), the entries of $\rho_V(g)$ are binomial coefficients (corresponding to the choice of factors in which the off-diagonal entry is taken when expanding the product), and the required equality between minors (that is, between the entries of $\rho_{\bigwedge^l V}(g)$ and $\rho_{\bigwedge^{d-l} V^*}(g)$) is

$$\det\left(\binom{b}{a}\right)_{a\in A, b\in B} = \det\left(\binom{a'}{b'}\right)_{a'\in A^{\mathrm{c}}, b'\in B^{\mathrm{c}}}$$

for subsets $A, B \subseteq [d-1]_0$. This identity, as well as being a consequence of Jacobi's formula, was proven combinatorially by Gessel and Viennot [GV85, Proposition 7] using a lattice path counting argument now known as the Lindström–Gessel–Viennot lemma (see [BC05] for an illuminating account of this lemma).

Motivated by the occurrence of these determinants in the action of $\mathrm{GL}_2(K)$ on symmetric powers, in [McD20] the author lifted the binomial identity above to q-binomials and to symmetric polynomials, and generalised further by allowing the number of indeterminates to vary, obtaining a duality theorem for *flagged Schur polynomials*. The proof again uses the Lindström–Gessel–Viennot lemma; the author shows that Jacobi's complementary minor formula is insufficient to prove the full generalisation.

8.2. Map between partition-labelled exterior powers

We use the map $\psi\colon \bigwedge^l V \to \bigwedge^{d-l} V^*$ of §8.1 to define a map $\bigwedge^{\lambda'} V \to \bigwedge^{\lambda^{\circ\prime}} V^*$ by applying ψ to each tensor factor. We describe this map explicitly using column tabloids (see §1.5). The group action on the basis vectors will not be needed for the rest of this section, and so for convenience we will view $\mathcal{B} = \mathcal{B}^* = [d]$.

Define a bijection $\mathrm{CSYT}_{[d]}(\lambda) \to \mathrm{CSYT}_{[d]}(\lambda^\circ)$ as follows. For each $1 \leqslant j \leqslant s$, let $j^\circ = c + 1 - j$ and observe that column j° of λ° has length $d - \lambda'_j$ (where we set $\lambda'_j = 0$ if j exceeds the greatest part of λ). Given a column standard tableau $t \in \mathrm{CSYT}_{[d]}(\lambda)$, let $t^\circ \in \mathrm{CSYT}_{[d]}(\lambda^\circ)$ be the column standard tableau whose entries in column j° are the the complement in $[d]$ of the entries of t in column j. Note that the assumption that t is column standard is essential so that t° has $d - \lambda'_i$ specified entries in column $\lambda_1 + 1 - i$.

Define the *surplus* of t to be $\mathrm{S}(t) = \sum_{(i,j)\in[\lambda]}(t(i,j) - i)$, or equivalently $\mathrm{S}(t) = \sum_{b\in[\lambda]} t(b) - \sum_{i=1}^{\lambda'_1} i\lambda_i$.

Definition 8.5. Let $\Psi\colon \bigwedge^{\lambda'} V \to \bigwedge^{\lambda^{\circ\prime}} V^*$ be the K-linear bijection defined by

$$\Psi(|t|) = (-1)^{\mathrm{S}(t)}|t^\circ|$$

for $t \in \mathrm{CSYT}_{[d]}(\lambda)$. Furthermore, let $\bar{\Psi}\colon \bigwedge^{\lambda} V \to \bigwedge^{\lambda^\circ} V^* \otimes (\det V)^{\otimes c}$ be the K-linear bijection defined by $\bar{\Psi}(x) = \Psi(x) \otimes 1$.

For example, suppose $d = 3$, $c = 4$ and $\lambda = (3, 1)$ with Young diagram $\ydiagram{3,1}$. Then $\lambda^\circ = (4, 3, 1)$ with Young diagram $\ydiagram{4,3,1}$. If $t = \ytableaushort{112,2}$, then $\mathrm{S}(t) = 0 + 0 + 1 + 0 = 6 - 5 = 1$ and

$$\Psi\left(\left| \begin{array}{c|c|c} 1 & 1 & 2 \\ 2 & & \end{array} \right| \right) = - \left| \begin{array}{c|c|c|c} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 & \\ 3 & & & \end{array} \right|$$

(where, as we are viewing $\mathcal{B} = \mathcal{B}^* = [d]$, the element i corresponds to the ith basis vector of V or V^* as appropriate).

We claim that applying the appropriate ψ to the ith tensor factor of $|t|$ yields the $(\lambda_1 + 1 - i)$th tensor factor of $(-1)^{\mathrm{S}(t)}|t^\circ|$, and hence by Proposition 8.2 applied to each column in the $d \times c$ rectangle in turn, that the map $\bar{\Psi}$ is a KG-isomorphism. The only difficulty is verifying that the sign arising from the maps on each factor is indeed given by the surplus of the tableau; this is achieved in the following lemma. Recall that for $l \in [d]$, the set $\Pi \subseteq S_d$ is the subset of permutations preserving the relative orders of $\{1, \dots, l\}$ and $\{l + 1, \dots, d\}$.

Lemma 8.6. *Let* $\sigma \in \Pi \subseteq S_d$. *Then* $\mathrm{sgn}(\sigma) = (-1)^{\mathrm{s}(\sigma)}$, *where* $\mathrm{s}(\sigma) = -\frac{1}{2}l(l+1) + \sum_{i=1}^{l} i\sigma$.

PROOF. We take l to be fixed and induct on d. If $d = l$, then σ is the identity permutation and $\mathrm{s}(\sigma) = 0$, establishing the base case.

Suppose $d > l$ and consider the value of $d\sigma$. If $d\sigma = d$, then σ can be viewed as an element of $\Pi \subseteq S_{d-1}$ and this does not change the value of $\mathrm{s}(\sigma)$, so the inductive hypothesis gives the claim.

If $d\sigma < d$, let $m \geqslant 1$ be such that $d\sigma = d - m$. Then, by definition of Π, we have that $(l - c)\sigma = d - c$ for each $0 \leqslant c \leqslant m - 1$; that is, the m largest elements of $\{1, \dots, d\}$ appear in the set $\{1\sigma, \dots, l\sigma\}$. Let $\tau = (d \;\; d{-}1 \;\; \cdots \;\; d{-}m{+}1 \;\; d{-}m)$, an $(m + 1)$-cycle. Observe that $d\sigma\tau = d$, and that $\sigma\tau$ preserves the relative orders of $\{1, \dots, l\}$ and $\{l + 1, \dots, d\}$, so $\sigma\tau \in \Pi$. Then viewing $\sigma\tau$ as an element of S_{d-1} as in the previous paragraph, by the inductive hypothesis we have $(-1)^{\mathrm{s}(\sigma\tau)} = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)$. But the set $\{1\sigma\tau, \dots, l\sigma\tau\}$ differs from $\{1\sigma, \dots, l\sigma\}$ only by the addition of $d - m$ and the removal of d, so $\mathrm{s}(\sigma) = \mathrm{s}(\sigma\tau) + m$, and τ is an $(m + 1)$-cycle so has sign $(-1)^m$. $\square$

8.3. Column sorting permutations

We need to know how permuting the boxes of a tableau t affects the image of its column tabloid under Ψ. The column sets of the resulting tabloid are clear, and permuting boxes does not change the value of the surplus $\mathrm{S}(t)$, but each column must be sorted into ascending order before the map $t \mapsto t^\circ$ can be applied, and more work is required to identify the sign which arises. Recall we view $\mathcal{B} = \mathcal{B}^* = [d]$, as we are now only interested in the linear structure of the map.

Fix $t \in \mathrm{CSYT}_{[d]}(\lambda)$ and two columns $1 \leqslant j < k \leqslant \lambda_1$. Let $j^\circ = c + 1 - j$ and $k^\circ = c + 1 - k$ be the columns in λ° complementary to the columns j and k in λ. Given a permutation $\tau \in S_{[\lambda]}$, the *support* of τ, denoted $\operatorname{supp} \tau$, is the set of points which are not fixed by τ.

Let $\tau \in S_{\mathrm{col}_j[\lambda] \sqcup \mathrm{col}_k[\lambda]}$ be a product of disjoint transpositions of the form $(a\ b)$ where $a \in \mathrm{col}_j[\lambda]$, $b \in \mathrm{col}_k[\lambda]$, such that the boxes in the support of τ have distinct entries in t. Suppose also that $|t \cdot \tau| \neq 0$; this precisely says that, in t, no box in column j in the support of τ has an entry which appears in column k, and vice versa. Observe that for each box in the support of τ, there is exactly one box in $\mathrm{col}_{j^\circ}[\lambda^\circ] \sqcup \mathrm{col}_{k^\circ}[\lambda^\circ]$ containing in t° the same entry: considering, for example, a box $a \in \mathrm{col}_j[\lambda]$ in the support of τ, the entry $t(a)$ does not appear in column k of t by the above assumptions, and so appears precisely once in column k° of t° (and does not appear in column j° of t° because it appears in column j of t). For $a \in \mathrm{col}_j[\lambda] \sqcup \mathrm{col}_k[\lambda]$ in the support of τ, denote this corresponding box $(t^\circ)^{-1}t(a)$. Then define $\tau^\circ \in S_{\mathrm{col}_{j^\circ}[\lambda^\circ] \sqcup \mathrm{col}_{k^\circ}[\lambda^\circ]}$ by replacing in every transposition the box a with the box $(t^\circ)^{-1}t(a)$.

It is clear that $\Psi(|t \cdot \tau|) = \pm|t^\circ \cdot \tau^\circ|$: by construction the permutation τ° swaps a pair of boxes between columns j° and k° if and only if the boxes containing their entries are swapped between columns j and k by τ. We claim that furthermore the correct sign is $(-1)^{\mathrm{S}(t)}$. To prove this, we require the following lemma.

Lemma 8.7. *Let $t \in \mathrm{CSYT}_{[d]}(\lambda)$. Let $x \in \mathrm{col}_j(t)$ and $y \in [d] \setminus \mathrm{col}_j(t)$. Let u be the tableau obtained from t by replacing in column j the entry x with the entry y, and let u' be the tableau obtained from t° by replacing in column j° the entry y with the entry x. The unique place permutation in $S_{[\lambda]}$ which sorts both column j of u and column j° of u' has sign $(-1)^{|x-y|-1}$.*

Proof. Let $Z = \{\min\{x, y\} + 1, \ldots, \max\{x, y\} - 1\}$. Column j of u is sorted by a cycle of length $1 + |Z \cap \mathrm{col}_j(t)|$, while column j° of u' is sorted by a cycle of length $1 + |Z \cap \mathrm{col}_{j^\circ}(t^\circ)|$. Let σ be the product of these disjoint cycles; this is the unique permutation in $S_{[\lambda]}$ which sorts both u and u'. Then σ has sign $(-1)^z$ where

$$z = |Z \cap \mathrm{col}_j(t)| + |Z \cap \mathrm{col}_{j^\circ}(t^\circ)|.$$

But by the definition of t° we have $\mathrm{col}_j(t) \sqcup \mathrm{col}_{j^\circ}(t^\circ) = [d]$. Thus $z = |Z| = |x - y| - 1$, as required. $\square$

Observe that in Lemma 8.7 the sign of the column sorting permutation depends only on the set $\{x, y\}$, and not on t (except through the requirement that $x \in \mathrm{col}_j(t)$ and $y \notin \mathrm{col}_j(t)$, which holds by hypothesis). Generalising, we obtain the following lemma.

Lemma 8.8. *Let $t \in \mathrm{CSYT}_{[d]}(\lambda)$. Let $\{x_1, \ldots, x_r\} \subseteq \mathrm{col}_j(t)$ and $\{y_1, \ldots, y_r\} \subseteq [d] \setminus \mathrm{col}_j(t)$. Let u be the tableau obtained from t by replacing in column j each entry x_i with the entry y_i, and let u' be the tableau obtained from t° by replacing in column j° each entry y_i with the entry x_i. The unique place permutation in $S_{[\lambda]}$ which sorts both column j of u and column j° of u' has sign depending only on the pairs $\{x_i, y_i\}$, and not on t.*

Proof. This follows by repeated application of Lemma 8.7. □

Proposition 8.9. *Let $t \in \mathrm{CSYT}_{[d]}(\lambda)$. Let $\tau \in S_{\mathrm{col}_j[\lambda] \sqcup \mathrm{col}_k[\lambda]}$ be a product of disjoint transpositions of the form $(a\ b)$ where $a \in \mathrm{col}_j[\lambda]$, $b \in \mathrm{col}_k[\lambda]$, such that the boxes in the support of τ have distinct entries in t. Suppose $|t \cdot \tau| \neq 0$. Then $\Psi(|t \cdot \tau|) = (-1)^{\mathrm{S}(t)} |t^\circ \cdot \tau^\circ|$.*

Proof. As has already been recorded, $\Psi(|t \cdot \tau|) = \pm |t^\circ \cdot \tau^\circ|$, or equivalently $[d] \setminus \mathrm{col}_j(t \cdot \tau) = \mathrm{col}_{j^\circ}(t^\circ \cdot \tau^\circ)$ and $[d] \setminus \mathrm{col}_k(t \cdot \tau) = \mathrm{col}_k(t^\circ \cdot \tau^\circ)$.

Let $\pi \in S_{\mathrm{col}_j[\lambda]}$, $\varphi \in S_{\mathrm{col}_k[\lambda]}$, $\pi' \in S_{\mathrm{col}_{j^\circ}[\lambda^\circ]}$, $\varphi' \in S_{\mathrm{col}_{k^\circ}}[\lambda^\circ]$ be the unique place permutations which sort, respectively, columns j and k of $t \cdot \tau$ and columns j° and k° of $t^\circ \cdot \tau^\circ$. By Lemma 8.8, the signs $\mathrm{sgn}(\pi\pi')$ and $\mathrm{sgn}(\varphi\varphi')$ depend only on the pairs $\{t(a), t(b)\}$ where $(a\ b)$ are the disjoint transpositions comprising τ, and therefore these signs are equal.

The tableaux $t \cdot \tau\pi\varphi$ and $t^\circ \cdot \tau^\circ\pi'\varphi'$ are column standard, their column sets are complementary as noted above, and they have the same surplus $\mathrm{S}(t)$. Thus we have $\Psi(|t \cdot \tau\pi\varphi|) = (-1)^{\mathrm{S}(t)} |t^\circ \cdot \tau^\circ\pi'\varphi'|$, and hence

$$\begin{aligned}
\Psi(|t \cdot \tau|) &= \mathrm{sgn}(\pi\varphi)\Psi(|t \cdot \tau\pi\varphi|) \\
&= \mathrm{sgn}(\pi\varphi)(-1)^{\mathrm{S}(t)} |t^\circ \cdot \tau^\circ\pi'\varphi'| \\
&= \mathrm{sgn}(\pi\varphi)\,\mathrm{sgn}(\pi'\varphi')(-1)^{\mathrm{S}(t)} |t^\circ \cdot \tau'| \\
&= (-1)^{\mathrm{S}(t)} |t^\circ \cdot \tau^\circ|
\end{aligned}$$

as claimed. □

8.4. Image of the Garnir relations

We now complete the strategy outlined at the start of this section by showing that the map $\Psi\colon \bigwedge^{\lambda'} V \to \bigwedge^{\lambda^{\circ\prime}} V^*$ sends Garnir relations to Garnir relations (the submodule $\mathsf{GR}^\lambda(V) \subseteq \bigwedge^{\lambda'} V$ of Garnir relations was introduced in §1.7). We then deduce Theorem A.

The proof of this key proposition is unavoidably somewhat long: after the setup it is split into three claims. Recall we view $\mathcal{B} = \mathcal{B}^* = [d]$, as we are now only interested in the linear structure of the map.

Proposition 8.10. *The map $\Psi\colon \bigwedge^{\lambda'} V \to \bigwedge^{\lambda^{\circ\prime}} V^*$ respects Garnir relations, in the sense that $\Psi(\mathsf{GR}^\lambda(V)) \subseteq \mathsf{GR}^{\lambda^\circ}(V^*)$.*

Proof. Let $\mathsf{R}_{(t,A,B)}$ be a Garnir relation (as defined in Definition 1.8). Thus $t \in \mathrm{CSYT}_{[d]}(\lambda)$, and $A \subseteq \mathrm{col}_j[\lambda]$ and $B \subseteq \mathrm{col}_k[\lambda]$ where $1 \leqslant j < k \leqslant \lambda_1$ and $|A| + |B| > \lambda'_j$. Our aim is to show that $\Psi(\mathsf{R}_{(t,A,B)}) \in \mathsf{GR}^{\lambda^\circ}(V^*)$. Note that place permutations do not change the value of $\mathrm{S}(t)$, so all signs arising from application of Ψ in the proof of this lemma will be $(-1)^{\mathrm{S}(t)}$.

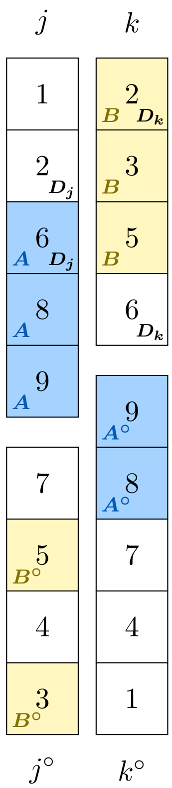
Recall that, by construction of t°, the entries in columns $j^\circ = c + 1 - j$ and $k^\circ = c + 1 - k$ of t° are complementary to the entries in columns j and k of t. By Lemma 1.9, we may assume that the entries of t in $A \sqcup B$ are distinct.

Let

$$
\begin{aligned}
A^\circ &= \{\, b \in \mathrm{col}_{k^\circ}[\lambda^\circ] \mid t^\circ(b) \in t(A) \,\} \\
B^\circ &= \{\, a \in \mathrm{col}_{j^\circ}[\lambda^\circ] \mid t^\circ(a) \in t(B) \,\} \\
D_j &= \{\, a \in \mathrm{col}_j[\lambda] \mid t(a) \in \mathrm{col}_k(t) \,\} \\
D_k &= \{\, b \in \mathrm{col}_k[\lambda] \mid t(b) \in \mathrm{col}_j(t) \,\}.
\end{aligned}
$$

The sets A° and B° are, respectively, the boxes in columns j° and k° of λ° whose entries in t° lie in the boxes A and B in t. The sets D_j and D_k are, respectively, the boxes in columns j and k of λ whose entries appear in both columns j and k of t. Note that $t^\circ(A^\circ) \subseteq t(A)$ and $t^\circ(B^\circ) \subseteq t(B)$, but equality need not hold because entries which appear in both columns of t do not appear in either column of t°. Thus $t^\circ(A^\circ)$ omits the entries in D_k and $t^\circ(B^\circ)$ omits the entries in D_j and

$$t^\circ(A^\circ) = t(A \setminus D_j), \quad t^\circ(B^\circ) = t(B \setminus D_k). \tag{8.10.1}$$



Since t and t° are injective on the sets of boxes appearing above, $|A^\circ| = |A| - |D_j|$, $|B^\circ| = |B| - |D_k|$.

An illustrative example in which $\lambda'_j = 5$, $\lambda'_k = 4$, $d = 9$ and $t(A) = \{6, 8, 9\}$, $t(B) = \{2, 3, 5\}$ is shown in the margin, with the sets introduced above indicated. See also Figure 8.1, which shows all the sets introduced in the course of the proof.

For each left coset of $S_A \times S_B$ in $S_{A \sqcup B}$, choose a coset representative which is a product of disjoint transpositions $(a\ b)$ with $a \in A$, $b \in B$. Let $\mathcal{T}$ be the subset of those representatives τ such that $|t \cdot \tau| \neq 0$; equivalently, $\mathcal{T}$ is the subset of coset representatives that fix all boxes in D_j and D_k. (Only entries in $t(D_j) = t(D_k)$ can be repeated in a column of $t \cdot \tau$, and since $t(A) \cap t(B) = \varnothing$, such an entry appears as a repeat in $t \cdot \tau$ if and only if it has changed column.) Thus the specified Garnir relation may be written as

$$\mathsf{R}_{(t,A,B)} = \sum_{\tau \in \mathcal{T}} |t \cdot \tau| \operatorname{sgn} \tau. \tag{8.10.2}$$

The chosen coset representatives $\mathcal{T}$ precisely meet the properties assumed in §8.3. Thus we can define for each $\tau \in \mathcal{T}$ a permutation $\tau^\circ \in S_{\mathrm{col}_{j^\circ}[\lambda^\circ] \sqcup \mathrm{col}_k[\lambda]}$ by, in every transposition comprising τ, replacing the box a with the unique box $(t^\circ)^{-1} t(a)$ in column j° or k° containing the entry $t(a)$. Let $\mathcal{T}^\circ = \{\, \tau^\circ \mid \tau \in \mathcal{T} \,\}$. Moreover, the conditions of Proposition 8.9 are met, and so we have

$$\Psi(|t \cdot \tau|) = (-1)^{\mathrm{S}(t)} |t^\circ \cdot \tau^\circ| \tag{8.10.3}$$

for all $\tau \in \mathcal{T}$.

Example 8.10.4. In the example shown in the margin, let τ be the place permutation $\big((4, j)\ (3, k)\big)$. Then τ is a permitted coset representative in $\mathcal{T}$ and $\tau^\circ = \big((3, j^\circ)\ (4, k^\circ)\big)$, both swapping the boxes containing 5 and 8. Since $t \cdot \tau$ and $t^\circ \cdot \tau^\circ$ are both sorted to column standard tableaux by applying two transpositions, $\Psi(|t \cdot \tau|) = (-1)^{\mathrm{S}(t)}|t^\circ \cdot \tau^\circ|$. If we instead take τ to be the place permutation $\big((3,j)\ (3,k)\big)$, then, since $(3,j) \in D_j$ and so its entry 6 appears in both column j and column k of t, we have $|t \cdot \tau| = 0$ and $\tau \notin \mathcal{T}$.

We now show that $\mathcal{T}^\circ$ has one of the properties required by the set $\mathcal{S}$ in the definition of Garnir relations.

Claim 8.10.5. *Excluding precisely those cosets whose place permutation actions send $|t^\circ|$ to 0, the set $\mathcal{T}^\circ$ is a complete irredundant set of left coset representatives of $S_{A^\circ} \times S_{B^\circ}$ in $S_{A^\circ \sqcup B^\circ}$.*

PROOF. If τ, $\theta \in \mathcal{T}$ are such that τ° and θ° represent the same coset of $S_{A^\circ} \times S_{B^\circ}$, then $|t^\circ \cdot \tau^\circ| = \pm|t^\circ \cdot \theta^\circ|$. Using Proposition 8.9 and that Ψ is a bijection, it follows that $|t \cdot \tau| = \pm|t \cdot \theta|$. Since the boxes $A \sqcup B$ have distinct entries in t, it follows that τ and θ represent the same coset of $S_A \times S_B$. Additionally, if $|t^\circ \cdot \tau^\circ| = 0$ then $|t \cdot \tau| = 0$, which contradicts $\tau \in \mathcal{T}$. Thus distinct elements of $\mathcal{T}^\circ$ are representatives of distinct cosets whose place permutation actions do not send $|t^\circ|$ to 0.

On the other hand, given any permutation in $S_{A^\circ \sqcup B^\circ}$, we may choose a coset representative σ that can be written as a product of disjoint transpositions $(a\ b)$ with $a \in A^\circ$ and $b \in B^\circ$. Because $t^\circ(A^\circ)$ and $t^\circ(B^\circ)$ are disjoint, the support of σ necessarily has distinct entries in t°. Supposing also that the place permutation action of this coset does not send $|t^\circ|$ to 0, then this representative satisfies the conditions of §8.3, and we may perform the construction symmetric to $\tau \mapsto \tau^\circ$. We thus obtain a permutation $\tau \in S_{A \sqcup B}$ such that $\tau \in \mathcal{T}$ and $\tau^\circ = \sigma$. We conclude that $\mathcal{T}^\circ$ is complete with the specified exclusions. $\square$

It follows from (8.10.2) and (8.10.3) that

$$\Psi(\mathsf{R}_{(t,A,B)}) = (-1)^{\mathrm{S}(t)} \sum_{\tau^\circ \in \mathcal{T}^\circ} |t^\circ \cdot \tau^\circ| \operatorname{sgn} \tau^\circ. \tag{8.10.6}$$

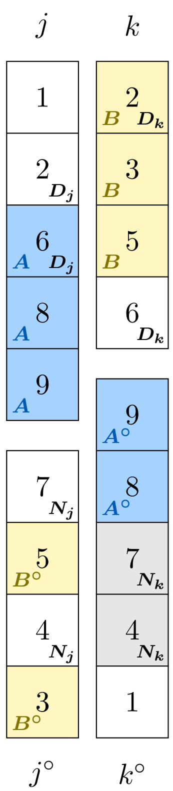
It would be very convenient to conclude from this and Claim 8.10.5 that $\Psi(\mathsf{R}_{(t,A,B)}) = (-1)^{\mathrm{S}(t)}\mathsf{R}_{(t^\circ,A^\circ,B^\circ)}$, finishing the proof. However, it may not be the case that $|A^\circ| + |B^\circ| > \lambda^{\circ\prime}_{k^\circ}$, and this is a requirement for $(t^\circ, A^\circ, B^\circ)$

to label a Garnir relation. We address this problem by expanding the subset A° of $\mathrm{col}_{k^\circ}[\lambda^\circ]$ in a way that does not affect the resulting relation: adding boxes which have entries lying also in column j° of t°.

Let

$$N_j = \{\, a \in \mathrm{col}_{j^\circ}[\lambda^\circ] \mid t^\circ(a) \in t^\circ(\mathrm{col}_k[\lambda^\circ]) \,\}$$
$$N_k = \{\, b \in \mathrm{col}_{k^\circ}[\lambda^\circ] \mid t^\circ(b) \in t^\circ(\mathrm{col}_j[\lambda^\circ]) \,\}$$

be the sets of boxes in columns j° and k° of λ° respectively whose entries appear in both columns j° and k° of t°. (Thus N_j and N_k are the analogues for t° of D_j and D_k.) In particular, N_k is disjoint from A° and N_j is disjoint from B°. These sets, and the sets introduced in the proof of the following claim, are shown in Figure 8.1.



Example 8.10.7. In our running example, shown in the margin now with full annotations, $A^\circ = \{(4,k^\circ),(5,k^\circ)\}$ and $B^\circ = \{(1,j^\circ),(3,j^\circ)\}$ so $|A^\circ| + |B^\circ| = 4 \not> \lambda^{\circ\prime}_{k^\circ} = 5$. Therefore A° and B° cannot be used directly to define a Garnir relation. We have $N_k = \{(2,k^\circ),(3,k^\circ)\}$, in bijection with $N_j = \{(2,j^\circ),(4,j^\circ)\}$, and $|A^\circ \sqcup N_k| + |B^\circ| = 6$. Therefore $A^\circ \sqcup N_k$ and B° define a Garnir relation. The relevant boxes are shaded in the margin. By Claim 8.10.9 at the end of this proof, $\Psi(\mathsf{R}_{(t,A,B)}) = (-1)^{\mathsf{S}(t)}\mathsf{R}_{(t^\circ, A^\circ \sqcup N_k, B^\circ)}$.

Claim 8.10.8. $|A^\circ \sqcup N_k| + |B^\circ| > \lambda^{\circ\prime}_{k^\circ}$.

Proof. Let $U = \mathrm{col}_j[\lambda] \setminus (A \cup D_j)$, and let $U^\circ = \{\, b \in \mathrm{col}_{k^\circ}[\lambda^\circ] \mid t^\circ(b) \in t(U) \,\}$. The entries in boxes in U do not appear in column k of t (because these boxes are in column j but not in D_j), and hence do appear in column k° of t°, so $|U| = |U^\circ|$. Furthermore, U° is disjoint from A° because U is disjoint from A, and we deduce that $\mathrm{col}_{k^\circ}[\lambda^\circ] = A^\circ \sqcup N_k \sqcup U^\circ$. We remind the reader that these subsets are illustrated in Figure 8.1.

Using $\mathrm{col}_{k^\circ}[\lambda^\circ] = A^\circ \sqcup N_k \sqcup U^\circ$, the inequality we are required to show becomes $|B^\circ| > |U^\circ|$. We observe that $|B^\circ| = |B| - |B \cap D_k|$ (for $t(B) = t(B^\circ) \sqcup t(B \cap D_k)$, and t is injective on these sets). Together with $|U| = |U^\circ|$ as noted above, our requirement becomes $|B| > |U| + |B \cap D_k|$.

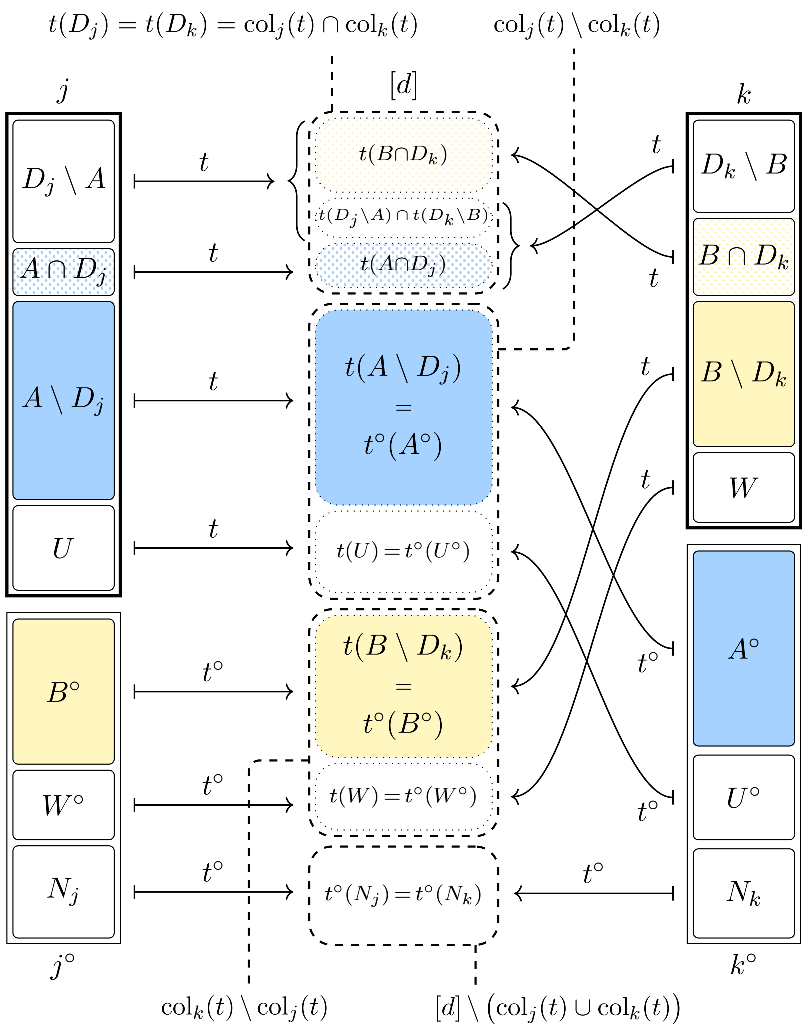


FIGURE 8.1. The sets of boxes and their entries considered in the proof of Proposition 8.10. Column j of $[\lambda]$ and column j° of $[\lambda^\circ]$ are shown on the left, column k of $[\lambda]$ and column k° of $[\lambda^\circ]$ are shown on the right, and the set $\{1, \ldots, d\}$ containing their entries is shown in the middle. The solid colouring indicates the boxes, and their entries, that may be moved by elements of $\mathcal{T}$; the dotted colouring indicates the boxes, and their entries, which lie in $A \sqcup B$ but which are fixed by $\mathcal{T}$. The sets $W = \mathrm{col}_k[\lambda] \backslash (B \cup D_k)$ and $W^\circ = \{\, a \in \mathrm{col}_{j^\circ}[\lambda^\circ] \mid t(a) \in t(W) \,\}$ are defined analogously to the sets of boxes U and U° used in the proof; they are indicated here only in order to complete the partition and are not used in the proof.

We next observe that $|B \cap D_k| \leqslant |D_j \setminus A|$. Indeed, $t(B \cap D_k) \subseteq t(D_j)$ because $t(D_j) = t(D_k)$; but also $t(B \cap D_k) \cap t(A) = \varnothing$ by the assumption that $t(A) \cap t(B) = \varnothing$, and thus $t(B \cap D_k) \subseteq t(D_j \setminus A)$. Then since t is injective on each of these sets, we have $|B \cap D_j| \leqslant |D_i \setminus A|$.

It now suffices to show that $|B| > |U| + |D_j \setminus A|$. Adding $|A|$ to each side and using that $\mathrm{col}_j[\lambda] = A \sqcup (D \setminus A) \sqcup U$, this requirement is equivalent to $|A| + |B| > \lambda'_j$. This was our initial assumption on A and B. $\square$

Claim 8.10.9. $\Psi(\mathsf{R}_{(t,A,B)}) = (-1)^{\mathrm{S}(t)}\mathsf{R}_{(t^\circ,A^\circ \sqcup N_k,B^\circ)}$.

Proof. Let $\mathcal{R}$ be a set of left coset representatives for $S_{A^\circ \sqcup N_k} \times S_{B^\circ}$ in $S_{A^\circ \sqcup N_k \sqcup B^\circ}$, chosen so that each representative that keeps all the boxes in N_k in column k° fixes all these boxes. Let $\mathcal{Q} \subseteq \mathcal{R}$ be this set of representatives fixing all the boxes in N_k; then $\mathcal{Q}$ forms a complete irredundant set of left coset representatives of $S_{A^\circ} \times S_{B^\circ}$ in $S_{A^\circ \sqcup B^\circ}$. By Claim 8.10.5 we have $\sum_{\sigma \in \mathcal{Q}} |t^\circ \cdot \sigma| \operatorname{sgn} \sigma = \sum_{\tau^\circ \in \mathcal{T}^\circ} |t^\circ \cdot \tau^\circ| \operatorname{sgn} \tau^\circ$. Thus

$$\mathsf{R}_{(t^\circ,A^\circ \sqcup N_k,B^\circ)} = \sum_{\sigma \in \mathcal{R} \setminus \mathcal{Q}} |t^\circ \cdot \sigma| \operatorname{sgn} \sigma + \sum_{\tau^\circ \in \mathcal{T}^\circ} |t^\circ \cdot \tau^\circ| \operatorname{sgn} \tau^\circ.$$

Each summand $|t^\circ \cdot \sigma|$ in the first sum is 0, because σ moves a box containing an entry in N_k into column j, in which this entry is already contained in a box in N_j. By (8.10.6) the second summand is $(-1)^{\mathrm{S}(t)}\Psi(\mathsf{R}_{(t,A,B)})$, as required. $\square$

We thus have $\Psi(\mathsf{R}_{(t,A,B)}) \in \mathsf{GR}^{\lambda^\circ}(V^*)$, finishing the proof of the proposition. $\square$

Remark 8.11. In the proof of Proposition 8.10, we could equally well have joined N_j to B° instead of N_k to A°, and shown instead that $|A^\circ| + |B^\circ \sqcup N_j| > \lambda^{\circ\prime}_{k^\circ}$ and $\Psi(\mathsf{R}_{(t,A,B)}) = \mathsf{R}_{(t^\circ,A^\circ,B^\circ \sqcup N_j)}$.

We can now deduce the main results of this section.

Proof of Theorem A. The quotient construction of the Schur endofunctor from Proposition 2.13 is:

$$\nabla^\lambda V \cong {\textstyle\bigwedge^{\lambda'} V} \big/ \mathsf{GR}^\lambda(V) \,.$$

By Proposition 8.10, the map Ψ descends to a linear map $\nabla^\lambda V \to \nabla^{\lambda^\circ} V^*$. Moreover, $\bar{\Psi}$ descends to a G-equivariant linear map

$$\begin{aligned}\nabla^\lambda V &\to \nabla^{\lambda^\circ} V^* \otimes (\det V)^{\otimes c}\\ \mathrm{e}(t) &\mapsto (-1)^{\mathrm{S}(t)}\,\mathrm{e}(t^\circ) \otimes 1\end{aligned}$$

for $t \in \mathrm{CSYT}_{[d]}(\lambda)$.

We observe that $t \mapsto t^\circ$ is a bijection $\mathrm{SSYT}_{[d]}(\lambda) \to \mathrm{SSYT}_{[d]}(\lambda^\circ)$ (and not just a bijection $\mathrm{CSYT}_{[d]}(\lambda) \to \mathrm{CSYT}_{[d]}(\lambda^\circ)$ as is immediate); this is shown in [PW21, Proposition 7.1]. Recalling from Proposition 2.12 that the semistandard tableaux label a basis of polytabloids, we deduce that the map above is bijective between bases and hence an isomorphism. $\square$

9. Wronskian isomorphism

In this section we prove Theorem C.

Theorem C (Characteristic-free Wronskian isomorphism)**.** *Let m, $l \in \mathbb{N}$. Let K be a field and let E be the natural 2-dimensional representation of $\mathrm{GL}_2(K)$. There is an isomorphism of $\mathrm{GL}_2(K)$-representations*

$$\mathrm{Sym}_m \mathrm{Sym}^l E \otimes (\det E)^{\otimes m(m-1)/2} \cong \bigwedge^m \mathrm{Sym}^{l+m-1} E$$

given by restriction of the K-linear map $(\mathrm{Sym}^l E)^{\otimes m} \to \bigwedge^m \mathrm{Sym}^{l+m-1} E$ defined on the canonical basis of $(\mathrm{Sym}^l E)^{\otimes m}$ by

$$X^{i_1}Y^{l-i_1} \otimes X^{i_2}Y^{l-i_2} \otimes \cdots \otimes X^{i_m}Y^{l-i_m} \\ \mapsto X^{i_1+m-1}Y^{l-i_1} \wedge X^{i_2+m-2}Y^{l-i_2+1} \wedge \cdots \wedge X^{i_m}Y^{l+m-1-i_m}.$$

In fact, we construct an injection between representations of the $n \times n$ general linear group $\mathrm{GL}_n(K)$, establishing the following theorem.

Theorem 9.1. *Let $n, m, l \in \mathbb{N}$, and let $\overline{m} = \binom{m+n-1}{m}$. There is an injective $\mathrm{GL}_n(K)$-equivariant linear map*

$$\mathrm{Sym}_{\overline{m}} \mathrm{Sym}^l E \otimes (\det E)^{m\overline{m}/n} \hookrightarrow \bigwedge^{\overline{m}} \mathrm{Sym}^{l+m} E.$$

When $n = 2$, the injective map we identify is the map described in Theorem C, as noted after its definition (Definition 9.3). In this case we have $\overline{m} = m+1$ and the dimensions of $\mathrm{Sym}_{m+1} \mathrm{Sym}^l E$ and $\bigwedge^{m+1} \mathrm{Sym}^{l+m} E$ agree, so the injective map is an isomorphism, yielding Theorem C (after shifting the parameter m by 1).

We adopt the notation of Theorem 9.1 throughout this section.

9.1. Construction of map

We begin by constructing a linear map $\mathrm{Sym}_{\overline{m}} \mathrm{Sym}^l E \otimes (\det E)^{m\overline{m}/n} \to \bigwedge^{\overline{m}} \mathrm{Sym}^{l+m} E$ which specialises to the required map when $n = 2$. We prove that our map is injective and $\mathrm{GL}_n(K)$-equivariant in §9.2 and §9.3 respectively.

Recall that $\{X_1, \dots, X_n\}$ denotes the natural basis for the natural representation E, so that for $g \in \mathrm{GL}_n(K)$ we have

$$gX_i = \sum_{j=1}^{n} g_{j,i} X_j.$$

For each $r \in \mathbb{N}$, we view $\operatorname{Sym}^r E$ as the space of homogeneous polynomials of degree r in $X_1, \dots, X_n$. Let $\mathsf{Mon}_r \subseteq \operatorname{Sym}^r E$ be the set of monomials of degree r, a basis for $\operatorname{Sym}^r E$.

We will make use of the lexicographical ordering on monomials. It is helpful to view this ordering as an injection into $\mathbb{N}$ as follows. Let $b > l + m$, and define an injection $\Xi\colon \bigsqcup_{r=0}^{b-1} \mathsf{Mon}_r \to \mathbb{N}$ by

$$\Xi(X_1^{a_1} \cdots X_n^{a_n}) = b^{n-1}a_1 + b^{n-2}a_2 + \ldots + ba_{n-1} + a_n.$$

For $0 \leqslant r < b$, we obtain the lexicographical ordering by totally ordering Mon_r via Ξ: define $f >_\Xi h$ if and only if $\Xi(f) > \Xi(h)$.

A basis of $\operatorname{Sym}_{\overline{m}} \operatorname{Sym}^l E$ is indexed by $\overline{m}$-tuples of monomials. For each $r \in \mathbb{N}$, let $(\mathsf{Mon}_r)^{\overline{m}}$ denote the set of $\overline{m}$-tuples whose entries are monomials in Mon_r. Denote componentwise multiplication of tuples by concatenation; that is, given $\boldsymbol{f} \in (\mathsf{Mon}_r)^{\overline{m}}$ and $\boldsymbol{h} \in (\mathsf{Mon}_{r'})^{\overline{m}}$, write $\boldsymbol{fh} \in (\mathsf{Mon}_{r+r'})^{\overline{m}}$ for the $\overline{m}$-tuple with ith entry $(\boldsymbol{fh})_i = \boldsymbol{f}_i \boldsymbol{h}_i$.

Define

$$(\mathsf{Mon}_r)^{\overline{m}}_{\geqslant} = \{\, \boldsymbol{f} \in (\mathsf{Mon}_r)^{\overline{m}} \mid \boldsymbol{f}_1 \geqslant_\Xi \ldots \geqslant_\Xi \boldsymbol{f}_{\overline{m}} \,\},$$
$$(\mathsf{Mon}_r)^{\overline{m}}_{>} = \{\, \boldsymbol{f} \in (\mathsf{Mon}_r)^{\overline{m}} \mid \boldsymbol{f}_1 >_\Xi \ldots >_\Xi \boldsymbol{f}_{\overline{m}} \,\}$$

to be the sets of weakly decreasing and strictly decreasing $\overline{m}$-tuples respectively. Note that $(\mathsf{Mon}_m)^{\overline{m}}_{>}$ contains a unique element, consisting of the monomials of degree m in decreasing order; call that element $\boldsymbol{w}$.

Example 9.2. Suppose $n = 2$, and write $X = X_1$, $Y = X_2$. Choose $b = 10$, and we have, for example,

$$\Xi(X^2Y^3) = 23.$$

On Mon_2, this gives the ordering $X^2 >_\Xi XY >_\Xi Y^2$. The unique element of $(\mathsf{Mon}_2)^3_{>}$ is $\boldsymbol{w} = (X^2, XY, Y^2)$.

Given $\boldsymbol{f} \in (\mathsf{Mon}_r)^{\overline{m}}$, define the *tensor product* of $\boldsymbol{f}$ to be

$$\boldsymbol{f}^{\otimes} = \boldsymbol{f}_1 \otimes \cdots \otimes \boldsymbol{f}_{\overline{m}} \in (\operatorname{Sym}^r E)^{\otimes \overline{m}},$$

and define the *alternating product* of $\boldsymbol{f}$ to be

$$\boldsymbol{f}^{\wedge} = \boldsymbol{f}_1 \wedge \cdots \wedge \boldsymbol{f}_{\overline{m}} \in \bigwedge^{\overline{m}} \operatorname{Sym}^r E.$$

Then $\{\, \boldsymbol{f}^{\otimes} \mid \boldsymbol{f} \in (\mathsf{Mon}_r)^{\overline{m}} \,\}$ is a basis for $(\operatorname{Sym}^r E)^{\otimes \overline{m}}$ and $\{\, \boldsymbol{f}^{\wedge} \mid \boldsymbol{f} \in (\mathsf{Mon}_r)^{\overline{m}}_{>} \,\}$ is a basis for $\bigwedge^{\overline{m}} \operatorname{Sym}^r E$.

Let the symmetric group $S_{\overline{m}}$ act (on the right) on $(\mathsf{Mon}_r)^{\overline{m}}$ by place permutation (given $\sigma \in S_{\overline{m}}$ and $\boldsymbol{f} \in (\mathsf{Mon}_r)^{\overline{m}}$, we say $(\boldsymbol{f} \cdot \sigma)_i = \boldsymbol{f}_{i\sigma^{-1}}$). Let $\operatorname{stab} \boldsymbol{f} \leqslant S_{\overline{m}}$ denote the stabiliser of $\boldsymbol{f}$ with respect to this action, and let $\operatorname{stab} \boldsymbol{f} \backslash S_{\overline{m}}$ denote a set of right coset representatives. Let $\boldsymbol{f}^{\mathrm{sym}}$ denote the symmetrisation of $\boldsymbol{f}^{\otimes}$; we can write this as

$$\boldsymbol{f}^{\mathrm{sym}} = \sum_{\sigma \in \operatorname{stab} \boldsymbol{f} \backslash S_{\overline{m}}} (\boldsymbol{f} \cdot \sigma)^{\otimes} = \sum_{\sigma \in \operatorname{stab} \boldsymbol{f} \backslash S_{\overline{m}}} \boldsymbol{f}_{1\sigma^{-1}} \otimes \cdots \otimes \boldsymbol{f}_{\overline{m}\sigma^{-1}} \in \operatorname{Sym}_{\overline{m}} \operatorname{Sym}^r E.$$

Then $\{\, \boldsymbol{f}^{\mathrm{sym}} \mid \boldsymbol{f} \in (\mathsf{Mon}_r)^{\overline{m}}_{\geqslant} \,\}$ is a basis for $\operatorname{Sym}_{\overline{m}} \operatorname{Sym}^r E$.

We can now define our K-linear map.

Definition 9.3. Let $\zeta\colon \operatorname{Sym}_{\overline{m}} \operatorname{Sym}^l E \otimes (\det E)^{\otimes m\overline{m}/n} \to \bigwedge^{\overline{m}} \operatorname{Sym}^{l+m} E$ be the K-linear map defined by extension of

$$\zeta(\boldsymbol{f}^{\mathrm{sym}} \otimes 1) = Z(\boldsymbol{f})$$

for each $\boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}_{\geqslant}$, where $Z(\boldsymbol{f}) \in \bigwedge^{\overline{m}} \operatorname{Sym}^{l+m} E$ is the sum

$$Z(\boldsymbol{f}) = \sum_{\sigma \in \operatorname{stab} \boldsymbol{f} \backslash S_{\overline{m}}} ((\boldsymbol{f} \cdot \sigma)\boldsymbol{w})^{\wedge} = \sum_{\sigma \in \operatorname{stab} \boldsymbol{f} \backslash S_{\overline{m}}} (\boldsymbol{f}_{1\sigma^{-1}} \boldsymbol{w}_1 \wedge \cdots \wedge \boldsymbol{f}_{\overline{m}\sigma^{-1}} \boldsymbol{w}_{\overline{m}}).$$

Note that when $n = 2$, and writing $X = X_1$ and $Y = X_2$, the tuple $\boldsymbol{w}$ is obtained by ordering the monomials of degree m by decreasing powers of X; that is, $\boldsymbol{w} = (X^m, X^{m-1}Y, \ldots, Y^m)$. Thus $X^i Y^{l-i} \boldsymbol{w}_j = X^{i+m+1-j} Y^{l-i-m-1+j}$, and it is clear that ζ is the map described in the statement of Theorem C.

Example 9.4. Suppose $l = 4$ and $m = 2$, and suppose $n = 2$ and write $X = X_1$ and $Y = X_2$. We have $\overline{m} = 3$ and $\boldsymbol{w} = (X^2, XY, Y^2)$. Let $\boldsymbol{f} = (X^4, XY^3, XY^3) \in (\mathsf{Mon}_4)^3_{\geqslant}$. Then

$$\boldsymbol{f}^{\mathrm{sym}} = (X^4 \otimes XY^3 \otimes XY^3) + (XY^3 \otimes XY^3 \otimes X^4) + (XY^3 \otimes X^4 \otimes XY^3)$$

and

$$Z(\boldsymbol{f}) = (X^6 \wedge X^2Y^4 \wedge XY^5) + (X^3Y^3 \wedge X^2Y^4 \wedge X^4Y^2) + (X^3Y^3 \wedge X^5Y \wedge XY^5).$$

Note that we have written each summand of $Z(\boldsymbol{f})$ below the summand of $\boldsymbol{f}^{\mathrm{sym}}$ from which it is obtained by componentwise multiplication with $\boldsymbol{w}$.

9.2. Injectivity of map

We prove that ζ is injective by showing that the set $\{\, Z(\boldsymbol{f}) \mid \boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}_{\geqslant} \,\}$ is K-linearly independent.

We make use of the following order on $(\mathsf{Mon}_{l+m})^{\overline{m}}$.

Definition 9.5. Define a partial order $<_\Sigma$ on $(\mathsf{Mon}_{l+m})^{\overline{m}}$ by $\boldsymbol{f} <_\Sigma \boldsymbol{h}$ if and only if $\sum_{i=1}^{\overline{m}} \Xi(\boldsymbol{f}_i)^2 < \sum_{i=1}^{\overline{m}} \Xi(\boldsymbol{h}_i)^2$. Extend $<_\Sigma$ to a total order on $(\mathsf{Mon}_{l+m})^{\overline{m}}$ arbitrarily.

Lemma 9.6. *Let $\boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}_{\geqslant}$, and let $\sigma \in S_{\overline{m}}$, $\sigma \notin \operatorname{stab} \boldsymbol{f}$. Then $(\boldsymbol{f} \cdot \sigma)\boldsymbol{w} <_\Sigma \boldsymbol{f}\boldsymbol{w}$.*

Proof. Observe that if f, h and fh are all in the domain of Ξ, then $\Xi(fh) = \Xi(f) + \Xi(h)$. Thus we have

$$\begin{aligned}
&\sum_{i=1}^{\overline{m}} \Xi((\boldsymbol{f}\boldsymbol{w})_i)^2 - \sum_{i=1}^{\overline{m}} \Xi(((\boldsymbol{f} \cdot \sigma)\boldsymbol{w})_i)^2 \\
&\qquad = \sum_{i=1}^{\overline{m}} \Big(\Xi(\boldsymbol{f}_i \boldsymbol{w}_i)^2 - \Xi(\boldsymbol{f}_{i\sigma^{-1}} \boldsymbol{w}_i)^2 \Big) \\
&\qquad = \sum_{i=1}^{\overline{m}} \Big((\Xi(\boldsymbol{f}_i) + \Xi(\boldsymbol{w}_i))^2 - (\Xi(\boldsymbol{f}_{i\sigma^{-1}}) + \Xi(\boldsymbol{w}_i))^2 \Big) \\
&\qquad = 2 \left(\sum_{i=1}^{\overline{m}} \Xi(\boldsymbol{f}_i)\Xi(\boldsymbol{w}_i) - \sum_{i=1}^{\overline{m}} \Xi(\boldsymbol{f}_{i\sigma^{-1}})\Xi(\boldsymbol{w}_i) \right).
\end{aligned}$$

This is positive, and the lemma follows, by the rearrangement inequality (which states that given real numbers $a_1 \geqslant \ldots \geqslant a_r$ and $b_1 \geqslant \ldots \geqslant b_r$ we have $\sum_{i=1}^r a_i b_i \geqslant \sum_{i=1}^r a_i b_{i\sigma^{-1}}$ for any permutation σ, with equality if and only if $b_i = b_{i\sigma^{-1}}$ for all i [HLP52, Theorem 368]). $\square$

Proposition 9.7. *The set* $\{\, Z(\boldsymbol{f}) \mid \boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}_{\geqslant} \,\}$ *is* K*-linearly independent.*

PROOF. Consider a K-linear combination $A = \sum_{\boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}_{\geqslant}} \alpha_{\boldsymbol{f}} Z(\boldsymbol{f})$, where $\alpha_{\boldsymbol{f}} \in K$ are not all zero. Let $\boldsymbol{f}^* \in (\mathsf{Mon}_l)^{\overline{m}}_{\geqslant}$ be $<_\Sigma$-maximal such that $\alpha_{\boldsymbol{f}^*} \neq 0$.

Let $\boldsymbol{f}$ be such that $\alpha_{\boldsymbol{f}} \neq 0$, and write $Z(\boldsymbol{f})$ with respect to the basis $\{\, \boldsymbol{h}^\wedge \mid \boldsymbol{h} \in (\mathsf{Mon}_{l+m})^{\overline{m}}_{>} \,\}$. If $\boldsymbol{f} <_\Sigma \boldsymbol{f}^*$, then also $\boldsymbol{f}\boldsymbol{w} <_\Sigma \boldsymbol{f}^*\boldsymbol{w}$, and hence for any $\boldsymbol{h}^\wedge$ with nonzero coefficient in $Z(\boldsymbol{f})$ we have by Lemma 9.6 that $\boldsymbol{h} \leqslant_\Sigma \boldsymbol{f}\boldsymbol{w} <_\Sigma \boldsymbol{f}^*\boldsymbol{w}$. Meanwhile, if $\boldsymbol{f} = \boldsymbol{f}^*$, we have by Lemma 9.6 that $(\boldsymbol{f}^* \cdot \sigma)\boldsymbol{w} <_\Sigma \boldsymbol{f}^*\boldsymbol{w}$ for all $\sigma \in S_{\overline{m}}$, $\sigma \notin \operatorname{stab} \boldsymbol{f}^*$, and hence $(\boldsymbol{f}^*\boldsymbol{w})^\wedge$ occurs with coefficient 1. Thus, when A is written with respect to this basis, the coefficient of $(\boldsymbol{f}^*\boldsymbol{w})^\wedge$ is $\alpha_{\boldsymbol{f}^*} \neq 0$. Thus $A \neq 0$. $\square$

9.3. Equivariance of map

We now show that ζ is $\mathrm{GL}_n(K)$-equivariant, completing the proof of Theorem 9.1 and hence Theorem C. Recall that $\mathrm{GL}_n(K)$ is generated by elementary transvections and diagonal matrices (Lemma 5.9), where an elementary transvection is a matrix that has 1s on the diagonal and a unique nonzero off-diagonal entry. Thus it suffices to show ζ respects the action of these matrices.

We first find an alternative expression for ζ. Consider the K-bilinear map defined by extension of

$$\begin{array}{ccc} (\operatorname{Sym}^l E)^{\otimes \overline{m}} \times (\operatorname{Sym}^m E)^{\otimes \overline{m}} & \to & \bigwedge^{\overline{m}} \operatorname{Sym}^{l+m} E \\ (\boldsymbol{f}^\otimes, \boldsymbol{h}^\otimes) & \mapsto & (\boldsymbol{f}\boldsymbol{h})^\wedge \end{array}$$

for $\boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}$ and $\boldsymbol{h} \in (\mathsf{Mon}_m)^{\overline{m}}$. This induces the following K-linear map $(\operatorname{Sym}^l E)^{\otimes \overline{m}} \otimes (\operatorname{Sym}^m E)^{\otimes \overline{m}} \to \bigwedge^{\overline{m}} \operatorname{Sym}^{l+m} E$.

Definition 9.8. Let $\omega\colon (\operatorname{Sym}^l E)^{\otimes\overline{m}} \otimes (\operatorname{Sym}^m E)^{\otimes\overline{m}} \to \bigwedge^{\overline{m}} \operatorname{Sym}^{l+m} E$ be the K-linear map defined on pure tensors by

$$\omega(f_1 \otimes \cdots f_{\overline{m}} \otimes h_1 \otimes \cdots \otimes h_{\overline{m}}) = f_1 h_1 \wedge \cdots \wedge f_{\overline{m}} h_{\overline{m}}$$

for $f_1, \ldots, f_{\overline{m}} \in \operatorname{Sym}^l E$ and $h_1, \ldots, h_{\overline{m}} \in \operatorname{Sym}^m E$ (not necessarily monomials).

It is easy to verify that ω is $\mathrm{GL}_n(K)$-equivariant.

Our map ζ can now be written as

$$\zeta(\boldsymbol{f}^{\mathrm{sym}} \otimes 1) = \omega(\boldsymbol{f}^{\mathrm{sym}} \otimes \boldsymbol{w}^{\otimes}).$$

for all $\boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}$.

Lemma 9.9. *Let* $\mathfrak{f} \in \operatorname{Sym}_{\overline{m}} \operatorname{Sym}^l E$ *and let* $\boldsymbol{h} \in (\mathsf{Mon}_m)^{\overline{m}}$. *If* $\boldsymbol{h}$ *has a repeated entry, then* $\omega(\mathfrak{f} \otimes \boldsymbol{h}^{\otimes}) = 0$.

Proof. By linearity, it suffices to prove this when $\mathfrak{f} = \boldsymbol{f}^{\mathrm{sym}}$ for some $\boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}$. For $\boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}$ and $\sigma \in S_{\overline{m}}$, observe that

$$\begin{aligned}
\omega((\boldsymbol{f} \cdot \sigma)^{\otimes} \otimes \boldsymbol{h}^{\otimes}) &= \boldsymbol{f}_{1\sigma^{-1}} \boldsymbol{h}_1 \wedge \cdots \wedge \boldsymbol{f}_{\overline{m}\sigma^{-1}} \boldsymbol{h}_{\overline{m}} \\
&= \operatorname{sgn}(\sigma) \boldsymbol{f}_1 \boldsymbol{h}_{1\sigma} \wedge \cdots \wedge \boldsymbol{f}_{\overline{m}} \boldsymbol{h}_{\overline{m}\sigma} \\
&= \operatorname{sgn}(\sigma) \omega(\boldsymbol{f}^{\otimes} \otimes (\boldsymbol{h} \cdot \sigma^{-1})^{\otimes}).
\end{aligned}$$

Since $\boldsymbol{h}$ has a repeated entry, there exists a transposition τ such that $\boldsymbol{h} \cdot \tau = \boldsymbol{h}$. Fix such a τ.

Since τ is a transposition, the orbits of the action of τ on the right cosets of $\operatorname{stab} \boldsymbol{f}$ in $S_{\overline{m}}$ are of size 1 or 2. Let $\mathcal{T}$ be the set of representatives $\sigma \in \operatorname{stab} \boldsymbol{f} \backslash S_{\overline{m}}$ for cosets in orbits of size 1 (that is, such that $\boldsymbol{f} \cdot \sigma\tau = \boldsymbol{f} \cdot \sigma$; equivalently, such that $\boldsymbol{f} \cdot \sigma$ has repeated entries at the positions swapped by τ). For $\sigma \in \mathcal{T}$, observe that $\omega((\boldsymbol{f} \cdot \sigma)^{\otimes} \otimes \boldsymbol{h}^{\otimes})$ has repeated entries (at the positions swapped by τ), and so is equal to 0.

Meanwhile, pick one coset from each orbit of size 2 of the action of τ on the right cosets of $\operatorname{stab} \boldsymbol{f}$ in $S_{\overline{m}}$, and let $\mathcal{A} \subseteq \operatorname{stab} \boldsymbol{f} \backslash S_{\overline{m}}$ be their set of representatives. Then using the observation from the beginning of the lemma we have

$$\begin{aligned}
&\sum_{\substack{\sigma\in\operatorname{stab}\boldsymbol{f}\backslash S_{\overline{m}}\\ \sigma\notin\mathcal{T}}} \omega((\boldsymbol{f}\cdot\sigma)^{\otimes}\otimes\boldsymbol{h}^{\otimes})\\
&\qquad\qquad = \sum_{\sigma\in\mathcal{A}} \big(\omega((\boldsymbol{f}\cdot\sigma)^{\otimes}\otimes\boldsymbol{h}^{\otimes}) + \omega((\boldsymbol{f}\cdot\sigma\tau)^{\otimes}\otimes\boldsymbol{h}^{\otimes})\big)\\
&\qquad\qquad = \sum_{\sigma\in\mathcal{A}} \Big(\omega((\boldsymbol{f}\cdot\sigma)^{\otimes}\otimes\boldsymbol{h}^{\otimes}) + \operatorname{sgn}(\tau)\omega((\boldsymbol{f}\cdot\sigma)^{\otimes}\otimes(\boldsymbol{h}\cdot\tau^{-1})^{\otimes})\Big)\\
&\qquad\qquad = 0.
\end{aligned}$$

Thus $\omega(\boldsymbol{f}^{\mathrm{sym}}\otimes\boldsymbol{h}^{\otimes}) = 0$. □

Lemma 9.10. *Let $g\in\mathrm{GL}_n(K)$ be an elementary transvection and let $r\in\mathbb{N}$. There exists a total order $\prec$ on Mon_r such that for all $f\in\mathsf{Mon}_r$ we have*

$$gf = f + \sum_{h\prec f}\gamma_{f,h}h.$$

for some constants $\gamma_{f,h}\in K$.

PROOF. Suppose g is the elementary transvection sending $X_i\mapsto X_i+\alpha X_j$ (and fixing all other variables), where $i,j\in[n]$ are distinct and $\alpha\in K$. Define a partial order on Mon_r by $h\prec f$ whenever the exponent of X_i in h is strictly lower than that in f, and extend to a total order (for example, we could take the lexicographical order where the exponent of X_i is the first to be compared). This has the required properties: given $f\in\mathsf{Mon}_r$ in which the exponent of X_i is a, the image gf is obtained from f by replacing X_i^a with $(X_i+\alpha X_j)^a$, so f occurs as a summand of the image exactly once and all other summands h have a strictly lower exponent of X_i. □

Lemma 9.11. *Let $\mathfrak{f}\in\mathrm{Sym}_{\overline{m}}\mathrm{Sym}^l E$ and let $g\in\mathrm{GL}_n(K)$ be an elementary transvection. Then*

$$\omega(\mathfrak{f}\otimes(g-1)\boldsymbol{w}^{\otimes}) = 0.$$

PROOF. Let $\prec$ be a total order on Mon_m which has the property of Lemma 9.10, and let $\hat{\boldsymbol{w}}\in(\mathsf{Mon}_m)^{\overline{m}}$ be the $\overline{m}$-tuple of distinct monomials in Mon_m in increasing order with respect to $\prec$. Thus $\hat{\boldsymbol{w}} = \boldsymbol{w}\cdot\sigma$ for some

$\sigma \in S_{\overline{m}}$. As in the proof of Lemma 9.9, we have $\omega(\mathfrak{f} \otimes (g-1)\hat{\boldsymbol{w}}^{\otimes}) = \operatorname{sgn}(\sigma)\omega(\mathfrak{f} \cdot \sigma^{-1} \otimes (g-1)\boldsymbol{w}^{\otimes})$, so it suffices to show that $\omega(\mathfrak{f} \otimes (g-1)\hat{\boldsymbol{w}}^{\otimes}) = 0$.

By the choice of $\prec$, we have for each $1 \leqslant i \leqslant \overline{m}$ that

$$g\hat{\boldsymbol{w}}_i = \sum_{j=1}^{i} \gamma_{i,j} \hat{\boldsymbol{w}}_j,$$

and hence

$$g\hat{\boldsymbol{w}}^{\otimes} = \bigotimes_{i=1}^{\overline{m}} \left(\sum_{j=1}^{i} \gamma_{i,j} \hat{\boldsymbol{w}}_j \right),$$

for some constants $\gamma_{i,j} \in K$ with $\gamma_{i,i} = 1$.

Observe that, when expanded out, the only summand without a repeated factor is $\hat{\boldsymbol{w}}^{\otimes}$ itself. Then by Lemma 9.9, $\omega(\mathfrak{f} \otimes \boldsymbol{h}) = 0$ for each summand $\boldsymbol{h}$ of $(g-1)\hat{\boldsymbol{w}}^{\otimes}$, and the result follows. $\square$

Proposition 9.12. *The map ζ is $\mathrm{GL}_n(K)$-equivariant.*

Proof. For $g \in \mathrm{GL}_n(K)$, $\boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}_{\geqslant}$, we have

$$g\zeta(\boldsymbol{f}^{\mathrm{sym}} \otimes 1) = g\omega(\boldsymbol{f}^{\mathrm{sym}} \otimes \boldsymbol{w}^{\otimes}) = \omega(g\boldsymbol{f}^{\mathrm{sym}} \otimes g\boldsymbol{w}^{\otimes}),$$
$$\zeta(g(\boldsymbol{f}^{\mathrm{sym}} \otimes 1)) = \zeta(g\boldsymbol{f}^{\mathrm{sym}} \otimes (\det g)^{m\overline{m}/n}) = (\det g)^{m\overline{m}/n}\omega(g\boldsymbol{f}^{\mathrm{sym}} \otimes \boldsymbol{w}^{\otimes}).$$

Thus we are required to show that

$$\omega(g\boldsymbol{f}^{\mathrm{sym}} \otimes g\boldsymbol{w}^{\otimes}) = \omega(g\boldsymbol{f}^{\mathrm{sym}} \otimes \boldsymbol{w}^{\otimes})(\det g)^{m\overline{m}/n}$$

for all $\boldsymbol{f} \in (\mathsf{Mon}_l)^{\overline{m}}_{\geqslant}$ and all $g \in \mathrm{GL}_n(K)$. It suffices to consider g an elementary transvection or a diagonal matrix, since these elements generate $\mathrm{GL}_n(K)$ (Lemma 5.9).

If g is an elementary transvection, then $\det g = 1$ and Lemma 9.11 leads immediately to $\omega(g\boldsymbol{f}^{\mathrm{sym}} \otimes g\boldsymbol{w}^{\otimes}) = \omega(g\boldsymbol{f}^{\mathrm{sym}} \otimes \boldsymbol{w}^{\otimes})$, giving the requirement.

If g is diagonal, writing g_i for the ith diagonal entry of g, we have that g acts on a monomial $X_1^{a_1} \cdots X_n^{a_n}$ by multiplication by $g_1^{a_1} \cdots g_n^{a_n}$. In $\boldsymbol{w}$, each variable occurs exactly $m\overline{m}/n$ many times, and so g acts on $\boldsymbol{w}^{\otimes}$ by multiplication by $g_1^{m\overline{m}/n} \cdots g_n^{m\overline{m}/n} = (\det g)^{m\overline{m}/n}$, as required. $\square$

10. Hermite reciprocity

We deduce Hermite reciprocity, restated below, from the complementary partition isomorphism and the Wronskian isomorphism. In fact we need only the special case Corollary 8.3 of the former isomorphism; this corollary was proved at the end of §8.1.

Theorem D (Characteristic-free Hermite reciprocity)**.** *Let $m, l \in \mathbb{N}$ and let E be the natural 2-dimensional representation of $\mathrm{GL}_2(K)$. Then*

$$\mathrm{Sym}_m \mathrm{Sym}^l E \cong \mathrm{Sym}^l \mathrm{Sym}_m E.$$

Proof. For convenience, we establish the isomorphism over $\mathrm{SL}_2(K)$. Since the representations are polynomial of equal degree, it follows from Proposition 5.10 that the isomorphism also holds over $\mathrm{GL}_2(K)$.

Recall from Proposition 3.3 and Proposition 3.7 that the contravariant dual $-^\circ$ satisfies $(\mathrm{Sym}^n V)^\circ \cong \mathrm{Sym}_n V^\circ$ and $(\bigwedge^n V)^\circ \cong \bigwedge^n V^\circ$. Note also that $E \cong E^\circ$. Using these relations, we have, as representations of $\mathrm{SL}_2(K)$,

$$\begin{aligned} \mathrm{Sym}_m \mathrm{Sym}^l E &\cong \textstyle\bigwedge^m \mathrm{Sym}^{l+m-1} E && \text{(by Theorem C)} \\ &\cong \textstyle\bigwedge^l \mathrm{Sym}_{l+m-1} E && \text{(by Corollary 8.3)} \\ &\cong \left(\textstyle\bigwedge^l \mathrm{Sym}^{l+m-1} E\right)^\circ && \\ &\cong (\mathrm{Sym}_l \mathrm{Sym}^m E)^\circ && \text{(by Theorem C)} \\ &\cong \mathrm{Sym}^l \mathrm{Sym}_m E, && \end{aligned}$$

as required. □

We illustrate how to explicitly compose the maps above with an example. (In practice it is convenient to address duality in a different order than in the proof of Theorem D.)

Example 10.1. Suppose $l = m = 2$, and write $E = \langle X, Y \rangle_K$ as in §9. In this example we identify the image in $\mathrm{Sym}^l \mathrm{Sym}_m E$ of the basis element $X^2 \otimes Y^2 + Y^2 \otimes X^2 \in \mathrm{Sym}_m \mathrm{Sym}^l E$.

We first apply the Wronskian isomorphism ζ (Theorem C), giving

$$\begin{gathered}\operatorname{Sym}_m \operatorname{Sym}^l E \to \textstyle\bigwedge^m \operatorname{Sym}^{l+m-1} E \\ X^2 \otimes Y^2 + Y^2 \otimes X^2 \mapsto X^3 \wedge Y^3 - X^2 Y \wedge XY^2.\end{gathered}$$

Next we apply the complementary partition isomorphism ψ (see Definition 8.1): we replace each summand with the wedge product of the duals of the complementary basis elements (and also pick up a sign, which in our example is $+$). Composing with the isomorphism $(\bigwedge^r V)^* \cong \bigwedge^r V^*$ of Proposition 3.3 we obtain

$$\begin{gathered}\textstyle\bigwedge^m \operatorname{Sym}^{l+m-1} E \to (\bigwedge^l \operatorname{Sym}^{l+m-1} E)^* \\ X^3 \wedge Y^3 - X^2 Y \wedge XY^2 \mapsto \big(X^2 Y \wedge XY^2\big)^* - \big(X^3 \wedge Y^3\big)^*.\end{gathered}$$

Now we apply the dual $\zeta^\star$ of the Wronskian isomorphism. To find the image $\zeta^*(x^*)$, we seek those basis elements y such that $\zeta(y)$ has x as a summand. For $x = X^2 Y \wedge XY^2$, there are two such basis elements: $XY \otimes XY$ and the symmetrisation of $X^2 \otimes Y^2$ (the latter appearing with sign -1); for $x = X^3 \wedge Y^3$, the symmetrisation of $X^2 \otimes Y^2$ is the only such basis element. Thus

$$\begin{gathered}(\textstyle\bigwedge^l \operatorname{Sym}^{l+m-1} E)^* \to (\operatorname{Sym}_l \operatorname{Sym}^m E)^* \\ \big(X^2 Y \wedge XY^2\big)^* - \big(X^3 \wedge Y^3\big)^* \mapsto \begin{array}{l} \big(XY \otimes XY\big)^* \\ \quad - 2\big(X^2 \otimes Y^2 + Y^2 \otimes X^2\big)^*. \end{array}\end{gathered}$$

The isomorphism $(\operatorname{Sym}^r V)^* \cong \operatorname{Sym}_r V^*$ in Proposition 3.7 is given by interchanging symmetrisations with products, yielding

$$\begin{gathered}(\operatorname{Sym}_l \operatorname{Sym}^m E)^* \to \operatorname{Sym}^l \operatorname{Sym}_m E^* \\ \begin{array}{l} \big(XY \otimes XY\big)^* \\ \quad - 2\big(X^2 \otimes Y^2 + Y^2 \otimes X^2\big)^* \end{array} \mapsto \begin{array}{r} (X^* \otimes Y^* + Y^* \otimes X^*) \cdot (X^* \otimes Y^* + Y^* \otimes X^*) \\ - 2(X^* \otimes X^*) \cdot (Y^* \otimes Y^*). \end{array}\end{gathered}$$

Finally we use Proposition 3.2: there is an isomorphism $E^* \cong E^\circ \cong E$ given by the basis change matrix $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, which in our case replaces X^* with

$-Y$ and Y^* with X. We have

$$\begin{array}{ccc} & \operatorname{Sym}^l \operatorname{Sym}_m E^* \to \operatorname{Sym}^l \operatorname{Sym}_m E & \\ \begin{array}{r}(X^*\otimes Y^* + Y^*\otimes X^*)\cdot(X^*\otimes Y^* + Y^*\otimes X^*) \\ -\,2(X^*\otimes X^*)\cdot(Y^*\otimes Y^*)\end{array} & \mapsto & \begin{array}{r}(X\otimes Y + Y\otimes X)\cdot(X\otimes Y + Y\otimes X) \\ -\,2(X\otimes X)\cdot(Y\otimes Y).\end{array}\end{array}$$

Thus our overall map sends

$$\begin{array}{ccc} & \operatorname{Sym}_m \operatorname{Sym}^l E \to \operatorname{Sym}^l \operatorname{Sym}_m E & \\ X^2\otimes Y^2 + Y^2\otimes X^2 & \mapsto & \begin{array}{r}(X\otimes Y + Y\otimes X)\cdot(X\otimes Y + Y\otimes X) \\ -\,2(X\otimes X)\cdot(Y\otimes Y).\end{array}\end{array}$$

Notice in particular that we have not merely interchanged symmetrisations and products. Thus this map is of interest even in characteristic 0, where it corresponds to a non-trivial automorphism of $\operatorname{Sym}^2 \operatorname{Sym}^2 E$.

11. Conjugate hook partition non-isomorphism

The goal of this section is to prove Theorem E, which rules out the existence of certain plethystic isomorphisms. We achieve this with the aid of a new invariant called the *defect set*.

Throughout we use the notation akin to that of §9 in which $E = \langle X, Y\rangle_K$ is the natural representation of $\mathrm{SL}_2(K)$.

11.1. Weight spaces and the defect set

Suppose, to begin, that K is infinite. Let T be the torus of diagonal matrices in $\mathrm{SL}_2(K)$. Let V be a representation of a subgroup of $\mathrm{SL}_2(K)$ containing T. For $r \in \mathbb{Z}$, we say the *r-weight space* of V is

$$V_r = \left\{ v \in V \;\middle|\; \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} v = \alpha^r v \text{ for all } \alpha \in K^\times \right\}. \tag{11.1}$$

An integer r such that $V_r \neq 0$ is called a *weight* of V; an element of an r-weight space is called a *weight vector* with weight r.

Remark 11.2. Weight spaces for representations of $\mathrm{GL}_n(K)$ were defined in Definition 6.1. Our definition for representations of $\mathrm{SL}_2(K)$ is slightly different, due to the restricted form of diagonal elements of $\mathrm{SL}_2(K)$. A weight vector with weight r of a representation of $\mathrm{SL}_2(K)$ (as defined above) corresponds to, in the notation of Definition 6.1, a vector of weight $(r+i, i)$ for some integer i. These weights spaces cannot be distinguished by diagonal matrices of the form $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$, whence the definition.

We say that T *acts diagonalisably* on V if $V = \bigoplus_{r\in\mathbb{Z}} V_r$, or equivalently if V has a basis of weight vectors. If V is a $K\mathrm{SL}_2(K)$-module on which T acts diagonalisably and $m \in \mathbb{Z}$ is maximal such that $V_m \neq 0$, then we say that V_m is the *highest weight space* of V, and that a nonzero $v \in V_m$ is a *highest weight vector*. We say $v \in V_m$ is a *unique highest weight vector* if V_m is one-dimensional.

Let B be the Borel subgroup of $\mathrm{SL}_2(K)$ consisting of lower triangular matrices. For $\gamma \in K$ we let

$$M_\gamma = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \in B.$$

We introduce the following invariant, which we will use to to distinguish non-isomorphic representations and hence obtain the results of this section.

Definition 11.3. Let V be a $K\mathrm{SL}_2(K)$-module on which T acts diagonalisably with unique highest weight vector v of weight m. Let Bv denote the KB-submodule of V generated by v. We define the *defect set* of V, denoted $\mathcal{D}(V)$, by

$$\mathcal{D}(V) = \{\, d \in \mathbb{N}_0 \mid (Bv)_{m-2d} \neq 0 \,\}.$$

Example 11.4. Let $\alpha \geqslant 1$. The module $\mathrm{Sym}^{p^\alpha} E$ has weight vector basis $\{X^{p^\alpha}, \ldots, X^{p^\alpha - i}Y^i, \ldots, Y^{p^\alpha}\}$, where $X^{p^\alpha - i}Y^i$ has weight $p^\alpha - 2i$. Thus the weights are $p^\alpha, \ldots, p^\alpha - 2i, \ldots, -p^\alpha$, and X^{p^α} is a unique highest weight vector. Observe that $M_\gamma X^{p^\alpha} = (X + \gamma Y)^{p^\alpha} = X^{p^\alpha} + \gamma^{p^\alpha} Y^{p^\alpha}$, and hence BX^{p^α} is spanned by X^{p^α} and Y^{p^α} whose weights are p^α and $-p^\alpha$ respectively. Hence the defect set is $\mathcal{D}(\mathrm{Sym}^{p^\alpha} E) = \{0, p^\alpha\}$.

We generalise this example to arbitrary upper and lower symmetric powers in Lemma 11.11.

Finite fields. To obtain the full version of Theorem E we need the extension of Definition 11.3 to $K\mathrm{SL}_2(K)$-modules when K is finite. Suppose that $|K| = q$. Since T is isomorphic to the cyclic group $K^\times$ of order $q-1$, a well-known generalisation of Maschke's Theorem implies that T acts diagonalisably on any $K\mathrm{SL}_2(K)$-module. Defining V_r as in (11.1), we have $V = \sum_{r \in \mathbb{Z}} V_r$. The sum is no longer direct in general, because the weight r is now well-defined only up to multiples of $q-1$. For the purposes of our work, we restrict the definition of weights to integers in the range $\left(\frac{q-1}{2}, \frac{q-1}{2}\right] \cap \mathbb{Z}$. Correspondingly, in the definition of the defect set, Definition 11.3, we take only those d in the range $0 \leqslant d < (q-1)/2$.

Example 11.5. We revisit Example 11.4, now supposing K is a finite field. For K sufficiently large ($|K| \geqslant p^{\alpha+2}$ suffices), all the weights written down in Example 11.4 are within the required range, and no changes are needed. However, when $|K| \leqslant 1 + 2m$, where m is the highest weight defined for an infinite field, the behaviour can be very different.

Consider $\mathrm{Sym}^4 E$ when $K = \mathbb{F}_8$. Weights are restricted to be between -3 and 3 (inclusive), and so X^4 has weight -3 (rather than 4 as in the infinite field case). A unique highest weight vector is Y^4 with weight 3 (the other weight vectors are X^3Y with weight 2, X^2Y^2 with weight 0, and XY^3 with weight -2). The submodule BY^4 is spanned by Y^4 and thus the defect set is $\mathcal{D}(\mathrm{Sym}^4 E) = \{0\}$.

Consider instead $\mathrm{Sym}^5 E$ when $K = \mathbb{F}_5$. Weights are restricted to be between -1 and 2 (inclusive), and so $\mathrm{Sym}^5 E$ has weights 1 (with weight vectors X^5, X^3Y^2 and XY^4) and -1 (with weight vectors X^4Y, X^2Y^3 and Y^5). In particular there is not a unique highest weight vector and so the defect set is not defined.

Identifying defect sets for images of Schur endofunctors. We first verify that defect sets are defined for the modules we wish to distinguish using them. We assume throughout that $|K| \geqslant 4$ (as otherwise weights are only permitted to be in the sets $\{0\}$ or $\{0, 1\}$, which is too restrictive).

The natural representation E has weight vector basis $\{X, Y\}$, where X is a unique highest weight vector of weight 1 and Y has weight -1. It is straightforward to identify weight vector bases for the images of E under iterated Schur endofunctors and their duals, and observe that there is a unique highest weight vector and hence that the defect set is defined.

Proposition 11.6. *Let V be a $K\mathrm{SL}_2(K)$-module with weight vector basis $\{v_1, \ldots, v_l\}$, where v_i has weight r_i, for some integers $r_1 \leqslant \cdots \leqslant r_{l-1} < r_l$.*

(i) The basis of $\nabla^\lambda V$ consisting of semistandard polytabloids is a weight vector basis, in which $\mathrm{e}(t)$ has weight $\sum_{b\in[\lambda]} r_{t(b)}$ (modulo $|K| - 1$). Let $t_{\max}$ be the semistandard tableau obtained by filling each column

from the bottom with integers decreasing from l, and suppose that $|K| > 1 + 2\sum_{b\in[\lambda]} r_{t_{\max}(b)}$. Then a unique highest weight vector is $\mathrm{e}(t_{\max})$.

(ii) *The basis $\{v_1^*, \ldots, v_l^*\}$ for V° dual to $\{v_1, \ldots, v_l\}$ is a weight vector basis, in which v_i^* has weight r_i. A unique highest weight vector is v_l^*, of weight r_l.*

Proof. The claimed weights are clear; that the semistandard polytabloids form a basis is Proposition 2.12. Since $r_{l-1} < r_l$, there is in each case a unique highest weight vector. □

Remark 11.7. It is also clear that, in the notation of Proposition 11.6, the basis $\{v_1^*, \ldots, v_l^*\}$ for V^* dual to $\{v_1, \ldots, v_l\}$ is a weight vector basis, in which v_i^* has weight $-r_i$ (where we view $-\frac{q-1}{2} = \frac{q-1}{2}$ if $|K| = q$ is finite and odd). But $V^* \cong V^\circ$ by Proposition 3.2, and so we deduce that the multiset of weights of V is symmetric about zero.

To identify which of the weight spaces intersect the KB-submodule generated by the highest weight vector, it suffices to consider the action of unipotent lower triangular matrices on the highest weight vector. This is made precise by the following lemma.

Lemma 11.8. *Let V be a $K\mathrm{SL}_2(K)$-module on which T acts diagonalisably, and let U be a KB-submodule of V generated by some weight vector $v \in V$. Then $U_r \neq 0$ if and only if there exists some $\gamma \in K$ such that the component of $M_\gamma v$ in V_r is nonzero.*

Proof. For the "if" direction, it suffices to prove that if $v_1, \ldots, v_n$ are nonzero weight vectors with distinct weights $r_1, \ldots, r_n$ such that $v_1 + \cdots + v_n \in U$, then each v_i lies in U. We use induction on n. The case $n = 1$ is clear. Suppose $n > 1$, and write $x = v_1 + \cdots + v_n$. Choose $\alpha \in K$ such that $\alpha^{r_1} \neq \alpha^{r_n}$ (when K is finite this is possible since $|K| > |r_1| + |r_n|$ by our definition of weights), and let $g = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \in B \leqslant \mathrm{SL}_2(K)$. Then

$$U \ni gx - \alpha^{r_n} x = (\alpha^{r_1} - \alpha^{r_n})v_1 + \ldots + (\alpha^{r_{n-1}} - \alpha^{r_n})v_{n-1}.$$

By the inductive hypothesis, $v_1 \in U$, and hence $x - v_1 \in U$. Then by the inductive hypothesis applied to $x - v_1$, we also have $v_2, \dots, v_n \in U$.

Conversely, suppose $U_r \neq 0$. Then there exists some $g \in B$ such that gv has nonzero component in V_r. An element of B can be written as $g = M_\gamma \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$ for some $\alpha, \gamma \in K$, and since v is a weight vector we have that $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) v$ is a nonzero scalar multiple of v. Thus $M_\gamma v$ has nonzero component in V_r. $\square$

Finally in this subsection we record a lemma which is of great use when ruling out certain elements from being in defect sets. Given subsets $I, J \subseteq \mathbb{N}_0$, let $I + J = \{\, i + j \mid i \in I,\, j \in J \,\}$.

Lemma 11.9. *Suppose V and W are $K\mathrm{SL}_2(K)$-modules on which T acts diagonalisably with a unique highest weight vector.*

(i) If $\varphi \colon V \to W$ is a homomorphism that does not annihilate the highest weight vector of V, then $\mathcal{D}(\operatorname{im}\varphi)$ is defined and $\mathcal{D}(\operatorname{im}\varphi) \subseteq \mathcal{D}(V)$. In particular, if W is a quotient of V, then $\mathcal{D}(W) \subseteq \mathcal{D}(V)$.

(ii) Suppose $|K| - 1$ is strictly greater than twice the sum of the highest weights of V and W. Then the set $\mathcal{D}(V \otimes W)$ is defined and $\mathcal{D}(V \otimes W) \subseteq \mathcal{D}(V) + \mathcal{D}(W)$.

Proof. Let $\{v_1, \dots, v_l\}$ and $\{w_1, \dots, w_m\}$ be weight vector bases for V and W respectively, with v_i having weight r_i and w_j having weight s_j, for some integers $r_1 \leqslant \cdots \leqslant r_{l-1} < r_l$ and $s_1 \leqslant \cdots \leqslant s_{m-1} < s_m$. We use the characterisation from Lemma 11.8 for the presence of elements in the defect sets.

For (i), observe that if $\varphi(v_i)$ is nonzero, then it is a weight vector of weight r_i; thus $\varphi(V_r) \subseteq W_r$ for all $r \in \mathbb{Z}$. Then $\{\varphi(v_1), \dots, \varphi(v_l)\}$ contains a weight vector basis for $\operatorname{im}\varphi$. Note that $\varphi(v_l)$ is in this basis since it is nonzero by assumption and is the unique element of the spanning set with weight r_l. Thus $\varphi(v_l)$ is the unique highest weight vector of $\operatorname{im}\varphi$, so $\mathcal{D}(\operatorname{im}\varphi)$ is defined. Furthermore, if $M_\gamma \varphi(v_l)$ has nonzero component in the weight space W_r, then $M_\gamma v_l$ has nonzero component in V_r.

For (ii), observe that $v_i \otimes w_j$ is a weight vector of weight $r_i + w_j$ (using the hypothesis on the field size). Moreover, the set $\{ v_i \otimes w_j \mid 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant m \}$ is a weight vector basis for $V \otimes W$, and $v_l \otimes w_m$ is the unique highest weight vector. Thus $\mathcal{D}(V \otimes W)$ is defined. The containment $\mathcal{D}(V \otimes W) \subseteq \mathcal{D}(V) + \mathcal{D}(W)$ is clear: if there exists γ such that $M_\gamma(v_l \otimes w_m)$ has nonzero component in the weight space $(V \otimes W)_{r_l + s_m - 2d}$, then, since $M_\gamma(v_l \otimes w_m) = (M_\gamma v_l) \otimes (M_\gamma w_m)$, there exists i, j such that $i + j = d$ and $M_\gamma v_l$ and $M_\gamma w_m$ have nonzero components in the weight spaces $V_{r_l - 2i}$ and $W_{s_m - 2j}$. $\square$

11.2. Symmetric powers and carry-free sums

In this subsection we identify the defect sets for iterated symmetric powers. This prepares the ground for the proof of Theorem E, and also yields Proposition 11.12, characterising when symmetric powers are isomorphic to their duals, and Proposition 11.14, demonstrating that our Theorem D is the unique modular generalisation of Hermite reciprocity.

For $a \in \{0, \ldots, l\}$, let $(X^{\otimes l-a} \otimes Y^{\otimes a})^{\mathrm{sym}} \in \mathrm{Sym}_l E$ be the sum of all $\binom{l}{a}$ pure tensors $Z_1 \otimes \cdots \otimes Z_l$ where exactly $l - a$ of the factors are X and the remaining a are Y.

Binomial and multinomial coefficients will frequently appear when expanding the action of matrices M_γ on symmetric powers. To determine when these coefficients are nonzero modulo p, we require the notion of carry-free sums.

Definition 11.10. Let $a_1, \ldots, a_s \in \mathbb{N}_0$, and write $a_i^{(j)}$ for the base p digit of a_i corresponding to the power of p^j. We say that the sum $a_1 + \cdots + a_s$ is *carry-free in base* p if $a_1^{(j)} + \cdots + a_s^{(j)} \leqslant p - 1$ for all j. For $a, l \in \mathbb{N}_0$, we say that a is a *carry-free summand* of l, denoted $a \trianglelefteq l$, if $a \leqslant l$ and the sum $a + (l - a)$ is carry-free.

Equivalently, $a_1 + \cdots + a_s$ is carry-free in base p if the sum can be computed in base p without carrying, by the usual algorithm taught in schools for base 10. Lucas's Theorem (see for instance [Jam78, Lemma 22.4]) states that the binomial coefficient $\binom{l}{a}$ is nonzero modulo p if and only if $a \trianglelefteq l$, and more generally that the multinomial coefficient $\binom{a_1+\cdots+a_s}{a_1,\ldots,a_s}$ is nonzero modulo p if and only if the sum $a_1 + \cdots + a_s$ is carry-free.

Lemma 11.11. *Let $l \in \mathbb{N}_0$. If K has prime characteristic p and $|K| > 1 + 2l$ then*

(i) $\mathcal{D}(\mathrm{Sym}_l\, E) = \{0, \ldots, l\}$*;*
(ii) $\mathcal{D}(\mathrm{Sym}^l E) = \{\, d \in \{0, \ldots, l\} \mid d \trianglelefteq l \,\}$*.*

Proof. A highest weight vector of $\mathrm{Sym}_l\, E$ is $X^{\otimes l}$ and a highest weight vector of $\mathrm{Sym}^l E$ is X^l. A simple calculation yields

$$M_\gamma(X^{\otimes l}) = \sum_{d=0}^{l} \gamma^d (X^{\otimes l-d} Y^{\otimes d})^{\mathrm{sym}},$$
$$M_\gamma(X^l) = \sum_{d=0}^{l} \gamma^d \binom{l}{d} X^{l-d} Y^d.$$

Note that $X^{\otimes l-d} \otimes Y^{\otimes d}$ and $X^{l-d}Y^d$ have weight $l - 2d$; using Lemma 11.8 and Lucas's Theorem mentioned above, the defect sets are then clear. $\square$

In the following proposition we use the defect set to distinguish non-isomorphic symmetric powers; this sharpens the well-known Proposition 3.5.

Proposition 11.12. *Let $l \in \mathbb{N}_0$. If K has prime characteristic p and $|K| > 1 + 2l$ then* $\mathrm{Sym}^l E \cong \mathrm{Sym}_l\, E$ *if and only if $l < p$ or $l = p^\varepsilon - 1$ for some $\varepsilon \in \mathbb{N}$. If K has characteristic zero then* $\mathrm{Sym}^l E \cong \mathrm{Sym}_l\, E$ *for any l.*

Proof. The condition that $l < p$ or $l = p^\varepsilon - 1$ for some $\varepsilon \in \mathbb{N}$ is equivalent to the condition that $a \trianglelefteq l$ for all $a \in \{0, \ldots, l\}$: if $l < p$ then we clearly have $a \trianglelefteq l$ for all $a \in \{0, \ldots, l\}$; if $l \geqslant p$ then $a \trianglelefteq l$ for all $a \in \{0, \ldots, l\}$ if and only if all base p digits of l are $p - 1$, which is if and only if $l = p^\varepsilon - 1$.

By Lemma 11.11, if $\mathrm{Sym}^l E \cong \mathrm{Sym}_l\, E$ then $a \trianglelefteq l$ for all $a \in \{0, \ldots, l\}$, as required. Conversely, consider the composition of the canonical maps

$$\mathrm{Sym}_l\, E \hookrightarrow E^{\otimes l} \twoheadrightarrow \mathrm{Sym}^l E$$

which sends $(X^{\otimes l-a} \otimes Y^{\otimes a})^{\mathrm{sym}} \in \mathrm{Sym}_l E$ to $\binom{l}{a} X^{l-a} Y^a$. Supposing $a \preccurlyeq l$ for all $a \in \{0, \dots, l\}$, or supposing instead the ground field has characteristic zero, we have that $\binom{l}{a} \neq 0$, and so this is an isomorphism. $\square$

Lemma 11.13. *Let m, $l \in \mathbb{N}_0$. Suppose that K has prime characteristic p and that $|K| > 1 + 2lm$. Then*

$$\mathcal{D}(\mathrm{Sym}_m \mathrm{Sym}_l E) = \{0, \dots, lm\};$$

$$\mathcal{D}(\mathrm{Sym}_m \mathrm{Sym}^l E) = \left\{ \sum_{j=0}^{l} j m_j \;\middle|\; \begin{array}{l} m_0, \dots, m_l \in \mathbb{N}_0,\ m_0 + \dots + m_l = m, \\ j \preccurlyeq l \textit{ for all } j \textit{ such that } m_j \neq 0 \end{array} \right\};$$

$$\mathcal{D}(\mathrm{Sym}^m \mathrm{Sym}_l E) = \left\{ \sum_{j=0}^{l} j m_j \;\middle|\; \begin{array}{l} m_0, \dots, m_l \in \mathbb{N}_0,\ m_0 + \dots + m_l = m, \\ m_0 + \dots + m_l \textit{ is carry-free} \end{array} \right\};$$

$$\mathcal{D}(\mathrm{Sym}^m \mathrm{Sym}^l E) = \left\{ \sum_{j=0}^{l} j m_j \;\middle|\; \begin{array}{l} m_0, \dots, m_l \in \mathbb{N}_0,\ m_0 + \dots + m_l = m, \\ m_0 + \dots + m_l \textit{ is carry-free,} \\ j \preccurlyeq l \textit{ for all } j \textit{ such that } m_j \neq 0 \end{array} \right\}.$$

Proof. We compute $\mathcal{D}(\mathrm{Sym}^m \mathrm{Sym}^l E)$. The highest weight vector is $(X^l)^m$ of weight lm, so it suffices to consider the expansion

$$\begin{aligned} M_\gamma(X^l)^m = \big((X + \gamma Y)^l\big)^m &= \left(\sum_{j=0}^{l} \binom{l}{j} \gamma^j X^{l-j} Y^j \right)^m \\ &= \sum_{\substack{m_0, \dots, m_l \in \mathbb{N}_0 \\ m_0 + \dots + m_l = m}} \binom{m}{m_0, \dots, m_l} \prod_{j=0}^{l} \left(\binom{l}{j} \gamma^j X^{l-j} Y^j \right)^{m_j}. \end{aligned}$$

The vectors of weight $lm - 2d$ are precisely the elements $\prod_{j=0}^{l} (X^{l-j} Y^j)^{m_j}$ where $\sum_{j=0}^{l} j m_j = d$, and such an element appears with nonzero coefficient in this expansion if and only if the corresponding binomial and multinomial coefficients are nonzero. Lucas's Theorem then yields the claimed defect set. The other parts follow similarly, with the binomial and/or multinomial coefficients not appearing in the expansion when the first and/or second symmetric powers are lower respectively. $\square$

Diverting our attention briefly from Theorem E, we conclude this subsection by showing that Theorem D is the unique modular generalisation of Hermite reciprocity.

Proposition 11.14. *Let $\varepsilon > 1$. Suppose that K has characteristic p and $|K| \geqslant p^{\varepsilon+3}$. The eight modules obtained from $\mathrm{Sym}^p \mathrm{Sym}^{p^\varepsilon} E$ by exchanging the order of the symmetric powers and replacing upper symmetric powers with lower symmetric powers are pairwise non-isomorphic, with the exceptions of*

$$\mathrm{Sym}_p \mathrm{Sym}^{p^\varepsilon} E \cong \mathrm{Sym}^{p^\varepsilon} \mathrm{Sym}_p E,$$

its dual

$$\mathrm{Sym}^p \mathrm{Sym}_{p^\varepsilon} E \cong \mathrm{Sym}_{p^\varepsilon} \mathrm{Sym}^p E,$$

and the possible exceptions of an isomorphism $\mathrm{Sym}^p \mathrm{Sym}^{p^\varepsilon} E \cong \mathrm{Sym}^{p^\varepsilon} \mathrm{Sym}^p E$ and its dual $\mathrm{Sym}_p \mathrm{Sym}_{p^\varepsilon} E \cong \mathrm{Sym}_{p^\varepsilon} \mathrm{Sym}_p E$. In particular there are either four or six isomorphism classes amongst these modules. If $p = 2$ the possible exceptions do not occur and there are precisely six isomorphism classes of modules.

Proof. Since $|K| > 1 + 2p^{\varepsilon+1}$, Lemma 11.13 applies. Routine applications give

$$\begin{aligned}
\mathcal{D}(\mathrm{Sym}_p \mathrm{Sym}_{p^\varepsilon} E) &= \mathcal{D}(\mathrm{Sym}_{p^\varepsilon} \mathrm{Sym}_p E) = \{0, 1, \ldots, p^{\varepsilon+1} - 1, p^{\varepsilon+1}\} \\
\mathcal{D}(\mathrm{Sym}^p \mathrm{Sym}_{p^\varepsilon} E) &= \mathcal{D}(\mathrm{Sym}_{p^\varepsilon} \mathrm{Sym}^p E) = \{0, p, 2p, \ldots, p^{\varepsilon+1} - p, p^{\varepsilon+1}\} \\
\mathcal{D}(\mathrm{Sym}_p \mathrm{Sym}^{p^\varepsilon} E) &= \mathcal{D}(\mathrm{Sym}^{p^\varepsilon} \mathrm{Sym}_p E) = \{0, p^\varepsilon, p^{\varepsilon+1}\} \\
\mathcal{D}(\mathrm{Sym}^p \mathrm{Sym}^{p^\varepsilon} E) &= \mathcal{D}(\mathrm{Sym}^{p^\varepsilon} \mathrm{Sym}^p E) = \{0, p^{\varepsilon+1}\}.
\end{aligned}$$

The third equality is expected from the isomorphism in Theorem D and the second from its dual; these are the two certain exceptions stated in the theorem. By Proposition 3.7, $\mathrm{Sym}_p \mathrm{Sym}_{p^\varepsilon} E \cong (\mathrm{Sym}^p \mathrm{Sym}^{p^\varepsilon} E)^\star$ and $\mathrm{Sym}_{p^\varepsilon} \mathrm{Sym}_p E \cong (\mathrm{Sym}^{p^\varepsilon} \mathrm{Sym}^p E)^\star$, so either both or neither of the possible exceptions occur. Therefore it remains only to prove, when $p = 2$, that $\mathrm{Sym}^2 \mathrm{Sym}^{2^\varepsilon} E \not\cong \mathrm{Sym}^{2^\varepsilon} \mathrm{Sym}^2 E$.

Again we use weight spaces, this time identifying a difference in the KB-submodules generated by the 0-weight space. The 0-weight space of

$\mathrm{Sym}^{2^\varepsilon}\,\mathrm{Sym}^2 E$ is spanned by all $(X^2)^{2^{\varepsilon-1}-a} \cdot (XY)^{2a} \cdot (Y^2)^{2^{\varepsilon-1}-a}$ for $0 \leqslant a \leqslant 2^{\varepsilon-1}$. Applying M_γ we get

$$(X^2 + \gamma^2 Y^2)^{2^{\varepsilon-1}-a} \cdot \big((X+\gamma Y)Y\big)^{2a} \cdot (Y^2)^{2^{\varepsilon-1}-a},$$

in which each factor has only even powers of X and Y. Thus the KB-submodule of $\mathrm{Sym}^{2^\varepsilon}\,\mathrm{Sym}^2 E$ generated by the 0-weight space has all weights congruent to 0 modulo 4. Meanwhile the 0-weight space of $\mathrm{Sym}^2\,\mathrm{Sym}^{2^\varepsilon} E$ contains $(X^{2^\varepsilon-1}Y) \cdot (XY^{2^\varepsilon-1})$; applying M_γ to this we get $(X+\gamma Y)^{2^\varepsilon-1}Y \cdot (X+\gamma Y)Y^{2^\varepsilon-1}$, whose expansion has $X^{2^\varepsilon-1}Y \cdot \gamma Y^{2^\varepsilon}$ with coefficient 1. Therefore the KB-submodule of $\mathrm{Sym}^2\,\mathrm{Sym}^{2^\varepsilon} E$ generated by the 0-weight space has a nonzero weight space for the weight -2. $\square$

If we work instead over the complex numbers, all eight modules in Proposition 11.14 are isomorphic (by classical Hermite reciprocity and Proposition 3.5).

11.3. Defect sets for hook Schur endofunctors

Our overall strategy is to use defect sets to distinguish the eight modules in Theorem E. The reader is invited to refer ahead to §11.5 to see how this is accomplished using the properties of defect sets identified in this subsection and the next. For the remainder of this section, K denotes a field of prime characteristic p. In this subsection we study the defect sets of the modules $\nabla^{(a+1,1^b)}\,\mathrm{Sym}^l E$ and $\nabla^{(a+1,1^b)}\,\mathrm{Sym}_l E$; in the next subsection, we do the same with Δ in place of ∇.

To identify elements of the defect sets, we need to evaluate the action of M_γ on the highest weight vectors. Working with $\nabla^{(a+1,1^b)}$, we can use the simple multilinear expansion rule for the polytabloids exemplified in Example 2.2. We also need the description of the action of M_γ on the canonical bases of $\mathrm{Sym}^l E$ and $\mathrm{Sym}_l E$, given by the following lemma.

Lemma 11.15. *We have*

(i) $M_\gamma (X^{\otimes i} \otimes Y^{\otimes l-i})^{\mathrm{sym}} = \sum_{j=0}^{i} \gamma^{i-j} \binom{l-j}{l-i} (X^{\otimes j} \otimes Y^{\otimes l-j})^{\mathrm{sym}},$

(ii) $M_\gamma(X^iY^{l-i}) = \sum_{j=0}^{i} \gamma^{i-j}\binom{i}{j} X^jY^{l-j}$.

Proof. Part (ii) is obvious from expanding $(X+\gamma Y)^iY^{l-i}$. For part (i), observe that $M_\gamma(X^{\otimes i}\otimes Y^{\otimes l-i})^{\mathrm{sym}}$ is the sum of all $\binom{l}{i}$ tensor products $Z_1\otimes\cdots\otimes Z_l$ where exactly i of the factors are $X+\gamma Y$ and the remaining $l-i$ are Y. Expanding into pure tensors in X and Y, there are $\binom{l}{i}\binom{i}{j}$ summands with j factors of X and $l-j$ factors of Y (each with coefficient γ^{i-j}). Then since $\binom{l}{j}$ such summands are required to form $(X^{\otimes j}\otimes Y^{\otimes l-j})^{\mathrm{sym}}$, the number of times this vector (each with coefficient γ^{i-j}) occurs is $\binom{l}{i}\binom{i}{j}\binom{l}{j}^{-1} = \binom{l-j}{l-i}$. $\square$

Lemma 11.16. *Let $a, b, l \in \mathbb{N}$ and suppose $|K| > 1+2(a+b+1)l - b(b+1)$. If $b \not\equiv -1$ mod p, then $1 \in \mathcal{D}(\nabla^{(a+1,1^b)}\operatorname{Sym}_l E)$.*

Proof. Let $t_{\max}$ be the tableau of shape $(a+1, 1^b)$ labelling the highest weight vector of $\nabla^{(a+1,1^b)}\operatorname{Sym}_l E$ identified in Proposition 11.6; by this proposition, its weight is $(a+1)l+(l-1)+\cdots+(l-b) = (a+b+1)l-b(b+1)/2$, whence the bound on $|K|$. Let s be the tableau obtained from $t_{\max}$ by reducing the entry in the top-left corner by 1. That is,

$$t_{\max} = \begin{array}{|c|c|c|c|}\hline l-b & l & \cdots & l \\ \hline l-b+1 \\ \cline{1-1} \vdots \\ \cline{1-1} l-1 \\ \cline{1-1} l \\ \cline{1-1}\end{array} \qquad \text{and} \quad s = \begin{array}{|c|c|c|c|}\hline l-b-1 & l & \cdots & l \\ \hline l-b+1 \\ \cline{1-1} \vdots \\ \cline{1-1} l-1 \\ \cline{1-1} l \\ \cline{1-1}\end{array}$$

where an entry of i corresponds to the basis vector $v_i = (X^{\otimes i}\otimes Y^{\otimes l-i})^{\mathrm{sym}}$.

We compute $M_\gamma\,\mathrm{e}(t_{\max})$ by acting on the entry in each box of $t_{\max}$, as in Example 2.2, and then using Garnir relations (see Definition 1.8) to express the result in the basis of semistandard polytabloids. Note that the Garnir relations do not change the multiset of entries of a tableau; thus to identify the coefficient of a semistandard polytabloid, it suffices to consider only those tableaux with the same multiset of entries.

By Lemma 11.15(i), $M_\gamma v_i = \sum_{j=0}^{l-i} \gamma^{i-j} \binom{l-j}{l-i} v_j$. The action of M_γ on the entries of $t_{\max}$ yields

$\sum\limits_{j=0}^{l-b} \gamma^{l-b-j} \binom{l-j}{b} v_j$	$\sum\limits_{j=0}^{l} \gamma^{l-j} v_j$	$\cdots$	$\sum\limits_{j=0}^{l} \gamma^{l-j} v_j$
$\sum\limits_{j=0}^{l-b+1} \gamma^{l-b+1-j} \binom{l-j}{b-1} v_j$			
$\vdots$			
$\sum\limits_{j=0}^{l-1} \gamma^{l-1-j} \binom{l-j}{1} v_j$			
$\sum\limits_{j=0}^{l} \gamma^{l-j} v_j$			

before multilinear expansion.

Consider how we can choose summands to obtain a tableau with the same multiset of entries as s. Since v_l must occur $a+1$ times, we must choose v_l from the sums in the $a+1$ boxes in which it appears; then v_{l-1} must occur once, so must be chosen in the only remaining sum in which it appears; and so on, until we choose v_{l-b+1} from the box immediately below the top-left box. Finally we must choose v_{l-b-1} from the box in the top-left. The coefficients arising from this choice are $\binom{b+1}{1}\gamma$ from the top-left box and 1s from every remaining box.

Since this sequence of choices gives the semistandard tableau s, no rewriting using Garnir relations is necessary, and it follows that the coefficient of $\mathrm{e}(s)$ in $M_\gamma\, \mathrm{e}(t_{\max})$ is $(b+1)\gamma$; this is nonzero by the hypothesis on b. $\square$

Lemma 11.17. *Let $\alpha, \beta, \varepsilon \in \mathbb{N}$ with $\alpha \neq \beta$ and $\alpha, \beta < \varepsilon$. Suppose $|K| > 1 + 2(p^\varepsilon + p^\beta)(p^\alpha + p^\beta + 1) - p^\beta(p^\beta + 1)$. Then*

(i) $p^{\beta+\varepsilon} - p^\varepsilon \in \mathcal{D}(\nabla^{(p^\alpha+1,1^{p^\beta})} \operatorname{Sym}^{p^\varepsilon + p^\beta} E)$;

(ii) 1, p^α, p^β, $p^{\alpha+\varepsilon} - p^\varepsilon \notin \mathcal{D}(\nabla^{(p^\alpha+1,1^{p^\beta})} \operatorname{Sym}^{p^\varepsilon + p^\beta} E)$.

Proof. For part (i), we consider (as in the proof of Lemma 11.16) how we can expand $M_\gamma\, \mathrm{e}(t_{\max})$ to obtain tableaux with certain multisets of entries. This time we choose the tableau s obtained from $t_{\max}$ by reducing all the entries in the first column by p^ε, except the first and last. That is,

$$t_{\max}=\begin{array}{|c|c|c|c|}\hline p^\varepsilon & p^\varepsilon+p^\beta & \cdots & p^\varepsilon+p^\beta \\ \hline p^\varepsilon+1 \\ \cline{1-1} \vdots \\ \cline{1-1} p^\varepsilon+p^\beta-1 \\ \cline{1-1} p^\varepsilon+p^\beta \\ \cline{1-1}\end{array} \qquad \text{and } s=\begin{array}{|c|c|c|c|}\hline p^\varepsilon & p^\varepsilon+p^\beta & \cdots & p^\varepsilon+p^\beta \\ \hline 1 \\ \cline{1-1} \vdots \\ \cline{1-1} p^\beta-1 \\ \cline{1-1} p^\varepsilon+p^\beta \\ \cline{1-1}\end{array}$$

where an entry of i corresponds to the basis vector $w_i = X^iY^{p^\varepsilon+p^\beta-i} \in \operatorname{Sym}^l E$. By Lemma 11.15(ii), $M_\gamma w_i = \sum_{j=0}^{l-i} \gamma^{i-j}\binom{i}{j} w_j$. Acting by M_γ on each entry of $t_{\max}$ yields

$$\begin{array}{|c|c|c|c|}\hline \displaystyle\sum_{j=0}^{p^\varepsilon} \gamma^{p^\varepsilon-j}\binom{p^\varepsilon}{j} w_j & \displaystyle\sum_{j=0}^{p^\varepsilon+p^\beta} \gamma^\star\binom{p^\varepsilon+p^\beta}{j} w_j & \cdots & \displaystyle\sum_{j=0}^{p^\varepsilon+p^\beta} \gamma^\star\binom{p^\varepsilon+p^\beta}{j} w_j \\ \hline \displaystyle\sum_{j=0}^{p^\varepsilon+1} \gamma^\star\binom{p^\varepsilon+1}{j} w_j \\ \cline{1-1} \vdots \\ \cline{1-1} \displaystyle\sum_{j=0}^{p^\varepsilon+p^\beta-1} \gamma^\star\binom{p^\varepsilon+p^\beta-1}{j} w_j \\ \cline{1-1} \displaystyle\sum_{j=0}^{p^\varepsilon+p^\beta} \gamma^\star\binom{p^\varepsilon+p^\beta}{j} w_j \\ \cline{1-1}\end{array}$$

before multilinear expansion, where $\gamma^\star$ denotes a power of γ omitted for reasons of space.

Consider how we can choose summands to obtain a tableau with the same multiset of entries as s. As before, since $w_{p^\varepsilon+p^\beta}$ must occur $p^\alpha+1$ many times, we must choose $w_{p^\varepsilon+p^\beta}$ from the sums in the $p^\alpha+1$ boxes in which it appears. Thus there is a unique choice in each box at the bottom of a column, and each such choice gives a coefficient of $\gamma^0=1$.

For the remaining p^β boxes, note that for $0 \leqslant i, j < p^\varepsilon$, we have $\binom{p^\varepsilon+i}{j} = \binom{i}{j}$ which is nonzero if and only if $j \trianglelefteqslant i$, which in particular requires $j \leqslant i$. Thus, since $\beta < \varepsilon$, the only remaining sum in which $w_{p^\beta-1}$ appears with nonzero coefficient is that in the penultimate box in the first column, so it must be chosen there; continuing, we must choose w_j from the sum in box $(j,1)$ for all $2 \leqslant j \leqslant p^\beta - 1$. Each of these choices gives a coefficient of γ^{p^ε}. Finally, in the top-left box w_{p^ε} must then be chosen. Thus there is a unique way to obtain a tableau with the same multiset of entries as s and the coefficient is $\gamma^{(p^\beta-1)p^\varepsilon}$. Therefore, writing s' for the semistandard tableau obtained from s by sorting the first column into ascending order, the coefficient of $\mathrm{e}(s')$ in $M_\gamma\, \mathrm{e}(t_{\max})$ is $\pm\gamma^{p^{\beta+\varepsilon}-p^\varepsilon} \neq 0$, as required.

For (ii), we recall that the module $\nabla^{(p^\alpha+1,1^{p^\beta})}\,\mathrm{Sym}^{p^\varepsilon+p^\beta}E$ is the image of the partition-labelled exterior power $\bigwedge^{p^\beta+1}\mathrm{Sym}^{p^\varepsilon+p^\beta}E \otimes (\mathrm{Sym}^{p^\varepsilon+p^\beta}E)^{\otimes p^\alpha}$ under the canonical quotient map $|t| \mapsto \mathrm{e}(t)$. Moreover, if t is a tableau and $\tau = \big((1,j)\ (1,j+1)\big)$ then, by the Garnir relation $\mathsf{R}_{(t,\{(1,j)\},\{(1,j+1)\}}$, we have $\mathrm{e}(t) = \mathrm{e}(t\cdot\tau)$; therefore if t and t' are tableaux differing only in the order of the entries in the top row (excluding the top-left box), then $\mathrm{e}(t) = \mathrm{e}(t')$. Hence the quotient map factors through

$$\textstyle\bigwedge^{p^\beta+1}\mathrm{Sym}^{p^\varepsilon+p^\beta}E \otimes \mathrm{Sym}^{p^\alpha}\mathrm{Sym}^{p^\varepsilon+p^\beta}E.$$

It follows, using both parts of Lemma 11.9, that

$$\begin{aligned}
&\mathcal{D}(\nabla^{(p^\alpha+1,1^{p^\beta})}\,\mathrm{Sym}^{p^\varepsilon+p^\beta}E)\\
&\quad\subseteq \mathcal{D}(\textstyle\bigwedge^{p^\beta+1}\mathrm{Sym}^{p^\varepsilon+p^\beta}E \otimes \mathrm{Sym}^{p^\alpha}\mathrm{Sym}^{p^\varepsilon+p^\beta}E)\\
&\quad\subseteq \mathcal{D}(\textstyle\bigwedge^{p^\beta+1}\mathrm{Sym}^{p^\varepsilon+p^\beta}E) + \mathcal{D}(\mathrm{Sym}^{p^\alpha}\mathrm{Sym}^{p^\varepsilon+p^\beta}E).
\end{aligned}$$

Applying the Wronskian isomorphism $\bigwedge^{p^\beta+1}\mathrm{Sym}^{p^\varepsilon+p^\beta}E \cong \mathrm{Sym}_{p^\beta+1}\mathrm{Sym}^{p^\varepsilon}E$ from Theorem C, this becomes

$$\begin{aligned}
&\mathcal{D}(\nabla^{(p^\alpha+1,1^{p^\beta})}\,\mathrm{Sym}^{p^\varepsilon+p^\beta}E)\\
&\quad\subseteq \mathcal{D}(\mathrm{Sym}_{p^\beta+1}\mathrm{Sym}^{p^\varepsilon}E) + \mathcal{D}(\mathrm{Sym}^{p^\alpha}\mathrm{Sym}^{p^\varepsilon+p^\beta}E).
\end{aligned}$$

From Lemma 11.13 we have

$$\begin{aligned}
\mathcal{D}(\mathrm{Sym}^{p^\alpha}\mathrm{Sym}^{p^\varepsilon+p^\beta}E) &= \{0, p^{\alpha+\beta}, p^{\alpha+\varepsilon}, p^{\alpha+\beta}+p^{\alpha+\varepsilon}\},\\
\mathcal{D}(\mathrm{Sym}_{p^\beta+1}\mathrm{Sym}^{p^\varepsilon}E) &= \{\, cp^\varepsilon \mid 0 \leqslant c \leqslant p^\beta+1\,\}.
\end{aligned}$$

It is clear that 1, p^α, p^β and $p^{\alpha+\varepsilon} - p^\varepsilon$ are not in this set. $\square$

11.4. Defect sets for hook Weyl endofunctors

In this subsection we show that $\Delta^\lambda V$, viewed as a submodule of the partition-labelled exterior power $\bigwedge^{\lambda'} V$, contains the highest weight vector of $\bigwedge^{\lambda'} V$. We use this fact to compute the defect set $\mathcal{D}(\Delta^{(a+1,1^b)}V)$ by working in $\bigwedge^{(a+1,1^b)'} V$, which has a canonical basis labelled by column standard column tabloids of shape $(a+1, 1^b)$ (see §1.5).

Lemma 11.18. *Let V be a $K\mathrm{SL}_2(K)$-module with a basis $\{v_1, \ldots, v_l\}$ of weight vectors, in which v_i has weight r_i, for some integers $r_1 \leqslant \cdots \leqslant r_{l-1} < r_l$. Let λ be any partition, and let $t_{\max}$ be the semistandard tableau obtained by filling each column from the bottom with integers decreasing from l. Suppose that $|K| > 1 + 2\sum_{b\in[\lambda]} r_{t_{\max}(b)}$. Then the unique highest weight vector $|t_{\max}|$ of $\bigwedge^{\lambda'} V$ is contained in $\Delta^\lambda V$. In particular, $\mathcal{D}(\Delta^\lambda V) = \mathcal{D}(\bigwedge^{\lambda'} V)$.*

Proof. Let $t'_{\max}$ be the tableau obtained from $t_{\max}$ by reversing the order of each column; thus $t'_{\max}(i,j) = l + 1 - i$ for all i, j. In particular, the row stabiliser of $t'_{\max}$ is trivial and hence $\text{ə}(t'_{\max}) = |t'_{\max}| = \pm|t_{\max}|$, so $|t_{\max}| \in \Delta^\lambda V$ as required. □

Remark 11.19. It is possible to deduce this result without the complete description of $\Delta^\lambda V$ from §3. It suffices to observe that $\Delta^\lambda V$, defined as the dual $(\nabla^\lambda V^\circ)^\circ$, has highest weight equal to the highest weight of $\bigwedge^{\lambda'} V$ (each with weight space of dimension 1), and that the map e^* from the proof of Proposition 3.13 used to view $\Delta^\lambda V$ as a submodule of $\bigwedge^{\lambda'} V$ does not annihilate the unique highest weight vector (see [McDW21, Lemma 6.16]).

Lemma 11.20. *Let $a, b, l \in \mathbb{N}$. Suppose that $|K| > 1+2(a+b+1)l - b(b+1)$. Then $1 \in \mathcal{D}(\Delta^{(a+1,1^b)} \mathrm{Sym}_l E)$.*

Proof. In light of Lemma 11.18, the claim is equivalent to

$$1 \in \mathcal{D}\big(\textstyle\bigwedge^{b+1} \mathrm{Sym}_l E \otimes (\mathrm{Sym}_l E)^{\otimes a}\big).$$

A unique highest weight vector of $\bigwedge^{b+1} \mathrm{Sym}_l E \otimes (\mathrm{Sym}_l E)^{\otimes a}$ is the column tabloid for the tableau $t_{\max}$ from Lemma 11.16; let s be the column standard tableau obtained from $t_{\max}$ by reducing the entry in box $(1,2)$ by 1. Then

$$\begin{aligned} |t_{\max}| &= \bigl((X^{\otimes l-b}\otimes Y^{\otimes b})^{\mathrm{sym}} \wedge \cdots \wedge X^{\otimes l}\bigr) \otimes \bigl(X^{\otimes l}\bigr)^{\otimes a}, \\ |s| &= \bigl((X^{\otimes l-b}\otimes Y^{\otimes b})^{\mathrm{sym}} \wedge \cdots \wedge X^{\otimes l}\bigr) \otimes (X^{\otimes l-1}\otimes Y)^{\mathrm{sym}} \otimes \bigl(X^{\otimes l}\bigr)^{\otimes a-1}. \end{aligned}$$

The coefficient of $|s|$ in $M_\gamma|t_{\max}|$ is the coefficient of $(X^{\otimes l-1} \otimes Y)^{\mathrm{sym}}$ in $M_\gamma X^{\otimes l}$, which is γ. Thus $|s|$ is in the KB-submodule generated by the highest weight vector, giving the required element of the defect set. $\square$

Lemma 11.21. *Let $\alpha, \beta, \varepsilon \in \mathbb{N}$ with $\alpha \neq \beta$ and $\alpha, \beta < \varepsilon$. Suppose that $|K| > 1 + 2(p^\varepsilon + p^\beta)(p^\alpha + p^\beta + 1) - p^\beta(p^\beta + 1)$. Then*
(i) $p^\beta \in \mathcal{D}(\Delta^{(p^\alpha+1,1^{p^\beta})} \operatorname{Sym}^{p^\varepsilon+p^\beta} E)$;
(ii) 1, $p^\alpha \notin \mathcal{D}(\Delta^{(p^\alpha+1,1^{p^\beta})} \operatorname{Sym}^{p^\varepsilon+p^\beta} E)$.

PROOF. As in the proof of Lemma 11.20, we use Lemma 11.18 to work in $\bigwedge^{p^\beta+1} \operatorname{Sym}^{p^\varepsilon+p^\beta} E \otimes (\operatorname{Sym}^{p^\varepsilon+p^\beta} E)^{\otimes p^\alpha}$ rather than $\Delta^{(p^\alpha+1,1^{p^\beta})} \operatorname{Sym}^{p^\varepsilon+p^\beta} E$.

The highest weight vector of $\bigwedge^{p^\beta+1} \operatorname{Sym}^{p^\varepsilon+p^\beta} E \otimes (\operatorname{Sym}^{p^\varepsilon+p^\beta} E)^{\otimes p^\alpha}$ is the column tabloid for the tableau $t_{\max}$ from Lemma 11.17; let s be the column standard tableau obtained from $t_{\max}$ by reducing the entry in box $(1,2)$ by p^β. Then

$$\begin{aligned} |t_{\max}| &= \bigl(X^{p^\varepsilon} Y^{p^\beta} \wedge \cdots \wedge X^{p^\varepsilon+p^\beta}\bigr) \otimes \bigl(X^{p^\varepsilon+p^\beta}\bigr)^{\otimes p^\alpha}, \\ |s| &= \bigl(X^{p^\varepsilon} Y^{p^\beta} \wedge \cdots \wedge X^{p^\varepsilon+p^\beta}\bigr) \otimes X^{p^\varepsilon} Y^{p^\beta} \otimes \bigl(X^{p^\varepsilon+p^\beta}\bigr)^{\otimes p^\alpha-1}. \end{aligned}$$

The coefficient of $|s|$ in $M_\gamma|t_{\max}|$ is the coefficient of $X^{p^\varepsilon} Y^{p^\beta}$ in $M_\gamma X^{p^\varepsilon+p^\beta}$, which is $\gamma^{p^\beta}\binom{p^\varepsilon+p^\beta}{p^\beta} \neq 0$. Thus $|s|$ is in the KB-submodule generated by the highest weight vector, proving (i).

For (ii), we use Lemma 11.9(ii) and the Wronskian isomorphism (Theorem C) to find that

$$\begin{aligned} &\mathcal{D}\bigl(\textstyle\bigwedge^{p^\beta+1} \operatorname{Sym}^{p^\varepsilon+p^\beta} E \otimes (\operatorname{Sym}^{p^\varepsilon+p^\beta} E)^{\otimes p^\alpha}\bigr) \\ &\qquad \subseteq \mathcal{D}(\textstyle\bigwedge^{p^\beta+1} \operatorname{Sym}^{p^\varepsilon+p^\beta} E) + \mathcal{D}(\operatorname{Sym}^{p^\beta+p^\varepsilon} E) + \cdots + \mathcal{D}(\operatorname{Sym}^{p^\beta+p^\varepsilon} E) \\ &\qquad = \mathcal{D}(\operatorname{Sym}_{p^\beta+1} \operatorname{Sym}^{p^\varepsilon} E) + \mathcal{D}(\operatorname{Sym}^{p^\beta+p^\varepsilon} E) + \cdots + \mathcal{D}(\operatorname{Sym}^{p^\beta+p^\varepsilon} E) \end{aligned}$$

where there are a copies of $\mathcal{D}(\mathrm{Sym}^{p^\beta+p^\varepsilon} E)$. Lemmas 11.11 and 11.13 give

$$\mathcal{D}(\mathrm{Sym}^{p^\beta+p^\varepsilon} E) = \{0, p^\beta, p^\varepsilon, p^\beta + p^\varepsilon\},$$
$$\mathcal{D}(\mathrm{Sym}_{p^\beta+1} \mathrm{Sym}^{p^\varepsilon} E) = \{\, cp^\varepsilon \mid 0 \leqslant c \leqslant p^\beta + 1 \,\}.$$

Since $\alpha < \varepsilon$ and $\alpha \neq \beta$, it is clear that 1 and p^α are not in this set. □

Remark 11.22. We remark on two interesting facts about $\Delta^\lambda V$ and its defect set, which can be used to give alternative proofs of the above lemmas in special cases.

(i) Suppose K is infinite and $V = E$ is the natural representation of $\mathrm{GL}_n(K)$. The module $\Delta^\lambda E$ is generated by its unique highest weight vector (as shown in [EGS08, (5.3b)], and noted earlier in Remark 3.14). By [Hum75, Proposition 31.2], the submodule of $\bigwedge^{\lambda'} E$ generated by its highest weight vector is the same whether we act by B or all of $\mathrm{SL}_2(K)$; thus in this case we have that every weight of $\Delta^\lambda E$ contributes to the defect set. That is, writing m for the highest weight, we have $\mathcal{D}(\Delta^\lambda V) = \{\, d \in \mathbb{N}_0 \mid (\Delta^\lambda E)_{m-2d} \neq 0 \,\}$.

(ii) Using Lemma 11.9 and the result from Lemma 11.18 that $\mathcal{D}(\Delta^\lambda V) = \mathcal{D}(\bigwedge^{\lambda'} V)$, we find that $\mathcal{D}(\Delta^\lambda V) \subseteq \sum_{j=1}^{\lambda_1} \mathcal{D}(\bigwedge^{\lambda'_j} V)$. When K is algebraically closed, it can be shown that this is an equality: indeed, under the conditions of Lemma 11.9, there is equality $\mathcal{D}(V \otimes W) = \mathcal{D}(V) + \mathcal{D}(W)$ because any two matrices M_γ and M_δ are conjugate in $\mathrm{SL}_2(K)$ by diagonal matrices, and so, up to a scalar, $M_\gamma v \otimes M_\delta w$ is equal to $M_\kappa(v \otimes w)$ for some suitable $\kappa \in K$.

11.5. Proof of Theorem E

We are now ready to prove the main theorem of this section.

Theorem E (Obstructions to the conjugate hook partition isomorphism)**.** *Let α, β, $\varepsilon \in \mathbb{N}$ with $\alpha < \beta < \varepsilon$. If K has characteristic p and $|K| > 1 + 2(p^\varepsilon + p^\beta)(p^\alpha + p^\beta + 1) - p^\alpha(p^\alpha + 1)$, then the eight representations of $\mathrm{SL}_2(K)$ obtained from $\Delta^{(p^\alpha+1,1^{p^\beta})} \mathrm{Sym}^{p^\varepsilon+p^\beta} E$ by any combination of*

- *replacing Δ with ∇,*
- *replacing Sym^- with Sym_-,*
- *swapping α and β,*

are pairwise non-isomorphic.

Proof. From Lemmas 11.17 and 11.21 we have

$$1,\, p^\alpha,\, p^\beta,\, p^{\alpha+\varepsilon} - p^\varepsilon \notin \mathcal{D}(\nabla^{(p^\alpha+1,1^{p^\beta})} \operatorname{Sym}^{p^\varepsilon+p^\beta}) \ni p^{\beta+\varepsilon} - p^\varepsilon$$

$$1,\, p^\alpha,\, p^\beta,\, p^{\beta+\varepsilon} - p^\varepsilon \notin \mathcal{D}(\nabla^{(p^\beta+1,1^{p^\alpha})} \operatorname{Sym}^{p^\varepsilon+p^\alpha}) \ni p^{\alpha+\varepsilon} - p^\varepsilon$$

$$1,\, p^\alpha \notin \mathcal{D}(\Delta^{(p^\alpha+1,1^{p^\beta})} \operatorname{Sym}^{p^\varepsilon+p^\beta}) \ni p^\beta$$

$$1,\, p^\beta \notin \mathcal{D}(\Delta^{(p^\beta+1,1^{p^\alpha})} \operatorname{Sym}^{p^\varepsilon+p^\alpha}) \ni p^\alpha$$

and from Lemmas 11.16 and 11.20 we have that 1 lies in each of the defect sets where Sym^- is replaced with Sym_-. Thus it is clear that the four modules whose defect sets are displayed above are pairwise non-isomorphic, and that none is isomorphic to any of the four modules obtained by replacing Sym^- with Sym_-. Finally, by applying contravariant duality to an isomorphism between any two of the latter four modules we would obtain an isomorphism between two modules defined using Sym^-. Therefore no two of the latter four modules are isomorphic. $\square$

CHAPTER IV

The Specht module under the inverse Schur functor

As usual, let K be a field, let $n, r \in \mathbb{N}$, and we consider modules for the symmetric group S_r and the general linear group $\mathrm{GL}_n(K)$.

The main results of this chapter are the following descriptions of the image of the Specht module under the inverse Schur functor $\mathcal{G}_\otimes = \mathcal{G}_\otimes^n$. Note that although the definition of the Schur functor $\mathcal{F}$ (Definition 6.5) requires $n \geqslant r$ and that K is infinite, the definition of $\mathcal{G}_\otimes$ (Definition 6.12) does not, and the main results of this chapter hold without these requirements.

Theorem F. *Suppose K has characteristic not 2 and let λ be a partition of r. Then there is an isomorphism $\mathcal{G}_\otimes(S^\lambda) \cong \nabla^\lambda E$.*

Theorem G. *Suppose K has characteristic 2 and let λ be a partition of r. There is a surjection $\mathcal{G}_\otimes(S^\lambda) \twoheadrightarrow \nabla^\lambda E$, which is an isomorphism if λ is 2-regular, or if $\lambda_1 = \lambda_2 \geqslant \lambda_3 + 2$ and λ minus its first part is 2-regular. Supposing also $n \geqslant r - 2$, if λ is not of this form then the surjection is not an isomorphism.*

Here E is the natural n-dimensional representation of $\mathrm{GL}_n(K)$ and S^λ is the Specht module for S_r. These results appear in the author's [McD21a].

Our approach utilises our constructions of the Specht and dual Weyl modules as quotients of suitable exterior powers by the Garnir relations. In §12 we obtain a description of the image of the Specht module as a quotient space similar to the dual Weyl module, with an important difference in characteristic 2: repeated entries in a column of a tableau do not cause the labelled element to vanish. We deduce that the isomorphism holds in characteristics other than 2 always, obtaining Theorem F, and that it holds in characteristic 2 if and only if every element labelled by a tableau with a repeated entry in a column can be written as a linear combination of the

'Garnir relations' in this setting (skew Garnir relations). Assessing for what partitions this condition holds is the goal of §13.

We prove various additional results about the image of the Specht module in characteristic 2 in §14: we demonstrate that the image need not have a filtration by dual Weyl modules, we bound the dimension of the kernel of the quotient map in Theorem G, and we give some explicit descriptions for particular cases. We also deduce the following corollary, identifying some new examples of indecomposable Specht modules in characteristic 2.

Corollary H. *Suppose K is infinite and has characteristic 2. Let λ be a partition such that $\lambda_1 = \lambda_2 \geqslant \lambda_3 + 2$ and such that λ minus its first part is 2-regular. Then S^λ is indecomposable.*

12. Quotient construction of the image of the Specht module

In this section we present an explicit model for $\mathcal{G}_{\otimes}(S^\lambda)$ in all characteristics. The isomorphism $\mathcal{G}_{\otimes}(S^\lambda) \cong \nabla^\lambda E$ stated in Theorem F for characteristics not 2 follows immediately.

12.1. Skew column tabloids

To describe the image of the Specht module under $\mathcal{G}_{\otimes}$, we require a modified notion of column tabloid, which we introduce in this subsection.

Recall that in §1.5 we defined a column tabloid as an element $t + J_{\mathsf{Alt}}$ in the quotient $\mathsf{Tbx}^\lambda(V)/J_{\mathsf{Alt}}$, where J_{Alt} is the subspace

$$J_{\mathsf{Alt}} = \langle\, x \in \mathsf{Tbx}^\lambda(V) \mid x \cdot \tau = x \text{ for some transposition } \tau \in \mathrm{CPP}(\lambda)\,\rangle_K.$$

We now consider a different subset to quotient by. Define

$$J_{\mathsf{Sk}} = \langle\, x \cdot \sigma - x \operatorname{sgn} \sigma \mid x \in \mathsf{Tbx}^\lambda(V),\, \sigma \in \mathrm{CPP}(\lambda)\,\rangle_K.$$

Note that $J_{\mathsf{Sk}} \subseteq J_{\mathsf{Alt}}$ with equality if $\operatorname{char} K \neq 2$. In characteristic 2, the additional elements of J_{Alt} are the tableaux with repeated entries in a column; that is:

$$J_{\mathsf{Alt}} = J_{\mathsf{Sk}} + \langle\, t \in \mathsf{Tbx}^\lambda(V) \mid t \text{ has a repeated entry in a column}\,\rangle_K.$$

Definition 12.1 (Skew column tabloid)**.** The *skew column tabloid* corresponding to a tableau t is the element $t + J_{\mathsf{Sk}}$ in the quotient $\mathsf{Tbx}^\lambda(V)/J_{\mathsf{Sk}}$.

When we wish emphasise that a column tabloid as defined in §1.5 is an element of $\mathsf{Tbx}^\lambda(V)/J_{\mathsf{Alt}}$ and not a skew column tabloid, we describe it as an *alternating column tabloid.*

We write the skew column tabloid corresponding to a tableau t as $||t||$, and draw a skew column tabloid by deleting the horizontal lines from a drawing of the corresponding tableau and double-striking the vertical lines, as depicted below in the case $\lambda = (3, 2)$.

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} \qquad \Longrightarrow \qquad ||t|| = \begin{array}{||c||c||c||} 1 & 2 & 4 \\ 3 & 5 & \end{array}$$

Depending on the characteristic, the space of skew column tabloids is isomorphic as a KG-module either to an exterior power or symmetric power:

$$\mathsf{Tbx}^{\lambda}(V)\big/J_{\mathsf{Sk}} \cong \begin{cases} \bigwedge^{\lambda'} V & \text{if } \operatorname{char} K \neq 2, \\ \operatorname{Sym}^{\lambda'} V & \text{if } \operatorname{char} K = 2. \end{cases}$$

For convenience, we define the *skew symmetric power* Sk^{-} to be the symmetric power Sym^{-} in characteristic 2 and the exterior power $\bigwedge^{-}$ otherwise. We use $\mathsf{Sk}^{\lambda'} V$ to denote the space of skew column tabloids.

As is already clear, the definitions of alternating column tabloids and skew column tabloids agree in characteristics other than 2. The definitions also agree if we restrict to tableaux of symmetric type.

Both alternating and skew column tabloids have the property that, for $\sigma \in \mathrm{CPP}(\lambda)$, the equalities

$$|t \cdot \sigma| = |t| \operatorname{sgn} \sigma$$
$$||t \cdot \sigma|| = ||t|| \operatorname{sgn} \sigma$$

hold. The key difference between the two definitions of tabloids is that alternating column tabloids furthermore have the property that if t has a repeated entry in a column then $|t| = 0$, whereas skew column tabloids do not have this property in characteristic 2. It is for these properties that the tabloids are named: an alternating column tabloid resembles an alternating multilinear form, whereas a skew column tabloid resembles a skew symmetric multilinear form.

There is a surjection $\delta\colon \mathsf{Sk}^{\lambda'} V \to \bigwedge^{\lambda'} V$ defined by K-linear extension of

$$\delta\colon ||t|| \mapsto |t|.$$

This map is easily seen to be G-equivariant. The kernel of δ is the subspace spanned by skew column tabloids with repeated column entries.

We define skew Garnir relations analogously to Garnir relations, as follows.

Definition 12.2 (Skew Garnir relations)**.** Let (t, A, B) and $\mathcal{S}$ be as in the definition of a Garnir relation from §1.7: t is a tableau of shape λ with entries in $\mathcal{B}$, $1 \leqslant j < j' \leqslant \lambda_1$, $A \subseteq \mathrm{col}_j(\lambda)$ and $B \subseteq \mathrm{col}_{j'}(\lambda)$ are such that $|A| + |B| > \lambda'_j$, and $\mathcal{S}$ is a set of left coset representatives for $S_A \times S_B$ in $S_{A \sqcup B}$. Then the *skew Garnir relation* labelled by (t, A, B) is

$$\mathsf{R}^{\mathsf{Sk}}_{(t,A,B)} = \sum_{\tau \in \mathcal{S}} ||t \cdot \tau|| \operatorname{sgn} \tau.$$

Let $\mathsf{SkGR}^\lambda(V)$ denote the subspace of $\mathsf{Sk}^{\lambda'} V$ which is spanned by the Garnir relations.

If we wish to emphasise that a Garnir relation as defined in Definition 1.8 is an element of $\bigwedge^{\lambda'} V$ and not a skew Garnir relation, we describe it as an *alternating Garnir relation.*

Just as for the alternating Garnir relations, a skew Garnir relation does not depend on the choice of coset representatives, and the K-subspace $\mathsf{SkGR}^\lambda(V)$ is moreover a KG-submodule because the group action commutes with the place permutation action.

We likewise define certain distinguished skew Garnir relations.

Definition 12.3 (Skew snake relations and basic skew snake relations)**.** A skew Garnir relation is called a *skew snake relation* under the same conditions described for Garnir relations in Definition 1.10: if, in the notation of Definition 12.2, $j' = j + 1$ and there exists i such that $A = \{\, (x, j) \mid i \leqslant x \leqslant \lambda'_j \,\}$ and $B = \{\, (x, j') \mid 1 \leqslant x \leqslant i \,\}$; in this case, we may also label the Garnir relation by (t, i, j). Given a function Φ on column semistandard tableaux which are not row semistandard whose output on such a tableau t is a box (i, j) such that $t(i, j) > t(i, j + 1)$, a skew snake relation labelled by (t, i, j) is called Φ-*basic* if t is column semistandard but not row semistandard and $(i, j) = \Phi(t)$ (that is, under the same conditions described for Garnir relations in Definition 2.14, with "column standard" replaced with "column semistandard").

The image of a skew Garnir relation $\mathsf{R}^{\mathsf{Sk}}_{(t,A,B)}$ under $\delta\colon \mathsf{Sk}^{\lambda'} V \to \bigwedge^{\lambda'} V$ is of course the Garnir relation $\mathsf{R}_{(t,A,B)}$. However, $\mathsf{R}^{\mathsf{Sk}}_{(t,A,B)}$ may have nonzero summands which vanish under δ.

12.2. The image of the Specht module

We can now use the skew column tabloids and the skew Garnir relations to model the image of the Specht module under the inverse Schur functor. Recall W denotes the natural permutation representation of S_r, with the property that $S^\lambda = \nabla^\lambda_{\mathrm{sym}} W$, and that E denotes the natural representation of $\mathrm{GL}_n(K)$. We view the basis of W as $[r]$; let $\mathcal{B}$ denote a basis for E.

Lemma 12.4. *Let n and r be any integers.*

(i) There is an isomorphism $\mathcal{G}_\otimes(\bigwedge^{\lambda'}_{\mathrm{sym}} W) \cong \mathsf{Sk}^{\lambda'} E$.

(ii) There is a short exact sequence

$$0 \longrightarrow \mathsf{SkGR}^\lambda(E) \longrightarrow \mathsf{Sk}^{\lambda'} E \longrightarrow \mathcal{G}_\otimes(S^\lambda) \longrightarrow 0$$

in the category of $K\mathrm{GL}_n(K)$-modules.

Proof. [(i)] Given a pure tensor $x = x_1 \otimes \cdots \otimes x_r \in E^{\otimes r}$ whose factors are basis elements in $\mathcal{B}$, and given also a tableau u of symmetric type of shape λ with entries in $[r]$, let x_u denote the tableau of shape λ with entries in $\mathcal{B}$ defined by

$$x_u(b) = x_{u(b)}$$

for all $b \in [\lambda]$.

Fix any tableau s of symmetric type with entries in $[r]$. We claim there are mutually inverse $K\mathrm{GL}_n(K)$-isomorphisms $\varphi\colon \mathcal{G}_\otimes(\bigwedge^{\lambda'}_{\mathrm{sym}} W) \to \mathsf{Sk}^{\lambda'} V$ and $\psi\colon \mathsf{Sk}^{\lambda'} V \to \mathcal{G}_\otimes(\bigwedge^{\lambda'}_{\mathrm{sym}} W)$ given by K-linear extension of

$$\varphi(x \otimes_{KS_r} |u|) = ||x_u||$$

and

$$\psi(||t||) = \bigotimes_{i\in[r]} t(s^{-1}i) \otimes_{KS_r} |s|$$

for all elements x and u as above and all tableaux t with entries in $\mathcal{B}$. For example, with $\lambda = (3,2)$ there is a correspondence between elements

$$x_1 \otimes \cdots \otimes x_5 \otimes_{KS_r} \left| \begin{array}{c} 1 \\ 3 \end{array} \middle| \begin{array}{c} 2 \\ 5 \end{array} \middle| \begin{array}{c} 4 \\ \end{array} \right| \quad \leftrightarrow \quad \left\| \begin{array}{c} x_1 \\ x_3 \end{array} \middle\| \begin{array}{c} x_2 \\ x_5 \end{array} \middle\| \begin{array}{c} x_4 \\ \end{array} \right\|$$

under φ and ψ.

Verifying that φ and ψ are well-defined and mutually inverse consists mostly of bookkeeping; we perform this task in the following pair of claims.

Recall that S_r acts on the left of W, and hence entrywise on tableaux with entries in $[r]$ on the left, and that we denote this action by concatenation; meanwhile $S_{[\lambda]}$ acts by place permutation on tableaux on the right, denoted by a central dot. To translate between these groups (which unfortunately requires a mix between writing tableaux as functions on the left and writing permutations on the right), we use the chosen tableau s: given an element $\tau \in S_{[\lambda]}$, we define $\tau^s \in S_r$ by

$$i\tau^s = s\Big((s^{-1}i)\tau\Big).$$

For convenience we write $\tau^{-s} = (\tau^{-1})^s = (\tau^s)^{-1}$. Note that $\tau^s s = s \cdot \tau$ (because $(\tau^s s)(b) = \big(s(b)\big)\tau^{-s} = s(b\tau^{-1}) = (s \cdot \tau)(b)$ for any $b \in [\lambda]$).

Claim 12.4.1. *The maps φ and ψ are well-defined.*

Proof. For φ, we use the universal property of the tensor product. Consider the K-bilinear map $E^{\otimes r} \times \bigwedge_{\mathrm{sym}}^{\lambda'} W \to \mathsf{Sk}^{\lambda'} V$ defined by extension of $(x, |u|) \mapsto ||x_u||$; indeed this is well-defined because for any $\tau \in S_{[\lambda]}$ we have $x_{u \cdot \tau}(b) = x_{u(b\tau^{-1})} = (x_u \cdot \tau)(b)$ for all $b \in [\lambda]$. The map is also S_r-balanced because for any $\sigma \in S_r$ we have $x_{\sigma u}(b) = x_{u(b)\sigma^{-1}} = (x \cdot \sigma)_{u(b)}$ for all $b \in [\lambda]$. This bilinear map induces the map φ.

For ψ, observe that for $\tau \in S_{[\lambda]}$ we have that

$$(t \cdot \tau)(s^{-1}i) = t\Big((s^{-1}i)\tau^{-1}\Big) = t\Big(s^{-1}(i\tau^{-s})\Big)$$

by definition of $-^s$, and hence

$$\begin{aligned}
\psi(||t \cdot \tau||) &= \bigotimes_{i \in [r]} t\Big(s^{-1}(i\tau^{-s})\Big) \otimes_{KS_r} |s| \\
&= \left(\bigotimes_{i \in [r]} t\big(s^{-1}i\big)\right) \cdot \tau^s \otimes_{KS_r} |s| \\
&= \bigotimes_{i \in [r]} t(s^{-1}i) \otimes_{KS_r} \tau^s|s| \\
&= \bigotimes_{i \in [r]} t(s^{-1}i) \otimes_{KS_r} |s \cdot \tau|.
\end{aligned}$$

Thus if $\tau \in \mathrm{CPP}(\lambda)$ then $\psi(||t \cdot \tau||) = \mathrm{sgn}(\tau)\psi(||t||)$ as required. □

Claim 12.4.2. *The maps φ and ψ are mutually inverse.*

PROOF. To see that $\varphi\psi = \mathrm{id}$, observe that if t is a tableau with entries in $\mathcal{B}$, then the element $x = \bigotimes_{i\in[r]} t(s^{-1}i)$ is such that $x_s(b) = x_{s(b)} = t(s^{-1}s(b)) = t(b)$, so $x_u = t$.

To see that $\psi\varphi = \mathrm{id}$, suppose $x \in E^{\otimes r}$ is a pure tensor of basis vectors and that u is any tableau of symmetric type with entries in $[r]$. Note that there exists a unique place permutation $\tau \in S_{[\lambda]}$ such that $s \cdot \tau = u$, that is, such that $u(b) = s(b\tau^{-1})$ for all $b \in [\lambda]$. Then

$$x_u(s^{-1}i) = x_{u(s^{-1}i)} = x_{s\left((s^{-1}i)\tau^{-1}\right)} = x_{i\tau^{-s}} = (x \cdot \tau^s)_i,$$

and thus

$$\begin{aligned}
\psi\varphi(x \otimes_{KS_r} |u|) &= \bigotimes_{i\in[r]} x_u(s^{-1}i) \otimes_{KS_r} |s| \\
&= (x \cdot \tau^s) \otimes_{KS_r} |s| \\
&= x \otimes_{KS_r} |s \cdot \tau| \\
&= x \otimes_{KS_r} |u|
\end{aligned}$$

as required. □

It is clear from multilinear expansion of tableaux that φ and ψ are $\mathrm{GL}_n(K)$-equivariant. This completes the proof of (i).

[(ii)] By Proposition 2.16, there is a short exact sequence

$$0 \longrightarrow \mathsf{GR}^{\lambda}_{\mathrm{sym}}(W) \xrightarrow{\iota} \textstyle\bigwedge^{\lambda'}_{\mathrm{sym}} W \xrightarrow{e|_{\mathrm{sym}}} S^{\lambda} \longrightarrow 0$$

where ι denotes the inclusion map. Since $\mathcal{G}_{\otimes}$ is right-exact, applying it to this sequence we obtain an exact sequence ending

$$\longrightarrow \mathcal{G}_{\otimes}(\mathsf{GR}^{\lambda}_{\mathrm{sym}}(W)) \xrightarrow{\mathcal{G}_{\otimes}(\iota)} \mathcal{G}_{\otimes}(\textstyle\bigwedge^{\lambda'}_{\mathrm{sym}} W) \xrightarrow{\mathcal{G}_{\otimes}(e|_{\mathrm{sym}})} \mathcal{G}_{\otimes}(S^{\lambda}) \longrightarrow 0.$$

Applying the isomorphism φ from (i), we have a short exact sequence

$$0 \longrightarrow \operatorname{im} \varphi\mathcal{G}_{\otimes}(\iota) \longrightarrow \mathsf{Sk}^{\lambda'} V \longrightarrow \mathcal{G}_{\otimes}(S^{\lambda}) \longrightarrow 0$$

and it suffices to show that $\operatorname{im} \varphi\mathcal{G}_{\otimes}(\iota) = \mathsf{SkGR}^{\lambda}(V)$.

The image $\operatorname{im} \varphi\mathcal{G}_{\otimes}(\iota)$ is spanned by elements of the form

$$\varphi(x \otimes_{KS_r} \mathsf{R}_{(t,A,B)})$$

where t is a tableau with entries in $[r]$, A and B are subsets of $[\lambda]$ as in the definition of a Garnir relation, and x is a pure tensor whose factors are basis elements of E. Fix such t, A, B and x, and let $\mathcal{S}$ be a set of left coset representatives for $S_A \times S_B$ in $S_{A \sqcup B}$. Then, using that $x_{t\cdot\sigma} = x_t \cdot \sigma$ for any $\sigma \in S_{[\lambda]}$, we have

$$\begin{aligned}
\varphi(x \otimes_{KS_r} \mathsf{R}_{(t,A,B)}) &= \sum_{\tau \in \mathcal{S}} \varphi(x \otimes_{KS_r} |t \cdot \tau| \operatorname{sgn} \tau) \\
&= \sum_{\tau \in \mathcal{S}} ||x_{t\cdot\tau}|| \operatorname{sgn} \tau \\
&= \sum_{\tau \in \mathcal{S}} ||x_t \cdot \tau|| \operatorname{sgn} \tau,
\end{aligned}$$

which is a skew Garnir relation labelled by (x_t, A, B). Since also any tableau with entries in $\mathcal{B}$ can be written in the form x_t for suitable x and t, we have that $\operatorname{im} \varphi\mathcal{G}_{\otimes}(\iota) = \mathsf{SkGR}^{\lambda}(V)$ as required. $\square$

Proposition 12.5. *The following diagram in the category of $K\mathrm{GL}_n(K)$-modules is commutative with exact rows and exact columns. In particular, there is a surjection $\mathcal{G}_{\otimes}(S^{\lambda}) \twoheadrightarrow \nabla^{\lambda} E$ which is an isomorphism if and only if $\ker \delta \subseteq \mathsf{SkGR}^{\lambda}(E)$.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker \delta|_{\mathsf{GR}} \arrow[r, hook] \arrow[d, hook] & \mathsf{SkGR}^{\lambda}(E) \arrow[r, "\delta|_{\mathsf{GR}}"] \arrow[d, hook] & \mathsf{GR}^{\lambda}(E) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \ker \delta \arrow[r, hook] \arrow[d] & \mathsf{Sk}^{\lambda'} E \arrow[r, "\delta"] \arrow[d] & \bigwedge^{\lambda'} E \arrow[r] \arrow[d, "e"] & 0 \\
0 \arrow[r] & {}^{\ker \delta}\!/\!_{\ker \delta|_{\mathsf{GR}}} \arrow[r] \arrow[d] & \mathcal{G}_{\otimes}(S^{\lambda}) \arrow[r] \arrow[d] & \nabla^{\lambda} E \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

PROOF. Clearly the first column and the first two rows are exact and the top two squares commute. The third column is exact by Proposition 2.13 and the second column is exact by Lemma 12.4.

The existence of the maps in the third row and the commutativity of the bottom two squares follow from the universal properties of the objects in the third row as cokernels. The map from $\mathcal{G}_{\otimes}(S^{\lambda})$ to $\nabla^{\lambda}E$ is surjective by surjectivity of $e\delta$ and the commutativity of the diagram. Exactness at the remaining two objects in the third row follows from (a degenerate case of) the snake lemma.

It is clear from the diagram that the surjection $\mathcal{G}_{\otimes}(S^{\lambda}) \twoheadrightarrow \nabla^{\lambda}E$ is injective if and only if $\ker\delta = \ker\delta|_{\mathsf{GR}}$, or equivalently $\ker\delta \subseteq \mathsf{SkGR}^{\lambda}(E)$. $\square$

From this proposition we can immediately identify the image of the Specht module in characteristics other than 2 (when δ is an isomorphism), obtaining the first main result of this chapter.

Theorem F. *Suppose K has characteristic not 2 and let λ be a partition of r. Then there is an isomorphism $\mathcal{G}_{\otimes}(S^{\lambda}) \cong \nabla^{\lambda}E$.*

13. Combinatorics of skew Garnir relations

The goal of this section is to identify for which partitions the necessary and sufficient condition from Proposition 12.5 holds in characteristic 2. This condition asserts that a skew column tabloid with a repeated column entry can be written as a linear combination of skew Garnir relations.

Our classification of partitions is proven in §13.2. To reach it, we must first identify a spanning set for the space of skew Garnir relations, which we do in §13.1.

Although our application concerns representations of the general linear group, in this subsection the group action is irrelevant, so we state our results for an arbitrary representation V of an arbitrary group G, of dimension d and with ordered basis $\mathcal{B}$. We work in characteristic 2 throughout.

13.1. Spanning set for the skew Garnir relations

We begin by observing that the basic skew snake relations are not sufficient to span the space of skew Garnir relations.

Example 13.1. Suppose $\lambda = (2,1)$ and $t = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$. Then

$$\mathsf{R}^{\mathsf{Sk}}_{(t,1,1)} = \left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\| \begin{matrix} 1 \\ \end{matrix} \left\| \vphantom{\begin{matrix} 1 \\ 1 \end{matrix}} \right. + \left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\| \begin{matrix} 1 \\ \end{matrix} \left\| \vphantom{\begin{matrix} 1 \\ 1 \end{matrix}} \right. + \left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\| \begin{matrix} 1 \\ \end{matrix} \left\| \vphantom{\begin{matrix} 1 \\ 1 \end{matrix}} \right. = \left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\| \begin{matrix} 1 \\ \end{matrix} \left\| \vphantom{\begin{matrix} 1 \\ 1 \end{matrix}} \right.$$

which is nonzero (in characteristic 2). By contrast, $\mathsf{R}_{(t,1,1)} = 0$.

Since t is the unique tableau of its weight, and since there is only one possible choice of subsets of $[\lambda]$ to define a Garnir relation, the relation $\mathsf{R}^{\mathsf{Sk}}_{(t,1,1)}$ is the unique (nonzero) skew Garnir relation of this weight. However, $\mathsf{R}^{\mathsf{Sk}}_{(t,1,1)}$ is not basic, and so cannot be written as a linear combination of basic skew snake relations.

The additional skew snake relations we require to form a spanning set are defined below. To prove that they span, we introduce additional symbols to force the tableaux to have distinct entries, then use the basis for Garnir relations of symmetric type identified in Proposition 2.17 and map back down to the case of interest.

Definition 13.2 (Supplementary skew snake relations)**.** A skew snake relation labelled by (t, i, j) is called *supplementary* if t is row-and-column semistandard and $t(i, j) = t(i, j + 1)$.

Proposition 13.3. *The basic and supplementary skew snake relations together span* $\mathsf{SkGR}^\lambda(V)$*.*

Proof. Recall that the set in which our tableaux take entries is an ordered basis $\mathcal{B}$ of V. Let $\mathcal{B}^\vee = \mathcal{B} \times [r]$, ordered lexicographically, and let $V^\vee$ be the K-vector space with basis $\mathcal{B}^\vee$. Let $\pi_1 \colon \mathcal{B}^\vee \to \mathcal{B}$ be the surjection defined by $\pi_1(x, k) = x$. Extend π_1 to a map on tableaux by acting entrywise. This map is also surjective, and remains so on restriction to tableaux of symmetric type: given any tableau t with entries in $\mathcal{B}$, there exists a tableau $t^\vee$ with entries in $\mathcal{B}^\vee$ such that $\pi_1(t^\vee) = t$, formed by replacing each $x \in \mathcal{B}$ with (x, k) for some $k \in [r]$, and these $k \in [r]$ can be chosen such that all entries in $t^\vee$ are distinct.

The map π_1 induces a K-linear surjection $\hat{\pi}_1 \colon \mathsf{GR}^\lambda_{\mathrm{sym}}(V^\vee) \to \mathsf{SkGR}^\lambda(V)$ defined by sending each column tabloid of symmetric type $|t^\vee|$ to the skew column tabloid $||\pi_1(t^\vee)||$; that is, $\hat{\pi}_1(\mathsf{R}_{(t^\vee, A, B)}) = \mathsf{R}^{\mathsf{Sk}}_{(\pi_1(t^\vee), A, B))}$ for any label $(t^\vee, A, B)$ for a Garnir relation in $\mathsf{GR}^\lambda_{\mathrm{sym}}(V^\vee)$. This is well-defined because for tableaux of symmetric type $t_1^\vee$ and $t_2^\vee$, there is equality $|t_1^\vee| = \pm|t_2^\vee|$ if and only if $t_1^\vee$ and $t_2^\vee$ have the same column sets (this is not the case for general tableaux: we may have equality $|t_1^\vee| = 0 = |t_2^\vee|$ in $\bigwedge^{\lambda'} V^\vee$ despite an inequality $||\pi_1(t_1^\vee)|| \neq ||\pi_1(t_2^\vee)||$ in $\mathsf{Sk}^{\lambda'} V$, when the tableaux have distinct column sets but some repeated column entries).

Let Φ be the function with respect to which we consider skew snake relations in $\mathsf{SkGR}^\lambda(V)$ basic. Choose a function $\Phi^\vee$ to consider snake relations in $\mathsf{GR}^\lambda_{\mathrm{sym}}(V^\vee)$ basic with respect to, chosen with the property that $\Phi^\vee(t^\vee) = \Phi(\pi_1(t^\vee))$ whenever $\Phi(\pi_1(t^\vee))$ is defined (that is, whenever $\pi_1(t^\vee)$ is not row semistandard). Indeed this is possible: when it is defined, the box $\Phi(\pi_1(t^\vee)) = (i, j)$ satisfies $t^\vee(i, j) > t^\vee(i, j + 1)$ by considering the first value of the pair in each box (that is, the image under π_1).

Proposition 2.17 tells us that, in $\mathsf{GR}^{\lambda}_{\mathrm{sym}}(V^{\vee})$, the $\Phi^{\vee}$-basic snake relations of symmetric type form a basis. Therefore the image of this set under $\hat{\pi}_1$ is a spanning set for $\mathsf{SkGR}^{\lambda}(V)$. It suffices to show that this image is the union of the sets of basic and supplementary skew snake relations.

Consider a skew snake relation $\mathsf{R}^{\mathsf{Sk}}_{(t,i,j)} \in \mathsf{SkGR}^{\lambda}(V)$ which is either Φ-basic or supplementary. We aim to show there exists a tableau $t^{\vee}$ with entries in $\mathcal{B}^{\vee}$ such that $\pi_1(t^{\vee}) = t$ and $(i,j) = \Phi^{\vee}(t^{\vee})$ (and hence $\mathsf{R}_{(t^{\vee},i,j)}$ is $\Phi^{\vee}$-basic and its image under $\hat{\pi}_1$ is $\mathsf{R}^{\mathsf{Sk}}_{(t,i,j)}$). If $\mathsf{R}^{\mathsf{Sk}}_{(t,i,j)}$ is Φ-basic, then choose any $t^{\vee}$ such that $\pi_1(t^{\vee}) = t$; since t is not row semistandard, neither is $t^{\vee}$, and so by choice of $\Phi^{\vee}$ we have $\Phi^{\vee}(t^{\vee}) = \Phi(t) = (i,j)$. If $\mathsf{R}^{\mathsf{Sk}}_{(t,i,j)}$ is supplementary, then t is row-and-column semistandard, and so for any choice of $t^{\vee}$ such that $\pi_1(t^{\vee}) = t$ we have that the first values of the entries of $t^{\vee}$ weakly increase along rows and columns. Choose the second values of the entries of $t^{\vee}$ such that $t^{\vee}(i,j) > t^{\vee}(i,j+1)$ and such that elsewhere the second values strictly increase along rows and columns (for example, by filling in the entries left to right of each row in turn, then swapping the entries of (i,j) and $(i,j+1)$). Then i and j are unique such that $t^{\vee}(i,j) > t^{\vee}(i,j+1)$, and hence $\Phi^{\vee}(t^{\vee}) = (i,j)$.

Now consider a $\Phi^{\vee}$-basic snake relation $\mathsf{R}_{(t^{\vee},i,j)} \in \mathsf{GR}^{\lambda}_{\mathrm{sym}}(V^{\vee})$. We aim to show that the skew snake relation $\hat{\pi}_1(\mathsf{R}_{(t^{\vee},i,j)}) = \mathsf{R}^{\mathsf{Sk}}_{(\pi_1(t^{\vee}),i,j)}$ is either Φ-basic or supplementary. If $\pi_1(t^{\vee})(i,j) > \pi_1(t^{\vee})(i,j+1)$, then $\pi_1(t^{\vee})$ is not row semistandard and, by choice of $\Phi^{\vee}$, we have that $(i,j) = \Phi^{\vee}(t^{\vee}) = \Phi(\pi_1(t^{\vee}))$ and hence $(\pi_1(t^{\vee}),i,j)$ labels a Φ-basic skew snake relation. If $\pi_1(t^{\vee})(i,j) = \pi_1(t^{\vee})(i,j+1)$, then $\pi_1(t^{\vee})$ is row semistandard (or else $\Phi(\pi_1(t^{\vee}))$ would be defined and not equal to $(i,j) = \Phi^{\vee}(t^{\vee})$), and so $(\pi_1(t^{\vee}),i,j)$ labels a supplementary skew snake relation. $\square$

The spanning set identified in Proposition 13.3 is in general not a basis: the supplementary skew snake relations may not be linearly independent. Indeed, a supplementary skew snake relation may even be zero, as evidenced in the following example.

Example 13.4. Suppose $\lambda = (2,1,1)$ and $t = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$. Then

$$\mathsf{R}^{\mathsf{Sk}}_{(t,1,1)} = \left\| \begin{array}{c} 1 \\ 2 \\ 2 \end{array} \right\| \begin{array}{c} 1 \\ \\ \\ \end{array} \Big\| + \left\| \begin{array}{c} 1 \\ 2 \\ 2 \end{array} \right\| \begin{array}{c} 1 \\ \\ \\ \end{array} \Big\| + \left\| \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right\| \begin{array}{c} 2 \\ \\ \\ \end{array} \Big\| + \left\| \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right\| \begin{array}{c} 2 \\ \\ \\ \end{array} \Big\| = 0.$$

It is plain to see that in fact all skew Garnir relations labelled by tableaux of this weight vanish; thus no tabloid of this weight appears with nonzero coefficient in any skew Garnir relation.

Nevertheless it is useful to have the following analogue for skew snake relations of Lemma 2.10 (which demonstrated the linear independence of the basic alternating snake relations). In particular, it shows that the basic skew snake relations are linearly independent.

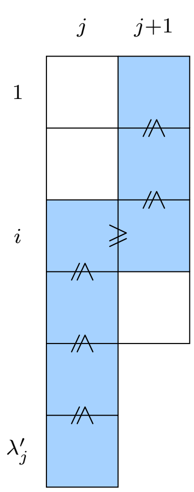
Lemma 13.5. *Let t be a column semistandard tableau, and suppose (i,j) in such that $t(i,j) \geqslant t(i,j+1)$. Then*

$$\mathsf{R}^{\mathsf{Sk}}_{(t,i,j)} = m_t||t|| + \sum_{u <_{\mathrm{c}} t} m_u||u||$$

for some elements m_u in the subring of K generated by 1. If $t(i,j) > t(i,j+1)$, then $m_t = 1$. If $t(i,j) = t(i,j+1)$ and a and b are the multiplicities of $t(i,j)$ in the sets defining the Garnir relation

$$A = \{\, (x,j) \mid i \leqslant x \leqslant \lambda'_j \,\} \text{ and } B = \{\, (x,j+1) \mid 1 \leqslant x \leqslant i \,\}$$

respectively, then $m_t = \binom{a+b}{a}$.



Proof. Analogously to the proof of Lemma 2.10, we observe that the sets A and B defining the Garnir relation satisfy

$$t(1,j+1) \leqslant \ldots \leqslant t(i,j+1) \leqslant t(i,j) \leqslant t(i+1,j) \leqslant \ldots \leqslant t(\lambda'_j,j),$$

and hence that $t \cdot \sigma \lesssim_{\mathrm{c}} t$ for any $\sigma \in S_{A \sqcup B}$. (The boxes in A and B and the inequalities between their entries are illustrated in the margin.) If $t(i,j) > t(i,j+1)$, then $t \cdot \sigma \sim_{\mathrm{c}} t$ holds if and only if $\sigma \in S_A \times S_B$. If $t(i,j) = t(i,j+1)$, then $t \cdot \sigma \sim_{\mathrm{c}} t$ holds for precisely those permutations which, modulo $S_A \times S_B$, permute only the boxes containing $t(i,j)$. The number of cosets of such permutations is the number of ways to choose a of the $a+b$ copies of the repeated entry to include in the left-hand column.

□

13.2. Writing column tabloids with the Garnir relations

In this subsection, we characterise when there is containment $\ker\delta \subseteq \mathsf{SkGR}^\lambda(V)$ in characteristic 2. When $V = E$, this containment is equivalent to the existence of an isomorphism $\mathcal{G}_\otimes(S^\lambda) \cong \nabla^\lambda E$ by Proposition 12.5.

Recall $\delta\colon \mathsf{Sk}^{\lambda'}V \to \bigwedge^{\lambda'} V$ is the map $||t|| \mapsto |t|$ (defined in §12.1). The kernel of δ is spanned by skew column tabloids with a repeated entry in a column. We have already seen that such a tabloid may or may not lie in the space of skew Garnir relations: Example 13.1 exhibited a skew column tabloid in the kernel of δ which is equal to a skew Garnir relation, whilst Example 13.4 exhibited a skew column tabloid in the kernel of δ that cannot be written as a linear combination of skew Garnir relation because all relations of that weight vanish. We further illustrate this behaviour with the following example.

Example 13.6. Fix an element $x \in \mathcal{B}$, and let t be the tableau whose entries are all x. Provided λ has at least two rows, we have $||t|| \in \ker\delta$. Meanwhile, t is the unique tableau of its weight, so it labels all skew Garnir relations of its weight. All summands of such relations are equal to $||t||$, so

$$\mathsf{R}^{\mathsf{Sk}}_{(t,A,B)} = \begin{cases} ||t|| & \text{if the number of summands } \binom{|A|+|B|}{|A|} \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose λ is a hook partition which has at least two rows and two columns. Let $a \geqslant 2$ and $l \geqslant 2$ be such that $\lambda = (a, 1^{l-1})$. Clearly the skew Garnir relations involving only columns of length 1 have exactly two summands and hence are zero. The number of summands in a skew Garnir relation involving the first column is $\binom{l+1}{1} = l+1$, which is odd if and only if l is even. Thus $||t|| \in \mathsf{SkGR}^\lambda(V)$ holds if and only if l is even.

We now proceed with classifying when $\ker\delta \subseteq \mathsf{SkGR}^\lambda(V)$. Recall we say that a partition is 2-regular if it has no repeated (positive) parts, and that it is 2-singular otherwise.

Lemma 13.7. *Suppose λ is 2-regular, or $\lambda_1 = \lambda_2 \geqslant \lambda_3 + 2$ and λ minus its first part is 2-regular. Then $\ker\delta \subseteq \mathsf{SkGR}^\lambda(V)$.*

Proof. First note that $\ker\delta = 0$ if λ has exactly one row; the lemma holds trivially in this case, so we may assume that λ has at least two rows, and hence that there exist tableaux t such that $||t|| \in \ker\delta$. Since λ has at least two rows, if λ is 2-regular then $\lambda_1 > \lambda_2 > \lambda_3$; thus under either hypothesis we have $\lambda_1 \geqslant \lambda_3 + 2$.

Let t be a tableau such that $||t|| \in \ker\delta$. Then t has at least one column with repeated entries; let j be the index of the rightmost column in which t has repeated entries. Let a_1 and a_2 be boxes in column j such that $t(a_1) = t(a_2)$. We proceed by downward induction on j.

Suppose $j > \lambda_3$. Since $\lambda_1 \geqslant \lambda_3 + 2$, there exists some $j' \neq j$ such that $\lambda_3 < j' \leqslant \lambda_1$. We have $\lambda'_j \leqslant 2$ and $\lambda'_{j'} \leqslant 2$. Let b be any box in column j', and set $A = \{a_1, a_2\}$ and $B = \{b\}$ (or vice versa if $j' < j$). Then (t, A, B) labels a Garnir relation, and

$$\begin{aligned}\mathsf{R}^{\mathsf{Sk}}_{(t,A,B)} &= ||t|| + ||t \cdot (a_1\ b)|| + ||t \cdot (a_2\ b)|| \\ &= ||t||\end{aligned}$$

since $t(a_1) = t(a_2)$. Thus $||t|| \in \mathsf{SkGR}^\lambda(V)$ as required.

Now suppose $j \leqslant \lambda_3$. Since λ minus its first part is 2-regular, we have that column j is at most one box longer than column $j+1$. Set $A = \{a_1, a_2\}$ and $B = \mathrm{col}_{j+1}[\lambda]$. Then (t, A, B) labels a Garnir relation, and

$$\mathsf{R}^{\mathsf{Sk}}_{(t,A,B)} = ||t|| + \sum_{\{b_1,b_2\}\subseteq B} ||t \cdot (a_1\ b_1)(a_2\ b_2)||$$

because the summands corresponding to permutations where only one box of A is moved cancel out. The tableaux $t \cdot (a_1\ b_1)(a_2\ b_2)$ in the above sum have a repeated entry in column $j+1$, so by the inductive hypothesis their skew column tabloids lie in $\mathsf{SkGR}^\lambda(V)$. Hence so does $||t||$. □

Lemma 13.8. *Suppose λ is such that λ minus its first part is 2-singular, and suppose $|\mathcal{B}| \geqslant r-2$. Then $\ker\delta \not\subseteq \mathsf{SkGR}^\lambda(V)$.*

PROOF. Let $d = |\mathcal{B}|$, and view $\mathcal{B} \cong [d]$. Pick any $k > 1$ such that $\lambda_k = \lambda_{k+1} > 0$. Set $x = 1 + \sum_{a=1}^{k-2} \lambda_a$, and let α be the weight in which x has multiplicity $\lambda_{k-1} + \lambda_k + \lambda_{k+1}$, and all other positive integers up to and including $r+1-(\lambda_{k-1}+\lambda_k+\lambda_{k+1})$ have multiplicity 1 (and all other integers have multiplicity 0). Let t be the $<_{\mathrm{c}}$-greatest row-and-column semistandard tableau with weight α; this indeed exists because the required inequality $|\mathcal{B}| \geqslant r+1-(\lambda_{k-1}+\lambda_k+\lambda_{k+1})$ follows from the assumption $|\mathcal{B}| \geqslant r-2$. Explicitly, t is defined by

$$t(i,j) = \begin{cases} j + \sum_{a=1}^{i-1} \lambda_a & \text{if } 1 \leqslant i \leqslant k-2; \\ x & \text{if } k-1 \leqslant i \leqslant k+1; \\ j + x + \sum_{a=k+2}^{i-1} \lambda_a & \text{if } k+2 \leqslant i \leqslant \lambda_1'. \end{cases}$$

For example, if $\lambda = (6,6,3,3,2,1)$ and $k=3$, then $x = 7$ and

$$t = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 7 & 7 & 7 & 7 & 7 & 7 \\ \hline 7 & 7 & 7 \\ \cline{1-3} 7 & 7 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2} 10 \\ \cline{1-1} \end{array}\,.$$

We aim to prove that $||t|| \in \ker\delta \setminus \mathsf{SkGR}^\lambda(V)$. Clearly $||t|| \in \ker\delta$. To show that $||t||$ is not an element of $\mathsf{SkGR}^\lambda(V)$, we require the following property of t. Given a skew Garnir relation, we say the *leading tableau* of the relation is the $<_{\mathrm{c}}$-greatest column semistandard tableau whose tabloid has nonzero coefficient in the relation.

Claim 13.8.1. *If u is the leading tableau of a supplementary skew snake relation and is of weight α, then $u <_{\mathrm{c}} t$ (where α and t are as defined above).*

PROOF. Consider a supplementary skew snake relation labelled by (s,i,j) (so that in particular s is row-and-column semistandard). The leading tableau of this skew snake relation is at most s by Lemma 13.5. If s is of weight α, then by maximality of t we have $s <_{\mathrm{c}} t$ or $s = t$. Thus it remains only to show that t is not the leading tableau of a supplementary skew snake relation labelled by (t, i, j).

Consider the sets A and B defining the skew Garnir relation $\mathsf{R}^{\mathsf{Sk}}_{(t,i,j)}$. Let a be the multiplicity of $t(i, j)$ in A and b the multiplicity of $t(i, j)$ in B. Using Lemma 13.5, the coefficient of $||t||$ in $\mathsf{R}^{\mathsf{Sk}}_{(t,i,j)}$ is $\binom{a+b}{a}$, so we are required to show that $\binom{a+b}{a}$ is even.

By construction of t, a supplementary skew snake relation labelled by (t, i, j) has $i \in \{k - 1, k, k + 1\}$. We assess each possibility:

- if $i = k - 1$ and $j \leqslant \lambda_k$, then $a = 3$ and $b = 1$, and indeed $\binom{4}{3} = 4$ is even;
- if $i = k - 1$ and $j > \lambda_k$, then $a = b = 1$, and indeed $\binom{2}{1} = 2$ is even;
- if $i = k$, then $a = b = 2$, and indeed $\binom{4}{2} = 6$ is even;
- if $i = k + 1$, then $a = 1$ and $b = 3$, and indeed $\binom{4}{1} = 4$ is even. □

Returning to the proof of the lemma, suppose towards a contradiction that $||t|| \in \mathsf{SkGR}^{\lambda}(V)$. Then there exists some linear combination γ of (nonzero) basic and supplementary skew snake relations of weight α such that $\gamma = ||t||$. Consider the basic and supplementary skew snake relations with nonzero coefficient in γ, and consider the set of their (column semistandard) leading tableaux. Let u be $<_{\mathrm{c}}$-greatest in this set. We cannot have $u <_{\mathrm{c}} t$ (or else $||t||$ does not occur in any of the relations with nonzero coefficient in γ), and so Claim 13.8.1 says that u is not the leading tableau of a supplementary skew snake relation. Hence u is the leading tableau of a (unique) basic skew snake relation, and furthermore labels that relation (since by Lemma 13.5 the leading tableau of a basic skew snake relation is its labelling tableau).

By maximality of u, the basic skew snake relation labelled by u is the unique relation with nonzero coefficient in γ which has $||u||$ as a summand. Thus $||u||$ has nonzero coefficient in γ, and hence $||u|| = ||t||$. Since u and t are both column semistandard, we have $u = t$. But t is row semistandard, which contradicts that u labels a basic skew snake relation. □

Lemma 13.9. *Suppose* $\lambda_1 = \lambda_2 = \lambda_3 + 1$, *and suppose that* $|\mathcal{B}| \geqslant r - 2$. *Then* $\ker \delta \not\subseteq \mathsf{SkGR}^\lambda(V)$.

PROOF. We argue as in the proof of Lemma 13.8 with a different choice of t. Let $d = |\mathcal{B}|$, and view $\mathcal{B} \cong [d]$. Let α be the weight in which 1 has multiplicity $\lambda_1 + \lambda_2 + \lambda_3$, and all other positive integers up to and including $r + 1 - (\lambda_1 + \lambda_2 + \lambda_3)$ have multiplicity 1 (and all other integers have multiplicity 0). Let t be the $<_\mathrm{c}$-greatest row-and-column semistandard tableau with weight α; indeed such tableaux exist as the required inequality $|\mathcal{B}| \geqslant r+1-(\lambda_1+\lambda_2+\lambda_3)$ follows from the assumption $|\mathcal{B}| \geqslant r-2$. Explicitly, t is defined by

$$t(i,j) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant 3; \\ j + 1 + \sum_{a=3}^{i-1} \lambda_a & \text{if } 4 \leqslant i \leqslant \lambda'_1. \end{cases}$$

For example, if $\lambda = (5,5,4,3,1)$, then

$$t = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-3} 5 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \,.$$

We deduce, using Lemma 13.5 as in the proof of Claim 13.8.1, that t satisfies $u <_\mathrm{c} t$ for any u of weight α which is the leading tableau of a supplementary skew snake relation. Then arguing as in the final paragraphs of Lemma 13.8, we conclude that $||t|| \in \ker \delta \setminus \mathsf{SkGR}^\lambda(V)$. $\square$

Combining Lemmas 13.7 to 13.9, we have the following characterisation of when $\ker \delta \subseteq \mathsf{SkGR}^\lambda(V)$.

Proposition 13.10. *There is containment* $\ker \delta \subseteq \mathsf{SkGR}^\lambda(V)$ *if* λ *is* 2-*regular, or if* $\lambda_1 = \lambda_2 \geqslant \lambda_3 + 2$ *and* λ *minus its first part is* 2-*regular. Supposing* $|\mathcal{B}| \geqslant r - 2$, *if* λ *is not of this form then* $\ker \delta \not\subseteq \mathsf{SkGR}^\lambda(V)$.

Remark 13.11. In Lemmas 13.8 and 13.9 and Proposition 13.10, the restriction on $|\mathcal{B}|$ is required to ensure that we can choose a tableau with entries all distinct except for in three specified rows. The restriction on

$|\mathcal{B}|$ can be weakened if we permit dependence on λ: in Lemma 13.8, it is sufficient to require that $|\mathcal{B}| \geqslant r + 1 - (\lambda_{k-1} + \lambda_k + \lambda_{k+1})$ where $k > 1$ is minimal such that $\lambda_k = \lambda_{k+1}$; in Lemma 13.9 it is sufficient to require that $|\mathcal{B}| \geqslant r + 1 - (\lambda_1 + \lambda_2 + \lambda_3)$.

14. The image of the Specht module in characteristic 2

The second main result of this chapter, restated below, is now clear by combining Propositions 12.5 and 13.10.

Theorem G. *Suppose K has characteristic 2 and let λ be a partition of r. There is a surjection $\mathcal{G}_{\otimes}(S^\lambda) \twoheadrightarrow \nabla^\lambda E$, which is an isomorphism if λ is 2-regular, or if $\lambda_1 = \lambda_2 \geqslant \lambda_3 + 2$ and λ minus its first part is 2-regular. Supposing also $n \geqslant r - 2$, if λ is not of this form then the surjection is not an isomorphism.*

In the remainder of this chapter, we establish a variety of other results concerning $\mathcal{G}_{\otimes}(S^\lambda)$ in characteristic 2. Each of the following subsections is logically independent.

In §14.1 we use our new knowledge of the module $\mathcal{G}_{\otimes}(S^\lambda)$ to deduce the indecomposability of some Specht modules in characteristic 2. In §14.2 we show that a lower bound on n that grows with r in Theorem G is necessary. In §14.3 we restrict the possible composition factors of the kernel of the surjection in Theorem G and bound its dimension growth. In §14.4 we identify the composition factors of $\mathcal{G}_{\otimes}(S^\lambda)$ when $r \leqslant 5$ and deduce that $\mathcal{G}_{\otimes}(S^\lambda)$ need not have a filtration by dual Weyl modules. In §14.5 we describe $\mathcal{G}_{\otimes}(S^\lambda)$ for some particular partitions and values of n.

14.1. Some indecomposable Specht modules

In characteristics other than 2, all Specht modules are known to be indecomposable, and in characteristic 2 those indexed by 2-regular partitions are known to be indecomposable [Jam78, Corollary 13.18]. Determining the decomposability of the remaining Specht modules is a difficult open problem. Families of decomposable Specht modules have been identified by Murphy [Mur80], Dodge and Fayers [DF12], and Donkin and Geranios [DG20]. Here we identify some new indecomposable Specht modules.

Corollary H. *Suppose K is infinite and has characteristic 2. Let λ be a partition such that $\lambda_1 = \lambda_2 \geqslant \lambda_3 + 2$ and such that λ minus its first part is 2-regular. Then S^λ is indecomposable.*

Proof. Suppose, towards a contradiction, that S^λ is decomposable; write $S^\lambda = V_1 \oplus V_2$ for V_1, V_2 nonzero submodules. Choose some $n \geqslant r$. The functor $\mathcal{G}_\otimes$ preserves direct sums, so applying it to this decomposition and using Theorem G we find that $\nabla^\lambda E \cong \mathcal{G}_\otimes(V_1) \oplus \mathcal{G}_\otimes(V_2)$. Note that $\mathcal{G}_\otimes(V_1)$ and $\mathcal{G}_\otimes(V_2)$ are nonzero, since they are mapped by $\mathcal{F}$ to the nonzero modules V_1 and V_2 respectively. This contradicts the indecomposability of $\nabla^\lambda E$. □

14.2. Requirement on n

In Theorem G, the restriction $n \geqslant r-2$ is required to ensure the existence of a certain tableau, though the restriction can be weakened if we permit dependence on λ (as noted in Remark 13.11). It is possible for the isomorphism $\mathcal{G}_\otimes(S^\lambda) \cong \nabla^\lambda E$ to fail for $n \geqslant r-2$ but hold for some $n < r-2$. Furthermore this may happen for arbitrarily large n, as demonstrated by Example 14.1 below, so a lower bound on n that grows with r is necessary. Bearing in mind that the composition factors of these modules are independent of n (using Corollary 7.3), this behaviour is due to $\mathcal{G}_\otimes(S^\lambda)$ having composition factors which $\nabla^\lambda E$ does not, but which vanish for small n.

Example 14.1. Fix $n \in \mathbb{N}$, and let $r = 1 + \frac{(n+2)(n+3)}{2}$. Let $\lambda = (n+2, n+1, n, \dots, 2, 1, 1)$; that is, λ is the partition of r obtained from the 2-core partition of length $n+2$ by adding a box to the first column. Clearly λ minus its first part is not 2-regular, so by Theorem G we have that $\mathcal{G}_\otimes^{n'}(S^\lambda) \ncong \nabla^\lambda E$ when $n' \geqslant r-2$. However, we claim that $\mathcal{G}_\otimes^{n}(S^\lambda) \cong \nabla^\lambda E$ (which is 0 in this case).

It suffices to show that $\ker \delta \subseteq \mathsf{SkGR}^\lambda(E)$. Let t be a tableau with entries in $\mathcal{B}$ such that $||t|| \in \ker \delta$. Then t has a repeated entry in some column, and moreover must have a repeated entry in the second column: there are $n+1$ boxes in the second column, so there are insufficiently many basis elements of E for all of them to have distinct entries. Then the argument of Lemma 13.7 can be applied: we induct downward on the index of the rightmost column in which t has a repeated entry; since this index is always at least 2, we do not require any constraint on the first column; all other columns satisfy the condition of being at most 1 longer than the next, so the argument goes through.

14.3. Restrictions on the kernel of the quotient map

We would like to know more of the structure of $\mathcal{G}_\otimes(S^\lambda)$ when $\mathcal{G}_\otimes(S^\lambda) \not\cong \nabla^\lambda E$. The missing information is a description of the kernel of the surjection $\mathcal{G}_\otimes(S^\lambda) \twoheadrightarrow \nabla^\lambda E$, which we denote U^λ (isomorphic to ${}^{\ker\delta}\!/\!_{\ker\delta|_{\mathsf{GR}}}$).

In this subsection we record some restrictions on U^λ. We show, when K is infinite and $n \geqslant r$, that U^λ does not have 2-restricted composition factors and does not have a dual Weyl module as a subquotient. We also bound the dimension growth of U^λ as n varies, finding that U^λ grows more slowly than $\nabla^\lambda E$, so informally $\nabla^\lambda E$ comprises "most" of $\mathcal{G}_\otimes(S^\lambda)$.

Recall $\mathcal{F}$ is the Schur functor defined in §6.

Proposition 14.2. *Suppose K is infinite and $n \geqslant r$.*

(i) $\mathcal{F}(U^\lambda) = 0$.

(ii) If $L^\mu(E)$ is a composition factor of U^λ, then μ is not 2-restricted.

(iii) $\nabla^\mu E$ is not a subquotient of U^λ for any partition μ of r.

Proof. Applying the exact functor $\mathcal{F}$ to the third row of the diagram in Proposition 12.5, we have a short exact sequence

$$0 \longrightarrow \mathcal{F}(U^\lambda) \longrightarrow \mathcal{F}\mathcal{G}_\otimes(S^\lambda) \longrightarrow \mathcal{F}(\nabla^\lambda E) \longrightarrow 0.$$

But $\mathcal{F}\mathcal{G}_\otimes(S^\lambda) \cong S^\lambda \cong \mathcal{F}(\nabla^\lambda E)$, so (i) follows. It is known that $\mathcal{F}(L^\mu(E)) = 0$ if and only if μ is not 2-restricted [EGS08, (6.4a),(6.4b)], so (ii) follows from (i).

Every dual Weyl module in characteristic p has a composition factor $L^\mu(E)$ with μ a p-restricted partition (this can be deduced by interpreting, as in [Jam80], the decomposition matrix for S_r as a submatrix of the decomposition matrix for $\mathrm{GL}_n(K)$). By (ii), such a composition factor cannot occur in U^λ, so (iii) follows. $\square$

We now bound the dimension of U^λ. We use big-O and big-Θ notation: given functions f and g, the statement $f(n) = O(g(n))$ means that the function f grows asymptotically at most as quickly as g, whilst $f(n) = \Theta(g(n))$ means f grows asymptotically at the same rate as g.

Lemma 14.3. *Fix r and allow n to vary. Let M be a K-vector space with basis labelled by (a subset of) tableaux with entries in $[n]$. Let U be a K-subspace of M. Let $l \geqslant 1$, and suppose all elements of U are linear combinations of basis elements labelled by tableaux with at most $r-l$ distinct entries. Then $\dim U = O(d^{r-l})$.*

Proof. Consider $L \leqslant M$ the K-subspace linearly spanned by basis elements labelled by tableaux with at most $r-l$ distinct entries. There are at most $\left(\!\binom{n}{r-l}\!\right) = \binom{n+r-l-1}{r-l} = O(n^{r-l})$ possibilities for the multiset of entries of such a tableau (where $\left(\!\binom{a}{b}\!\right)$ denotes the number of multisubsets of size b in a set of size a), and there are at most $(r-l)^r$ possibilities for the arrangement of a given $(r-l)$-multiset of entries into a tableau. Thus $\dim L = O(n^{r-l})$. By assumption, U is a subspace of L, and so $\dim U = O(n^{r-l})$. □

Proposition 14.4. *Fix r and allow n to vary. Then $\dim U^\lambda = O(n^{r-1})$.*

Proof. Skew column tabloids in U^λ have a repeated entry in a column, and so have at most $r-1$ distinct entries; the proposition then follows from Lemma 14.3. □

Remark 14.5. The dimensions of the dual Weyl modules are known (given by the hook content formula [Sta01, Theorem 7.21.2]), and in particular $\dim \nabla^\mu E = \Theta(n^r)$ for all partitions μ of r. Thus Proposition 14.4 tells us that U^λ grows more slowly than any dual Weyl module, and in particular more slowly than $\nabla^\lambda E \cong \mathcal{G}_\otimes(S^\lambda)/U^\lambda$. This fact also offers an alternative proof of Proposition 14.2(iii) when n is sufficiently large but which holds also for finite fields: for large n, U^λ is too small to have $\nabla^\mu E$ as a subquotient.

14.4. Composition factors of the image of the Specht module

In this subsection, we identify the composition factors of $\mathcal{G}_\otimes(S^\lambda)$ when $r \leqslant 5$ and K is infinite. The composition factors of $\nabla^\lambda E$ are recorded in, for example, [Jam80, Appendix], so we record only the composition factors of the kernel of the surjection $\mathcal{G}_{\otimes}(S^{\lambda}) \twoheadrightarrow \nabla^{\lambda}E$, which we denote U^{λ} as in §14.3. By Corollary 7.3, the composition factors are independent of n, though some may vanish for small n.

We use a dimension counting argument to identify the composition factors. We show the argument explicitly for the case $\lambda = (2,2,1)$ below. The same approach yields the composition factors of all partitions of $r \leqslant 5$ which we record following this example. The example of $\lambda = (2,2,1)$ also demonstrates that $\mathcal{G}_{\otimes}(S^{\lambda})$ need not have a ∇-filtration (that is, a filtration by dual Weyl modules), since the multiset of composition factors we identify does not permit a ∇-filtration.

Example 14.6 (Composition factors of $\mathcal{G}_{\otimes}(S^{\lambda})$ when $\lambda = (2,2,1)$ do not permit a ∇-filtration)**.** Let $r = 5$ and $\lambda = (2,2,1)$. Suppose $n \geqslant r-1$. We view $\mathcal{B} \cong [n]$. It can be shown directly that for any tableau t whose skew column tabloid lies in $\ker\delta$, given any other tableau t' of the same weight there exist skew Garnir relations γ also lying in $\ker\delta$ such that $||t|| + \gamma = ||t'||$. For example, if $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}$ and $t = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$, we can choose $\gamma = \mathsf{R}^{\mathsf{Sk}}_{(t,1,1)}$. Furthermore, no skew column tabloid lies in $\mathsf{SkGR}^{\lambda}(E)$, because all the skew snake relations have an even number of summands (and so every linear combination of snake relations is either zero or has at least two distinct column tabloids with nonzero coefficients). Therefore, in U^{λ} there is exactly one distinct element $||t|| + \ker\delta|_{\mathsf{GR}}$ for each weight of tableau that permits at least one repeated column entry, and these elements are linearly independent. The number of such weights are enumerated in Table 14.1. This allows us to compute $\dim U^{\lambda} = \frac{1}{6}n^4 + \frac{5}{6}n^2$.

The dimensions of the simple modules for $r = 5$ can be computed from the dimensions of the dual Weyl modules (found using the hook content formula [Sta01, Theorem 7.21.2]) and the decomposition matrix for $\mathrm{GL}_n(K)$ (see [Jam80, Appendix]). These dimensions are recorded in Table 14.2 below. By Corollary 7.3, the partitions labelling the composition factors of U^{λ} are independent of n for $n \geqslant r$. Thus $\dim U^{\lambda} = \frac{1}{6}n^4 + \frac{5}{6}n^2$ is a positive linear combination of the dimensions in this table.

dominant weight	example tabloid	number of weights
$(2,1^3)$	$\begin{Vmatrix}1\\1\\4\end{Vmatrix}\begin{Vmatrix}2\\3\end{Vmatrix}$	$4\binom{n}{4}$
$(2^2,1)$	$\begin{Vmatrix}1\\1\\2\end{Vmatrix}\begin{Vmatrix}2\\3\end{Vmatrix}$	$3\binom{n}{3}$
$(3,1^2)$	$\begin{Vmatrix}1\\1\\1\end{Vmatrix}\begin{Vmatrix}2\\3\end{Vmatrix}$	$3\binom{n}{3}$
$(3,2)$	$\begin{Vmatrix}1\\1\\1\end{Vmatrix}\begin{Vmatrix}2\\2\end{Vmatrix}$	$2\binom{n}{2}$
$(4,1)$	$\begin{Vmatrix}1\\1\\1\end{Vmatrix}\begin{Vmatrix}1\\2\end{Vmatrix}$	$2\binom{n}{2}$
(5)	$\begin{Vmatrix}1\\1\\1\end{Vmatrix}\begin{Vmatrix}1\\1\end{Vmatrix}$	n

TABLE 14.1. The number of weights of tableaux with entries in $[n]$ which have at least one repeated entry in a column.

λ	$\dim L^{\lambda}(E)$
(1^5)	$\frac{1}{120}n^5 - \frac{1}{12}n^4 + \frac{7}{24}n^3 - \frac{5}{12}n^2 + \frac{1}{5}n$
$(2,1^3)$	$\frac{1}{30}n^5 - \frac{1}{6}n^4 + \frac{1}{6}n^3 + \frac{1}{6}n^2 - \frac{1}{5}n$
$(2^2,1)$	$\frac{1}{30}n^5 - \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{1}{5}n$
$(3,1^2)$	$\frac{1}{6}n^4 - \frac{1}{2}n^3 + \frac{1}{3}n^2$
$(3,2)$	$\frac{1}{2}n^3 - \frac{1}{2}n^2$
$(4,1)$	$\frac{1}{3}n^4 - \frac{1}{3}n^2$
(5)	n^2

TABLE 14.2. The dimensions of the simple $K\mathrm{GL}_n(K)$-modules of polynomial degree $r=5$.

This allows us to deduce the composition factors of U^{λ}. Considering the coefficient of n^4, we first deduce that $L^{(3,1,1)}(E)$ must be a composition factor. Subtracting these dimensions and considering the highest remaining powers of n in turn, we deduce that the composition factors of U^λ are $L^{(3,1,1)}(E)$, $L^{(3,2)}(E)$ and $L^{(5)}(E)$.

These composition factors, together with those of $\nabla^\lambda E$, can then be compared with the possible composition series of dual Weyl modules (found from the decomposition matrix for $\mathrm{GL}_n(K)$). Doing so reveals that the composition factors of $\mathcal{G}_\otimes(S^\lambda)$ cannot be partitioned into sets of composition factors for dual Weyl modules, and hence that $\mathcal{G}_\otimes(S^\lambda)$ has no ∇-filtration.

We now state the results for the remaining partitions of $r \leqslant 5$.

For $r \leqslant 3$, the only partitions for which $\mathcal{G}_\otimes(S^\lambda) \ncong \nabla^\lambda E$ are the columns (1^2) and (1^3). If $\lambda = (1^2)$, then $U^\lambda \cong L^{(2)}(E)$; if $\lambda = (1^3)$, then $U^\lambda \cong L^{(3)}(E)$. (In fact, in these cases the image $\mathcal{G}_\otimes(S^\lambda)$ is easily identified: see Proposition 14.7.)

For $r = 4$, the partitions for which $\mathcal{G}_\otimes(S^\lambda) \ncong \nabla^\lambda E$ are (1^4) and $(2,1^2)$. For $r = 5$, the partitions for which $\mathcal{G}_\otimes(S^\lambda) \ncong \nabla^\lambda E$ are (1^5), $(2,1^3)$, $(2^2,1)$ and $(3,1^2)$. The composition factors of U^λ in these cases are given in Table 14.3.

	(2^2)	$(3,1)$	(4)
(1^4)	1	1	1
$(2,1^2)$	2	1	1

(a) $r = 4$

	$(3,1^2)$	$(3,2)$	$(4,1)$	(5)
(1^5)	1	1		1
$(2,1^3)$			1	
$(2^2,1)$	1	1		1
$(3,1^2)$	1	2		1

(b) $r = 5$

TABLE 14.3. The composition factors of U^λ for partitions of $r = 4, 5$. The composition factors of U^λ are given by the row labelled λ; the multiplicities of the simple module $L^\mu(E)$ by the column labelled μ.

14.5. Descriptions in particular cases

In this subsection, we describe the module $\mathcal{G}_\otimes(S^\lambda)$ for some particular tractable examples. In particular, we:

- fully describe $\mathcal{G}_\otimes(S^\lambda)$ when λ is a column, row, or two-row partition (Proposition 14.7);

- fully describe $\mathcal{G}_{\otimes}(S^{\lambda})$ when $n = 1$ (Proposition 14.10);
- compute the dimension of $\mathcal{G}_{\otimes}(S^{\lambda})$ when $n = 2$ and λ is a hook partition, and furthermore for hook partitions of even length identify $\mathcal{G}_{\otimes}(S^{\lambda})$ as a tensor product of known representations (Proposition 14.12).

Proposition 14.7 (Columns, rows and two-row partitions)**.**

(i) Suppose $\lambda = (1^r)$ is a single column. Then $\mathcal{G}_{\otimes}(S^{\lambda}) \cong \mathsf{Sk}^r E$.

(ii) Suppose $\lambda = (r)$ is a single row. Then $\mathcal{G}_{\otimes}(S^{\lambda}) \cong \mathrm{Sym}^r E$.

(iii) Suppose $\lambda = (r-m, m)$ is a two-row partition and $\lambda \neq (1,1)$. Then $\mathcal{G}_{\otimes}(S^{\lambda}) \cong \nabla^{(r-m,m)} E$.

Proof. When λ is a column, we observe that $\mathsf{SkGR}^{\lambda}(E) = 0$ and so $\mathcal{G}_{\otimes}(S^{\lambda}) \cong \mathsf{Sk}^{\lambda'} E$. When λ consists of at most two rows (and $\lambda \neq (1,1)$), the claim follows from Theorems F and G (or, in the case of a single row, can be seen clearly from the fact that the skew Garnir relations become relations exchanging the entries along the row). $\square$

It is interesting that even the case of $n = 1$ is nontrivial. When $n = 1$, the dual Weyl module is easy to describe: $\nabla^{\lambda} E = 0$ unless λ is a single row, in which case $\nabla^{\lambda} E \cong \mathrm{Sym}^r E \cong E^{\otimes r}$ of dimension 1. For $\mathcal{G}_{\otimes}(S^{\lambda})$, again 0 and $E^{\otimes r}$ are the only two possibilities, but both can occur for partitions of arbitrary length, and the dichotomy of partitions is not straightforward to describe.

To distinguish between the two possibilities, we require the following characterisation of the parity of binomial coefficients. This is the case $p = 2$ of Definition 11.10 and Lucas's Theorem, for which the results can be stated particularly concisely.

Definition 14.8. We say the binary addition of integers a and b is *carry-free* if, for all i, the ith binary digits of a and b are not both 1.

Lemma 14.9. *Let $a, b, c \in \mathbb{N}$.*

(i) The binomial coefficient $\binom{a+b}{a}$ is odd if and only if the binary addition of a and b is carry-free.

(ii) There exists $1 \leqslant i \leqslant c-1$ *such that* $\binom{c}{i}$ *is odd if and only if* c *is not a power of* 2*. When this is the case, the minimal* $i \geqslant 1$ *such that* $\binom{c}{i}$ *is odd is the maximal power of* 2 *that divides* c*.*

Proof. Part (i) is a consequence of Lucas's Theorem, as given (for example) in [Jam78, Lemma 22.4]. Part (ii) follows from part (i) by writing c in binary. □

Proposition 14.10. *Suppose* $n = 1$ *and* $\operatorname{char} K = 2$*. Then* $\mathcal{G}_{\otimes}(S^{\lambda}) = 0$ *if and only if there exists* $1 \leqslant j < \lambda_1$ *such that:*

- $\lambda'_j + 1$ *is not a power of* 2*; and*
- $\lambda'_{j+1} \geqslant 2^m$*, where* $m \geqslant 0$ *is maximal such that* 2^m *divides* $\lambda'_j + 1$*.*

When $\mathcal{G}_{\otimes}(S^{\lambda}) \neq 0$*, we have* $\mathcal{G}_{\otimes}(S^{\lambda}) \cong E^{\otimes r}$*.*

Proof. Since $n = 1$, the set $\mathcal{B}$ is a singleton and there is a unique tableau t with entries in $\mathcal{B}$ (having all entries the same). We therefore have that $\mathcal{G}_{\otimes}(S^{\lambda}) = 0$ if and only if $||t|| \in \mathsf{SkGR}^{\lambda}(E)$, and $\mathcal{G}_{\otimes}(S^{\lambda}) \cong E^{\otimes r}$ otherwise.

Place permutations leave t unchanged, so the skew Garnir relation labelled by sets A and B is the sum of $|S_{A \sqcup B} : S_A \times S_B|$ copies of $||t||$. That is,

$$\mathsf{R}^{\mathsf{Sk}}_{(t,A,B)} = \frac{(|A| + |B|)!}{|A|!|B|!}||t||.$$

Focusing on skew snake relations, this becomes

$$\mathsf{R}^{\mathsf{Sk}}_{(t,i,j)} = \binom{\lambda'_j + 1}{i}||t||.$$

Thus $||t|| \in \mathsf{SkGR}^{\lambda}(E)$ if and only if there exists j such that $\binom{\lambda'_j+1}{i}$ is odd for some $1 \leqslant i \leqslant \lambda'_{j+1}$. The proposition now follows from Lemma 14.9(ii). □

This proof of Proposition 14.10 generalises the argument for hook partitions given in Example 13.6. The following corollary of the proposition can in fact be deduced from that example alone.

Corollary 14.11 (Hooks when $n = 1$)**.** *Let $a, l \geqslant 2$ with $r = a + l - 1$. Suppose $n = 1$, $\operatorname{char} K = 2$, and $\lambda = (a, 1^{l-1})$ is a hook partition. Then*

$$\mathcal{G}_{\otimes}(S^{\lambda}) \cong \begin{cases} 0 & \text{if } l \text{ is even,} \\ E^{\otimes r} & \text{if } l \text{ is odd.} \end{cases}$$

Our final example concerns hooks when $n = 2$. Our description includes a *Frobenius twist*. Recall that if K is a field of characteristic p, then the map $x \mapsto x^p$ is a field endomorphism called the *Frobenius endomorphism*. This yields a group endomorphism of $\mathrm{GL}_n(K)$ defined by acting entrywise. Composing this map with the representing group homomorphism of a representation V over K yields a new representation, called the *Frobenius twist* of V, which we denote $\mathsf{Fr}(V)$. Given an element $v \in V$, we denote the corresponding element of $\mathsf{Fr}(V)$ by $\mathsf{Fr}(v)$.

Proposition 14.12 (Hooks when $n = 2$)**.** *Let $a, l \geqslant 2$ with $r = a + l - 1$. Suppose $n = 2$, $\operatorname{char} K = 2$, and $\lambda = (a, 1^{l-1})$.*

(i) Suppose l is even. Then

$$\mathcal{G}_{\otimes}(S^{\lambda}) \cong \mathsf{Fr}(\operatorname{Sym}^{\frac{l}{2}-1} E) \otimes \operatorname{Sym}^{a-1} E \otimes \det E,$$

of dimension $\frac{1}{2}al$.

(ii) Suppose l is odd. Then $\dim \mathcal{G}_{\otimes}(S^{\lambda}) = \frac{1}{2}(a+1)(l+1)$.

Proof. Write $\mathcal{B} = \{X, Y\}$, with $X < Y$, for the basis of E. Given a tableau t, write $\mathrm{e}^{\mathsf{Sk}}(t)$ for the image of $||t||$ in $\mathcal{G}_{\otimes}(S^{\lambda})$.

We consider the spanning set for the skew Garnir relations identified in Proposition 13.3, with the function Φ defined on column semistandard but not row semistandard tableaux by choosing the right-most box eligible box in the first row (noting there is no other row with more than one box).

A skew Garnir relation involving only columns other than the first tells us precisely that in $\mathcal{G}_{\otimes}(S^{\lambda})$ the entries, except the first, of the first row can be permuted freely.

The remaining elements of our spanning set we must consider are labelled by $(t, 1, 1)$ for some t, where either: t is row-and-column semistandard and

$t(1,1) = t(1,2)$; or the first column of t has all entries Y, $t(1,2) = X$, and the remainder of the first row is weakly increasing.

For $0 \leqslant b \leqslant l$ and $0 \leqslant m \leqslant a-1$, let $t_{b,m}$ be the (column semistandard) tableau of shape λ where X appears b times in the first column and m times in the remaining columns, with the Xs in the first column at the top, and the Xs in the first row at the left (except possibly the first column). The tableaux identified in the previous paragraph, labelling the snake relations we are still to consider, are all of the form $t_{b,m}$ for some $0 \leqslant b \leqslant l$ and $0 \leqslant m \leqslant a-1$. Additionally, if $t_{b,m}$ is one of the identified tableaux and $m = 0$, then also $b = 0$. In these cases, we have:

$$\mathsf{R}^{\mathsf{Sk}}_{(t_{b,m},1,1)} = \begin{cases} (b+1)||t_{b,m}|| + (l-b)||t_{b+1,m-1}|| & \text{if } m > 0, \\ (l+1)||t_{0,0}|| & \text{if } b = m = 0. \end{cases} \tag{14.12.1}$$

[(i)] Suppose l is even. Then each relation (14.12.1) above has an odd total number of summands, and thus is equal to a single tabloid. If $||t_{b,m}||$ appears as a relation, which is precisely if b is even, then its image in $\mathcal{G}_{\otimes}(S^{\lambda})$ is zero; if it does not, which is precisely if b is odd, then its image in $\mathcal{G}_{\otimes}(S^{\lambda})$ is nonzero and is linearly independent of the images of all other tabloids of that form. Thus

$$\{\, \mathrm{e}^{\mathsf{Sk}}(t_{b,m}) \mid 0 \leqslant b \leqslant l,\, b \text{ odd},\, 0 \leqslant m \leqslant a-1 \,\}$$

is a basis for $\mathcal{G}_{\otimes}(S^{\lambda})$. The dimension follows.

Let $\varphi\colon \mathcal{G}_{\otimes}(S^{\lambda}) \to \mathsf{Fr}(\mathrm{Sym}^{\frac{l}{2}-1} E) \otimes \mathrm{Sym}^{a-1} E \otimes \det E$ be the map defined by K-linear extension of

$$\varphi(\mathrm{e}^{\mathsf{Sk}}(t_{b,m})) = \mathsf{Fr}(X^{\frac{b-1}{2}} Y^{\frac{l-b-1}{2}}) \otimes X^m Y^{a-1-m} \otimes 1$$

for $0 \leqslant b \leqslant l$, b odd, $0 \leqslant m \leqslant a-1$. Since φ is a bijection between bases, it is a linear isomorphism. It is easy to verify that φ respects the action of diagonal elements of $\mathrm{GL}_2(K)$: the element $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ acts on both $\mathrm{e}^{\mathsf{Sk}}(t_{b,m})$ and its image by multiplication by $\alpha^{b+m}\beta^{r-b-m}$. By Lemma 5.9, it then suffices to show φ respects the action of elementary transvections.

Let $0 \leqslant b \leqslant l$, b odd, $0 \leqslant m \leqslant a-1$. Let $g = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ for some $\alpha \in K$; that is, g is the elementary transvection fixing Y and acting on X as

$gX = X + \alpha Y$. Then

$$gX^m Y^{a-1-m} = \sum_{j=0}^{m} \binom{m}{j} \alpha^j X^{m-j} Y^{a-1-m+j}$$

and

$$\begin{aligned} g\mathsf{Fr}(X^{\frac{b-1}{2}} Y^{\frac{l-b-1}{2}}) &= \mathsf{Fr}(\left(\begin{smallmatrix} 1 & 0 \\ \alpha^2 & 1 \end{smallmatrix}\right) X^{\frac{b-1}{2}} Y^{\frac{l-b-1}{2}}) \\ &= \sum_{k=0}^{\frac{b-1}{2}} \binom{\frac{b-1}{2}}{k} \alpha^{2k} \mathsf{Fr}(X^{\frac{b-1}{2}-k} Y^{\frac{l-b-1}{2}+k}). \end{aligned}$$

Also $\det g = 1$, so

$$g\varphi(\mathrm{e}^{\mathsf{Sk}}(t_{b,m})) = \sum_{k=0}^{\frac{b-1}{2}} \sum_{j=0}^{m} \binom{\frac{b-1}{2}}{k} \binom{m}{j} \alpha^{2k+j} \varphi(\mathrm{e}^{\mathsf{Sk}}(t_{b-2k,m-j})).$$

Meanwhile,

$$\begin{aligned} g\,\mathrm{e}^{\mathsf{Sk}}(t_{b,m}) &= \sum_{i=0}^{b} \sum_{j=0}^{m} \binom{b}{i} \binom{m}{j} \alpha^{i+j}\,\mathrm{e}^{\mathsf{Sk}}(t_{b-i,m-j}) \\ &= \sum_{k=0}^{\frac{b-1}{2}} \sum_{j=0}^{m} \binom{b}{2k} \binom{m}{j} \alpha^{2k+j}\,\mathrm{e}^{\mathsf{Sk}}(t_{b-2k,m-j}) \end{aligned}$$

where the second equality holds because $\mathrm{e}^{\mathsf{Sk}}(t_{b-i,m-j}) = 0$ when i is odd, so we can relabel via $i = 2k$. Equivariance is then clear provided that $\binom{\frac{b-1}{2}}{k} \equiv \binom{b}{2k} \pmod 2$. Indeed this follows from the Lemma 14.9(i) by noting that the binary addition of c and d is carry-free if and only if the binary addition of $2c$ and $2d+1$ is carry-free. Showing that φ respects the action of $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ is analogous, and completes the proof.

[(ii)] Suppose l is odd. Then each relation (14.12.1) above has an even total number of summands, and thus is either zero or the sum of two distinct tableaux. The $b = m = 0$ relation is clearly zero. When $m > 0$, the relation is nonzero if and only if b is even. We thus have that $\mathrm{e}^{\mathsf{Sk}}(t_{b,m}) = \mathrm{e}^{\mathsf{Sk}}(t_{b+1,m-1})$ for b even and $m > 0$, and furthermore that

$$\{\, \mathrm{e}^{\mathsf{Sk}}(t_{b,m}) \mid 0 \leqslant b \leqslant l,\ b \text{ even},\ 0 \leqslant m \leqslant a-1 \,\} \\ \sqcup \{\, \mathrm{e}^{\mathsf{Sk}}(t_{b,a-1}) \mid 0 \leqslant b \leqslant l,\ b \text{ odd} \,\}$$

is a basis for $\mathcal{G}_{\otimes}(S^{\lambda})$. The dimension follows. $\square$

CHAPTER V

Tensor products of representations of $\mathrm{SL}_2(\mathbb{F}_p)$

This chapter studies tensor products of representations of the finite group $\mathrm{SL}_2(\mathbb{F}_p)$ over an algebraically closed field K of characteristic p. We decompose tensor products into indecomposable modules, and investigate a random walk on the simple representations defined via the tensor product. This draws from the author's [McD21b].

The relevant modules are introduced in §15.

In §16 we identify decompositions of tensor products of simple modules, of tensor products of projective indecomposable modules, and of symmetric and exterior squares.

In §17 we investigate Markov chains defined by tensoring by a fixed simple module and choosing a non-projective indecomposable summand of the result. We show our chains are reversible and find their connected components and their stationary distributions, and draw connections between these properties of the chain and the representation theory of $\mathrm{SL}_2(\mathbb{F}_p)$, emphasising symmetries of the tensor product.

In this chapter, our representations are over an algebraically closed field K with prime subfield $\mathbb{F}_p$, and our group is $G = \mathrm{SL}_2(\mathbb{F}_p)$, except possibly in §15 and §16.1 when we sometimes permit G to be another subgroup of $\mathrm{GL}_2(K)$. We define the following notation for a family of sets that will index the summands of tensor products.

Definition 15.0. For $n \geqslant m \geqslant 1$, let the (n,m)-*string* be the set

$$\langle n, m\rangle = \{n+m-1, n+m-3, \dots, n-m+3, n-m+1\},$$

and let $\langle n, 0\rangle = \varnothing$.

15. Representations of $\mathrm{SL}_2(\mathbb{F}_p)$

In this section we define the modules whose tensor products we will study in subsequent sections, and provide some additional background on the representation theory of $\mathrm{SL}_2(\mathbb{F}_p)$.

15.1. Simple modules

Write $V_m = \mathrm{Sym}^{m-1} E$ for $m \geqslant 1$, where E is the natural $K\mathrm{SL}_2(\mathbb{F}_p)$-module (or with some other matrix subgroup $G \leqslant \mathrm{GL}_2(K)$ in place of $\mathrm{SL}_2(\mathbb{F}_p)$). Since $\dim E = 2$, the parameter shift means that we are labelling each module V_m by its dimension m.

By writing X and Y for the standard basis vectors of E, we can model V_m as the space of homogeneous polynomials over K of degree $m-1$ in two variables X and Y, with G-action given by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) f(X,Y) = f(aX + cY, bX + dY).$$

When $G \geqslant \mathrm{SL}_2(\mathbb{F}_p)$, the modules $V_1, \ldots, V_p$ are simple [Alp86, pp. 14–16]. Since the number of p-regular conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_p)$ is precisely p, we deduce that when $G = \mathrm{SL}_2(\mathbb{F}_p)$ the set $\{\, V_m \mid 1 \leqslant m \leqslant p \,\}$ is a complete set of simple KG-modules.

In particular, we deduce that there is a unique simple KG-module of each dimension less than or equal to p. Thus the simple modules are self-dual.

15.2. Projective indecomposable modules

When G is finite, let P_m be the projective cover of V_m. Then when $G = \mathrm{SL}_2(\mathbb{F}_p)$, the set $\{\, P_m \mid 1 \leqslant m \leqslant p \,\}$ is a complete set of projective indecomposable KG-modules.

The projective indecomposable $K\mathrm{SL}_2(\mathbb{F}_p)$-modules are constructed in [Alp86, pp. 48–52] (using the special case $m = 2$ of our Proposition 16.2). We give their descriptions here.

The module $P_p \cong V_p$ is projective and simple.

When $p = 2$, there is only one other projective indecomposable module: P_1, which is of composition length 2 (and hence has composition factors only V_1). For $p > 2$, all other projective indecomposable modules have

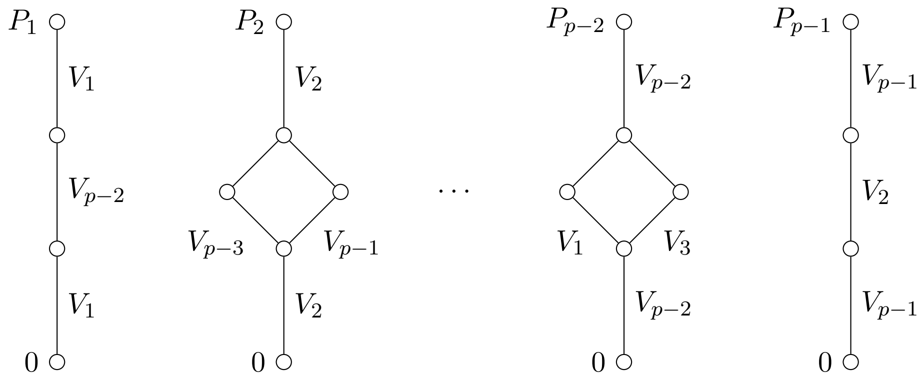


FIGURE 15.1. The structures of the projective indecomposable representations of $\mathrm{SL}_2(\mathbb{F}_p)$ in defining characteristic, when $p > 2$.

composition length 3, and so the only structural information which is missing is their heart. The heart of P_1 is V_{p-2}, the heart of P_{p-1} is V_2, and for $2 \leqslant n \leqslant p-2$ the heart of P_n is $V_{p-n-1} \oplus V_{p-n+1}$; these structures are illustrated in Figure 15.1.

Note that P_1 and P_p are both p-dimensional, while all other projective indecomposable KG-modules are $2p$-dimensional.

15.3. Block structure of $\mathrm{SL}_2(\mathbb{F}_p)$

From the structure of the projective indecomposable modules, we can describe the blocks of $\mathrm{SL}_2(\mathbb{F}_p)$ and write down their Brauer trees and the Cartan matrix. These descriptions are given for interest and completeness; they are not required in subsequent sections. For definitions of Brauer trees and Cartan matrices, see [Alp86, Section 17] and [CR62, p. 593] respectively.

The module $V_p \cong P_p$ is projective, and hence lies in its own block of defect 0.

For $p = 2$, there is only one other block: the principal block, having Brauer tree a single edge with multiplicity 1. For $p > 2$, there are two other blocks: the principal block containing the simple modules of odd dimension, and a block containing the simple modules of even dimension. It can be seen directly from the structure of the projective indecomposable modules that

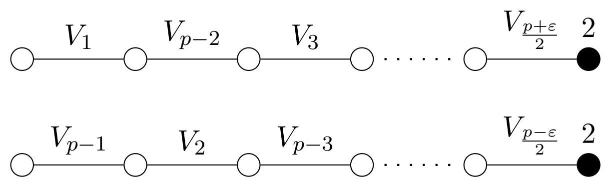


FIGURE 15.2. Brauer trees for the blocks of $\mathrm{SL}_2(\mathbb{F}_p)$ of nonzero defect, when $p > 2$. Here $\varepsilon \in \{\pm 1\}$ and $\varepsilon \equiv p \pmod 4$.

each of these blocks is a Brauer tree algebra, with the trees illustrated in Figure 15.2. These trees are described in [Alp86, p. 123].

Since they contain non-projective simple modules, these blocks have nontrivial defect, and then since the unique nontrivial p-subgroup of G is cyclic of order p, this is the defect group of each block (alternatively, given these Brauer trees, the defect groups must be cyclic of order p by the classification of blocks of cyclic defect).

To write down the Cartan matrix, it is most convenient to give the simple modules and their covers the ordering

$$V_1, V_{p-2}, V_3, \dots, V_{\frac{p+\varepsilon}{2}}, V_{p-1}, V_2, V_{p-3}, \dots, V_{\frac{p-\varepsilon}{2}}, V_p$$

where $\varepsilon \in \{\pm 1\}$ and $\varepsilon \equiv p \pmod 4$. For $p = 2$, the Cartan matrix is simply $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For $p > 2$, let C be the $\frac{p-1}{2} \times \frac{p-1}{2}$ matrix

$$\begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & 3 \end{pmatrix}$$

and let $C = \begin{pmatrix} 3 \end{pmatrix}$ when $p = 3$. Then, in block diagonal form,

$$\begin{pmatrix} C & & \\ & C & \\ & & 1 \end{pmatrix}$$

is the Cartan matrix.

16. Decompositions of tensor products

In this section we prove the Clebsch–Gordan rule, which describes the decomposition of tensor products of simple modules. We then use it to decompose tensor products of projective indecomposable modules and to decompose symmetric and exterior squares.

16.1. Maps between tensor products

Our proof of the Clebsch–Gordan rule requires two families of maps between tensor products.

Multiplication map. The first map we require is the multiplication map. Identifying its kernel leads to a filtration of tensor products of the symmetric powers.

Definition 16.1. Let $\mu\colon V_n \otimes V_m \to V_{n+m-1}$ be the multiplication map, defined by K-linear extension of $\mu(f \otimes g) = fg$. The dependence of μ on n and m is suppressed, since it is always clear from context.

It is easily seen that μ is surjective and $\mathrm{GL}_2(K)$-equivariant. The following result identifying the kernel of μ is well-known (see [Glo78, (5.1)], or for the case $m = 2$ [Alp86, Lemma 5, p. 50–51] or [Kou90a, Proposition 1.2(a)]).

Proposition 16.2. *Suppose $G \leqslant \mathrm{SL}_2(K)$ and suppose $n, m \geqslant 2$. Then the kernel of μ is isomorphic to $V_{n-1} \otimes V_{m-1}$, and hence there is a short exact sequence*

$$0 \longrightarrow V_{n-1} \otimes V_{m-1} \longrightarrow V_n \otimes V_m \xrightarrow{\mu} V_{n+m-1} \longrightarrow 0.$$

Proof. Consider the map $\theta\colon V_{n-1} \otimes V_{m-1} \to V_n \otimes V_m$ defined by K-linear extension of $\theta(f \otimes g) = Xf \otimes Yg - Yf \otimes Xg$. Observe that θ is

$\mathrm{SL}_2(K)$-equivariant: for $t = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(K)$, we have

$$\begin{aligned} t\theta(f \otimes g) &= t(Xf \otimes Yg - Yf \otimes Xg) \\ &= (aX + cY)tf \otimes (bX + dY)tg - (bX + dY)tf \otimes (aX + cY)tg \\ &= (ad - bc)Xtf \otimes Ytg - (ad - bc)Ytf \otimes Xtg \\ &= \det(t)(Xsf \otimes Ytg - Ytf \otimes Xtg) \\ &= Xtf \otimes Ytg - Ytf \otimes Xtg \\ &= \theta(t(f \otimes g)). \end{aligned}$$

It is easy to see that $\operatorname{im}\theta \leqslant \ker\mu$. Because μ is surjective, we have that $\dim(\ker\mu) = \dim(V_n \otimes V_m) - \dim(V_{n+m-1}) = \dim(V_{n-1} \otimes V_{m-1})$, and so it remains only to show that θ is injective.

Let

$$e_{i,j} = X^i Y^{n-2-i} \otimes X^j Y^{m-2-j} \in V_{n-1} \otimes V_{m-1},$$

so that $\{\, e_{i,j} \mid 0 \leqslant i \leqslant n-2,\ 0 \leqslant j \leqslant m-2 \,\}$ is a linear basis for $V_{n-1} \otimes V_{m-1}$. For $0 \leqslant r \leqslant n + m - 4$, let

$$U_r = \langle\, e_{i,j} \mid i + j = r \,\rangle_K \subseteq_K V_{n-1} \otimes V_{m-1},$$

and note that as vector spaces $V_{n-1} \otimes V_{m-1} = \bigoplus_{r=0}^{n+m-4} U_r$.

Similarly, let

$$f_{i,j} = X^i Y^{n-1-i} \otimes X^j Y^{m-1-j} \in V_n \otimes V_m,$$

so that $\{\, f_{i,j} \mid 0 \leqslant i \leqslant n-1,\ 0 \leqslant j \leqslant m-1 \,\}$ is a linear basis for $V_n \otimes V_m$. For $0 \leqslant r \leqslant n + m - 2$, let

$$W_r = \langle\, f_{i,j} \mid i + j = r \,\rangle_K \subseteq_K V_n \otimes V_m,$$

and note that as vector spaces $V_n \otimes V_m = \bigoplus_{r=0}^{n+m-2} W_r$.

Observe that $\theta(e_{i,j}) = f_{i+1,j} - f_{i,j+1}$. Then $\theta(U_r) \subseteq_K W_{r+1}$, and thus it suffices to show that $\theta|_{U_r}$ is injective for each $0 \leqslant r \leqslant n + m - 4$. Fix r in this range, and let $i_0 = \max\{0, r - (m - 2)\}$ and $j_0 = \max\{0, r - (n - 2)\}$ so that $U_r = \langle\, e_{i,r-i} \mid i_0 \leqslant i \leqslant r - j_0 \,\rangle_K$. Then the images under θ of these basis vectors for U_r are as follows.

$$\begin{aligned}\theta(e_{i_0,r-i_0}) &= f_{i_0+1,r+1-(i_0+1)} - f_{i_0,r+1-i_0}\\ \theta(e_{i_0+1,r-(i_0+1)}) &= f_{i_0+2,r+1-(i_0+2)} - f_{i_0+1,r+1-(i_0+1)}\\ \theta(e_{i_0+2,r-(i_0+2)}) &= f_{i_0+3,r+1-(i_0+3)} - f_{i_0+2,r+1-(i_0+2)}\\ &\vdots\\ \theta(e_{r-(j_0+1),j_0+1}) &= f_{r+1-(j_0+1),j_0+1} - f_{r+1-(j_0+2),j_0+2}\\ \theta(e_{r-j_0,j_0}) &= f_{r+1-j_0,j_0} - f_{r+1-(j_0+1),j_0+1}\end{aligned}$$

Thus the $(r-i_0-j_0+1)\times(r-i_0-j_0)$ matrix representing θ with respect to these bases is

$$\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1\\ & & & -1\end{pmatrix},$$

which is of full (column) rank. Thus $\theta|_{U_r}$ is injective as required. $\square$

Remark 16.3. Unlike μ, the map θ used in the proof of Proposition 16.2 is *not* $\mathrm{GL}_2(K)$-equivariant. Since $t\theta(f\otimes g)=\det(t)\theta(t(f\otimes g))$ for $t\in\mathrm{GL}_2(K)$, we see that θ is not G-equivariant for any subgroup G which contains a matrix with determinant not equal to 1. For an extension of this proposition to such subgroups, see [Glo78, (5.1)].

Corollary 16.4. *Suppose $G\leqslant\mathrm{SL}_2(K)$ and suppose $1\leqslant m\leqslant n$. Then $V_n\otimes V_m$ has a filtration*

$$0=M_m\subseteq M_{m-1}\subseteq\cdots\subseteq M_1\subseteq M_0=V_n\otimes V_m$$

where $M_i\cong V_{n-i}\otimes V_{m-i}$ and ${}^{M_i}\!/\!{}_{M_{i+1}}\cong V_{n+m-1-2i}$.

Proof. By induction on m. The case $m=1$ is immediate. For $m\geqslant 2$, the short exact sequence involving μ gives that there is $M_1\subseteq V_n\otimes V_m$ such that

$$M_1\cong V_{n-1}\otimes V_{m-1}$$

and

$$V_n \otimes V_m \big/ M_1 \cong V_{n+m-1}.$$

Applying the inductive hypothesis to M_1 gives the rest of the filtration. $\square$

Remark 16.5. The proof of Proposition 16.2 holds equally well if K is of characteristic 0. In this case the simple modules are also projective and so the short exact sequences split, and we obtain $V_n \otimes V_m \cong \bigoplus_{i \in \langle n,m\rangle} V_i$ (recovering the well-known Clebsch–Gordan rule for $\mathrm{SU}_2(\mathbb{C})$). The same decomposition is obtained when $G \leqslant \mathrm{SL}_2(K)$ is finite with $p \nmid |G|$.

Separation map. The second family of maps we require was introduced by the author in [McD21b], generalising the map δ defined in [Glo78, (5.2)] (corresponding to $n = 1$ below). These maps allow us to see the inclusion of the bottom layer of the above filtration into $V_n \otimes V_m$, and they split in more cases than μ does.

Definition 16.6. For $n \geqslant 1$ and $m \geqslant 2$, let $\lambda\colon V_n \otimes V_m \to V_{n+1} \otimes V_{m-1}$ be the map defined by K-linear extension of

$$\lambda(f \otimes g) = Xf \otimes \frac{\partial g}{\partial X} + Yf \otimes \frac{\partial g}{\partial Y}.$$

The dependence of λ on n and m is suppressed, since it is always clear from context.

Lemma 16.7. *The map λ is $\mathrm{GL}_2(K)$-equivariant.*

Proof. Let $t = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$, and let $f \in V_n$ and $g \in V_m$. Then

$$\begin{aligned} t\lambda(f \otimes g) &= t\left(Xf \otimes \frac{\partial g}{\partial X} + Yf \otimes \frac{\partial g}{\partial Y}\right) \\ &= (aX + cY)tf \otimes t\frac{\partial g}{\partial X} + (bX + dY)tf \otimes t\frac{\partial g}{\partial Y} \\ &= Xtf \otimes \left(at\frac{\partial g}{\partial X} + bt\frac{\partial g}{\partial Y}\right) + Ytf \otimes \left(ct\frac{\partial g}{\partial X} + dt\frac{\partial g}{\partial Y}\right) \end{aligned}$$

and

$$\lambda(t(f \otimes g)) = Xtf \otimes \frac{\partial(tg)}{\partial X} + Ytf \otimes \frac{\partial(tg)}{\partial Y}.$$

So it suffices to show that $\frac{\partial(tg)}{\partial X} = at\frac{\partial g}{\partial X} + bt\frac{\partial g}{\partial Y}$ and that $\frac{\partial(tg)}{\partial Y} = ct\frac{\partial g}{\partial X} + dt\frac{\partial g}{\partial Y}$.

Without loss of generality, suppose g is a monomial; write $g = X^i Y^j$ (where $i + j = m - 1$). Then $tg = (aX + cY)^i (bX + dY)^j$, and

$$\begin{aligned} \frac{\partial(tg)}{\partial X} &= \frac{\partial(aX+cY)^i}{\partial X}(bX+dY)^j + (aX+cY)^i \frac{\partial(bX+dY)^j}{\partial X} \\ &= ia(aX+cY)^{i-1}(bX+dY)^j + jb(aX+cY)^i(bX+dY)^{j-1} \\ &= at\frac{\partial g}{\partial X} + bt\frac{\partial g}{\partial Y} \end{aligned}$$

and similarly $\frac{\partial(tg)}{\partial Y} = ct\frac{\partial g}{\partial X} + dt\frac{\partial g}{\partial Y}$. □

Lemma 16.8. *Suppose $n \geqslant m$ and $2 \leqslant m \leqslant p$. Then the map λ is surjective.*

Proof. Let $f = X^i Y^{i'} \in V_{n+1}$, $g = X^j Y^{j'} \in V_{m-1}$ be monomials. We have $i + i' + j + j' = n + m - 2 \geqslant 2(m-1)$, and hence either $i + j \geqslant m - 1$ or $i' + j' \geqslant m - 1$. We show that $f \otimes g \in \operatorname{im}\lambda$ by downward induction on j whenever $i + j \geqslant m - 1$; then by analogy the same holds whenever $i' + j' \geqslant m - 1$.

Note first that if $i + j \geqslant m - 1$, then $i \geqslant 1$ (since $0 \leqslant j \leqslant m - 2$) and so $\frac{1}{X}f$ is a polynomial (in V_n).

If $j = m - 2$, then $g = X^{m-2}$ so $\frac{\partial(Xg)}{\partial X} = (m-1)X^{m-2}$ and $\frac{\partial g}{\partial Y} = 0$. Then

$$\lambda\left(\tfrac{1}{X}f \otimes Xg\right) = (m-1) f \otimes g$$

and $m - 1$ is invertible (since $2 \leqslant m \leqslant p$), so $f \otimes g \in \operatorname{im}\lambda$.

Now suppose $0 \leqslant j < m - 2$. Then

$$\lambda\left(\tfrac{1}{X}f \otimes Xg\right) = (j+1) f \otimes g + \frac{Y}{X} f \otimes X\frac{\partial g}{\partial Y}.$$

But by the inductive hypothesis $\frac{Y}{X}f \otimes X\frac{\partial g}{\partial Y} \in \operatorname{im}\lambda$ (since $X\frac{\partial g}{\partial Y}$ has a higher power of X than g, and the sum of the powers of X in $\frac{Y}{X}f$ and $X\frac{\partial g}{\partial Y}$ is $i + j \geqslant m - 1$). Then since $j + 1$ is invertible, we have $f \otimes g \in \operatorname{im}\lambda$. □

Proposition 16.9. *Suppose $G \leqslant \mathrm{SL}_2(K)$ and suppose $n \geqslant m$ and $2 \leqslant m \leqslant p$. Then the kernel of λ is isomorphic to V_{n-m+1}, and hence there is a short exact sequence*

$$0 \longrightarrow V_{n-m+1} \longrightarrow V_n \otimes V_m \xrightarrow{\lambda} V_{n+1} \otimes V_{m-1} \longrightarrow 0.$$

Proof. Define variations on the multiplication map by

$$\begin{aligned} \mu^{(r)} \colon V_{n_1} \otimes V_{m_1} \otimes \cdots \otimes V_{n_r} \otimes V_{m_r} &\to V_{N-(r-1)} \otimes V_{M-(r-1)} \\ f_1 \otimes g_1 \otimes \cdots \otimes f_r \otimes g_r \quad &\mapsto \quad f_1 \cdots f_r \otimes g_1 \cdots g_r \end{aligned}$$

extended K-linearly, where $N = \sum_{i=1}^r n_i$ and $M = \sum_{i=1}^r m_i$. Let $g_m \in V_m \otimes V_m$ be the element

$$\begin{aligned} g_m &= \mu^{(m-1)}\left((X \otimes Y - Y \otimes X) \otimes \cdots \otimes (X \otimes Y - Y \otimes X)\right) \\ &= \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} X^i Y^{m-1-i} \otimes X^{m-1-i} Y^i. \end{aligned}$$

By the first expression, it is clear that $tg_m = (\det t)^{m-1} g_m$ for any $t \in \mathrm{GL}_2(K)$.

Now define a K-linear map $\eta \colon V_{n-m+1} \to V_n \otimes V_m$ by

$$\begin{aligned} \eta(f) &= \mu^{(2)}(f \otimes 1 \otimes g_m) \\ &= \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} f X^i Y^{m-1-i} \otimes X^{m-1-i} Y^i. \end{aligned}$$

Then for any $t \in \mathrm{GL}_2(K)$, we have $t\eta(f) = (\det t)^{m-1} \eta(tf)$, and so η is G-equivariant. Clearly the expression above is zero if and only if $f = 0$, so η is injective. Furthermore,

$$\begin{aligned} \lambda\eta(f) = &\sum_{i=0}^{m-2} (-1)^{m-1-i} \binom{m-1}{i} (m-1-i) f X^{i+1} Y^{m-1-i} \otimes X^{m-2-i} Y^i \\ &+ \sum_{i=1}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} i f X^i Y^{m-i} \otimes X^{m-1-i} Y^{i-1} \\ = 0, \end{aligned}$$

where the final equality can be seen by replacing i with $i-1$ in the first sum, and noting that $\binom{m-1}{i}i = (m-1)\binom{m-2}{i-1} = \binom{m-1}{i-1}(m-i)$. Thus $V_{n-m+1} \cong \operatorname{im}\eta \leqslant \ker\lambda$.

Since $n \geqslant m$ and $2 \leqslant m \leqslant p$, by Lemma 16.8 we have that λ is surjective, and then by counting dimensions we have $V_{n-m+1} \cong \ker\lambda$. □

Remark 16.10. Using Corollary 16.4 and comparing the filtrations of $V_n \otimes V_m$ and $V_{n+1} \otimes V_{m-1}$, we see immediately that $\ker\lambda$ and V_{n-m+1} have the same multiset of composition factors (provided λ is surjective). In the case $n-m+1 \leqslant p$ and $G \geqslant \mathrm{SL}_2(\mathbb{F}_p)$, we have that V_{n-m+1} is simple, and we could then deduce this proposition immediately without considering η.

Power map. When decomposing tensor products involving projective modules, we require also the following isomorphism. We will only require the case $n=2$, $q=p$ and $G = \mathrm{SL}_2(\mathbb{F}_p)$, but we prove it more generally as it is no more difficult. This isomorphism, for representations of the semigroup of 2×2 matrices over $\mathbb{F}_p$, is established in [Glo78, (5.3)].

Lemma 16.11. *Suppose $\mathbb{F}_q \leqslant K$ is a finite subfield of order q (where q is a power of p) and $G \leqslant \mathrm{GL}_2(\mathbb{F}_q)$. Then there is an isomorphism $V_n \otimes V_q \cong V_{nq}$.*

PROOF. Let $\psi\colon V_n \to V_{nq-q+1}$ be the map defined by $\psi(f(X,Y)) = f(X^q, Y^q)$. It is K-linear (indeed, it is the K-linear extension of $X^iY^j \mapsto X^{qi}Y^{qj}$). Then let $\varphi\colon V_n \otimes V_q \to V_{nq}$ be the map defined by K-linear extension of

$$\varphi(f \otimes g) = \psi(f)g.$$

We immediately see that φ is surjective: given $X^rY^{nq-1-r} \in V_{nq}$, write $r = iq + j$ with $0 \leqslant j \leqslant q-1$, and then $\varphi(X^iY^{n-1-i} \otimes X^jY^{q-1-j}) = X^rY^{nq-1-r}$. Then φ is also injective, since $\dim(V_n \otimes V_q) = nq = \dim(V_{nq})$. To obtain an isomorphism $V_n \otimes V_q \cong V_{nq}$, it remains only to show that φ is G-equivariant. For this it suffices to show that ψ is G-equivariant.

Let $t = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. Recall that $x^q = x$ for any $x \in \mathbb{F}_q$, and that $(y+z)^q = y^q + z^q$ for any y, z in any ring of characteristic p. Then

$$\begin{aligned} t\psi(f(X,Y)) &= tf(X^q, Y^q) \\ &= f((aX+cY)^q, (bX+dY)^q) \\ &= f(aX^q + cY^q, bX^q + dY^q) \\ &= \psi(f(aX+cY, bX+dY)) \\ &= \psi(tf(X,Y)) \end{aligned}$$

as required. □

16.2. Clebsch–Gordan rule

We now obtain the decomposition of tensor products of simple modules. Here and for the remainder of this chapter we take $G = \mathrm{SL}_2(\mathbb{F}_p)$.

Theorem 16.12 (Clebsch–Gordan rule for $\mathrm{SL}_2(\mathbb{F}_p)$ in characteristic p)**.** *Suppose $G = \mathrm{SL}_2(\mathbb{F}_p)$ and $1 \leqslant m \leqslant n \leqslant p$. Then*

$$V_n \otimes V_m \cong \bigoplus_{\substack{i \in \langle n,m\rangle \cap [p] \\ 2p-i \notin \langle n,m\rangle}} V_i \oplus \bigoplus_{\substack{i \in \langle n,m\rangle \cap [p] \\ 2p-i \in \langle n,m\rangle}} P_i \oplus \mathbb{1}[n=m=p] V_p.$$

Remark 16.13. We make several immediate observations about the tensor product of simple modules V_n and V_m (where $1 \leqslant m \leqslant n \leqslant p$):

(i) all non-projective indecomposable summands of $V_n \otimes V_m$ are simple;

(ii) $V_n \otimes V_m$ is semisimple if and only if $n + m \leqslant p + 1$, in which case $V_n \otimes V_m \cong \bigoplus_{i \in \langle n,m\rangle} V_i$, which is exactly the rule for analogously defined representations of $\mathrm{SU}_2(\mathbb{C})$ over $\mathbb{C}$;

(iii) $V_n \otimes V_m$ is projective if and only if $n = p$, in which case $V_p \otimes V_m \cong \bigoplus_{i \in \langle p,m\rangle \cap [p]} P_i \oplus \mathbb{1}[m=p] V_p$;

(iv) in the sense of indecomposable summands, $V_n \otimes V_m$ is multiplicity-free unless $n = m = p$ (when V_p occurs with multiplicity 2, and all other indecomposable summands occur only once).

Proof. Our strategy is to establish two implications. Implication (i) is that if the theorem holds for $(n+1, m-1)$, then it holds for (n, m) (where

$2 \leqslant m \leqslant n \leqslant p-1$), which we prove by showing that the short exact sequence involving λ splits in this case. Implication (ii) is that if the theorem holds for $(p-1, m-1)$, then it holds for (p, m) (where $2 \leqslant m \leqslant p$), which we prove using the short exact sequence involving μ. With these implications, it suffices to show the theorem holds for $(n, 1)$ for $1 \leqslant n \leqslant p$ (as illustrated in the case $p = 7$ in Figure 16.1). But these cases are trivial, since $V_n \otimes V_1 \cong V_n$ (and $P_p \cong V_p$), so the theorem follows.

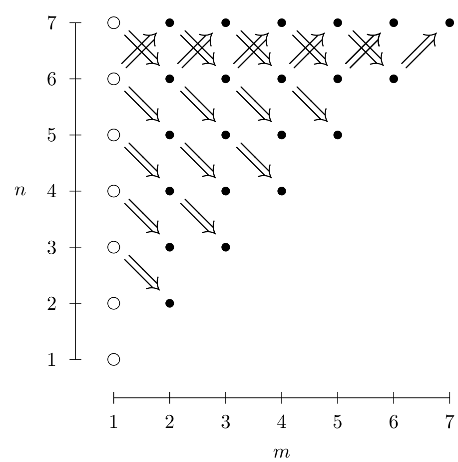


FIGURE 16.1. An illustration of how the implications we prove suffice to prove the entire theorem, in the case $p = 7$. The dot in position (n, m) represents the theorem holding for that pair of values, the hollow dots being the trivial cases with $m = 1$; the arrows represent the implications we prove here.

Implication (i)

Suppose the theorem holds for $(n+1, m-1)$ (where $2 \leqslant m \leqslant n \leqslant p-1$); that is,

$$V_{n+1} \otimes V_{m-1} \cong \bigoplus_{\substack{i \in \langle n+1, m-1 \rangle \cap [p] \\ 2p-i \notin \langle n+1, m-1 \rangle}} V_i \quad \oplus \bigoplus_{\substack{i \in \langle n+1, m-1 \rangle \cap [p] \\ 2p-i \in \langle n+1, m-1 \rangle}} P_i.$$

Observe that the proposed decomposition of $V_n \otimes V_m$ differs from that of $V_{n+1} \otimes V_{m-1}$ only by an additional summand of V_{n-m+1}. Thus to show the theorem holds for (n, m), it suffices to show that the short exact sequence

$$0 \longrightarrow V_{n-m+1} \longrightarrow V_n \otimes V_m \xrightarrow{\lambda} V_{n+1} \otimes V_{m-1} \longrightarrow 0$$

from Proposition 16.9 splits.

Let $Q \cong \bigoplus_{\substack{i \in \langle n+1, m-1\rangle \cap [p] \\ 2p-i \in \langle n+1, m-1\rangle}} P_i$ be the projective part of $V_{n+1} \otimes V_{m-1}$. Then the projection of λ onto Q splits, and so there is a module W such that

$$V_n \otimes V_m \cong W \oplus Q$$

and such that there is a short exact sequence

$$0 \longrightarrow V_{n-m+1} \longrightarrow W \longrightarrow \bigoplus_{\substack{i \in \langle n+1, m-1\rangle \cap [p] \\ 2p-i \notin \langle n+1, m-1\rangle}} V_i \longrightarrow 0.$$

It now suffices to show that this sequence splits. Indeed, suppose, towards a contradiction, the sequence does not split. Then W, and hence $V_n \otimes V_m$, has as an indecomposable summand some non-split extension T of V_{n-m+1} by a module with composition factors a nonempty subset of $\{ V_i \mid i \in \langle n+1, m-1\rangle \cap [p] \}$. This set of composition factors does not contain V_{n-m+1} itself, so T is not self-dual. Furthermore, the dual of T is not a summand of W, since V_{n-m+1} occurs only once as a composition factor of W, and nor is it a summand of Q, since V_{n-m+1} does not occur as the head of any of the projective summands of Q. Thus the dual of T is not a summand of $V_n \otimes V_m$, contradicting the self-duality of $V_n \otimes V_m$. So the sequence splits as required.

Implication (ii)

Suppose the theorem holds for $(p-1, m-1)$ (where $2 \leqslant m \leqslant p$). Then, using that $\langle p-1, m-1\rangle \cap [p] = \langle p, m\rangle \cap [p]$, we have

$$V_{p-1} \otimes V_{m-1} \cong V_{p-m+1} \oplus \bigoplus_{\substack{i \in \langle p, m\rangle \cap [p] \\ i \neq p-m+1}} P_i.$$

Then by Proposition 16.2 we have a short exact sequence

$$0 \longrightarrow V_{p-m+1} \oplus \bigoplus_{\substack{i \in \langle p, m\rangle \cap [p] \\ i \neq p-m+1}} P_i \longrightarrow V_p \otimes V_m \xrightarrow{\mu} V_{p+m-1} \longrightarrow 0.$$

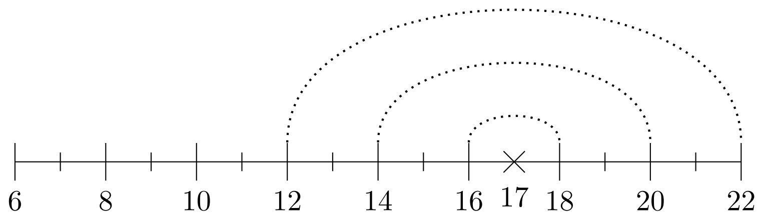
Thus $\bigoplus_{i\in\langle p,m\rangle\cap[p]} V_i$ is isomorphic to a submodule of $\mathrm{soc}(V_p \otimes V_m)$. But since V_p is projective, so is $V_p \otimes V_m$ (as the tensor product of a projective module with any other module is projective [Alp86, Lemma 4, p. 47]). Thus $\bigoplus_{i\in\langle p,m\rangle\cap[p]} P_i$ is isomorphic to a submodule of $V_p \otimes V_m$.

The proof is completed by counting dimensions to show that this submodule is the entire tensor product, unless $m = p$ when we must identify one additional summand. Recall from §15 that the projective indecomposable $K\mathrm{SL}_2(\mathbb{F}_p)$-modules are $2p$-dimensional, except for P_1 and $P_p \cong V_p$ which are p-dimensional.

Suppose $m \neq p$. Then $1 \notin \langle p,m\rangle$ and also $p > 2$. If m is even, then $p \notin \langle p,m\rangle$ and $|\langle p,m\rangle \cap [p]| = \frac{m}{2}$, so $\dim(\bigoplus_{i\in\langle p,m\rangle\cap[p]} P_i) = \frac{m}{2}\cdot 2p = mp = \dim(V_p \otimes V_m)$. If m is odd, then $p \in \langle p,m\rangle$ and $|\langle p,m\rangle \cap [p]| = \frac{m+1}{2}$, so $\dim(\bigoplus_{i\in\langle p,m\rangle\cap[p]} P_i) = p + \frac{m-1}{2}\cdot 2p = mp = \dim(V_p \otimes V_m)$. Thus, in either case, $V_p \otimes V_m \cong \bigoplus_{i\in\langle p,m\rangle\cap[p]} P_i$.

Now suppose $m = p$. Then $1 \in \langle p,p\rangle$, and so in the count above one of the $2p$-dimensional modules is replaced with a p-dimensional module, which leaves us with $\dim(\bigoplus_{i\in\langle p,p\rangle\cap[p]} P_i) = \dim(V_p \otimes V_p) - p$ (and if $p = 2$ then $\langle p,p\rangle = \{1,3\}$ and $\bigoplus_{i\in\langle p,p\rangle\cap[p]} P_i = P_1$ is of dimension $p = 2 = p^2 - p$ as well). Since $V_p \otimes V_p$ is projective, these p dimensions must be accounted for by an additional copy of either P_1 or V_p. Since V_p is self-dual, we have $V_p \otimes V_p \cong \mathrm{Hom}_K(V_p, V_p)$. Noting that the direct sum of all trivial submodules of $\mathrm{Hom}_K(V_p, V_p)$ is $\mathrm{Hom}_{KG}(V_p, V_p)$, which is isomorphic to V_1 by Schur's Lemma, we deduce that V_1 occurs in the socle of $V_p \otimes V_p$ with multiplicity 1. Thus the missing summand is V_p. □

Example 16.14. Let $G = \mathrm{SL}_2(\mathbb{F}_p)$ and $p = 17$, and we consider $V_{14} \otimes V_9$. We draw the $(14,9)$-string below, and indicate those elements i for which $2p - i \in \langle 14,9\rangle$ by joining i and $2p - i$ with a dotted line. The unpaired elements give rise to simple summands, while the paired elements give rise to projective indecomposable summands; the summand of $V_{14} \otimes V_9$ that arises out of each element of $\langle 14,9\rangle \cap [17]$ is written below it.



$$V_{14} \otimes V_9 \;\cong\; V_6 \;\oplus\; V_8 \;\oplus V_{10} \oplus P_{12} \oplus P_{14} \oplus P_{16}$$

The pairing-up of i and $2p-i$ in fact corresponds to an isomorphism

$$V_{2p-i} \cong {}^{P_i}\!\diagup_{V_i} \oplus \mathbb{1}[i=1]V_p$$

proved in Lemma 16.16.

16.3. Decompositions of tensor products of projective modules

We now use Theorem 16.12 to decompose tensor products of combinations of simple and projective indecomposable representations of $\mathrm{SL}_2(\mathbb{F}_p)$. We establish the following decompositions (where we permit descriptions in terms of tensor products whose decompositions have already been identified).

Theorem 16.15. *Suppose $G = \mathrm{SL}_2(\mathbb{F}_p)$. For $2 \leqslant n \leqslant p-1$ and $2 \leqslant m \leqslant p$, we have*

$$P_n \otimes V_m \cong \begin{cases} \displaystyle\bigoplus_{\substack{i \in \langle n,m\rangle \\ i \leqslant p}} P_i \oplus \bigoplus_{\substack{i \in \langle n,m\rangle \\ i \geqslant p}} P_{2p-i} \oplus \mathbb{1}[n=m]V_p & \text{if } n \geqslant m, \\ \displaystyle\bigoplus_{\substack{i \in \langle m,n\rangle \\ i \leqslant p}} P_i \oplus \bigoplus_{\substack{i \in \langle m,n\rangle \\ i \geqslant p}} P_{2p-i} \oplus P_{p-(m-n)} \oplus \bigoplus_{\substack{i \in \langle p,m-n-1\rangle \\ i \leqslant p}} P_i^{\oplus 2} & \text{if } n < m < p, \\ \displaystyle\bigoplus_{\substack{i \in \langle p,n\rangle \\ i \leqslant p}} P_i^{\oplus 2} \oplus \bigoplus_{\substack{i \in \langle p,p-n-1\rangle \\ i \leqslant p}} P_i^{\oplus 2} \oplus P_n & \text{if } m = p, \end{cases}$$

while for all $1 \leqslant m \leqslant p$ we have

$$P_1 \otimes V_m \cong P_m \;\oplus\; \mathbb{1}[m>2](V_p \otimes V_{m-2}) \;\oplus\; \mathbb{1}[m=p]V_p.$$

For $2 \leqslant n, m \leqslant p-1$ *we have*

$$P_n \otimes P_m \cong (P_n \otimes V_m) \oplus (P_m \otimes V_n)$$
$$\oplus \begin{cases} (P_{p-1} \otimes V_{2p-(n+m)}) & \text{if } n+m \geqslant p, \\ (P_{p-1} \otimes V_{p+1-(n+m)}) \oplus (P_{n+m} \otimes V_{p-1}) & \text{if } n+m < p, \end{cases}$$

while for all $1 \leqslant m \leqslant p$ *we have*

$$P_1 \otimes P_m \cong P_m^{\oplus 2} \oplus \mathbb{1}[p>2](V_{p-2} \otimes P_m).$$

Our proof relies on two key facts: a tensor product involving a projective module is itself projective [Alp86, Lemma 4, p. 47], and a projective module is uniquely determined by its multiset of composition factors. The latter follows from the invertibility of the Cartan matrix. Thus to determine the decompositions in Theorem 16.15, it suffices to show that the given tensor products have the claimed composition factors.

One approach to this is to apply our Clebsch–Gordan rule to every pair of composition factors from the modules we are tensoring together (we know the composition factors of the projective indecomposable modules, as recorded in §15.2). This would allow us to find all the composition factors of the tensor product, and multiplying by the inverse of the Cartan matrix would then yield the multiplicities of the projective indecomposable summands. However, we wish to avoid these onerous calculations.

The approach we take is to use the result below to pair up classes of (not necessarily simple) modules into classes of projective modules (such pairings are also made when applying our Clebsch–Gordan rule in the manner described in Example 16.14). This method avoids using the structure of the projective indecomposable modules in most cases.

Lemma 16.16. *Suppose* $1 \leqslant n \leqslant p-1$. *Then*

$$V_{2p-n} \cong {P_n}/{V_n} \oplus \mathbb{1}[n=1]V_p.$$

Remark 16.17. The structure of the projective indecomposable modules is known (see §15.2), so this corollary gives us the structure of V_i for $p+1 \leqslant i \leqslant 2p-1$.

PROOF. Let $2 \leqslant m \leqslant p$. Via μ, we have an isomorphism

$$V_{p+m-1} \cong \frac{V_p \otimes V_m}{V_{p-1} \otimes V_{m-1}}.$$

Then, applying Theorem 16.12, we have

$$\begin{aligned} V_{p+m-1} &\cong \frac{\bigoplus_{i \in \langle p,m \rangle \cap [p]} P_i \oplus \mathbb{1}[m=p] V_p}{V_{p-m+1} \oplus \bigoplus_{\substack{i \in \langle p,m \rangle \cap [p] \\ i \neq p-m+1}} P_i} \\ &\cong {P_{p-m+1}}/{V_{p-m+1}} \oplus \mathbb{1}[m=p] V_p. \end{aligned}$$

Taking $n = p - m + 1$ gives the result. $\square$

Whilst identifying composition factors in the propositions that follow, it is convenient to use the notation of the Grothendieck group.

Definition 16.18. For an algebra $\mathcal{A}$, the *Grothendieck group* $G_0(\mathcal{A})$ is the abelian group with:

- a generator $[V]$ for every $\mathcal{A}$-module V, and
- a relation $[W] = [U] + [V]$ for every short exact sequence $0 \to U \to W \to V \to 0$.

The important property of the Grothendieck group for our purposes is that $[U] = [V]$ if and only if U and V have the same multiset of composition factors. Thus to decompose a projective module, it suffices to write its image in the Grothendieck group as a sum of classes of projective indecomposable modules.

Product of a simple and a projective. We begin with the case of a product of a simple and a projective.

Proposition 16.19. *Suppose* $2 \leqslant n, m \leqslant p-1$ *(and in particular* $p > 2$*). Then:*

$$P_n \otimes V_m \cong \begin{cases} \bigoplus\limits_{\substack{i \in \langle n,m \rangle \\ i \leqslant p}} P_i \oplus \bigoplus\limits_{\substack{i \in \langle n,m \rangle \\ i \geqslant p}} P_{2p-i} \oplus \mathbb{1}[n=m] V_p & \text{if } m \leqslant n, \\ \bigoplus\limits_{\substack{i \in \langle m,n \rangle \\ i \leqslant p}} P_i \oplus \bigoplus\limits_{\substack{i \in \langle m,n \rangle \\ i \geqslant p}} P_{2p-i} \oplus P_{p-(m-n)} \oplus \bigoplus\limits_{\substack{i \in \langle p, m-n-1 \rangle \\ i \leqslant p}} P_i^{\oplus 2} & \text{if } m > n. \end{cases}$$

Proof. We have that $V_n \otimes V_m$ is isomorphic to a submodule of $P_n \otimes V_m$. Using Lemma 16.16, for $2 \leqslant n \leqslant p-1$ we have

$$\begin{aligned} \frac{P_n \otimes V_m}{V_n \otimes V_m} &\cong P_n\!\big/\!V_n \otimes V_m \\ &\cong V_{2p-n} \otimes V_m. \end{aligned}$$

That is, in the Grothendieck group,

$$[P_n \otimes V_m] = [V_n \otimes V_m] + [V_{2p-n} \otimes V_m].$$

Suppose first that $m \leqslant n$. Then by Corollary 16.4, and observing that $\langle 2p-n, m \rangle = 2p - \langle n, m \rangle$, we have

$$\begin{aligned} [P_n \otimes V_m] &= \sum_{i \in \langle n,m \rangle} [V_i] + \sum_{i \in \langle 2p-n,m \rangle} [V_i] \\ &= \sum_{i \in \langle n,m \rangle} [V_i] + [V_{2p-i}]. \end{aligned}$$

But Lemma 16.16 tells us that $[V_i] + [V_{2p-i}] = [P_{\min\{i,2p-i\}}] + \mathbb{1}[i \in \{1, 2p-1\}][V_p]$ for $1 \leqslant i \leqslant 2p-1$ and $i \neq p$. Thus

$$\begin{aligned} [P_n \otimes V_m] &= \sum_{\substack{i \in \langle n,m \rangle \\ i \leqslant p}} [P_i] + \sum_{\substack{i \in \langle n,m \rangle \\ i \geqslant p}} [P_{2p-i}] + \mathbb{1}[1 \in \langle n,m \rangle][V_p] \\ &= \bigoplus_{\substack{i \in \langle n,m \rangle \\ i \leqslant p}} [P_i] \oplus \bigoplus_{\substack{i \in \langle n,m \rangle \\ i \geqslant p}} [P_{2p-i}] \oplus \mathbb{1}[n=m][V_p], \end{aligned}$$

which completes the first case.

Now suppose $m > n$. As before, we use Lemma 16.16 and Corollary 16.4, and this time we find

$$[P_n \otimes V_m] = \sum_{i \in \langle m,n \rangle} [V_i] + \sum_{i \in \langle 2p-n,m \rangle} [V_i]$$

and we cannot pair up the summands as we did in the case $m \leqslant n$. However, we do find that

$$\begin{aligned}\langle 2p-n, m\rangle &= \{2p-n-m+1,\ 2p-n-m+3,\ \ldots,\ 2p-n-m+(2n-1),\\ &\qquad 2p-n-m+(2n+1),\ \ldots,\ 2p-n+m-3,\ 2p-n+m-1\}\\ &= \{2p-(m+n-1),\ 2p-(m+n-3),\ \ldots,\ 2p-(m-n+1),\\ &\qquad 2p-(m-n)+1,\ \ldots,\ 2p+(m-n)-3,\ 2p+(m-n)-1\}\\ &= (2p-\langle m,n\rangle) \sqcup \langle 2p, m-n\rangle.\end{aligned}$$

Thus

$$\begin{aligned}[P_n \otimes V_m] &= \sum_{i\in\langle m,n\rangle}([V_i]+[V_{2p-i}]) + \sum_{i\in\langle 2p,m-n\rangle}[V_i]\\ &= [P_m \otimes V_n] + [V_{2p}\otimes V_{m-n}]\\ &= [P_m \otimes V_n] + [P_{p-1}\otimes V_{m-n}],\end{aligned}$$

where the final equality holds because $V_{2p} \cong V_2 \otimes V_p$ by Lemma 16.11 and $V_2 \otimes V_p \cong P_{p-1}$ for $p > 2$ by Theorem 16.12.

We can now use the first case to decompose each of the products in this sum (or, if $m - n = 1$, simply using $P_{p-1} \otimes V_1 \cong P_{p-1}$). The second product becomes

$$\begin{aligned}[P_{p-1}\otimes V_{m-n}] &= \sum_{\substack{i\in\langle p-1,m-n\rangle\\ i\leqslant p}}[P_i] + \sum_{\substack{i\in\langle p-1,m-n\rangle\\ i\geqslant p}}[P_{2p-i}]\\ &= [P_{p-(m-n)}] + \sum_{i\in\langle p,m-n-1\rangle\cap[p]} 2[P_i],\end{aligned}$$

as required. $\square$

Proposition 16.20. *Suppose* $2 \leqslant m \leqslant p-1$ *(and in particular* $p > 2$*). Then*

$$V_p \otimes P_m \cong \bigoplus_{\substack{i\in\langle p,m\rangle\\ i\leqslant p}} P_i^{\oplus 2} \oplus \bigoplus_{\substack{i\in\langle p,p-m-1\rangle\\ i\leqslant p}} P_i^{\oplus 2} \oplus P_m.$$

PROOF. We have

$$V_p \otimes P_m \big/ V_p \otimes V_m \cong V_p \otimes V_{2p-m}.$$

Now

$$\begin{aligned}\langle 2p-m, p\rangle &= \{p-m+1, p-m+3, \dots, 3p-m-1\}\\ &= \langle p, m\rangle \sqcup \{p+m+1, \dots, 3p-m-1\}\\ &= \langle p, m\rangle \sqcup \langle 2p, p-m\rangle\end{aligned}$$

and so $[V_{2p-m} \otimes V_p] = [V_p \otimes V_m] + [V_{2p} \otimes V_{p-m}]$. But $V_{2p} \cong P_{p-1}$, so we have

$$V_p \otimes P_m \cong (V_p \otimes V_m)^{\oplus 2} \oplus (P_{p-1} \otimes V_{p-m}).$$

Using the modular Clebsch–Gordan rule and Proposition 16.19 gives the decomposition into indecomposable modules. $\square$

Product of two projectives. Next we deal with the case of a product of two (non-simple) projectives.

Proposition 16.21. *Suppose* $2 \leqslant m \leqslant n \leqslant p-1$ *(and in particular* $p > 2$*). Then*

$$\begin{aligned}P_n \otimes P_m \cong{}& (P_n \otimes V_m) \oplus (P_m \otimes V_n)\\ &\oplus \begin{cases}(P_{p-1} \otimes V_{2p-(n+m)}) & \text{if } n+m \geqslant p,\\ (P_{p-1} \otimes V_{p+1-(n+m)}) \oplus (P_{n+m} \otimes V_{p-1}) & \text{if } n+m < p.\end{cases}\end{aligned}$$

PROOF. We have

$$\frac{P_n \otimes P_m}{P_n \otimes V_m} \cong P_n \otimes V_{2p-m}$$

and

$$\frac{P_n \otimes V_{2p-m}}{V_n \otimes V_{2p-m}} \cong V_{2p-n} \otimes V_{2p-m},$$

and so

$$[P_n \otimes P_m] = [P_n \otimes V_m] + [V_n \otimes V_{2p-m}] + [V_{2p-n} \otimes V_{2p-m}].$$

Now,

$$\begin{aligned}\langle 2p-m, 2p-n\rangle &= \{n-m+1, m-n+3, \ldots, 4p-n-m-1\}\\ &= \langle n,m\rangle \sqcup \{n+m+1, \ldots, 4p-n-m-1\}\\ &= \langle n,m\rangle \sqcup \langle 2p, 2p-(n+m)\rangle.\end{aligned}$$

Thus $[V_{2p-m}\otimes V_{2p-n}] = [V_n\otimes V_m] + [V_{2p}\otimes V_{2p-(n+m)}]$. But $[V_n\otimes V_{2p-m}] + [V_n\otimes V_m] = [V_n\otimes P_m]$ and $V_{2p}\cong P_{p-1}$ (for $p>2$), so

$$[P_n\otimes P_m] = [P_n\otimes V_m] + [P_m\otimes V_n] + [P_{p-1}\otimes V_{2p-(n+m)}].$$

If $n+m\geqslant p$, we are done.

If $n+m<p$, and since also $n+m>1$, we have $V_{2p-(n+m)}\cong P_{n+m}/V_{n+m}$. Then $[P_{p-1}\otimes V_{2p-(n+m)}] = [P_{p-1}\otimes P_{n+m}] - [P_{p-1}\otimes V_{n+m}]$. We use the first case to decompose

$$P_{p-1}\otimes P_{n+m}\cong (P_{p-1}\otimes V_{n+m})\oplus(P_{n+m}\otimes V_{p-1})\oplus(P_{p-1}\otimes V_{p+1-(n+m)}),$$

and so $[P_{p-1}\otimes V_{2p-(n+m)}] = [P_{p-1}\otimes V_{p+1-(n+m)}] + [P_{n+m}\otimes V_{p-1}]$ giving the result. □

Product with P_1. We have so far avoided using the structure of the projective indecomposable modules. Nevertheless, for the case of tensoring with P_1 it is most convenient to make use of our knowledge of their composition factors. As described in §15.2, for $p=2$ we have $[P_1]=2[V_1]$ whilst for $p>2$ we have:

$$\begin{aligned}[P_1] &= 2[V_1]+[V_{p-2}],\\ [P_{p-1}] &= 2[V_{p-1}]+[V_2],\\ [P_i] &= 2[V_i]+[V_{p-i-1}]+[V_{p-i+1}]\quad\text{for } 2\leqslant i\leqslant p-2.\end{aligned}$$

Proposition 16.22. *Suppose* $1\leqslant m\leqslant p$. *Then*

$$P_1\otimes P_m\cong P_m^{\oplus 2}\oplus \mathbb{1}[p>2](V_{p-2}\otimes P_m).$$

PROOF. Immediate from the structure of P_1. □

Proposition 16.23. *Suppose* $1 \leqslant m \leqslant p-1$. *Then*

$$P_1 \otimes V_m \cong P_m \oplus \mathbb{1}[m > 2](V_p \otimes V_{m-2}).$$

Proof. The case $m = 1$ is trivial. For the remaining cases, we have $p > 2$. Observe that

$$[P_1 \otimes V_m] = 2[V_m] + [V_{p-2} \otimes V_m].$$

For $m = 2$, we have $V_{p-2} \otimes V_2 \cong V_{p-3} \oplus V_{p-1}$, and so $[P_1 \otimes V_2] = 2[V_2] + [V_{p-3}] + [V_{p-1}] = [P_2]$.

Next suppose $3 \leqslant m \leqslant p-2$. Then

$$\begin{aligned} 2[V_m] + [V_{p-2} \otimes V_m] &= 2[V_m] + \sum_{i \in \langle p-2, m\rangle} [V_i] \\ &= 2[V_m] + [V_{p-m-1}] + [V_{p-m+1}] + \sum_{i \in \langle p, m-2\rangle} [V_i] \\ &= [P_m] + [V_p \otimes V_{m-2}]. \end{aligned}$$

Finally, for $m = p-1$, we have

$$\begin{aligned} 2[V_{p-1}] + [V_{p-1} \otimes V_{p-2}] &= 2[V_{p-1}] + \sum_{i \in \langle p-1, p-2\rangle} [V_i] \\ &= 2[V_{p-1}] + [V_2] + \sum_{i \in \langle p, p-3\rangle} [V_i] \\ &= [P_{p-1}] + [V_p \otimes V_{p-3}] \end{aligned}$$

as required. □

This completes the proof of Theorem 16.15.

16.4. Decompositions of symmetric and exterior squares

Invoking the Wronskian isomorphism (Theorem C from Chapter III), we use our Clebsch–Gordan rule to inductively decompose symmetric and exterior squares. We find that, for $1 \leqslant m \leqslant p$, the symmetric square $\mathrm{Sym}^2 V_m$ contains those summands of $V_m \otimes V_m$ indexed by elements congruent to $2m-1$ modulo 4, while the exterior square $\bigwedge^2 V_m$ contains those indexed by elements congruent to $2m+1$ modulo 4.

We first note that the two symmetric squares coincide in this setting.

Lemma 16.24. *Suppose* $1 \leqslant m \leqslant p$. *Then* $\operatorname{Sym}^2 V_m \cong \operatorname{Sym}_2 V_m$.

PROOF. This is immediate for $p > 2$ from Proposition 3.5, and for $m = 1$ from observing $\operatorname{Sym}^2 V_1 \cong V_1 \cong \operatorname{Sym}_2 V_1$. For $m = p = 2$, we have $V_2 = E$ and hence we are required to show that $\operatorname{Sym}^2 E \cong \operatorname{Sym}_2 E$. But $(\operatorname{Sym}^2 E)^* \cong (\operatorname{Sym}^2 E)^\circ \cong \operatorname{Sym}_2 E$ by Propositions 3.2 and 3.7, so this is equivalent to the statement that $\operatorname{Sym}^2 E \cong V_3$ is self-dual. Indeed, from Lemma 16.16, we have $V_3 \cong V_1 \oplus V_2$ which is self-dual. $\square$

With the notation $V_m = \operatorname{Sym}^{m-1} E$, the Wronskian isomorphism for squares here becomes

$$\operatorname{Sym}_2 V_m \cong \textstyle\bigwedge^2 V_{m+1} \tag{16.25}$$

for any $m \geqslant 1$. Combining with Lemma 16.24, this becomes

$$\operatorname{Sym}^2 V_m \cong \textstyle\bigwedge^2 V_{m+1} \tag{16.26}$$

for $1 \leqslant m \leqslant p$.

Remark 16.27. An isomorphism $\operatorname{Sym}^2 V_m \to \bigwedge^2 V_{m+1}$ can be identified explicitly for $p \neq 2$ without appealing to Theorem C: it can be shown that the map $f \cdot h \mapsto Xf \wedge Yh - Yf \wedge Xh$, where $f, h \in V_m$, is bijective when $p \neq 2$. Up to a scalar, this map can be seen as the inverse to the isomorphism $\bigwedge^r V_{m+r-1} \to \operatorname{Sym}^r V_m$ of representations of $\mathrm{SL}_2(\mathbb{C})$ described in [AC07, §2.5]. Indeed, in the case $r = 2$, Abdesselam and Chipalkatti's isomorphism is given by

$$f \wedge h \mapsto \frac{1}{2m^2}\Big(\frac{\partial f}{\partial X} \cdot \frac{\partial h}{\partial Y} - \frac{\partial f}{\partial Y} \cdot \frac{\partial h}{\partial X}\Big)$$

where $f, h \in V_{m+1}$; if the scalar factor is removed, this defines an $\mathrm{SL}_2(K)$-equivariant map, and composing with the map above yields $f \cdot h \mapsto 2m\, f \cdot h$.

We need also the well-known short exact sequence

$$0 \longrightarrow \textstyle\bigwedge^2 U \longrightarrow U \otimes U \longrightarrow \operatorname{Sym}^2 U \longrightarrow 0 \tag{16.28}$$

for any module U, which is obtained by identifying $\bigwedge^2 U$ as the kernel of the canonical surjection $U \otimes U \twoheadrightarrow \operatorname{Sym}^2 U$. In characteristics other than 2, this becomes the familiar decomposition $U \otimes U \cong \operatorname{Sym}^2 U \oplus \bigwedge^2 U$ (as a small consequence of the following theorem, we see that this decomposition of the tensor square also holds in characteristic 2 when U is a simple representation of $\mathrm{SL}_2(\mathbb{F}_p)$).

Theorem 16.29. *Suppose $G = \mathrm{SL}_2(\mathbb{F}_p)$ and $1 \leqslant m \leqslant p$. Then*

$$\operatorname{Sym}_2 V_m \cong \operatorname{Sym}^2 V_m \cong \bigoplus_{\substack{i \in \langle m,m\rangle \cap [p] \\ 2p-i \notin \langle m,m\rangle \\ i \equiv 2m-1 \ (\mathrm{mod}\ 4)}} V_i \quad \oplus \bigoplus_{\substack{i \in \langle m,m\rangle \cap [p] \\ 2p-i \in \langle m,m\rangle \\ i \equiv 2m-1 \ (\mathrm{mod}\ 4)}} P_i \quad \oplus \mathbb{1}[m=p]V_p$$

and

$$\bigwedge^2 V_m \cong \bigoplus_{\substack{i \in \langle m,m\rangle \cap [p] \\ 2p-i \notin \langle m,m\rangle \\ i \equiv 2m+1 \ (\mathrm{mod}\ 4)}} V_i \quad \oplus \bigoplus_{\substack{i \in \langle m,m\rangle \cap [p] \\ 2p-i \in \langle m,m\rangle \\ i \equiv 2m+1 \ (\mathrm{mod}\ 4)}} P_i.$$

Proof. We induct on m. The case $m = 1$ is immediate: $\operatorname{Sym}_2 V_1 \cong V_1$ and $\bigwedge^2 V_1 = 0$.

Suppose the proposition holds for some m where $1 \leqslant m \leqslant p-1$. Then using (16.26) we have

$$\bigwedge^2 V_{m+1} \cong \operatorname{Sym}^2 V_m \cong \bigoplus_{\substack{i \in \langle m,m\rangle \cap [p] \\ 2p-i \notin \langle m,m\rangle \\ i \equiv 2m-1 \ (\mathrm{mod}\ 4)}} V_i \quad \oplus \bigoplus_{\substack{i \in \langle m,m\rangle \cap [p] \\ 2p-i \in \langle m,m\rangle \\ i \equiv 2m-1 \ (\mathrm{mod}\ 4)}} P_i.$$

Observe that $\langle m+1, m+1\rangle \setminus \langle m, m\rangle = \{2m+1, 2p-(2m+1)\}$ has no elements congruent to $2m-1$ modulo 4. Thus replacing $\langle m, m\rangle$ with $\langle m+1, m+1\rangle$ in the above decomposition does not alter the summands; that is

$$\bigwedge^2 V_{m+1} \cong \bigoplus_{\substack{i \in \langle m+1,m+1\rangle \cap [p] \\ 2p-i \notin \langle m+1,m+1\rangle \\ i \equiv 2m-1 \ (\mathrm{mod}\ 4)}} V_i \quad \oplus \bigoplus_{\substack{i \in \langle m+1,m+1\rangle \cap [p] \\ 2p-i \in \langle m+1,m+1\rangle \\ i \equiv 2m-1 \ (\mathrm{mod}\ 4)}} P_i,$$

as required. Then using the short exact sequence (16.28) and the Clebsch–Gordan rule, we have

$$\begin{aligned}
\mathrm{Sym}^2 V_{m+1} &\cong V_{m+1} \otimes V_{m+1} \big/ {\textstyle\bigwedge}^2 V_{m+1} \\
&\cong \frac{\displaystyle\bigoplus_{\substack{i\in\langle m+1,m+1\rangle\cap[p] \\ 2p-i\notin\langle m+1,m+1\rangle}} V_i \oplus \bigoplus_{\substack{i\in\langle m+1,m+1\rangle\cap[p] \\ 2p-i\in\langle m+1,m+1\rangle}} P_i \oplus \mathbb{1}[m+1=p]V_p}{\displaystyle\bigoplus_{\substack{i\in\langle m+1,m+1\rangle\cap[p] \\ 2p-i\notin\langle m+1,m+1\rangle \\ i\equiv 2m-1 \pmod 4}} V_i \oplus \bigoplus_{\substack{i\in\langle m+1,m+1\rangle\cap[p] \\ 2p-i\in\langle m+1,m+1\rangle \\ i\equiv 2m-1 \pmod 4}} P_i} \\
&\cong \bigoplus_{\substack{i\in\langle m+1,m+1\rangle\cap[p] \\ 2p-i\notin\langle m+1,m+1\rangle \\ i\equiv 2m+1 \pmod 4}} V_i \oplus \bigoplus_{\substack{i\in\langle m+1,m+1\rangle\cap[p] \\ 2p-i\in\langle m+1,m+1\rangle \\ i\equiv 2m+1 \pmod 4}} P_i \oplus \mathbb{1}[m+1=p]V_p
\end{aligned}$$

as required. $\square$

17. Random walk on indecomposable summands of tensor products

In this section we investigate a new family of Markov chains, defined by tensoring by a fixed simple module and choosing a non-projective summand of the result. In §17.1 we investigate the graph on which the walk takes place; in §17.2 we investigate the walk itself. Let $G = \mathrm{SL}_2(\mathbb{F}_p)$ and $p \neq 2$ throughout this section.

17.1. Tables of multiplicities

We examine the table of multiplicities of simple modules as indecomposable summands of tensor products of simple modules, as well as the graph which has this table as its adjacency matrix. This table has symmetries that reveal properties of the tensor products of representations of G. Furthermore, the Markov chain defined in Definition 17.14 is shown to be a walk on this graph, so our observations here aid our understanding of that Markov chain. We use $[\,:\,]$ to denote multiplicity as an indecomposable summand.

Definition 17.1. For $n \in [p-1]$, let $A^{(n)}$ be the matrix with entries $A^{(n)}_{i,j} = [V_i \otimes V_n : V_j]$. Let $\mathcal{G}^{(n)}$ be the (directed) graph (with loops) whose adjacency matrix is $A^{(n)}$. The parameter n is suppressed unless there is need to emphasise it.

The matrix A is depicted in Figure 17.1. It is visually apparent that A is symmetric; this motivates our next result.

Lemma 17.2. *Suppose $1 \leqslant i, j, k \leqslant p-1$. The following are equivalent:*

(i) V_k is a summand of $V_i \otimes V_j$;

(ii) V_i is a summand of $V_j \otimes V_k$;

(iii) V_j is a summand of $V_k \otimes V_i$;

(iv) $i + j + k \equiv 1 \pmod 2$, $i + j + k < 2p$, and $k < i + j$, $i < j + k$ and $j < k + i$.

In particular, A is a symmetric matrix.

Proof. Observe that (iv) is symmetric in i, j and k, and so it suffices to show that (i) and (iv) are equivalent. Indeed, Theorem 16.12 tells us that

$$
\begin{array}{ccc}
 & \begin{array}{cccccccccccccc} 1 & 2 & \cdots & & & n & & & & & & & & \end{array} & \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ \\ \\ n \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} &
\left(\begin{array}{cccccccccccccc}
 & & & & & 1 & & & & & & & & \\
 & & & & 1 & & 1 & & & & & & & \\
 & & & 1 & & 1 & & 1 & & & & & & \\
 & & \iddots & & \iddots & & \ddots & & \ddots & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & & & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & & & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & & & & \ddots & & \ddots & & \iddots & & \iddots & & \\
 & & & & & & 1 & & 1 & & 1 & & & \\
 & & & & & & & 1 & & 1 & & & & \\
 & & & & & & & & 1 & & & & &
\end{array}\right) &
\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ p-n \\ \\ \\ \\ \\ p-1 \end{array} \\
 & \begin{array}{cccccccccccccc} & & & & & & & & p-n & & & & & p-1 \end{array} &
\end{array}
$$

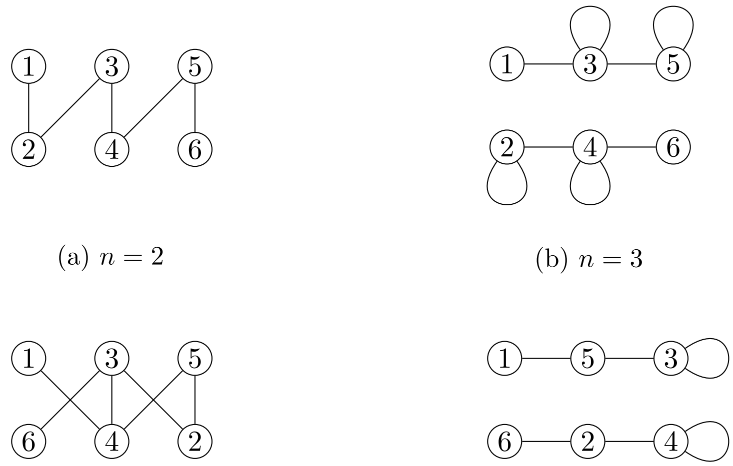
FIGURE 17.1. The matrix A (here with $n < p - n$)

(i) holds if and only if $k \equiv i + j - 1 \pmod 2$ and $\max\{i - j, j - i\} < k < \min\{i + j, 2p - (i + j)\}$, which easily rearranges to (iv). $\square$

Thus $\mathcal{G}$ can be viewed as an *undirected* graph (with loops); we do so from now on. Some small examples of $\mathcal{G}$ are depicted in Figure 17.2.

There is another visually apparent symmetry of the adjacency matrix A: it is invariant under rotation by 180 degrees. We give various interpretations of this fact in Proposition 17.6. To give these interpretations, we make the following definitions.

Definition 17.3. Let T be the $(p-1) \times (p-1)$ matrix defined by $T_{i,j} = [i + j = p]$.



(a) $n = 2$ (b) $n = 3$

(c) $n = 4$ (d) $n = 5$

FIGURE 17.2. The graphs $\mathcal{G}^{(n)}$, for $p = 7$ and $2 \leqslant n \leqslant p - 2$

That is, T is the matrix with 1s on the antidiagonal:

$$T = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

It is the basis-change matrix for reversing the order of the basis, and is self-inverse. Also:

- left-multiplying by T reflects a matrix in the horizontal midline;
- right-multiplying by T reflects a matrix in the vertical midline;
- conjugating by T rotates a matrix by 180 degrees.

Definition 17.4. Let $\Omega^0(-)$ denote the projective-free part of a module.

Definition 17.5. Let $\mathfrak{p}$ be the subgroup of the Grothendieck group $G_0(KG)$ generated by classes of projective modules.

Note that $G_0(KG)$ can be made into a (commutative) ring via tensoring, and that $\mathfrak{p}$ is an ideal of this ring. Recall that a quotient ring is naturally a (left) module for the original ring by (left) multiplication.

Proposition 17.6 (Interpretations of rotational symmetry of A)**.**

(a) *V_k is a summand of $V_i \otimes V_j$ if and only if it is a summand of $V_{p-i} \otimes V_{p-j}$, for all $1 \leqslant i, j, k \leqslant p-1$.*

(b) *$A^{(n)} = TA^{(p-n)} = A^{(p-n)}T$.*

(c) *$TAT = A$.*

(d) *The map $i \mapsto p-i$ is a graph automorphism of $\mathcal{G}$, and hence the induced subgraph on even vertices is isomorphic to the induced subgraph on odd vertices.*

(e) *$\Omega^0 (V_i \otimes V_j) \cong \Omega^0 (V_{p-i} \otimes V_{p-j})$ for all $1 \leqslant i, j \leqslant p-1$.*

(f) *$[V_i \otimes V_j] + \mathfrak{p} = [V_{p-i} \otimes V_{p-j}] + \mathfrak{p}$ for all $1 \leqslant i, j \leqslant p-1$.*

(g) *The K-linear automorphism ξ of $G_0(KG)/\mathfrak{p}$ defined by $\xi \colon [V_i] + \mathfrak{p} \mapsto [V_{p-i}] + \mathfrak{p}$ is $G_0(KG)$-equivariant.*

Proof. Statement (a) and the first equality in (b) are equivalent, and the second equality in (b) follows from the first since A and T are symmetric. The statements (c) and (d) are equivalent, and are implied by (b). The statement (e) clearly implies (f), and given that the projective-free parts of the tensor products of simple modules are multiplicity-free sums of simple modules, both are equivalent to (a).

To see that (g) follows from (b), let $I \subseteq [p-1]$ be such that $\Omega^0(V_j \otimes V_k) \cong \bigoplus_{i\in I} V_i$. Then, by the second equality in (b), we have $\Omega^0(V_j \otimes V_{p-k}) \cong \bigoplus_{i \in I} V_{p-i}$; thus

$$\begin{aligned} \xi([V_j \otimes V_k] + \mathfrak{p}) &= \xi\left(\sum_{i\in I}[V_i] + \mathfrak{p}\right) \\ &= \sum_{i\in I}[V_{p-i}] + \mathfrak{p} \\ &= [V_j \otimes V_{p-k}] + \mathfrak{p}, \end{aligned}$$

as required.

Thus it suffices to show (a) holds. Indeed, condition (iv) of Lemma 17.2 is invariant under taking both $i \mapsto p-i$ and $j \mapsto p-j$. $\square$

Remark 17.7. The observations of Lemma 17.2 and Proposition 17.6 can be seen as observations about the fusion category corresponding to the algebraic group $\mathrm{SL}_2(K)$ described in [AP95, Section 2]. The tensor product in this category is "reduced", in a sense which in our setting means "modulo projectives". The quotient ring ${G_0(K)}/{\mathfrak{p}}$ is known as the fusion ring for this category. The fusion rules state how reduced tensor products decompose in this category, and thus are specified by our Lemma 17.2. The observed symmetries of these fusion rules can be deduced either axiomatically [Mat00, Axiom 3, p. 183] or as a consequence of the modular Verlinde formula [Mat00, Theorem 9.5].

We next observe that a certain submatrix of A contains all the information of A, and use the resulting simplification of the structure of A to identify the connected components of $\mathcal{G}$.

Definition 17.8. Let $\overline{A}^{(n)}$ be the $\frac{p-1}{2} \times \frac{p-1}{2}$ submatrix of (a conjugate of) A defined by

$$\overline{A}^{(n)}_{i,j} = \begin{cases} A^{(n)}_{2i-1,2j-1} & \text{if } n \text{ is odd;} \\ A^{(n)}_{2i-1,p+1-2j} & \text{if } n \text{ is even.} \end{cases}$$

That is, if the vertices are reordered to $1, 3, \ldots, p-2, p-1, p-3, \ldots, 4, 2$ (the odd integers followed by the even integers, with the former in ascending order and the latter in descending order), then $\overline{A}$ is the upper-left block of A when n is odd and is the upper-right block of A when n is even.

Lemma 17.9. *The matrix $\overline{A}$ has the following properties:*

(a) *under the ordering $1, 3 \ldots p-2, p-1, \ldots, 4, 2$, the matrix A is of the form*

$$A = \begin{cases} \begin{pmatrix} \overline{A} & * \\ * & \overline{A} \end{pmatrix} & \textit{if } n \textit{ is odd,} \\[2ex] \begin{pmatrix} * & \overline{A} \\ \overline{A} & * \end{pmatrix} & \textit{if } n \textit{ is even,} \end{cases}$$

where $*$ denotes an unspecified matrix;

(b) $\overline{A}^{(n)}_{i,j} = 1$ *if and only if* $2|i-j| < r < 2(i+j-1) < 2p-r$, *where* $r = n$ *if* n *is odd and* $r = p - n$ *if* n *is even.*

(c) $\overline{A}^{(p-n)} = \overline{A}^{(n)}$;

(d) $\bar{A}$ *is symmetric;*

(e) for $1 < n < p-1$*, the graph with adjacency matrix* $\bar{A}$ *is connected.*

Proof. By Proposition 17.6(c) we have $A_{2i-1,2j-1} = A_{p+1-2i,p+1-2j}$, and so (under the new ordering) the upper-left and lower-right blocks of A are the same. Similarly the upper-right and lower-left blocks are the same, and (a) follows.

The condition for $\bar{A}_{i,j}$ to be nonzero is obtained from condition (iv) of Lemma 17.2 with the appropriate values of i and j substituted. Properties (c) and (d) are easily verified using this condition.

It follows from (b) that $\bar{A}$ has nonzero entries precisely in a rectangle bounded by the straight lines determined by these inequalities; we draw matrix $\bar{A}$ in Figure 17.3. The connectedness of its graph is then clear provided $r \neq 1$. □

Lemma 17.10.

(a) If n *is odd, then* $\mathcal{G}$ *is disconnected, with each connected component a subset of either the odd integers or the even integers.*

(b) If n *is even, then* $\mathcal{G}$ *is bipartite, with classes the odd integers and the even integers.*

(c) When the vertices are ordered as $1, 3, \ldots, p-2, p-1, p-3, \ldots, 4, 2$*, we have*

$$A = \bar{A} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{n+1}.$$

Proof. Let $1 \leqslant i \leqslant p-1$. Observe that the neighbours of i are all elements of $\langle i, n\rangle$ or $\langle n, i\rangle$ (according to whether $i \geqslant n$ or $i \leqslant n$). Furthermore, elements of these strings are all of the same parity, which is the parity of $i+n-1$. Thus if n is odd, the neighbours of i are of the same parity as i, whilst if n is even, the neighbours of i are of different parity to i. The statements (a) and (b) are then immediate.

That is, under the new ordering, when n is even the diagonal blocks of A are zero, and when n is odd the off-diagonal blocks are zero. The expression as a Kronecker product then follows from Lemma 17.9(a). □

$$
\begin{array}{ccc}
 & \begin{array}{cccccccccccccc} 1 & 2 & \cdots & & & \frac{r+1}{2} & & & & & & & & \end{array} & \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ \\ \\ \frac{r+1}{2} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} &
\left(\begin{array}{cccccccccccccc}
 & & & & & 1 & & & & & & & & \\
 & & & & 1 & 1 & 1 & & & & & & & \\
 & & & 1 & 1 & 1 & 1 & 1 & & & & & & \\
 & & \iddots & & \iddots & & \ddots & & \ddots & & & & & \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & \ddots & & \ddots & & & \iddots & & \iddots & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & & & & & 1 & 1 & 1 & 1 & & & \\
 & & & & & & & & 1 & 1 & & & &
\end{array}\right) &
\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \frac{p-r}{2} \\ \frac{p-r+2}{2} \\ \\ \\ \\ \frac{p-1}{2} \end{array} \\
 & \begin{array}{cccccccccccccc} & & & & & & & & \frac{p-r}{2} & \frac{p-r+2}{2} & & & & \frac{p-1}{2} \end{array} &
\end{array}
$$

FIGURE 17.3. The matrix $\overline{A}$ (here with $r < p - r$), where $r = n$ if n is odd and $r = p - n$ if n is even.

Proposition 17.11.

(a) If n is odd and $n > 1$, then $\mathcal{G}$ has precisely two connected components, the odd integers and the even integers, and they are isomorphic.

(b) If n is even and $n < p - 1$, then $\mathcal{G}$ is connected.

PROOF. For n odd, $\overline{A}$ is the adjacency matrix for the subgraphs of $\mathcal{G}$ on odd vertices and on even vertices, so (a) follows immediately from Lemma 17.9(e).

For n even, $\overline{A}$ is the adjacency matrix for the quotient graph of $\mathcal{G}$ with i and $p - i$ identified. Again using Lemma 17.9(e), since $\mathcal{G}$ is bipartite (with each of i and $p - i$ in a distinct class), to show (b) it suffices to show that i is reachable from $p - i$ for some i. Indeed, $\overline{A}$ has a nonzero diagonal entry (at $\frac{n+1}{2}$), and so the two vertices identified to form the corresponding vertex of the quotient are adjacent. $\square$

We conclude this section by finding the degrees of the vertices in $\mathcal{G}$. The degree of i in $\mathcal{G}$ is also the number of nonzero entries in the ith row of A, and is the number of non-projective indecomposable summands of $V_i \otimes V_n$.

Definition 17.12. For $1 \leqslant i \leqslant p-1$, let $d(i)$ be the degree of i in $\mathcal{G}$ (where a loop is considered to contribute 1 to the degree). The dependence of d on n is suppressed, since it is always clear from context.

Lemma 17.13. *For $1 \leqslant i \leqslant p-1$, we have*

$$d(i) = \min\{i, p-i, n, p-n\}.$$

Furthermore,

$$\sum_{i=1}^{p-1} d(i) = n(p-n).$$

PROOF. Clearly $d(i)$ is symmetric in i and n, so for the first equality it suffices to show that $d(i) = \min\{i, p-n\}$ when $i \leqslant n$. By Theorem 16.12, the number of simple non-projective summands of $V_n \otimes V_i$ is the number of elements j of $\langle n, i\rangle$ for which $2p - j \notin \langle n, i\rangle$.

If $i + n - 1 < p$ (equivalently, $i \leqslant p - n$) then this is all the elements of $\langle n, i\rangle$, of which there are i.

If $i + n - 1 \geqslant p$ (equivalently, $i > p - n$), then the number of $j \in \langle n, i\rangle$ such that $2p - j \in \langle n, i\rangle$ is

$$2\left\lfloor \frac{i+n-1-p}{2} \right\rfloor + \mathbb{1}[i+n-1 \text{ is odd}] = i+n-p,$$

and so $d(i) = i - (i + n - p) = p - n$.

We now find the sum of the $d(i)$. Let $m = \min\{n, p-n\}$. We have:

$$\begin{aligned}\sum_{i=1}^{p-1} d(i) &= \sum_{i=1}^{p-1} \min\{i, p-i, n, p-n\} \\ &= 2\sum_{i=1}^{\frac{p-1}{2}} \min\{i, m\} \\ &= 2\left(\frac{p-1}{2} - m\right) m + 2\sum_{i=1}^{m} i \\ &= m(p-1-2m) + m(m+1) \\ &= m(p-m) \\ &= n(p-n)\end{aligned}$$

as required. □

17.2. The random walk

We now introduce the random walk itself.

Definition 17.14 (Non-projective summand random walk)**.** Fix $n \in [p-1]$, w a function that assigns a positive weighting to each non-projective indecomposable KG-module, and ν a distribution on the non-projective simple KG-modules. Let the *non-projective summand random walk* be the (discrete time) Markov chain on the set of non-projective indecomposable KG-modules with initial distribution ν in which the probability of a step from U to V is

$$Q^{(n)}_{UV} = \frac{w(V)[U \otimes V_n : V]}{\sum_M w(M)[U \otimes V_n : M]},$$

where the sum is over all non-projective indecomposable modules M (and $[\ :\ ]$ denotes multiplicity as an indecomposable summand, as in §17.1). The parameter n is suppressed unless there is need to emphasise it.

Remark 17.15.

(i) If U is a simple non-projective KG-module, Theorem 16.12 implies that $U \otimes V_n$ indeed has non-projective indecomposable summands, and that these summands are simple. Thus the chain is well-defined and remains on simple non-projective KG-modules throughout. The states of the chain can therefore be labelled with the dimensions of the modules, taking values in the finite set $[p-1]$.

(ii) Theorem 16.12 implies that the non-projective part of a tensor product of simple modules is multiplicity-free, so $[U \otimes V_n : M] \in \{0, 1\}$ for all M.

(iii) If we were to allow steps to projective indecomposable modules, these modules would form an absorbing set (in the sense that once the chain hit a projective module it would stay on projective modules for all time). This definition allows us to consider a recurrent chain on the (non-projective) simple modules.

(iv) There are two trivial cases to be excluded: if $n = 1$, we never step away from the initial state; if $n = p-1$, then V_{p-i} is the unique non-projective indecomposable summand of $V_i \otimes V_{p-1}$, so at each step we switch between the initial state i and $p - i$. From now on we assume $2 \leqslant n \leqslant p-2$.

(v) If we were to replace $\mathrm{SL}_2(\mathbb{F}_p)$ with an arbitrary group and define the non-projective summand walk analogously, it may not be clear how many states our chain has, or whether that number is finite. For $\mathrm{SL}_2(\mathbb{F}_p)$, however, we could have deduced that there are only finitely many states without explicitly knowing any decompositions: since $\mathrm{SL}_2(\mathbb{F}_p)$ has a cyclic Sylow p-subgroup, all its simple representations are algebraic (in the sense of satisfying a polynomial over $\mathbb{Z}$ in the Green ring), and hence their tensor powers collectively have only finitely many summands [Cra07, Lemma 1.1, Corollary 1.6].

An illustrative example of our chain is given below. Note that when $w \equiv 1$, the summands are chosen uniformly at random; this case, and the case where $w(i) = i$ (in which modules are weighted by their dimension), are described for general n at the end of this section.

Example 17.16. Suppose $w \equiv 1$ and $n = 2$. We have that

$$V_i \otimes V_2 \cong \begin{cases} V_2 & \text{if } i = 1, \\ V_{i-1} \oplus V_{i+1} & \text{if } 2 \leqslant i \leqslant p-2, \\ V_{p-2} \oplus P_p & \text{if } i = p-1. \end{cases}$$

Thus the non-projective summand random walk is a symmetric random walk in one dimension with reflecting boundaries. The transition matrix is

$$\begin{pmatrix} & 1 & & & & & & \\ \frac{1}{2} & & \frac{1}{2} & & & & & \\ & \frac{1}{2} & & \frac{1}{2} & & & & \\ & & \frac{1}{2} & & \frac{1}{2} & & & \\ & & & \ddots & & \ddots & & \\ & & & & \frac{1}{2} & & \frac{1}{2} & \\ & & & & & \frac{1}{2} & & \frac{1}{2} \\ & & & & & & 1 & \end{pmatrix}$$

and the stationary distribution is $\frac{1}{2(p-2)}(1, 2, 2, \ldots, 2, 1)$.

Our key observation while studying the non-projective summand random walk is that it is the random walk on the graph $\mathcal{G}$ (defined in §17.1) in which the probability of moving from a vertex i to a neighbour j is proportional to $w(j)$. Indeed, the transition matrix Q has nonzero entries precisely where A (the adjacency matrix for $\mathcal{G}$) does, and the transition probabilities are proportional to the weighting of the destination. That is,

$$Q_{i,j} = \frac{w(j)}{\sum_{il \in E(\mathcal{G})} w(l)} A_{i,j}.$$

We use the properties of $\mathcal{G}$ given in §17.1 to shed light on the non-projective summand random walk. The first relevant property of $\mathcal{G}$ is that it is undirected, which implies that the chain is reversible and diagonalisable and allows us to find a stationary distribution as demonstrated in the following proposition.

Proposition 17.17. *Let π be the distribution defined by*

$$\pi_i = \frac{w(i) \sum_{il \in E(\mathcal{G})} w(l)}{C}.$$

where $C = \sum_{x \in V(\mathcal{G})} \sum_{xy \in E(\mathcal{G})} w(y)$.

Then π is a stationary distribution in detailed balance with Q, and hence the random walk is reversible and diagonalisable.

PROOF. It suffices to verify the detailed balance equations for π (noting that diagonalisability follows from reversibility [PR13, Section 2.4]). Observe:

$$\begin{aligned}\pi_i Q_{i,j} &= \frac{w(j)}{\sum_{ik\in E(\mathcal{G})} w(k)} \frac{w(i)\sum_{ik\in E(\mathcal{G})} w(k)}{C} \mathbb{1}[ij \in E(\mathcal{G})] \\ &= \frac{w(i)w(j)}{C} \mathbb{1}[ij \in E(\mathcal{G})] \\ &= \pi_j Q_{j,i},\end{aligned}$$

as required. □

That our walk takes place on an undirected graph also implies that the communicating classes of our Markov chain are all closed (that is, they are irreducible chains themselves) and they are precisely the connected components of $\mathcal{G}$. Making use of our results about the connectedness and periodicity of $\mathcal{G}$, we obtain the following proposition.

Proposition 17.18.

(a) If n is odd, then the non-projective summand random walk is reducible into two chains, one on the even states and one on the odd states, which are each irreducible and aperiodic.

(b) If n is even, then the non-projective summand random walk is irreducible and periodic with period 2.

PROOF. The description of the irreducible components follows immediately from the description of the connected components of $\mathcal{G}$ in Proposition 17.11.

A walk on an undirected graph necessarily has period at most 2 (since any vertex can be revisited after two steps). The walk has period equal to 2 if and only if the graph contains no odd cycles and no loops, which is if and only if the graph is bipartite—and the walk is aperiodic otherwise. Thus the periodicity claims follow from Lemma 17.10(b) and the observation that when n is odd, each component of $\mathcal{G}$ has loops (at $\frac{p-1}{2}$ and $\frac{p+1}{2}$). □

Remark 17.19. Thus for n even, the chain has a *unique* stationary distribution but it does not necessarily converge to it. Meanwhile, for n odd, each subchain has a unique stationary distribution which it converges to, and the stationary distributions of the entire chain are precisely the convex combinations of these distributions.

If w satisfies $w(i) = w(p-i)$ for all i, then Q has the same rotational symmetry as A, and several of the results from §17.1 carry over. Some of these results are helpful for identifying the eigenvalues of Q; the rate of convergence to equilibrium is determined by the second-largest (in absolute value) eigenvalue, so this in turn is helpful for finding the mixing time for the Markov chain.

Let $\overline{Q}$ be the submatrix of (a conjugate of) Q defined analogously to $\overline{A}$.

Proposition 17.20. *Suppose $w(i) = w(p-i)$ for all i. Then:*

(a) $Q^{(n)} = TQ^{(p-n)} = Q^{(p-n)}T$*;*
(b) $TQT = Q$*;*
(c) the non-projective summand random walk is invariant under the relabelling $i \mapsto p-i$;
(d) if n is odd, the two irreducible subchains are isomorphic;
(e) $\overline{Q}^{(p-n)} = \overline{Q}^{(n)}$*;*
(f) with the vertices are ordered as $1, 3, \ldots, p-2, p-1, p-3, \ldots, 4, 2$, we have

$$Q = \overline{Q} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{n+1};$$

(g) if n is odd, every eigenvalue of Q has even multiplicity; if n is even, the eigenvalues of Q come in signed pairs;
(h) the non-projective summand random walk has mixing time bounded by

$$t_{\text{mix}}(\varepsilon) \leqslant \frac{1}{1-\lambda_\star} \log\left(\frac{1}{\varepsilon \min_i(\pi_i)}\right)$$

where $\lambda_\star = \max\{\, |\lambda| \;\mid\; \lambda \neq 1 \text{ is an eigenvalue of } \overline{Q}\,\}$ and π is the stationary distribution from Proposition 17.17.

PROOF. Statements (a) to (f) are entirely analogous to results in §17.1, using $w(i) = w(p-i)$ to deduce that the entries in the desired places of Q are not only nonzero but also equal.

Once we have the Kronecker product expression in (f), we see immediately that if $\overline{Q}$ has eigenvector-eigenvalue pairs $\{(v_1, \lambda_1), \dots, (v_{\frac{p-1}{2}}, \lambda_{\frac{p-1}{2}})\}$, then the eigenvector-eigenvalue pairs of Q are, if n is odd

$$\{\, (v_i \otimes \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \lambda_i) \mid 1 \leqslant i \leqslant \tfrac{p-1}{2} \,\} \sqcup \{\, (v_i \otimes \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \lambda_i) \mid 1 \leqslant i \leqslant \tfrac{p-1}{2} \,\},$$

and if n is even

$$\{\, (v_i \otimes \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \lambda_i) \mid 1 \leqslant i \leqslant \tfrac{p-1}{2} \,\} \sqcup \{\, \big(v_i \otimes \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right), -\lambda_i\big) \mid 1 \leqslant i \leqslant \tfrac{p-1}{2} \,\}.$$

Both parts of (g) then follow.

Statement (h) is obtained by applying [LP17, Theorem 12.4, p. 163] to each irreducible component with transition matrix $\overline{Q}$ when n is odd, or to the lazy chain with transition matrix $\frac{1}{2}(Q + I)$ when n is even. Indeed, when n is even, the eigenvalues of $\overline{Q}$ coming in signed pairs implies that the second-largest eigenvalue of $\frac{1}{2}(Q + I)$ is $\frac{\lambda_\star + 1}{2}$; halving the resulting mixing time to account for the fact that the lazy chain converges at half the rate of the original yields the required value. $\square$

In fact, for n even, the eigenvalues still come in signed pairs, regardless of the weighting: it is always the case that Q has nonzero entries only in the off-diagonal $\frac{p-1}{2} \times \frac{p-1}{2}$ blocks, and if $\left(\begin{smallmatrix}u\\v\end{smallmatrix}\right)$ is an eigenvector with eigenvalue λ for such a matrix, then $\left(\begin{smallmatrix}u\\-v\end{smallmatrix}\right)$ is an eigenvector with eigenvalue $-\lambda$.

We conclude by exhibiting our results in the cases $w \equiv 1$ and $w(i) = i$. Recall from Definition 17.12 that $d(i)$ is the degree of i in $\mathcal{G}$.

Example 17.21. Let $w \equiv 1$. Then

$$Q_{i,j} = \frac{A_{i,j}}{d(i)}.$$

This transition matrix is shown explicitly in Figure 17.4. Trivially $w(i) = w(p - i)$, and so Q satisfies $TQT = Q$, and for n odd the the two irreducible subchains are isomorphic.

By Lemma 17.13 and Proposition 17.17, a stationary distribution is

$$\pi_i = \frac{\min\{i, p - i, n, p - n\}}{n(p - n)}.$$

$$
\begin{array}{ccc}
 & \begin{array}{cccccccccccccc} 1 & 2 & \cdots & & & n & & & & & & & & \end{array} & \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ \\ \\ n \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} &
\left(\begin{array}{cccccccccccccc}
 & & & & & 1 & & & & & & & & \\
 & & & & \frac{1}{2} & & \frac{1}{2} & & & & & & & \\
 & & & \frac{1}{3} & & \frac{1}{3} & & \frac{1}{3} & & & & & & \\
 & & \iddots & & \iddots & & \ddots & & \ddots & & & & & \\
 & \frac{1}{n-1} & & \frac{1}{n-1} & & \frac{1}{n-1} & & \frac{1}{n-1} & & \frac{1}{n-1} & & & & \\
\frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & & \\
 & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \\
 & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \\
 & & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} & & \frac{1}{n} \\
 & & & & \frac{1}{n-1} & & \frac{1}{n-1} & & \frac{1}{n-1} & & \frac{1}{n-1} & & \frac{1}{n-1} & \\
 & & & & & \ddots & & \ddots & & \iddots & & \iddots & & \\
 & & & & & & \frac{1}{3} & & \frac{1}{3} & & \frac{1}{3} & & & \\
 & & & & & & & \frac{1}{2} & & \frac{1}{2} & & & & \\
 & & & & & & & & 1 & & & & &
\end{array}\right) &
\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ p-n \\ \\ \\ \\ \\ p-1 \end{array} \\
 & \begin{array}{cccccccccccccc} & & & & & & & & p-n & & & & & p-1 \end{array} &
\end{array}
$$

$$
\begin{array}{ccc}
 & \begin{array}{cccccccccccccc} 1 & 2 & \cdots & & & & & & n & & & & & \end{array} & \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ \\ \\ \\ \\ \\ n \\ \\ \\ \\ \\ \\ \end{array} &
\left(\begin{array}{cccccccccccccc}
 & & & & & & & & 1 & & & & & \\
 & & & & & & & \frac{1}{2} & & \frac{1}{2} & & & & \\
 & & & & & & \frac{1}{3} & & \frac{1}{3} & & \frac{1}{3} & & & \\
 & & & & & \iddots & & \iddots & & \ddots & & \ddots & & \\
 & & & & \frac{1}{p-n-1} & & \frac{1}{p-n-1} & & \frac{1}{p-n-1} & & \frac{1}{p-n-1} & & \frac{1}{p-n-1} & \\
 & & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} \\
 & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & \\
 & \iddots & & \iddots & & \iddots & & \iddots & & \iddots & & \iddots & & \\
\frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & \frac{1}{p-n} & & & \\
 & \frac{1}{p-n-1} & & \frac{1}{p-n-1} & & \frac{1}{p-n-1} & & \frac{1}{p-n-1} & & \frac{1}{p-n-1} & & & & \\
 & & \ddots & & \ddots & & \iddots & & \iddots & & & & & \\
 & & & \frac{1}{3} & & \frac{1}{3} & & \frac{1}{3} & & & & & & \\
 & & & & \frac{1}{2} & & \frac{1}{2} & & & & & & & \\
 & & & & & 1 & & & & & & & &
\end{array}\right) &
\begin{array}{c} \\ \\ \\ \\ \\ p-n \\ \\ \\ \\ \\ \\ \\ \\ p-1 \end{array} \\
 & \begin{array}{cccccccccccccc} & & & & & p-n & & & & & & & & p-1 \end{array} &
\end{array}
$$

FIGURE 17.4. The transition matrix Q when $w \equiv 1$, in the cases $2n < p$ (top) and $2n > p$ (bottom). This choice of w is discussed in Example 17.21.

Observe that $\pi T = \pi$. In particular, this stationary distribution assigns equal probability to being on an even or an odd state; that is,

$$\sum_{i \equiv 0 \pmod 2} \pi_i = \sum_{i \equiv 1 \pmod 2} \pi_i = \frac{1}{2}.$$

Thus, for n even, the chain converges to the stationary distribution, provided that the initial distribution ν has equal weighting for even and odd states or that the chain is made lazy by taking the transition matrix to be $\frac{1}{2}(Q+I)$. Meanwhile, for n odd, π is the stationary distribution with equal weighting given to the even-state and odd-state walks.

If $n \in \{\frac{p-1}{2}, \frac{p+1}{2}\}$, it can be shown that the eigenvalues of $\overline{Q}$ are

$$\{1, -\tfrac{1}{2}, \tfrac{1}{3}, \ldots, (-1)^{\frac{p+1}{2}} \tfrac{2}{p-1}\}.$$

If n is odd, the eigenvalues of Q are the eigenvalues in this set each with multiplicity 2; if n is even, the eigenvalues are $\{\pm 1, \pm\frac{1}{2}, \ldots \pm \frac{2}{p-1}\}$ (both cases by the proof of Proposition 17.20(g)). Then by Proposition 17.20(h), the mixing time of the walk is bounded by

$$t_{\mathrm{mix}}(\varepsilon) \leqslant 2 \log\left(\tfrac{p^2-1}{4\varepsilon}\right).$$

The case of $w \equiv 1$ and $n \in \{\frac{p-1}{2}, \frac{p+1}{2}\}$, n odd, is an example of an *involutive walk* defined by Britnell and Wildon [BW21]: up to relabelling the states, each of our irreducible chains is in their notation the $\gamma^{(0,0)}$-involutive random walk on n, and the occurrence of the eigenvalues stated above is a consequence of [BW21, Theorem 1.5].

Example 17.22. Suppose $w(i) = i$ for each i; that is, each module has a chance of being chosen proportional to its dimension. Of course, $w(i) \neq w(p-i)$, and so the results of Proposition 17.20 may not hold (in particular, the two irreducible chains when n is odd are not isomorphic). Nevertheless we can compute the transition probabilities and the stationary distribution.

For fixed i we have

$$\begin{aligned}\sum_{ij \in E(\mathcal{G})} j &= (\text{number of neighbours of } i) \times (\text{mean value of the neighbours of } i)\\ &= d(i) \times \mathrm{mean}\{\, j \mid V_j \text{ is a summand of } V_i \otimes V_n\,\}.\end{aligned}$$

If $i+n \leqslant p$, all of the composition factors of $V_i \otimes V_n$ are summands, and so their average dimension is $\max\{i,n\}$, the midpoint of the (i,n)-string or the (n,i)-string (as appropriate). If $i+n \geqslant p$, the midpoint of the relevant section of the string is instead

$$\frac{(|i-n|+1)+(2p-(i+n-1))}{2} = p - \min\{i,n\}.$$

Also, by Lemma 17.13,

$$d(i) = \begin{cases} \min\{i,n\} & \text{if } i+n \leqslant p, \\ p-\max\{i,n\} & \text{if } i+n \geqslant p. \end{cases}$$

Thus

$$\begin{aligned} \sum_{ij \in E(\mathcal{G})} j &= \begin{cases} d(i)\max\{i,n\} & \text{if } i+n \leqslant p, \\ d(i)(p-\min\{i,n\}) & \text{if } i+n \geqslant p \end{cases} \\ &= \begin{cases} in & \text{if } i+n \leqslant p, \\ (p-i)(p-n) & \text{if } i+n \geqslant p. \end{cases} \end{aligned}$$

Then

$$Q_{i,j} = \begin{cases} \frac{j}{in} & \text{if } i+n \leqslant p \text{ and } A_{i,j} \neq 0, \\ \frac{j}{(p-i)(p-n)} & \text{if } i+n \geqslant p \text{ and } A_{i,j} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that $\sum_{i \in [p-1]} \sum_{ij \in E(\mathcal{G})} j = \frac{1}{6}np(p-n)(2p-n)$. Then by Proposition 17.17 a stationary distribution is

$$\pi_i = \begin{cases} \dfrac{6i^2}{p(p-n)(2p-n)} & \text{if } i+n \leqslant p, \\ \dfrac{6i(p-i)}{np(2p-n)} & \text{if } i+n \geqslant p. \end{cases}$$

Glossary of symbols

General		
G	a group	
K	a field	
p	a prime; characteristic of K	
q	a power of p; size of K (if K finite)	
n	a positive integer; a parameter for matrix groups	
r	a positive integer; a parameter for the symmetric group	
$\mathbb{F}_p$	the finite field of order p; the prime subfield of K	
U, V	K-vector spaces; representations of G over K	
d	dimension of V	
$\mathcal{B}$	set of entries for a tableaux; basis for V (frequently viewed as $[d]$)	
ρ_V	homomorphism representing the action of G on V	
$\mathbb{1}[-]$	indicator function for propositions (evaluates to 1 if proposition is true, 0 otherwise)	
G/H	set of (representatives of) the left cosets gH of H in G	
$H\backslash G$	set of (representatives of) the right cosets Hg of H in G	
$F\backslash G/H$	set of (representatives of) the double cosets FgH of F and H in G	

General linear and symmetric groups		
$\mathrm{GL}_n(K)$	general linear group of invertible $n \times n$ matrices over K	
$\mathrm{SL}_n(K)$	special linear group of $n \times n$ matrices over K with determinant 1	
S_r	symmetric group on r symbols	
E	natural representation of $\mathrm{GL}_n(K)$	§0.6, p. 13
X_i	basis element of E	
$\Delta^\lambda E$	a Weyl module for $\mathrm{GL}_n(K)$	
$\nabla^\lambda E$	a dual Weyl module for $\mathrm{GL}_n(K)$	
$L^\lambda(E)$	$= \operatorname{soc} \nabla^\lambda E = \Delta^\lambda E / \operatorname{rad} \Delta^\lambda E$, a simple $K\mathrm{GL}_n(K)$-module	§4.2, p. 56
$-_\nu$	ν-weight space of a representation of $\mathrm{GL}_n(K)$	Definition 6.1, p. 66
W	natural permutation representation of S_r	§4.1, p. 55
S^λ	$= \nabla^\lambda_{\mathrm{sym}} W$, a Specht module for S_r	§4.1, p. 55

$\mathcal{F}$	Schur functor	Definition 6.5, p. 67
$\mathcal{G}_{\otimes}$	left-adjoint inverse Schur functor	Definition 6.12, p. 74
$\mathcal{G}_{\mathrm{Hom}}$	right-adjoint inverse Schur functor	Definition 6.16, p. 78

Multilinear algebra

Sym^r	(upper) symmetric power (a quotient of a tensor power)	§0.6, p. 14
Sym_r	lower symmetric power (a submodule of a tensor power)	§3.2, p. 36
$\bigwedge^r$	exterior power (a quotient of a tensor power)	§0.6, p. 14
$-_{\mathrm{sym}}$	restriction to subspace of symmetric type	§1.2, p. 18
$-^*$	(usual) dual (for representations of any group)	§3.1, p. 34
$-^\circ$	contravariant dual (for representations of matrix groups)	§3.1, p. 34
∇^λ	Schur endofunctor	Definition 2.3, p. 27
Δ^λ	Weyl endofunctor	Definition 3.9, p. 40
e	the map $\bigwedge^{\lambda'} V \twoheadrightarrow \nabla^\lambda V$ defined by $\lvert t\rvert \mapsto \mathrm{e}(t)$	§2.2, p. 28
$\wedge$	the map $\mathsf{Tbx}^\lambda V \twoheadrightarrow \bigwedge^{\lambda'} V$ defined by $t \mapsto \lvert t\rvert$; restricts to a map $\mathrm{Sym}_\lambda V \twoheadrightarrow \Delta^\lambda V$	

Tableaux combinatorics

$\lvert\lambda\rvert$	size of λ, the sum of its parts	§1.1, p. 16
$\ell(\lambda)$	length of λ, the number of its parts	§1.1, p. 16
$[\lambda]$	Young diagram of λ	§1.1, p. 16
Tbx^λ	space of tableaux of shape λ, isomorphic to the tensor power $-^{\otimes\lvert\lambda\rvert}$	§1.1, p. 17
$\mathrm{CSYT}(\lambda)$	column standard tableaux of shape λ	§1.1, p. 17
$\mathrm{RSSYT}(\lambda)$	row semistandard tableaux of shape λ	§1.1, p. 17
$\mathrm{SSYT}(\lambda)$	semistandard tableaux of shape λ	§1.1, p. 17
$\mathrm{RPP}(\lambda)$	$=\prod_{i=1}^{\lambda'_1} S_{\mathrm{row}_i[\lambda]}$, group of row preserving permutations	
$\mathrm{rstab}(t)$	$=\mathrm{stab}\, t \cap \mathrm{RPP}(\lambda)$, row stabiliser of t	
$\mathrm{CPP}(\lambda)$	$=\prod_{j=1}^{\lambda_1} S_{\mathrm{col}_j[\lambda]}$, group of column preserving permutations	
$[t]$	row tabloid of t	§1.4, p. 19
$\mathrm{rsym}(t)$	row symmetrisation of t	Definition 3.8, p. 40
$\lvert t\rvert$	(alternating) column tabloid of t	§1.5, p. 20
$<_{\mathrm{r}}, \sim_{\mathrm{r}}$	row ordering on tableaux	§1.6, p. 21
$<_{\mathrm{c}}, \sim_{\mathrm{c}}$	column ordering on tableaux	§1.6, p. 21

$\mathrm{e}(t)$	polytabloid of t	Definition 2.1, p. 26
ə(t)	copolytabloid of t	Definition 3.11, p. 42
$\mathsf{R}_{(t,A,B)}$	a Garnir relation	Definition 1.8, p. 23
$\mathsf{R}_{(t,i,j)}$	a snake relation	Definition 1.10, p. 25
GR^{λ}	space of Garnir relations	
ꓤ$_{(t,A,B)}$	a row Garnir relation	Definition 3.16, p. 46
ꓤ$_{(t,i,(j,j'))}$	a row snake relation	Definition 3.16, p. 46
Gꓤ$^{\lambda}$	space of row Garnir relations	
Φ	a function with respect to which snake relations are considered basic	

Chapter III, §8

λ°	box complement of λ in the $d \times c$ rectangle	
j°	$= c+1-j$, label of the column in λ° corresponding to column j in λ	
t°	a tableau of shape λ° complementary to t	§8.2, p. 92
τ°	permutation in $S_{[\lambda^{\circ}]}$ obtained from $\tau \in S_{[\lambda]}$	§8.3, p. 94
$\psi\,(\bar{\psi})$	the map $\bigwedge^{l} V \to \bigwedge^{d-l} V^{*}\,(\otimes \det V)$	Definition 8.1, p. 89
$\Psi\,(\bar{\Psi})$	the map $\bigwedge^{\lambda} V \to \bigwedge^{\lambda^{\circ\prime}} V^{*}\,(\otimes (\det V)^{\otimes s})$	Definition 8.5, p. 93
S, s	surplus (of a tableau or a permutation)	§8.2, p. 93

Chapter III, §9

$\overline{m}$	$= \binom{m+n-1}{m}$	
Mon_r	basis of $\mathrm{Sym}^r E$ consisting of monomials in $X_1, \ldots, X_n$ of degree r	
$(\mathsf{Mon}_r)^{\overline{m}}$	set of $\overline{m}$-tuples of elements of Mon_r (indexing a basis for $(\mathrm{Sym}^r E)^{\otimes \overline{m}}$)	
Ξ	an injection $\bigsqcup_{r=0}^{b-1} \mathsf{Mon}_r \to \mathbb{N}$	§9.1, p. 105
$<_{\Xi}$	lexicographical ordering on monomials	§9.1, p. 105
$(\mathsf{Mon}_r)^{\overline{m}}_{\geqslant}$	set of weakly decreasing $\overline{m}$-tuples of elements of Mon_r (indexing a basis for $\mathrm{Sym}_{\overline{m}} \mathrm{Sym}^r E$)	
$(\mathsf{Mon}_r)^{\overline{m}}_{>}$	set of strictly decreasing $\overline{m}$-tuples of elements of Mon_r (indexing a basis for $\bigwedge^{\overline{m}} \mathrm{Sym}^r E$)	
$\boldsymbol{w}$	unique element of $(\mathsf{Mon}_m)^{\overline{m}}_{>}$	
$\boldsymbol{f}^{\otimes}$	$= \boldsymbol{f}_1 \otimes \cdots \otimes \boldsymbol{f}_{\overline{m}}$, the tensor product of $\boldsymbol{f}$	§9.1, p. 106
$\boldsymbol{f}^{\wedge}$	$= \boldsymbol{f}_1 \wedge \cdots \wedge \boldsymbol{f}_{\overline{m}}$, the alternating product of $\boldsymbol{f}$	§9.1, p. 106
$\boldsymbol{f}^{\mathrm{sym}}$	symmetrisation of $\boldsymbol{f}^{\otimes}$	§9.1, p. 106
$Z(\boldsymbol{f})$	an element of $\bigwedge^{\overline{m}} \mathrm{Sym}^{l+m} E$	Definition 9.3, p. 106

ζ	a map $\mathrm{Sym}_{\overline{m}} \mathrm{Sym}^l E \otimes (\det E)^{\otimes mM/n} \to \bigwedge^M \mathrm{Sym}^{l+m} E$	Definition 9.3, p. 106
$<_\Sigma$	an ordering on tuples of monomials	Definition 9.5, p. 107
ω	a map $(\mathrm{Sym}^l E)^{\otimes \overline{m}} \otimes (\mathrm{Sym}^m E)^{\otimes \overline{m}} \to \bigwedge^{\overline{m}} \mathrm{Sym}^{l+m} E$	Definition 9.8, p. 109

Chapter III, §11

T	subgroup of $\mathrm{SL}_2(K)$ of diagonal matrices	
B	subgroup of $\mathrm{SL}_2(K)$ of lower triangular matrices	
$-_r$	r-weight space of a representation of $\mathrm{SL}_2(K)$	(11.1), p. 115
M_γ	$= \left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right)$, an element of B	
$\mathcal{D}$	defect set	Definition 11.3, p. 116
$t_{\max}$	a tableau labelling a highest weight vector	
$\trianglelefteq$	carry-free summand relation	Definition 11.10, p. 120

Chapter IV

Sk^r	skew symmetric power	§12.1, p. 135
$\|\|t\|\|$	skew column tabloid of t	Definition 12.1, p. 135
$\mathsf{R}^{\mathsf{Sk}}_{(t,A,B)}$	a skew Garnir relation	Definition 12.2, p. 136
$\mathsf{R}^{\mathsf{Sk}}_{(t,i,j)}$	a skew snake relation	Definition 12.3, p. 137
SkGR^λ	space of skew Garnir relations	Definition 12.2, p. 136
δ	the map $\mathsf{Sk}^{\lambda'} V \twoheadrightarrow \bigwedge^{\lambda'} V$ defined by $\|\|t\|\| \mapsto \|t\|$	§12.1, p. 136

Chapter V

$\langle n, m \rangle$	$= \{n+m-1, n+m-3, \ldots, n-m+3, n-m+1\}$, the (n,m)-string	Definition 15.0, p. 165
V_m	$= \mathrm{Sym}^{m-1} E$; for $m \leqslant p$, the simple representation of $\mathrm{SL}_2(\mathbb{F}_p)$ of dimension m	
P_m	projective cover of V_m	
μ	multiplication map $V_n \otimes V_m \to V_{n+m-1}$	Definition 16.1, p. 169
λ	a map $V_n \otimes V_m \to V_{n+1} \otimes V_{m-1}$	Definition 16.6, p. 172
$G_0(-)$	Grothendieck group	Definition 16.18, p. 182
$[U]$	isomorphism class of U in the Grothendieck group	
$[V : U]$	multiplicity of U as an indecomposable summand of V	
$A = A^{(n)}$	table of multiplicities: matrix with entries $A^{(n)}_{i,j} = [V_j : V_i \otimes V_n]$	Definition 17.1, p. 191
$\mathcal{G} = \mathcal{G}^{(n)}$	graph with adjacency matrix $A^{(n)}$	Definition 17.1, p. 191

$\overline{A} = \overline{A}^{(n)}$	a submatrix of $A^{(n)}$	Definition 17.8, p. 195
T	matrix with 1s on the antidiagonal	Definition 17.3, p. 192
$d(i)$	degree of vertex i in $\mathcal{G}$	Definition 17.12, p. 198
w	a weighting function	Definition 17.14, p. 199
$Q = Q^{(n)}$	transition matrix for the non-projective summand random walk	Definition 17.14, p. 199
$\overline{Q} = \overline{Q}^{(n)}$	a submatrix of $Q^{(n)}$ defined analogously to $\overline{A}^{(n)}$	

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