SMALL MAHLER MEASURES FROM DIGRAPHS

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Abstract. We attach Mahler measures to digraphs and find combinatorial realisations of nearly all of the known low-degree (≤ 180) small (< 1.3) one-variable Mahler measures. We find one new such measure not on either of the lists maintained by Mossinghoff and Sac-Épée. Considering limits of sequences of measures attached to families of digraphs, we get combinatorial explanations for 57 of the 61 known irreducible two-variable measures below 1.37.

1. Introduction

For $Q(z_1, \ldots, z_r) \in \mathbb{C}[z_1, \ldots, z_r]\backslash\{0\}$, the Mahler measure of Q, written $M(Q)$, is defined by the integral formula

$$\log M(Q) = \int_{0 \leq t_1, \ldots, t_r \leq 1} \log |Q(e^{2\pi i t_1}, \ldots, e^{2\pi i t_r})| \, \mathrm{d}t_1 \cdots \mathrm{d}t_r \,. \tag{1}$$

In the special case $r = 1$, if $Q(z) = a \prod_{i=1}^{d} (z - \alpha_i)$ with the α_i in $\mathbb{C}$, then Jensen's Theorem (see, e.g., [17]) can be used to derive the alternative formula

$$M(Q) = |a| \prod_{i=1}^{d} \max(1, |\alpha_i|) \,. \tag{2}$$

For $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$, define

$$\mu(\mathbf{n}) = \min_{\substack{\mathbf{c}=(c_1,\ldots,c_r)\in\mathbb{Z}^r\backslash\{\mathbf{0}\},\\ \mathbf{c}\cdot\mathbf{n}=0}} \max_{1 \leq i \leq r} |c_i| \,. \tag{3}$$

Lawton [5] (and see Boyd [1] for the case $r = 2$) showed that for sequences $\mathbf{n}_i = (n_{i,1}, \ldots, n_{i,r}) \in \mathbb{Z}^r$ for which $\mu(\mathbf{n}_i) \to \infty$ as $i \to \infty$,

$$M(Q(z^{n_{i,1}}, \ldots, z^{n_{i,r}})) \to M(Q(z_1, \ldots, z_r))$$

as $i \to \infty$. We do not address the question of whether the limits of the particular sequences that we consider are in fact limit points of the set of Mahler measures. See [2] for some discussion of this point.

Let $\mathbf{c} = (c_1 : c_2 : \cdots : c_r) \in \mathbb{P}^{r-1}(\mathbb{Q})$. Scaling $\mathbf{c}$, we can suppose that all the c_i are in $\mathbb{Z}$, and their greatest common divisor is 1. Then the **height** of $\mathbf{c}$ is defined to be the maximum of the $|c_i|$. Such points $\mathbf{c}$ correspond precisely to hyperplanes through the origin in $\mathbb{Q}^r$, with the hyperplane corresponding to $\mathbf{c}$ being the set of points $\mathbf{x}$ satisfying $\mathbf{c} \cdot \mathbf{x} = 0$. We define the **height** of the hyperplane to be the height of the corresponding point $\mathbf{c}$.

One way to interpret $\mu(\mathbf{n})$ in (3) is as the minimum height of a hyperplane through the origin that contains the point $\mathbf{n}$ (viewing $\mathbb{Z}^r$ as a subset of the vector

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space $\mathbb{Q}^r$). The condition $\mu(\mathbf{n}_i) \to \infty$ in the Boyd and Lawton theorems then immediately translates as the following lemma.

Lemma 1. *Let $(\mathbf{n}_i)$ be a sequence of elements of $\mathbb{Z}^r$. Then $\mu(\mathbf{n}_i) \to \infty$ as $i \to \infty$ if and only if there is no hyperplane in $\mathbb{Q}^r$ through the origin (i.e., no $(r-1)$-dimensional subspace of $\mathbb{Q}^r$) containing infinitely many terms of the sequence.*

Proof. Suppose that for some constant $\mathbf{c} = (c_1, \ldots, c_r) \neq \mathbf{0}$ the hyperplane $\mathbf{c}\cdot\mathbf{x} = 0$ contains $\mathbf{n}_i$ for infinitely many i, say for $i = i_1 < i_2 < \cdots$. Clearing denominators, we can suppose that $\mathbf{c} \in \mathbb{Z}^r$. Then $\mu(\mathbf{n}_{i_j}) \leq \max_{1\leq k\leq r} |c_k|$ for all j, and so $\mu(\mathbf{n}_{i_j}) \not\to \infty$ as $j \to \infty$, and $\mu(\mathbf{n}_i) \not\to \infty$ as $i \to \infty$.

Conversely, if $\mu(\mathbf{n}_i)$ has a bounded subsequence, then infinitely many of the $\mathbf{n}_i$ lie on a finite set of hyperplanes, and so there is some hyperplane containing infinitely many of the $\mathbf{n}_i$. □

A special case of Lawton's Theorem is due to Boyd [1]:

$$\lim_{n\to\infty} M(P(z, z^n)) = M(P(x, y)) \,. \tag{4}$$

Here the points $(1, n)$ lie on a line, but not on a line through the origin.

We propose the following approach for searching for small Mahler measures of single-variable polynomials, and for limits of sequences of such. For the moment this is vaguely expressed: more detail is given in Sections 2–6.

- Find a good $(r + 1)$-variable polynomial $Q(z_0, z_1, \ldots, z_r) \in \mathbb{Z}[z_0, \ldots, z_r]$.
- For each of a selection of good hyperplanes through the origin in $\mathbb{Q}^{r+1}$ (or the intersection of several), choose a sequence of $\mathbf{n}_i = (1, n_{i,1}, \ldots, n_{i,r})$ on the hyperplane (or the intersection) and consider:
  - (i) $M(Q(z, z^{n_{i,1}}, \ldots, z^{n_{i,r}}))$ for small Mahler measures;
  - (ii) $\lim_{i\to\infty} M(Q(z, z^{n_{i,1}}, \ldots, z^{n_{i,r}}))$ for small limits.

In practice we choose the n_i in such a way that the limit point can be written as a 2-variable Mahler measure as in (4), which can then be computed numerically reasonably quickly. For more detail, see Section 2.

But what makes a polynomial Q good, and what makes a (hyper)plane good? We know due to Breusch [3] and Smyth [15] that in the single-variable case, small Mahler measures only occur from reciprocal polynomials. As such, our approach for finding candidates for Q is to go via reciprocal polynomials attached to integer matrices, viewing these matrices as the adjacency matrices of $\mathbb{Z}$-weighted digraphs. The digraphs we choose are combinatorially close to ones for which the reciprocal polynomial has Mahler measure 1. In particular, this means that our methods use only matrices with entries 0, 1 or −1, and which are close to being symmetric. In this way we produce an abundant supply of potentially useful Q. The detail of this appears in Sections 3–6, but first, in Section 2. we outline the method that we use to find good sequences $\mathbf{n}_i$, given Q.

Our experiments so far have found 97% of the thousands of known 1-variable Mahler measures below 1.3 and degree at most 180: see the lists [10] and [14]. We find all but one of the 236 known measures below 1.25. We also find one new measure not on the above lists. These small measures have therefore been given a combinatorial 'explanation': they are attached to digraphs that are 'combinatorially close' to those that have Mahler measure 1. From limits of sequences of digraph measures, we find 57 of the 61 known small limits below 1.37. The results are detailed in Section 7.

2. The search for good lines on planes

Before discussing a possible strategy for choosing a good polynomial Q, we address, in this section, the question of what we will do with any candidate polynomial. This may inform what we want from our polynomial Q if it is to be useful, and the work in this section equally applies to any technique for choosing Q.

All our computations were done using PARI/GP [16]. Computing single-variable Mahler measures can be done easily via formula (2). For two-variable Mahler measures we combine root finding with numerical integration. The integral formula (1) for the Mahler measure of $P(x, y)$ can be written

$$\log M(P(x,y)) = \int_{t=0}^{1} \log M(P(x, e^{2\pi i t}))\,\mathrm{d}t\,. \tag{5}$$

Given t, the single-variable Mahler measure $M(P(x, e^{2\pi i t}))$ can be computed using (2). The integrand in (5) may have discontinuous derivative, which can make numerical integration less accurate, so we identified the discontinuities and split the range of integration accordingly. In this way we could compute numerical estimates of two-variable Mahler measures to good accuracy, in agreement with known values. See [11], [2] and references therein for other approaches. Computing r-variable Mahler measures to high accuracy for $r > 2$ is in general considerably more challenging.

Let $Q(z_0, \ldots, z_r) \in \mathbb{Z}[z_0, \ldots, z_r]$ be a candidate polynomial. For $n_1, \ldots, n_r$ positive integers, and putting $\mathbf{n} = (n_1, \ldots, n_r)$, we define

$$P_{\mathbf{n}}(z) = Q(z, z^{n_1}, \ldots, z^{n_r})\,.$$

Take a sequence of $\mathbf{n}_i = (n_{1,i}, \ldots, n_{r,i})$ with all the $n_{j,i}$ positive integers. Provided there is no hyperplane through the origin containing infinitely many of the points $(1, n_{1,i}, \ldots, n_{r,i})$, Lawton's Theorem tells us that

$$\lim_{i\to\infty} M(P_{\mathbf{n}_i}) = M(Q(z_0, \ldots, z_r))\,.$$

We shall deliberately break the constraints of this theorem by choosing points that lie on hyperplanes through the origin, but if we seek small Mahler measures it is reasonable for us to seek Q for which the $(r+1)$-variable Mahler measure is not large: although we avoid the generic limit, it would seem more surprising to find small Mahler measures if the generic limit were large. Given the difficulties of computing such Mahler measures, we shall need an indirect way of encouraging the Mahler measure to be small. We return to this in Sections 3–6.

For fixed integers a_1, b_1, a_2, b_2, …, a_r, b_r, the Laurent polynomial

$$Q(z, z^{a_1 t + b_1}, z^{a_2 t + b_2}, \ldots, z^{a_r t + b_r})$$

can be written as a Laurent polynomial in z and z^t, say

$$Q(z, z^{a_1 t + b_1}, z^{a_2 t + b_2}, \ldots, z^{a_r t + b_r}) = P_{\mathbf{a},\mathbf{b}}(z, z^t)\,,$$

where $\mathbf{a} = (a_1, \ldots, a_r)$ and $\mathbf{b} = (b_1, \ldots, b_r)$. In practice we will take $a_i \geq 0$ for all i, and if $a_i = 0$ then $b_i \geq 0$, but there is no need in theory for such restrictions. Note that for $r \geq 2$ and $t \in \mathbb{Z}$ the line of points

$$(n_0, \ldots, n_r)_t = (1, a_1 t + b_1, \ldots, a_r t + b_r)$$

lies on a plane through the origin in $\mathbb{Q}^{r+1}$, namely the intersection of the $r-1$ hyperplanes

$$(-a_ib_1 + a_1b_i)n_0 + a_in_1 - a_1n_i = 0 \tag{6}$$

$(2 \le i \le r)$. Thus we are not constrained by the generic limit in Lawton's Theorem, and the limits (using (4))

$$\lim_{t\to\infty} M(P_{\mathbf{a},\mathbf{b}}(z, z^t)) = M(P_{\mathbf{a},\mathbf{b}}(x, y)) \tag{7}$$

might be different for different choices of **a** and **b** (and indeed generally are different). From a single polynomial Q we have the prospect of finding more than one small limit. How should we arrange this search?

One option would be to search over pairs $(\mathbf{a}, \mathbf{b})$ of bounded height. Our alternative, less painful as r grows, was the following three phase strategy.

Phase 1: searching for points

Choose a bound H for the heights of points, and a bound B for the largest Mahler measure to be tolerated. When searching for small Mahler measures, we might chose $B = 1.3$; when searching for small limits, we might choose $B = 1.38$. Loop over the r-dimensional box $0 \le n_i \le H$, $1 \le i \le r$, and store points $(n_1, \ldots, n_r)$ for which $M(Q(z, z^{n_1}, \ldots, z^{n_r})) \le B$.

Phase 2: searching for lines

For each triple of points found in Phase 1, see if they lie on a line in $\mathbb{Q}^r$. If so, write this line in canonical form $\mathbf{a}t + \mathbf{b}$, where $\mathbf{a}$, $\mathbf{b} \in \mathbb{Z}^r$, the content of **a** is 1 and the first nonzero entry in **a** is positive. Store distinct lines found.

Phase 3: check the lines

For each line found in Stage 2, perform one of two tests, depending on whether one is seeking small Mahler measures or small limit points. If seeking small limit points, compute $M(P_{\mathbf{a},\mathbf{b}}(x, y))$. If seeking small single-variable Mahler measures, then we are interested in the precise values obtained as we approach the limit, whilst constraining the degree, and so we compute $M(P_{\mathbf{a},\mathbf{b}}(z, z^t))$ for a range of t, constrained by a target bound for the degree (we chose a degree bound of 180, in order to compare our results with those in Mossinghoff's and Sac-Épée's online tables [10] and [14]). We performed two reductions before assessing the degree of $P_{\mathbf{a},\mathbf{b}}(z, z^t)$: (i) we removed any cyclotomic factors; (ii) if the polynomial was a polynomial in z^k for any $k > 1$, then we replaced z^k by z (preserving the Mahler measure, but lowering the degree).

There remains the question of how to select candidates for Q. To answer this question we take a short digression to discuss the Mahler measure of a digraph: this involves attaching a reciprocal polynomial to a digraph and then computing its Mahler measure. Our choices for Q were factors of reciprocal polynomials of digraphs that are close in a combinatorial sense to digraphs known to have Mahler measure 1. We considered families for which the generic limit was the Mahler measure of a polynomial having at most six variables.

3. The Mahler measure of a digraph

Let A be an $n \times n$ integer matrix. The **reciprocal polynomial** of A, denoted R_A, is defined by

$$R_A(z) = z^n \det((z + 1/z)I - A)\,,$$

where I is the $n \times n$ identity matrix. This is a transform of the **characteristic polynomial** of A, namely $\chi_A(x) = \det(xI - A)$. Generalising [6], we define the Mahler measure of A to be the Mahler measure of its reciprocal polynomial.

Restricting to symmetric matrices with integer entries, the complete spectrum of Mahler measures below 1.3 was computed in [8]. Only 16 polynomials with Mahler measures below 1.3 arise as Mahler measures coming from the reciprocal polynomials of integer symmetric matrices (including Lehmer's number), compared to more than 8000 known of polynomial degree up to 180. One reason for this small yield is because for symmetric A, any root α of R_A must have $\alpha + 1/\alpha$ real. As such, this severely restricts the number of reciprocal polynomials one can find. Moving to more general matrices we can find many more Mahler measures, but we lose the powerful property of interlacing of eigenvalues (delete row i and column i from symmetric A to produce symmetric B; the eigenvalues of A and B interlace). In order to make progress, we considered small deformations of symmetric matrices. We chose matrices that had Mahler measure 1, and nudged them slightly to break the symmetry. Although no longer able to use interlacing, we were able to obtain explicit formulas for the reciprocal polynomials of families of matrices produced in this way.

The matrices we considered had all their entries either 1, -1 or 0. Such matrices can be viewed as the adjacency matrices of **charged signed digraphs**, and we refer to these simply as **digraphs**. A digraph comprises (for the purposes of this paper): (i) a set of n **vertices**, labelled 1, 2, …, n, which may be positively charged, negatively charged, or neutral; (ii) a set of signed **arcs**, either positively or negatively signed, each going from some vertex i to some other (different) vertex j. No ordered pair of vertices has more than one arc going from the first to the second, but we allow there to be both an arc from i to j and an arc from j to i, possibly of different signs. If there are arcs from i to j and from j to i and the signs are the same, then we speak of a **signed edge** joining i and j (positive or negative). The adjacency matrix $A = (a_{ij})$ of such a digraph is an $n \times n$ matrix that has diagonal entries a_{ii} corresponding to the charge of the ith vertex (1, -1, or 0), and entries a_{ij} for $i \neq j$ corresponding to the sign of the arc from i to j (1 or -1), or 0 if there is no arc. Moreover, we define the **reciprocal polynomial of a digraph** as the reciprocal polynomial of its associated adjacency matrix.

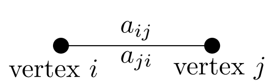
We can draw a digraph as follows. We draw positive, negative, and neutral vertices as

⊕ , ⊖ , and ●

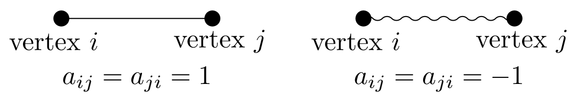
respectively. If we wish to indicate the presence of a vertex whose charge is not specified, we draw it

⊛ .

If either there is an arc from i to j or from j to i (or both), then we draw a line between i and j, and we then record the values of a_{ij} and a_{ji} as shown (here with neutral vertices):

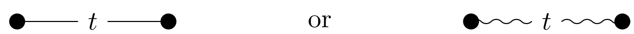


If we go along the line from vertex i to vertex j, we see a_{ij} on the left of the line. If both a_{ij} and a_{ji} equal 1, then we omit the labels and simply draw a line between the two vertices. If both a_{ij} and a_{ji} equal -1, then we omit the labels and draw a wavy line. Indeed, these signed edges arise frequently in our digraphs, so this convention will regularly be used.

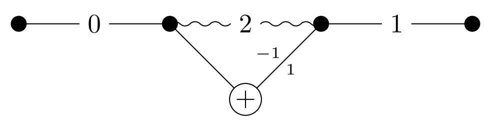


The drawing of a digraph does not usually record the labels of the vertices 1, . . . , n (such as vertices i and j in the above pictures), so that from a given drawing there is a choice as to how we label them. On the other hand, the reciprocal polynomial of the digraph is invariant under these choices of labels, so we do not usually care that this information is lost in the drawing, and prefer to omit the vertex labels to avoid clutter.

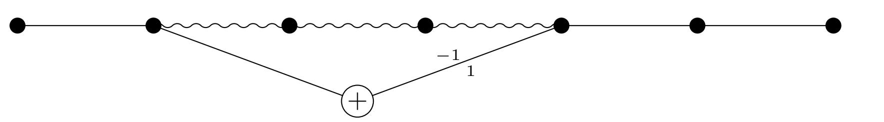
Finally, we shall be interested in families of digraphs where we repeatedly subdivide a signed edge between two neutral vertices, adding new neutral vertices along the edge. Depending on sign, we use the picture



to indicate that we have added t vertices, so that for example

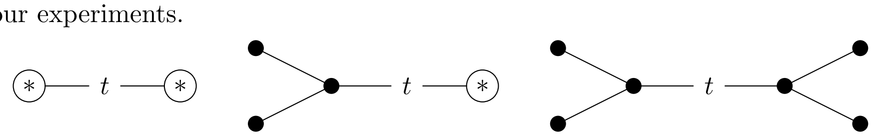


is a compression of



We allow $t = -1$:

In [7] a complete description was given of which symmetric digraphs had Mahler measure equal to 1 (and see [9] for symmetrizable matrices). There are many possibilities for such digraphs that we could take as a starting point, and then we can try deforming them slightly to break the symmetry and give a digraph that is a plausible candidate for one that has small Mahler measure. In the experiments reported here, we considered a particularly simple subset of the myriad possibilities; other variants are explored in [4]. The digraphs shown below have Mahler measure 1 (see, e.g., [7, Theorems 1, 2]), and these are the building blocks for the digraphs in our experiments.



The precise families considered are described fully in Section 6. Once we have broken the symmetry, we no longer have the tool of interlacing to help us, but we can produce explicit formulas for the families of reciprocal polynomials that we consider, and use these to search for small Mahler measures and limit points reasonably efficiently.

4. Growing paths

Our aim in this section is to establish the following Proposition concerning the general shape of the reciprocal polynomial of a digraph that has one or more of its signed edges subdivided. This will be one of our main tools for associating digraphs with small two-variable measures.

Proposition 2. *Let H be a digraph with r distinguished signed edges $e_1, \ldots, e_r$, each joining two neutral vertices. For integers $t_1, \ldots, t_r \geq 0$, let $H_{[t_1,\ldots,t_r]}$ be the digraph obtained from H by subdividing each edge e_i by adding t_i new vertices. Thus e_i is replaced by*

●—— t_i ——●   *or*   ●∼∼∼ t_i ∼∼∼●

according as its sign is positive or negative. Then there is a polynomial

$$Q(z_0, z_1, \ldots, z_r) \in \mathbb{Z}[z_0, z_1, \ldots, z_r]\,,$$

depending on H but independent of $t_1, \ldots, t_r$, such that $H_{[t_1,\ldots,t_r]}$ has reciprocal polynomial

$$\frac{Q(z, z^{t_1}, z^{t_2}, \ldots, z^{t_r})}{z^{2r}(z^2-1)^r}\,.$$

If the growing paths are all pendant (one end is a **leaf**, i.e., a vertex that has only one adjacent vertex), then this result is contained in [6, Lemma 4.2].

A tool in our proof will be a tiny modification of a deletion-contraction theorem due to Rowlinson [13]. Rowlinson's theorem was stated and proved for multigraphs, but an examination of the proof shows that it easily extends to $\mathbb{Z}$-weighted digraphs (more general than the digraphs we need), and so we begin by stating and proving this generalisation. In a $\mathbb{Z}$-weighted digraph, the weights attached to arcs can be arbitrary elements of $\mathbb{Z}$, and the 'charges' on vertices (directed weighted loops if you prefer) can also be arbitrary elements of $\mathbb{Z}$. The integer charges appear as diagonal entries of the adjacency matrix, and the integer arc weights appear as off-diagonal entries. If $A = (a_{ij})$ is the adjacency matrix, and $a_{ij} = a_{ji} \neq 0$ for some i and j, then we say that there is a **signed multiedge** between i and j.

Let G be a $\mathbb{Z}$-weighted digraph for which there is a signed multiedge between vertices u and v. Thus if G has adjacency matrix $A = (a_{ij})$, then $a_{uv} = a_{vu} \neq 0$. The $\mathbb{Z}$-weighted digraph $G - [uv]$ ('deleting' the multiedge) has adjacency matrix obtained from A be setting $a_{uv} = a_{vu} = 0$. The $\mathbb{Z}$-weighted digraph G^* ('contracting' the multiedge; of course this depends on u and v but the given notation is standard) has adjacency matrix obtained from A by: (i) adding row u to row v; (ii) adding column u to column v; (iii) subtracting $2a_{uv}$ from the new (v, v) entry; (iv) deleting row u and column u. Thus the vertices u and v are 'amalgamated': the new charge is the sum of the old charges; the weight of the old signed multiedge does not contribute to the new charge. The $\mathbb{Z}$-weighted digraph $G - u$ (deleting the vertex u) has adjacency matrix obtained from A by deleting row u and column u; if we also delete row v and column v then we obtain $G - u - v$.

Rowlinson's deletion-contraction theorem generalises as follows.

Theorem 3 (Compare [13, Theorem 1.3])**.** *Let G be a $\mathbb{Z}$-weighted digraph with at least three vertices for which there is a signed multiedge between (distinct) vertices u and v, and let $c \in \mathbb{Z}$ be the weight of this multiedge. Writing $\chi_H = \chi_H(x)$ for the* *characteristic polynomial of any given digraph H, we have*

$$\chi_G = \chi_{G-[uv]} + c\chi_{G^*} + c(x-c)\chi_{G-u-v} - c\chi_{G-u} - c\chi_{G-v}\,. \tag{8}$$

Proof. We follow Rowlinson's proof, commenting on the tiny differences needed for the generalisation.

We may as well assume that $u = 1$ and $v = 2$. Then the adjacency matrix (a_{ij}) of G has the shape

$$\begin{pmatrix} a & c & \mathbf{r} \\ c & b & \mathbf{s} \\ \mathbf{p} & \mathbf{q} & A \end{pmatrix}.$$

(In the multigraph setting, $c > 0$, $\mathbf{p} = \mathbf{r}^T$, $\mathbf{q} = \mathbf{s}^T$, and A is symmetric. The only symmetry we require in this generalisation is that $a_{12} = a_{21} = c$.) The adjacency matrices of $G - [uv]$, $G - u$, $G - v$, $G - u - v$, and G^* are then:

$$\underset{G-[uv]}{\begin{pmatrix} a & 0 & \mathbf{r} \\ 0 & b & \mathbf{s} \\ \mathbf{p} & \mathbf{q} & A \end{pmatrix}}, \quad \underset{G-u}{\begin{pmatrix} b & \mathbf{s} \\ \mathbf{q} & A \end{pmatrix}}, \quad \underset{G-v}{\begin{pmatrix} a & \mathbf{r} \\ \mathbf{p} & A \end{pmatrix}}, \quad \underset{G-u-v}{(A)}, \quad \underset{G^*}{\begin{pmatrix} a+b & \mathbf{r}+\mathbf{s} \\ \mathbf{p}+\mathbf{q} & A \end{pmatrix}}.$$

We have, using that the determinant is a linear function of any row or column, and that swapping rows or columns changes the sign,

$$\begin{aligned}
\chi_G(x) &= \begin{vmatrix} x-a & -c & -\mathbf{r} \\ -c & x-b & -\mathbf{s} \\ -\mathbf{p} & -\mathbf{q} & xI-A \end{vmatrix} \\
&= \begin{vmatrix} x-a & 0 & -\mathbf{r} \\ -c & x-b & -\mathbf{s} \\ -\mathbf{p} & -\mathbf{q} & xI-A \end{vmatrix} + \begin{vmatrix} 0 & -c & \mathbf{0} \\ -c & x-b & -\mathbf{s} \\ -\mathbf{p} & -\mathbf{q} & xI-A \end{vmatrix} \\
&= \begin{vmatrix} x-a & 0 & -\mathbf{r} \\ 0 & x-b & -\mathbf{s} \\ -\mathbf{p} & -\mathbf{q} & xI-A \end{vmatrix} + \begin{vmatrix} x-a & 0 & -\mathbf{r} \\ -c & 0 & \mathbf{0} \\ -\mathbf{p} & -\mathbf{q} & xI-A \end{vmatrix} + \begin{vmatrix} 0 & -c & \mathbf{0} \\ -c & x-b & -\mathbf{s} \\ -\mathbf{p} & -\mathbf{q} & xI-A \end{vmatrix} \\
&= \chi_{G-[uv]} + c\begin{vmatrix} 1 & 0 & \mathbf{0} \\ x-a & 0 & -\mathbf{r} \\ -\mathbf{p} & -\mathbf{q} & xI-A \end{vmatrix} + c\begin{vmatrix} 1 & 0 & \mathbf{0} \\ x-b & -c & -\mathbf{s} \\ -\mathbf{q} & -\mathbf{p} & xI-A \end{vmatrix} \\
&= \chi_{G-[uv]} + c\begin{vmatrix} 0 & -\mathbf{r} \\ -\mathbf{q} & xI-A \end{vmatrix} + c\begin{vmatrix} 0 & -\mathbf{s} \\ -\mathbf{p} & xI-A \end{vmatrix} + c\begin{vmatrix} -c & \mathbf{0} \\ -\mathbf{p} & xI-A \end{vmatrix} \\
&= \chi_{G-[uv]} - c^2\chi_{G-u-v} + c\begin{vmatrix} 0 & -\mathbf{r} \\ -\mathbf{q} & xI-A \end{vmatrix} + c\begin{vmatrix} 0 & -\mathbf{s} \\ -\mathbf{p} & xI-A \end{vmatrix}. \qquad (9)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\chi_{G^*} &= \begin{vmatrix} x-(a+b) & -\mathbf{r}-\mathbf{s} \\ -\mathbf{p}-\mathbf{q} & xI-A \end{vmatrix} \\
&= \begin{vmatrix} x-(a+b) & \mathbf{0} \\ -\mathbf{p}-\mathbf{q} & xI-A \end{vmatrix} + \begin{vmatrix} 0 & -\mathbf{r} \\ -\mathbf{p}-\mathbf{q} & xI-A \end{vmatrix} + \begin{vmatrix} 0 & -\mathbf{s} \\ -\mathbf{p}-\mathbf{q} & xI-A \end{vmatrix} \\
&= (x-a-b)\chi_{G-u-v} \\
&\qquad + \begin{vmatrix} x-a & -\mathbf{r} \\ -\mathbf{p} & xI-A \end{vmatrix} - \begin{vmatrix} x-a & \mathbf{0} \\ -\mathbf{p} & xI-A \end{vmatrix} + \begin{vmatrix} 0 & -\mathbf{s} \\ -\mathbf{p} & xI-A \end{vmatrix} \\
&\qquad + \begin{vmatrix} x-b & -\mathbf{s} \\ -\mathbf{q} & xI-A \end{vmatrix} - \begin{vmatrix} x-b & \mathbf{0} \\ -\mathbf{q} & xI-A \end{vmatrix} + \begin{vmatrix} 0 & -\mathbf{r} \\ -\mathbf{q} & xI-A \end{vmatrix} \\
&= (x-a-b)\chi_{G-u-v} + \chi_{G-v} - (x-a)\chi_{G-u-v} + \chi_{G-u} - (x-b)\chi_{G-u-v} \\
&\qquad + \begin{vmatrix} 0 & -\mathbf{r} \\ -\mathbf{q} & xI-A \end{vmatrix} + \begin{vmatrix} 0 & -\mathbf{s} \\ -\mathbf{p} & xI-A \end{vmatrix}
\end{aligned}
$$

$$
\chi_{G-u} + \chi_{G-v} - x\chi_{G-u-v} + \begin{vmatrix} 0 & -\mathbf{r} \\ -\mathbf{q} & xI-A \end{vmatrix} + \begin{vmatrix} 0 & -\mathbf{s} \\ -\mathbf{p} & xI-A \end{vmatrix}. \tag{10}
$$

Multiplying (10) by c and then subtracting from (9) gives the result. □

The next ingredient is a simple result about the shape of the reciprocal polynomial of a digraph with a pendant path attached. A **pendant path** in a digraph G is a sequence of neutral vertices $v_1, \ldots, v_t$ where there are positive edges between v_i and v_{i+1} $(1 \le i < t)$, and if w is any vertex of G that is not one of the v_i, then there are no arcs in either direction between w and any of $v_1, \ldots, v_{t-1}$, although arcs between w and v_t are allowed. We call v_t the **base** of the path, and note that v_1 is a leaf.

Given a digraph G with a distinguished vertex v, we define a family of digraphs G_t for $t \ge 0$ by letting $G_0 = G$, and for $t \ge 1$ letting G_t be formed by adding a pendant path with base v, and t new vertices.

Lemma 4. *Let G be a digraph with a distinguished vertex v. Let G_t be as above, and let R_t be the reciprocal polynomial of G_t. Then*

$$
R_t(z) = \frac{z^{2t}-1}{z^2-1}R_1(z) - \frac{z^{2t-2}-1}{z^2-1}z^2R_0(z) \tag{11}
$$

for all $t \ge 1$.

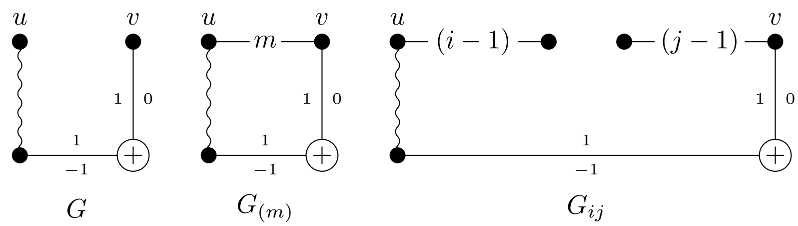
Proof. This is contained in the proof of [6, Lemma 4.1]. □

Next we consider subdividing a signed edge between two neutral vertices. To facilitate this we define two modifications of a digraph G that has distinct vertices u and v for which there are no signed arcs in either direction between u and v.

For $m \ge 2$ let $G_{(m)}$ be the digraph obtained from G by adding a positive edge between u and v and then subdividing this by adding a further m new vertices along that edge. Thus $G_{(m)}$ has m more vertices than G.

For $i \ge 0$ and $j \ge 0$, let G_{ij} be the digraph obtained by identifying one end of an $(i+1)$-vertex path (all edges positive, all vertices neutral) with u, and one end of a $(j+1)$-vertex path (all edges positive, all vertices neutral) with v. Thus G_{ij} has $i+j$ more vertices than G and $G_{00} = G$. Choosing first u and then v as the distinguished vertices in Lemma 4, we have $G_{ij} = (G_i)_j$. If it is clearer, we use $G_{i,j}$ as an alternative for G_{ij}.

It may help to see an example:



Lemma 5. *Let G be a digraph with two distinguished neutral vertices u and v that have no signed arc between them in either direction.*

Let $R_{(m)} = R_{(m)}(z)$ and $R_{i,j} = R_{ij} = R_{ij}(z)$ be the reciprocal polynomials of $G_{(m)}$ and G_{ij} respectively. Then, for $m \geq 2$,

$$\begin{aligned}(z^2-1)R_{(m)} = {} & z^{m-2}(z^2-1)R_{(2)} \\ & + (z^{m-2}-1)(z^m+1)R_{11} \\ & - z^2(z^{m-2}-1)(z^{m-2}+1)(R_{10}+R_{01}) \\ & + z^4(z^{m-2}-1)(z^{m-4}+1)R_{00}\,. \qquad (12)\end{aligned}$$

Proof. Let v_1 be the vertex on our subdivided edge that is adjacent to u, and let v_2 be that which is adjacent to v_1. We shall apply the deletion-contraction formula (8) for the digraph $G_{(m)}$ with the edge between v_1 and v_2:

(13)

$$\chi_{G_{(m)}}(x) = \chi_{G_{1,m-1}}(x) + \chi_{G_{(m-1)}}(x) + (x-1)\chi_{G_{0,m-2}}(x) - \chi_{G_{0,m-1}}(x) - \chi_{G_{1,m-2}}(x)\,.$$

We replace x by $z + 1/z$ and multiply by z^m, noting that $G_{1,m-1}$ has the same number of vertices as $G_{(m)}$, $G_{0,m-2}$ has two vertices fewer, and all the other digraphs whose characteristic polynomial appears on the right in (13) have one fewer vertex:

$$R_{(m)} = R_{1,m-1} + zR_{(m-1)} + z(z^2-z+1)R_{0,m-2} - zR_{0,m-1} - zR_{1,m-2}\,. \qquad (14)$$

Now we use (11) to write each R_{0j} in terms of R_{00} and R_{01} and each R_{1j} in terms of R_{10} and R_{11}, simplifying by multiplying by (z^2-1):

$$\begin{aligned}(z^2-1)R_{(m)} = {} & z(z^2-1)R_{(m-1)} \\ & + (z-1)(z^{2m-3}+1)R_{11} \\ & - z^2(z-1)(z^{2m-5}+1)(R_{10}+R_{01}) \\ & + z^4(z-1)(z^{2m-7}+1)R_{00}\,. \qquad (15)\end{aligned}$$

Now we claim that for $0 \leq t \leq m-2$ we have:

$$\begin{aligned}(z^2-1)R_{(m)} = {} & z^t(z^2-1)R_{(m-t)} \\ & + (z^t-1)(z^{2m-2-t}+1)R_{11} \\ & - z^2(z^t-1)(z^{2m-4-t}+1)(R_{10}+R_{01}) \\ & + z^4(z^t-1)(z^{2m-6-t}+1)R_{00}\,. \qquad (16)\end{aligned}$$

This is trivial for $t = 0$, and for $t = 1$ it is precisely (15). If (16) holds for some $t < m-2$, then applying (15) (with m replaced by $m-t$) to the right hand side of (16) gives (after some manipulation) (16) for $t+1$. Inductively, our claim is established.

Putting $t = m-2$ in (16) gives (12). □

Explicit formulas for R_{0j} and R_{1j} in terms of R_{00}, R_{10}, R_{01} and R_{00} can be found in [4], as well as more explicit computations and manipulations of the above formulas.

To deal with subdivisions of negative signed edges we use a process known as **switching**. To switch at a vertex i means to change the signs of all arcs going in and out of i (the charge at i is not changed). Switching at a vertex preserves the characteristic polynomial and hence preserves the reciprocal polynomial. If we have a pendant path with some or all of its edges replaced by negative edges, then a judicious choice of switching vertices along the path transforms all the negative edges back into positive ones, and hence the characteristic and reciprocal polynomials are independent of the signs of the edges. Hence Lemma 4 and (11) still hold for pendant paths that have negative edges.

For the analogue of Lemma 5 when we subdivide a negative edge, we note that Theorem 3 still applies to give (14) unchanged. Then, having noted that (11) still holds regardless of signs of edges on any pendant paths, we get all the subsequent formulas of Lemma 5 too. Of course the analogue of $R_{(2)}$ for the subdivision of a negative edge may be different from that for the subdivision of a positive edge.

Now we can prove Proposition 2.

Proof of Proposition 2. We proceed by induction on the number, r, of subdivided signed edges: the case $r = 0$ being trivial.

Inductively, if we subdivide $r-1$ chosen signed edges of any digraph K, adding t_i new vertices to the ith chosen signed edge to produce $K_{[t_1,\dots,t_{r-1}]}$, we have

$$z^{2(r-1)}(z^2-1)R_{K_{[t_1,\dots,t_{r-1}]}} = Q_{K_{[t_1,\dots,t_{r-1}]}}(z, z^{t_1}, \dots, z^{t_{r-1}})\,,$$

for some polynomial $Q_{K_{[t_1,\dots,t_{r-1}]}}(z) \in \mathbb{Z}[z_0, z_1, \dots, z_{r-1}]$, depending on K and the chosen signed edges. In particular, if we subdivide the first $r-1$ chosen signed edges of H, and then apply Lemma 5 to the final signed edge, each recpiprocal polynomial on the right of (12) has this form for various choices of K. Multiplying (12) by $z^{2r}(z^2-1)^{r-1}$, we deduce that $z^{2r}(z^2-1)^r R_{H_{[t_1,\dots,t_r]}}$ has the desired shape. □

5. Explicit formulas for two pendant paths, and decorations

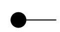
Let H be a digraph with two distinguished neutral vertices u and v. For $i, j \geq 0$, let H_{ij} be obtained from H by merging the endvertex of an $(i+1)$-vertex path with u (all vertices neutral, all edges positive) and merging the endvertex of a $(j+1)$-vertex path with v, as in the preamble to Lemma 5. Let R_{ij} be the reciprocal polynomial of H_{ij}. Applying (11) repeatedly, we get (as in [6, Lemma 4.2], but made more explicit here)

$$(z^2-1)^2 R_{\ell,r}(z) = z^{2(\ell+r)}P_{11}(z) - z^{2\ell}P_{10}(z) - z^{2r}P_{01}(z) + P_{00}(z)\,,$$

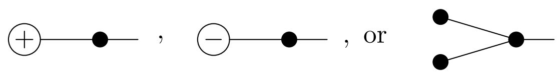
where

$$\begin{aligned} P_{11} &= R_{11} - R_{10} - R_{01} + R_{00}\,, \\ P_{10} &= R_{11} - z^2 R_{10} - R_{01} + z^2 R_{00}\,, \\ P_{01} &= R_{11} - R_{10} - z^2 R_{01} + z^2 R_{00}\,, \\ P_{00} &= R_{11} - z^2 R_{10} - z^2 R_{01} + z^4 R_{00}\,. \end{aligned} \tag{17}$$

We considered decorating the ends of the attached pendant paths, replacing



by one of



at the end of one or both paths (or neither). This gives us sixteen possible decoration combinations: we have four possible choices for each end of our path (the three decorations shown, or no decoration). In Table 1 we indicate these decorations by $+, -, >$ respectively (or $+, -, <$ if the decorations appear on the right hand end of a path), and use $\bullet$ to indicate no decoration. In all cases we found that there were polynomials $h(z)$ and $k(z)$, both with Mahler measure 1, such that (with suitable interpretations of ℓ and r for the decorated paths) the minimal polynomial $R_{\ell r}$ of $H_{\ell r}$ with decorated endvertices had the shape

(18)
$$\frac{h(z)R_{\ell r}(z)}{k(z)} = z^{2(\ell+r)}P_{11}(z) + c_{10}(z)z^{2\ell}P_{10}(z) + c_{01}(z)z^{2r}P_{01}(z) + c_{00}(z)P_{00}(z)\,,$$

where

$$c_{10}(z),\ c_{01}(z) \in \{1, -1, z, -z\}\,, \tag{19}$$

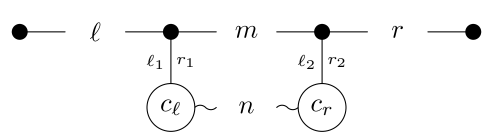
$$c_{00}(z) = c_{10}(z)c_{01}(z)\,. \tag{20}$$

Indeed the sixteen different decorations cover precisely the sixteen patterns in (18) produced by the sixteen choices in (19). For example, we see from (17) that the case of undecorated paths gives $c_{10} = c_{01} = -1$ and $c_{00} = 1$.

Given a 'core' digraph H with distinguished neutral vertices u and v, we can compute the P_{ij} in (17) and then immediately have formulas for the $R_{\ell r}$ for sixteen different families of digraphs. In the next section we detail the core digraphs that we used, and how we arranged the search for small Mahler measures and their limits.

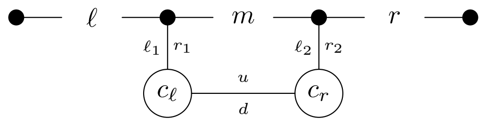
6. A family of digraphs

For the experiments reported in this paper, we used digraphs of the following shape:

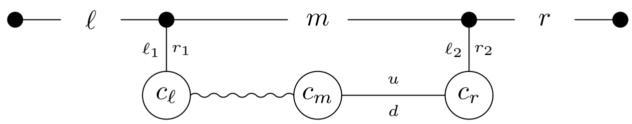


We also added 'decorations' to either of the growing ℓ and r paths, as just described in Section 5. We have that $m \geq 0$, and ℓ, r, $n \geq -1$, and l_1, l_2, r_1, $r_2 \in \{-1, 0, 1\}$. The charges c_ℓ and c_r can be any combination of $\{-1, 0, 1\}$, but if $n = -1$ then we must have $c_\ell = c_r$. For $n \geq 0$, we also considered making exactly one of the $n+1$ edges on the lower horizontal path a positive edge. Note that deleting the vertical arcs leaves a union of two cyclotomic digraphs, so that in a combinatorial sense any digraph in this family is close to being cyclotomic (more discussion of cyclotomic digraphs can be found in [7]). We wished to break symmetry, so insisted that at least one of $l_1 \neq r_1$ or $l_2 \neq r_2$ was satisfied.

In the case $n = 0$, we also considered other options for the lower horizontal arcs, introducing parameters u, $d \in \{-1, 0, 1\}$:



and in the case $n = 1$ we considered not only different arc weights for part of the lower horizontal path, but also a possible charge on the central vertex:



As in the previous cases, we wished to break symmetry. So in these cases, we insisted that at least one of $l_1 \neq r_1$, $l_2 \neq r_2$ or $u \neq d$ was satisfied.

Again decorations were allowed on either growing pendant path. Note that unlike the symmetric case, the Mahler measure is not a monotonic function of the path lengths: there is no longer interlacing of eigenvalues when a vertex is added.

For each choice of ℓ_1, r_1, etc., we organised our search for small Mahler measures as follows.

- Loop over m and n up to some bound, determined by our patience.
- Given m and n (and a decision as to whether one of the $n+1$ edges is to be positive), compute the P_{ij} in (17) (with $\ell = r = 0$).
- Loop over ℓ and r and for each pair (ℓ, r) compute the polynomial on the right hand side of (18) with the sixteen possibilities given by (19) to give sixteen reciprocal polynomials (multiplied and/or divided by measure 1 polynomials) corresponding to the different decorations for the ends of the paths. Compute the Mahler measure, and keep if it is small.

That was Phase 1 of the general strategy outlined in Section 2. Note that Proposition 2 tells us that for each decoration there is a polynomial $Q(v, w, x, y, z)$ such that for each ℓ, m, r, n our reciprocal polynomial has the shape

$$Q(z, z^\ell, z^m, z^r, z^n)\,.$$

But we do not in fact compute Q explicitly. We merely compute various instances of $Q(z, z^\ell, z^m, z^r, z^n)$.

Phase 2 is done precisely as described in Section 2.

For Phase 3, if we are after small limits, how do we compute P, given that we have not computed Q explicitly? We could in principle work through the detail to get explicit formulas for all sixteen Q, but in practice, for a line given by $\ell = a_1 t + b_1$, $m = a_2 t + b_2$, $r = a_3 t + b_3$, $n = a_4 t + b_4$, we simply computed $Q(z, z^\ell, z^m, z^r, z^n)$ with $t = 50$ to obtain $P(z, z^{50})$. Then writing the monomials in $P(z, z^{50})$ in the shape cz^{50a+b} and replacing this monomial by $cy^a z^b$ we expect to have recovered $P(z, y)$, and indeed for the relatively small digraphs used we can be certain that we have done so.

7. The results of our search

7.1. **Small Mahler measures, including a new one.** We experimented with the family of digraphs described in the previous section. We describe first the parameters used in the search for small Mahler measures, and then in Section 7.2 the changes when searching for limit points.

Putting a bound of 10 on ℓ, m, r, n, we tried all possibilities for ℓ_1, ℓ_2, r_1, r_2, c_ℓ, c_r. For those that were especially fruitful, we pushed the bound on ℓ, m, r, n up to 20. In each case we looped over all sixteen patterns of decorations, and employed our three-phase campaign: search for points (with a threshold of 1.3 for their Mahler measure); search for lines passing through triples of points; test the lines.

In Mossinghoff's list [10] there are 8458 small measures, and in Sac Épée's list [14] (along with two additional examples in the paper [12]) there are a further 115. We found 8334 of these, and also 1 new one. There are 236 known Mahler measures below 1.25: we found 235 of these (pushing the bound on ℓ, m, n, r up to 30 for a particularly good family, and reducing the Mahler measure bound to 1.25).

The new measure is $1.252826882865\ldots$, the Mahler measure of a degree-180 irreducible polynomial. This polynomial is a factor of the short polynomial $z^{221} - z^{160} - z^{155} + z^{66} + z^{61} - 1$.

7.2. **Small limits.** When searching for limit points, the bound on Mahler measures used in Phase 1 of our strategy was increased to 1.38 (at least for promising values of ℓ_1, ℓ_2, r_1, r_2, c_ℓ, c_r), but the bound for ℓ, m, r, n was necessarily lower: 15. We found 57 of the 61 known small limit points (Table 1).

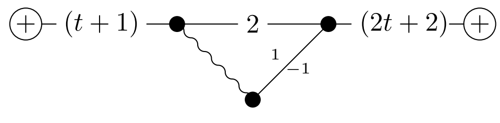
The rows of Table 1 contain the following data. First there is a list of the parameters ℓ_1, r_1, ℓ_2, r_2, c_ℓ, c_r for the family in Section 6. In the next column, the line of (ℓ, m, r, n) values is shown, parametrized by the variable t. Many of the examples were found with $n = -1$ (and then $c_\ell = c_r$). The decorations D_L and D_R are indicated as follows:

- no decoration: $\bullet$;
- a charged vertex: $+$ or $-$ to indicate the charge;
- a fork of two neutral leaves: $>$ if on the left and $<$ if on the right.

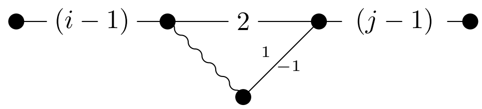
For example, the entry

ℓ_1, r_1	ℓ_2, r_2	c_ℓ, c_r D_L, D_R	u, d c_m	ℓ, m r, n	Limit point
-1, -1	1, -1	0, 0 $+$, $+$		$t+1$, $0t+2$ $2t+2$, -1	$1.36443581\ldots$

represents this family of digraphs:



We recall that (18) gives us a general shape for the reciprocal polynomial of the digraph. So, our first step is to calculate this polynomial; that is, we calculate the P_{ij} and the c_{ij}. The c_{ij} are determined by the decorations at the ends of the pendant paths. The P_{ij} are linear combinations of the R_{ij}, which come from calculating the reciprocal polynomial of the basic core digraph:



We compute that:

$$R_{11} = z^{10} + 2z^8 + 2z^6 + 2z^4 + 2z^2 + 1\,,$$
$$R_{10} = z^8 + 2z^6 + z^4 + 2z^2 + 1\,,$$
$$R_{01} = z^8 + 2z^6 + 3z^4 + 2z^2 + 1\,,$$
$$R_{00} = z^6 + 2z^4 + 2z^2 + 1\,. \tag{21}$$

Following (17), we have that:

$$P_{11} = z^{10} - z^6\,,$$
$$P_{10} = z^6 - z^4\,,$$
$$P_{01} = -z^6 + z^4\,,$$
$$P_{00} = -z^4 + 1\,. \tag{22}$$

Furthermore, for the given decoration, we have that $c_{10} = -z$, $c_{01} = -z$ and so $c_{00} = z^2$. So, (18) gives us:

$$\frac{h(z)R_{t+1,2t+2}(z)}{k(z)} = z^2(z^2-1)\big(z^{6t+10}(z^2+1) + z^{4t+7} - z^{2t+5} - (z^2+1)\big)\,. \tag{23}$$

Since $h(z)$ and $k(z)$ are both polynomials with Mahler measure 1 (they are 1 and $(z-1)^2$ respectively), we do not need actually to include them in our calculations, and work solely with the right hand side of (23) rather than computing the reciprocal polynomial itself. Furthermore, we can ignore the $z^2(z^2-1)$ term, since this also has Mahler measure 1.

We now need to calculate the two-variable polynomial whose Mahler measure gives the limit of the sequence of Mahler measures obtained as t varies, as explained at the end of Section 6. For this particular example, however, we can perform the manipulation directly. Noting that only even powers of z appear, we put $y = z^{2t+3}$, where the '3' is chosen to produce a polynomial of smallest degree in z. This gives us

$$P(z,y) = y^3z^3 - (y+1)z^2 + (y^3+y^2)z - 1\,. \tag{24}$$

Calculating the Mahler measure of (24) gives us the limit point $1.36443581\ldots$ as stated. The polynomial $-P(-z,y)$ agrees with that found by Boyd and Mossinghoff [2].

The parameters in Table 1 are as in Section 6. The column for the parameters u and d is needed only if $n = 0$ or 1, and c_m is only relevant if $n = 1$.

ℓ_1, r_1	ℓ_2, r_2	c_ℓ, c_r D_L, D_R	u, d c_m	ℓ, m r, n	Limit point
-1, -1	1, -1	0, 0 $-$, $\bullet$		$t+2$, $0t+2$ $2t+1$, -1	$1.25543386\ldots$
-1, -1	1, -1	0, 0 $+$, $+$		$t+1$, $0t+2$ $3t+5$, -1	$1.28573486\ldots$

$\ell_1,\ r_1$	$\ell_2,\ r_2$	$c_\ell,\ c_r$ $D_L,\ D_R$	$u,\ d$ c_m	$\ell,\ m$ $r,\ n$	Limit point
$-1,\ 0$	$0,\ -1$	$-1,\ 0$ $-,\ <$	$0,\ -1$ 0	$0t+1,\ t+0$ $0t+2,\ 1$	$1.30909838\ldots$
$-1,\ -1$	$1,\ -1$	$0,\ 0$ $\bullet,\ -$		$2t+3,\ 0t+2$ $3t+3,\ -1$	$1.31569270\ldots$
$0,\ -1$	$0,\ -1$	$0,\ 0$ $+,\ +$		$t+1,\ 0t+2$ $t+1,\ -1$	$1.32471795\ldots$
$-1,\ -1$	$1,\ -1$	$0,\ 0$ $+,\ +$		$t+2,\ 0t+2$ $3t+3,\ -1$	$1.32537249\ldots$
$-1,\ -1$	$1,\ -1$	$0,\ 0$ $\bullet,\ \bullet$		$t+2,\ 0t+4$ $2t+2,\ -1$	$1.33205110\ldots$
$-1,\ -1$	$1,\ 0$	$0,\ 0$ $-,\ +$		$t+1,\ 0t+5$ $3t+2,\ -1$	$1.33239612\ldots$
$-1,\ -1$	$1,\ -1$	$0,\ 0$ $+,\ +$		$t+2,\ 0t+5$ $3t+3,\ -1$	$1.33813743\ldots$
$-1,\ -1$	$-1,\ 0$	$0,\ 0$ $\bullet,\ -$	$1,\ 0$	$0t+1,\ t+0$ $0t+2,\ 0$	$1.33999992\ldots$
$-1,\ -1$	$1,\ -1$	$0,\ 0$ $+,\ +$		$t+1,\ 0t+2$ $2t+3,\ -1$	$1.34050688\ldots$
$-1,\ -1$	$1,\ 0$	$0,\ 0$ $-,\ -$		$t+2,\ 0t+5$ $3t+2,\ -1$	$1.34865199\ldots$
$-1,\ 0$	$0,\ -1$	$0,\ 0$ $\bullet,\ -$	$1,\ 1$	$0t+3,\ 0t+2$ $t+0,\ 0$	$1.34971610\ldots$
$-1,\ 0$	$0,\ -1$	$-1,\ 1$ $-,\ +$	$0,\ 1$	$0t+2,\ t+0$ $0t+2,\ 0$	$1.35001483\ldots$
$-1,\ -1$	$1,\ 0$	$0,\ 0$ $-,\ \bullet$		$t+1,\ 0t+5$ $4t+6,\ -1$	$1.35031697\ldots$
$-1,\ -1$	$1,\ 0$	$0,\ 0$ $-,\ \bullet$		$t+2,\ 0t+5$ $4t+2,\ -1$	$1.35114589\ldots$
$-1,\ -1$	$1,\ 0$	$0,\ 0$ $\bullet,\ \bullet$		$4t+4,\ 0t+5$ $5t+1,\ -1$	$1.35246806\ldots$
$-1,\ -1$	$1,\ 0$	$0,\ 0$ $-,\ +$		$t+1,\ 0t+5$ $t+5,\ -1$	$1.35369764\ldots$
$-1,\ -1$	$-1,\ 0$	$-1,\ 0$ $\bullet,\ +$	$-1,\ 0$	$0t+0,\ 2t+0$ $t+2,\ 0$	$1.35674810\ldots$
$-1,\ -1$	$1,\ -1$	$0,\ 0$ $+,\ +$		$3t+2,\ 0t+2$ $5t+1,\ -1$	$1.35678598\ldots$
$-1,\ -1$	$1,\ -1$	$0,\ 0$ $+,\ +$		$3t+2,\ 0t+2$ $5t+4,\ -1$	$1.35854559\ldots$
$-1,\ -1$	$-1,\ 0$	$0,\ 0$ $\bullet,\ +$	$1,\ 0$	$0t+1,\ t+0$ $0t+3,\ 0$	$1.35920806\ldots$
$-1,\ -1$	$1,\ 0$	$0,\ 0$ $\bullet,\ -$		$t+2,\ 0t+5$ $2t+1,\ -1$	$1.35937564\ldots$

ℓ_1, r_1	ℓ_2, r_2	c_ℓ, c_r D_L, D_R	u, d c_m	ℓ, m r, n	Limit point
$-1, -1$	$1, 0$	$0, 0$ $\bullet, \bullet$		$5t+5, 0t+5$ $6t+2, -1$	$1.35981177\ldots$
$-1, -1$	$1, 0$	$0, 0$ $-, \bullet$		$t+1, 0t+5$ $6t+6, -1$	$1.35981589\ldots$
$-1, -1$	$-1, 0$	$0, 0$ $\bullet, +$	$-1, -1$	$0t+1, 0t+7$ $t+0, 0$	$1.35991414\ldots$
$-1, 0$	$0, -1$	$-1, 1$ $-, +$	$0, -1$ 0	$0t+2, t+0$ $0t+0, 1$	$1.36022084\ldots$
$-1, -1$	$-1, 0$	$0, 0$ $\bullet, +$	$1, 0$	$0t+0, t+7$ $3t-3, 0$	$1.36195645\ldots$
$-1, -1$	$1, -1$	$0, 0$ $+, +$		$t+1, 0t+2$ $5t+1, -1$	$1.36272428\ldots$
$-1, -1$	$1, 0$	$0, 0$ $-, +$		$5t+6, 0t+5$ $3t+1, -1$	$1.36365149\ldots$
$-1, -1$	$1, 0$	$0, 0$ $\bullet, -$		$2t+2, 0t+5$ $3t+1, -1$	$1.36419954\ldots$
$-1, -1$	$1, -1$	$0, 0$ $+, +$		$t+1, 0t+2$ $2t+2, -1$	$1.36443581\ldots$
$-1, 0$	$0, -1$	$-1, 1$ $-, +$	$0, -1$ 0	$0t+4, t+0$ $0t+0, 1$	$1.36465572\ldots$
$-1, 0$	$0, -1$	$-1, -1$ $+, +$		$0t+4, t-2$ $t+1, t+3$	$1.36506231\ldots$
$-1, -1$	$1, -1$	$0, 0$ $\bullet, -1$		$4t+4, 0t+2$ $5t+2, -1$	$1.36526954\ldots$
$-1, 0$	$0, -1$	$-1, 1$ $\bullet, +$	$-1, -1$	$2t-1, 0t+5$ $t+1, 0$	$1.36546873\ldots$
$-1, 0$	$0, -1$	$-1, 1$ $-, +$	$0, -1$ 0	$0t+0, t+0$ $0t+0, 1$	$1.36614596\ldots$
$-1, -1$	$1, -1$	$0, 0$ $+, +$		$t+1, 0t+2$ $5t+4, -1$	$1.36629907\ldots$
$-1, -1$	$1, 0$	$0, 0$ $-, \bullet$		$3t+1, 0t+5$ $2t+1, -1$	$1.36640199\ldots$
$-1, -1$	$1, 0$	$0, 0$ $\bullet, -1$		$2t+1, 0t+5$ $t+2, -1$	$1.36643553\ldots$
$-1, 0$	$0, -1$	$-1, 1$ $\bullet, +$	$0, -1$ 0	$0t+0, t+0$ $0t+0, 1$	$1.36657097\ldots$
$-1, -1$	$1, -1$	$0, 0$ $+, +$		$2t+3, 0t+2$ $3t+5, -1$	$1.36680788\ldots$
$-1, -1$	$1, -1$	$0, 0$ $+, +$		$t+2, 0t+6$ $t+3, -1$	$1.36688307\ldots$
$-1, 0$	$0, -1$	$-1, -1$ $-, \bullet$		$0t+3, t+1$ $0t+6, 0t+6$	$1.36699091\ldots$

ℓ_1, r_1	ℓ_2, r_2	c_ℓ, c_r D_L, D_R	u, d c_m	ℓ, m r, n	Limit point
-1, -1	1, 0	0, 0 $-$, $\bullet$		$t+2$, $0t+5$ $4t+3$, -1	$1.36751103\ldots$
-1, 0	0, -1	0, 0 $-$, $-$	-1, -1 0	$t+3$, $t-3$ $0t+6$, 1	$1.36779885\ldots$
-1, 0	0, -1	-1, 1 $-$, $\bullet$	-1, -1	$t+0$, $0t+8$ $0t+0$, 0	$1.36785463\ldots$
-1, 1	1, 0	1, 1 $-$, $\bullet$	1, 1	$0t+2$, $0t+1$ $t+0$, 0	$1.36813222\ldots$
-1, 0	0, -1	0, 0 $\bullet$, $+$	-1, -1	$t+3$, $2t-2$ $0t+8$, 0	$1.36819625\ldots$
-1, -1	-1, 0	0, 1 $\bullet$, $+$	-1, -1	$0t+0$, $t+0$ $0t+3$, 0	$1.36821400\ldots$
-1, 0	0, -1	1, 0 $>$, $<$	0, -1 -1	$0t+5$, $t+0$ $0t+2$, 1	$1.36834343\ldots$
-1, -1	1, -1	0, 0 $+$, $+$		$3t+1$, $0t+2$ $5t+3$, -1	$1.36839671\ldots$
-1, 0	0, -1	0, 0 $-$, $\bullet$	1, 1	$0t+5$, $2t+0$ $t+3$, 0	$1.36874744\ldots$
-1, -1	1, 0	0, 0 $\bullet$, $-$		$3t+2$, $0t+5$ $2t+1$, -1	$1.36892221\ldots$
-1, -1	1, 0	0, 0 $-$, $+$		$t+3$, $0t+5$ $3t+1$, -1	$1.36897873\ldots$
-1, 0	0, -1	-1, -1 $+$, $<$		$t+0$, $0t+2$ $3t+2$, $2t+1$	$1.36948937\cdots$
-1, 0	0, -1	-1, 1 $+$, $<$	0, -1 0	$0t+0$, $t+1$ $4t-1$, 1	$1.36978231\cdots$

Table 1: Small limits of sequences of Mahler measures from families of digraphs

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