Sums of Reciprocals of Fractional Parts and Aspects of Multiplicative Diophantine Approximation

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Declaration

These doctoral studies were conducted under the supervision of Dr. Martin Widmer.

The work presented in this thesis is the result of original research carried out by myself, in collaboration with others, whilst enrolled in the Department of Mathematics as a candidate for the degree of Doctor of Philosophy. This work has not been submitted for any other degree or award in any other university or educational establishment.

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[1] I am aware that this is more than one word.

Abstract

Motivated by a question of Lê and Vaaler, we study upper bounds and asymptotic estimates for sums of reciprocals of fractional parts in the multi-dimensional setting. These sums often occur in Diophantine approximation, and a better understanding of their growth is of key importance in multiple instances.

We obtain upper bounds by reducing the problem to counting lattice points in certain subsets of $\mathbb{R}^n$. These subsets are not easily treated because of their highly distorted shape and the presence of "hyperbolic spikes". We develop an older partitioning method of Widmer to prove a sophisticated counting result for weakly admissible lattices and sets with "hyperbolic spikes". This counting result is the core of our thesis.

To apply our lattice-point counting result in the context of sums of reciprocals, we need to introduce weaker types of multiplicative bad approximability for matrices, where the decay of the product of fractional parts is controlled by a non-increasing function. We use this function to make our bounds for sums of reciprocals explicit.

This raises questions on the existence of matrices with a prescribed type of multiplicative bad approximability (in terms of the decay of the non-increasing function). In fact, it is well-known that classical mutiplicatively badly approximable matrices are unlikely to exist, in connection with the Littlewood conjecture. We prove the existence of matrices with a logarithmically decaying type of bad approximability, by using an adaptation of a Cantor-type set construction introduced by Badziahin and Velani. Matrices of this type have very low sums of reciprocals in view of our previous estimate, and hence, provide a partial answer to Lê and Vaaler's question.

Our bounds for sums of reciprocals can also be used to produce new examples of Khintchine and strong Khintchine type vector subspaces of the Euclidean space, thus improving on certain results of Huang and Liu.

The final chapter of this thesis is devoted to the solution of a problem posed by Lambert A' Campo on the boundedness of certain exponential sums. We use a theorem of Duffin and Schaeffer to settle the problem and we further investigate conditions under which such boundedness is achieved. This involves the use of Cantor-type set constructions somewhat similar to those of Badziahin and Velani.

Contents

Introduction

In this thesis we prove a precise and general lattice-point counting result for weakly admissible lattices and subsets of $\mathbb{R}^n$ that involve "hyperbolic spikes". Such lattices and sets naturally occur in multiplicative Diophantine approximation.

We apply our counting result to establish new and nearly sharp upper bounds for sums of reciprocals of fractional parts and, as a consequence, make significant progress on a problem raised by Lê and Vaaler. As a second application, we establish the existence of new Khintchine-type subspaces, extending on results of Huang and Liu. These results in turn provide new cases for the generalised Baker-Schmidt problem.

Motivated by the technical hypothesis in our lattice-point counting result, we also establish the existence of multiplicatively badly approximable matrices of a certain type. This result generalises the work of Moshchevitin from 2 to arbitrary dimension. However, here we use different methods to those of Moshchevitin, that can be traced back to Badziahin and Velani.

Finally, partially inspired by the work of Badziahin and Velani, we solve a problem proposed by Lambert A'Campo on the boundedness of certain exponential sums.

1 Sums of reciprocals

Let $\alpha \in \mathbb{R}$ be irrational and let

$$S_\alpha(Q) := \sum_{q=1}^{Q} \frac{1}{\|\alpha q\|}, \tag{1}$$

where $\|x\|$ stands for the distance from $x \in \mathbb{R}$ to the nearest integer. Sums of this form are known as sums of reciprocals of fractional parts, and appear in different areas of number theory. In particular, it is often the case that one needs to find "workable" functions $l_\alpha, u_\alpha : \mathbb{N} \to (0, +\infty)$

such that

$$l_\alpha(Q) \leq S_\alpha(Q) \leq u_\alpha(Q)$$

for all $Q \in \mathbb{N}$. Bounds for sums of reciprocals have a wide range of applications, e.g., in metric Diophantine approximation, lattice-point counting, and uniform distribution theory (see [5],[9],[17],[18],[26],[28],[39]).

Nowadays, optimal choices for the functions l_α and u_α are known. In particular, several different results have been obtained by imposing conditions on the Diophantine type of the number α. Walfisz [39, Hilfssatz 11] showed that for any given $\varepsilon > 0$ we have

$$S_\alpha(Q) \ll_{\alpha,\varepsilon} Q(\log Q)^{1+\varepsilon}$$

for almost all $\alpha \in \mathbb{R}$. Hardy and Littlewood [17],[18] showed that if α is badly approximable, then

$$Q \log Q \ll_\alpha S_\alpha(Q) \ll_\alpha Q \log Q, \tag{2}$$

although the upper bound was already known to Chowla [9]. Lang [28, Chapter III, Theorem 2] proved that if α is of principal cotype[2] $\leq f$, then

$$S_\alpha(Q) \ll Q \log Q + Qf(Q)$$

for all $Q \in \mathbb{N}$. An unconditional lower bound for $S_\alpha(Q)$ was eventually proved by Lê and Vaaler. In [29, Theorem 1.1] they showed that

$$S_\alpha(Q) \gg Q \log Q \tag{3}$$

for all irrational values of $\alpha \in \mathbb{R}$. They also found large sets of numbers α for which inequality (3) is sharp. Finally, very precise estimates (involving the denominators of the principal convergents of α) can also be found in the earlier work of Kruse [26, Equations (75),(76)].

For state-of-the-art estimates on sums of reciprocals and further references, the reader may consult the work of Beresnevich, Haynes, and Velani [5, Chapter 1], where multi-dimensional versions of sums of reciprocals are also considered.

Arguably, the authors who first investigated the multidimensional case were Lê and Vaaler. The starting point of this thesis is precisely Lê and Vaaler's [29, Theorem 1.1], which provides lower

[2]Let $f : [1, +\infty) \to [1, +\infty)$. We say that $\alpha \in \mathbb{R}$ is of principal cotype $\leq f$ if for all large enough integers B there exists a principal convergent p_i/q_i of α such that $B < q_i \leq Bf(B)$.

bounds for general multidimensional sums of reciprocals. For a matrix $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ and a box $X_{\boldsymbol{Q}} := \prod_{j=1}^{\mathcal{N}}[-Q_j, Q_j]$ in $\mathbb{R}^{\mathcal{N}}$, these sums take the shape

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) := \sum_{\boldsymbol{q}\in X_{\boldsymbol{Q}}\cap\mathbb{Z}^{\mathcal{N}}\setminus\{\boldsymbol{0}\}} \prod_{i=1}^{\mathcal{M}} \frac{1}{\|L_i\boldsymbol{q}\|},$$

where $L_i\boldsymbol{q}$ stands for $\sum_{j=1}^{\mathcal{N}} L_{ij}q_j$. Note that for the sum to be well-defined, we need to assume that the entries L_{ij} of each row L_i, along with 1, are linearly independent over $\mathbb{Z}$ for $i = 1,\ldots,\mathcal{M}$. Lê and Vaaler prove [29, Theorem 1.1] that independently of $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ we have

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \gg_{\mathcal{M},\mathcal{N}} Q^{\mathcal{N}} \log(Q)^{\mathcal{M}},$$

where $Q := (Q_1\cdots Q_{\mathcal{N}})^{1/\mathcal{N}}$.

They also ask if this bound is sharp (see [29, (1.14) and (1.15)]).

Question 1.1 (Lê and Vaaler)**.** Are there matrices $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ such that

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{\mathcal{M},\mathcal{N}} Q^{\mathcal{N}} \log(Q)^{\mathcal{M}} \tag{4}$$

for all $\boldsymbol{Q}$, where $Q := (Q_1\cdots Q_{\mathcal{N}})^{1/\mathcal{N}}$?

In analogy with Hardy and Littlewood's result for the one-dimensional case [17],[18], Lê and Vaaler prove that the bound in (4) holds for multiplicatively badly approximable matrices [29, Theorem 2.1]. However, for $\mathcal{M}+\mathcal{N} \geq 3$ each such matrix would provide a counterexample to the famous Littlewood conjecture (see [7]), and hence, is unlikely to exist. Therefore, [29, Theorem 2.1] does not fully answer Question 1.1.

In Chapter I we improve on a very special case of [29, Theorem 2.1]. For $\mathcal{M} = 1$ and $Q_1 = \cdots = Q_{\mathcal{N}}$ we give upper bounds for $S_{\boldsymbol{\alpha}}(\boldsymbol{Q})$ in terms of the Diophantine type of the vector[3] $\boldsymbol{\alpha} \in \mathbb{R}^{\mathcal{N}}$. We introduce a non-increasing function $\phi : [1,+\infty) \to (0,1]$ and we say that a vector $\boldsymbol{\alpha}$ is ϕ-badly approximable if

$$|\boldsymbol{q}|_\infty^{\mathcal{N}}\|\boldsymbol{q}\cdot\boldsymbol{\alpha}\| \geq \phi(|\boldsymbol{q}|_\infty) \tag{5}$$

for all $\boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}}$. For such vectors, the following result holds.

Theorem 1.2. *Let $\phi : [1,+\infty) \to (0,1]$ be a non-increasing function and let $\boldsymbol{\alpha} \in \mathbb{R}^{1\times\mathcal{N}}$ be a ϕ-badly approximable vector. Then, we have*

$$Q^{\mathcal{N}} \log(Q\phi(Q)) \ll_{\mathcal{N}} \sum_{\boldsymbol{q}\in[-Q,Q]^{\mathcal{N}}\cap\mathbb{Z}^{\mathcal{N}}\setminus\{\boldsymbol{0}\}} \frac{1}{\|\boldsymbol{\alpha}\cdot\boldsymbol{q}\|} \ll_{\mathcal{N}} Q^{\mathcal{N}}\log(Q) + \frac{Q^{\mathcal{N}}}{\phi(Q)}$$

[3] We use the word vector here for a $1\times\mathcal{N}$ matrix.

for all $Q \geq 1$.

This shows that if $\boldsymbol{\alpha} \in \mathbb{R}^{\mathcal{N}}$ is ϕ-badly approximable, with $\phi(x) \gg \log(x)^{-1}$, then $S_{\boldsymbol{\alpha}}(\boldsymbol{Q})$ satisfies (4). Condition (5) with $\phi(x) \gg \log(x)^{-1}$ is much weaker compared to multiplicative bad approximability.

We remark that Theorem 1.2 follows form a lattice-point counting result of Widmer for generalised aligned boxes [41, Theorem 1.2].

Another interesting improvement on [29, Theorem 2.1] was made by Widmer himself [40, Corollary 1.2]. For $\mathcal{M} = 2$, $\mathcal{N} = 1$ he relaxed the multiplicative bad approximability condition, showing that, if for some real numbers α, β we have

$$q\|\alpha q\|\|\beta q\| \geq \phi(|q|) > 0 \tag{6}$$

for all $q \in \mathbb{Z} \setminus \{0\}$ (ϕ non-increasing), then

$$\sum_{q=1}^{\lfloor Q \rfloor} \frac{1}{\|\alpha q\|\|\beta q\|} \ll Q \log\left(\frac{Q}{\phi(Q)}\right)^2 + \frac{Q}{\phi(Q)} \log\left(\frac{Q}{\phi(Q)}\right)$$

for all $Q \geq 1$.

Now, if $\phi(x) \gg \log(x)^{-1}$, the vector (α, β) satisfies (4), and multiplicative bad appoximability is again no longer a necessary condition.

A similar argument can be applied for general values of $\mathcal{M}$ and $\mathcal{N}$, provided one generalises (6) to arbitrary matrices $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M} \times \mathcal{N}}$. We say that a matrix $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M} \times \mathcal{N}}$ is ϕ-multiplicatively badly approximable if for some non-increasing function ϕ it holds

$$\prod_{j=1}^{\mathcal{N}} \max\{1, |q_j|\} \prod_{i=1}^{\mathcal{M}} \|L_i \boldsymbol{q}\| \geq \phi\left(\prod_{j=1}^{\mathcal{N}} \max\{1, |q_j|\}^{1/\mathcal{N}}\right) \tag{7}$$

for all $\boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}$. If ϕ is a constant function, this reduces to usual multiplicative bad approximability.

In this thesis we show that, with this general definition, Widmer's result can be extended to arbitrary dimension matrices under the same symmetry assumptions that we made in Theorem 1.2. In particular, we consider sums $S_{\boldsymbol{L}}(\boldsymbol{Q})$ where $Q_1 = \cdots = Q_{\mathcal{N}}$, i.e., sums over a symmetric counting domain $X_{\boldsymbol{Q}}$. This is necessary in order to apply the counting results that we develop in Chapter II, which generalise Widmer's [41, Theorem 1.2].

As in the case $\mathcal{M} = 1$, these symmetry assumptions allow us to weaken the multiplicative bad approximability hypothesis of [29, Theorem 2.1]. However, for $\mathcal{N} > 1$ we do not just let the product of the fractional parts in (7) decay (ϕ-multiplicative bad approximability), but we additionally weaken the multiplicative bad approximability condition by introducing a new condition, that we call ϕ-semimultiplicative bad approximability. For $\mathcal{N} = 1$ this condition coincides with ϕ-multiplicative bad approximability, whereas for $\mathcal{M} = 1$ it coincides with ϕ-bad approximability (see Theorem 1.2). More precisely, we say that a matrix $\boldsymbol{L}$ is ϕ-semimultiplicatively badly approximable if in (7) the product $\prod_{j=1}^{\mathcal{N}} \max\{1, |q_j|\}$ can be replaced by the larger quantity $|\boldsymbol{q}|_\infty^{\mathcal{N}}$.

Our generalisation of [40, Corollary 1.2] to the case of arbitrary $\mathcal{M}$ and $\mathcal{N}$ can be found in Chapter II, Corollary 1.8, and states the following.

Theorem 1.3. *Let $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ be a ϕ-semimultiplicatively badly approximable matrix. Then, we have*

$$\sum_{\substack{\boldsymbol{q}\in[-Q,Q]^{\mathcal{N}} \\ \cap\mathbb{Z}^{\mathcal{N}}\setminus\{\boldsymbol{0}\}}} \prod_{i=1}^{\mathcal{M}} \frac{1}{\|L_i\boldsymbol{q}\|} \ll_{\mathcal{M},\mathcal{N}} Q^{\mathcal{N}} \log\left(\frac{Q^{\mathcal{N}}}{\phi(Q)}\right)^{\mathcal{M}} + \frac{Q^{\mathcal{N}}}{\phi(Q)} \log\left(\frac{Q^{\mathcal{N}}}{\phi(Q)}\right)^{\mathcal{M}-1}$$

for all $Q \geq 2$.

Theorem 1.3 implies that any ϕ-semimultiplicatively badly approximable matrix $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$, with $\phi(x) \gg_{\mathcal{M},\mathcal{N}} \log(x)^{-1}$, satisfies (4) in the case $Q_1 = \cdots = Q_{\mathcal{N}}$.

In Section 3 of the Introduction we will discuss an application of Theorem 1.3 to identify new Khintchine-type subspaces.

It is now reasonable to ask for which choices of the function ϕ we can guarantee the existence of ϕ-multiplicatively or ϕ-semimultiplicatively badly approximable matrices, and whether ϕ can be chosen to grow more slowly than $\log(x)^{-1}$. This we treat in Chapter III.

For simplicity, let us assume that $\mathcal{M} = 2$ and $\mathcal{N} = 1$. For $\lambda \geq 0$ we consider the set

$$\mathrm{Mad}^{\lambda}(2,1) := \left\{(\alpha,\beta)\in\mathbb{R}^2 : \liminf_{q\to+\infty} q(\log q)^{\lambda}\|q\alpha\|\|q\beta\| > 0\right\}.$$

Then, any vector in $\mathrm{Mad}^1(2,1)$ is ϕ-multiplicatively (but also ϕ-semimultiplicatively) badly approximable with $\phi(x) \gg_{\alpha,\beta} (\log x)^{-1}$, and thus, satisfies (4) by Widmer's [40, Corollary 2]. To answer Question 1.1, we are therefore interested in showing that $\mathrm{Mad}^\lambda(2,1)$ is non-empty.

Unfortunately, we know very little about the set $\mathrm{Mad}^1(2,1)$. By Gallagher's Theorem [16], we have

$$\mathscr{L}\left(\mathrm{Mad}^\lambda(2,1)\right) = \begin{cases} 0 & \text{if } \lambda \leq 2 \\ 1 & \text{if } \lambda > 2 \end{cases} \quad ,$$

where $\mathscr{L}$ stands for the 2-dimensional Lebesgue measure. For $\lambda \leq 2$ the set $\mathrm{Mad}^\lambda(2,1)$ could however be empty.

Motivated by volume arguments, originally proposed by Pollington and Velani, and by comparing the measure-theoretic properties of $\mathrm{Mad}^\lambda(2,1)$ with those of its additive analogue, i.e., the set

$$\mathrm{Bad}^\lambda(2,1) := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \liminf_{q\to+\infty} q(\log q)^\lambda \max\{\|q\alpha\|,\|q\beta\|\}^2 > 0\right\},$$

Badziahin and Velani made the following conjecture [2, Statements L1-L3].

Conjecture 1.4 (Badziahin-Velani)**.**

$$\mathrm{Mad}^\lambda(2,1) = \begin{cases} \emptyset & \text{if } \lambda < 1 \\ \text{full Hausdorff dimension but zero Lebesgue measure set} & \text{if } 1 \leq \lambda \leq 2 \, . \\ \text{full Lebesgue measure set} & \text{if } \lambda > 2 \end{cases}$$

The cases $\lambda > 1$ of this conjecture have been settled, whereas for $\lambda \leq 1$ the conjecture is still open (note that the case $\lambda = 0$ corresponds to the Littlewood Conjecture). Initial progress was made by Moshchevitin [31]. By using the Peres-Schlag method, Moshchevitin showed that $\mathrm{Mad}^2(2,1) \neq \emptyset$. Subsequently, Moshchevitin and Bugeaud [8] showed that $\mathrm{Mad}^2(2,1)$ has full Hausdorff dimension. Finally, Badziahin [1] proved that $\mathrm{Mad}^\lambda(2,1)$ has full Hausdorff dimension for $\lambda > 1$. The case $\lambda = 1$ is still unsolved.

Here, we do not solve the case $\lambda = 1$, but, in Chapter III, we extend Moshchevitin's result to arbitrary dimension matrices, thus finding some new upper bounds for the sums $S_{\boldsymbol{L}}(\boldsymbol{Q})$ for general values of $\mathcal{M}$ and $\mathcal{N}$ (in view of Theorem 1.3). We also generalise this existence result to the inhomogeneous case. For simplicity, we set

$$\prod(\boldsymbol{q}) := \prod_{j=1}^{\mathcal{N}} \max\{1, |q_j|\}$$

for any $\boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}}$, and we define $\log^*(x) := \log(\max\{e, x\})$ for $x \in [0, +\infty)$, where $e = 2.71828\ldots$ is the base of the natural logarithm. Then, we have the following result, which we state as a corollary (deriving from Chapter III, Proposition 1.5).

Corollary 1.5. *Let $\mathcal{M} + \mathcal{N} \geq 3$. Then, for all $\boldsymbol{\gamma} \in \mathbb{R}^{\mathcal{M}}$ the set*

$$\mathrm{Mad}^{\mathcal{M}+\mathcal{N}-1}(\mathcal{M}, \mathcal{N}, \boldsymbol{\gamma}) := \left\{ \boldsymbol{A} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}} : \liminf_{|\boldsymbol{q}|_\infty \to +\infty} \prod(\boldsymbol{q}) \left(\log \prod(\boldsymbol{q})\right)^{\mathcal{M}+\mathcal{N}-1} \prod_{i=1}^{\mathcal{M}} \|A_i \boldsymbol{q} + \gamma_i\| > 0 \right\}$$

is everywhere dense in $\mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ and does not lie on a countable union of hyperplanes.

It follows from the case $\boldsymbol{\gamma} = \mathbf{0}$ of Corollary 1.5 that the set of $c \log^*(x)^{-(\mathcal{M}+\mathcal{N}-1)}$- multiplicatively badly approximable matrices in $\mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ (c small enough) is always non-empty.

It is worth noting that to prove Corollary 1.5 we do not follow Moshchevitin's method [31], i.e., the Peres-Schlag method. This method relies on estimates of the form

$$\sum_{q=1}^{Q} \frac{1}{q\|q\alpha\|} \ll (\log Q)^2$$

for $\alpha \in \mathbb{R}$. To inductively apply Moshchevitin's argument, we would need estimates such as

$$\sum_{q=1}^{Q} \frac{1}{q\|q\alpha_1\| \cdots \|q\alpha_{\mathcal{M}}\|} \ll_{\mathcal{M}} (\log Q)^{\mathcal{M}}$$

for vectors $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{\mathcal{M}}) \in \mathbb{R}^{\mathcal{M}}$. However, as opposed to the case $\mathcal{M} = 1$, these estimates are not available for $\mathcal{M} > 1$, unless the matrix $(\alpha_1, \ldots, \alpha_{\mathcal{M}})$ is multiplicatively badly approximable. Since multiplicatively badly approximable matrices are not known to exist, a different approach is required. Instead of using induction, we work directly in higher dimension. The main tool to prove Corollary 1.5 is a simple generalisation of a method introduced by Badziahin and Velani in [2]. We recursively construct a Cantor-like set contained in $\mathrm{Mad}_{\mathcal{M},\mathcal{N}}(\log^*(x)^{\mathcal{M}+\mathcal{N}-1}, \boldsymbol{\gamma}, c)$ and we prove that this set is non-empty for small enough values of c. For more details we refer the reader to Chapter III.

Corollary 1.5, along with Theorem 1.3, yields the following.

Corollary 1.6. *For any $\mathcal{M}, \mathcal{N} \in \mathbb{N}$ there exist uncountably many matrices $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ such that*

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{\mathcal{M},\mathcal{N}} Q^{\mathcal{N}} (\log Q)^{2\mathcal{M}+\mathcal{N}-2}$$

for all $\boldsymbol{Q} = (Q, \ldots, Q) \in \mathbb{R}^{\mathcal{N}}$ with $Q \geq 2$.

Corollary 1.6 does not answer Lê and Vaaler's question, but it does provide new upper bounds for generalised sums of reciprocals.

2 Weak admissibility and lattice-point counting

In this subsection we give a quick insight into the main tools used to prove Theorem 1.3. The proof of this result relies on a general counting theorem for weakly admissible lattices and sets with "hyperbolic spikes" that can be found in Chapter II.

Let $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ be an $\mathcal{M} \times \mathcal{N}$ matrix. Let $\varepsilon, T \in (0, +\infty)$ and let $Q \in [1, +\infty)$. We consider the set

$$M(\boldsymbol{L}, \varepsilon, T, Q) := \Bigg\{ (\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^{\mathcal{M}} \times \left(\mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}\right) : \prod_{i=1}^{\mathcal{M}} |L_i \boldsymbol{q} + p_i| < \varepsilon, \\ |L_i \boldsymbol{q} + p_i| \leq T,\ i = 1, \dots, \mathcal{M},\ |q_j| \leq Q,\ j = 1, \dots, \mathcal{N} \Bigg\},$$

where $L_i \boldsymbol{q} := \sum_{j=1}^{\mathcal{N}} L_{ij} q_j$. To prove Theorem 1.3 we need a precise asymptotic estimate for the cardinality of the set $M(\boldsymbol{L}, \varepsilon, T, Q)$. Assuming that for some non-increasing function $\phi : [0, +\infty) \to (0, 1]$ the matrix $\boldsymbol{L}$ is ϕ-semimultiplicatively badly approximable, we can estimate $\# M(\boldsymbol{L}, \varepsilon, T, Q)$ in terms of ε, T, Q and ϕ. More precisely, we have the following asymptotic result.

Proposition 2.1. *Let $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ be a ϕ-semimultiplicatively badly approximable matrix and suppose that $T^{\mathcal{M}}/\varepsilon \geq e^{\mathcal{M}}$, where $e = 2,71828\cdots$ is the base of the natural logarithm. Then,*

$$|\# M(\boldsymbol{L}, \varepsilon, T, Q) - \mathscr{V}| \ll_{\mathcal{M},\mathcal{N}} (1 + T)^{\mathcal{M}+\mathcal{N}-1} \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} \left(\frac{\varepsilon Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{\mathcal{M}+\mathcal{N}-1}{\mathcal{M}+\mathcal{N}}},$$

where

$$\mathscr{V} = 2^{\mathcal{M}+\mathcal{N}} Q^{\mathcal{N}} \left[\varepsilon \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} + T^{\mathcal{M}} \left(1 - \left(1 - \frac{\varepsilon}{T^{\mathcal{M}}}\right)^{\mathcal{M}-1}\right)\right].$$

Estimating $\# M(\boldsymbol{L}, \varepsilon, T, Q)$ can be reduced to a lattice-point counting problem. We show how in the next few lines. Consider the lattice

$$\Lambda_{\boldsymbol{L}} := \left\{ (L_1 \boldsymbol{q} + p_1, \dots, L_{\mathcal{M}} \boldsymbol{q} + p_{\mathcal{M}}, \boldsymbol{q}) : \boldsymbol{p} \in \mathbb{Z}^{\mathcal{M}}, \boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}} \right\},$$

and the set

$$Z := \left\{ \boldsymbol{x} \in \mathbb{R}^{\mathcal{M}} : \prod_{i=1}^{\mathcal{M}} |x_i| < \varepsilon,\ |x_i| \leq T,\ i = 1, \ldots, \mathcal{M} \right\} \times [-Q, Q]^{\mathcal{N}}. \tag{8}$$

Then, we have

$$\#M(\boldsymbol{L}, \varepsilon, T, Q) = \#((\Lambda_{\boldsymbol{L}} \cap Z) \setminus C),$$

where $C := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{\mathcal{M}+\mathcal{N}} : \boldsymbol{y} = \boldsymbol{0}\}$. Since $\Lambda_{\boldsymbol{L}} \cap C = \mathbb{Z}^{\mathcal{M}} \times \{\boldsymbol{0}\}$, to estimate $\#M(\boldsymbol{L}, \varepsilon, T, Q)$ it suffices to estimate $\#(\Lambda_{\boldsymbol{L}} \cap Z)$. Hence, we simply need to compute asymptotics for $\#(\Lambda_{\boldsymbol{L}} \cap Z)$.

We know that the matrix $\boldsymbol{L}$ is ϕ-semimultiplicatively badly approximable. Thus, for all $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^{\mathcal{M}} \times \left(\mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}\right)$ we have

$$|\boldsymbol{q}|_2^{\mathcal{N}} \prod_{i=1}^{\mathcal{M}} |L_i \boldsymbol{q} + p_i| \geq |\boldsymbol{q}|_\infty^{\mathcal{N}} \prod_{i=1}^{\mathcal{M}} \|L_i \boldsymbol{q}\| \geq \phi(|\boldsymbol{q}|_\infty) \geq \phi(|\boldsymbol{q}|_2),$$

i.e., for all $(\boldsymbol{x}, \boldsymbol{y}) \in \Lambda_{\boldsymbol{L}}$ with $\boldsymbol{y} \neq \boldsymbol{0}$ we have

$$|\boldsymbol{y}|_2^{\mathcal{N}} \prod_{i=1}^{\mathcal{M}} |x_i| \geq \phi(|(\boldsymbol{x}, \boldsymbol{y})|_2) > 0. \tag{9}$$

This inequality expresses a precise geometric property of the lattice $\Lambda_{\boldsymbol{L}}$, that can be defined for general lattices $\Lambda \subset \mathbb{R}^{\mathcal{M}+\mathcal{N}}$. This property is known as weak admissibility.

The notion of weak admissibility already appears in [36, Section 4]. Skriganov defines a lattice to be weakly admissible when none of its points lie on any of the coordinate subspaces. In [36, Theorem 6.1] he proves a counting result for general polyhedra, assuming that certain "dual" lattices (with respect to the polyhedron) are weakly admissible.

In our context, lattice vectors are allowed to have 0 components. However, in view of (9), certain "subvectors" of the lattice vectors are never null. We therefore need a slightly more general definition of weak admissibility. Such a definition is provided by Widmer in [41, Definition 1]. Inequality (9) corresponds precisely to Widmer's definition of weak admissibility for the lattice $\Lambda_{\boldsymbol{L}} \subset \mathbb{R}^{\mathcal{M}+\mathcal{N}}$.

In [41] Widmer also proves a counting result for weakly admissible lattices and generalised aligned boxes, i.e., subsets of an aligned box whose boundary is parametrisable by a bounded number of Lipschitz maps (with some control over the Lipschitz constant). [41, Theorem 2.1] is the key tool to prove Theorem 1.2, i.e., the case $\mathcal{M} = 1$ of Theorem 1.3. Indeed, for $\mathcal{M} = 1$ the set $M(\boldsymbol{L}, \varepsilon, T, Q)$ is precisely an aligned box.

For $\mathcal{M} > 1$, however, the set $M(\boldsymbol{L}, \varepsilon, T, Q)$ has some "hyperbolic spikes" deriving from the bound on the product of the $\boldsymbol{x}$ components. The estimate provided by [41, Theorem 2.1] is thus useless in this case, as it does not take into account the constant ε appearing in the definition of $M(\boldsymbol{L}, \varepsilon, T, Q)$. We therefore need to develop an analogue of this result for sets with this new hyperbolic shape. This is done in Chapter II. More specifically, Theorem 1.5 in Chapter II is a very general lattice-point counting estimate for weakly admissible lattices and sets with "hyperbolic spikes", and constitutes the core of our thesis. We derive Proposition 2.1 as a corollary of this result.

As with generalised aligned boxes in [41, Theorem 2.1], we do not just want our result to hold for sets with "hyperbolic spikes", but for all "structured enough" subsets of such sets. Instead of using parametrisability by Lipschitz maps, to make this assertion precise, we use *o*-minimality.

The concept of *o*-minimal expansions of the real numbers was first introduced by van den Dries [38] in the context of mathematical logic. The term *o*-minimality however does not explicitly appear in his work. The full theory of these structures was subsequently developed by Knight, Pillay, and Steinhorn [24],[32]. In essence, *o*-minimality is a very general language that captures certain topological properties of subsets of $\mathbb{R}^n$ (see also [43] for a general overview). In Theorem 1.5 (Chapter II) we assume that the counting domain we work with is a definable set in an *o*-minimal structure expanding Wilkie's $\mathbb{R}_{\exp}$ [42]. Roughly speaking, this means that our counting domain is allowed to have "exponentially" defined components. This is essential, since the set Z itself has to be definable, and its "hyperbolic spikes" are "exponentially" defined.

For the sake of clarity, we briefly explain how the proof of Theorem 1.5 (Chapter II) is structured in the case of the set Z itself. The geometry of the set Z is distorted and nothing can be said for arbitrary lattices. To tackle the problem, we split Z into "slices" and we apply a different diagonal linear map to each slice obtaining ball-like shaped sets. We then count the points of the corresponding transformation of $\Lambda_{\boldsymbol{L}}$ lying in each of these ball-like shaped sets. To perform the counting within each such set, we use a theorem of Barroero and Widmer [3, Theorem 1.3]. This theorem simply requires that the counting domain is definable in some *o*-minimal structure.

[3, Theorem 1.3] allows us to give sufficiently good estimates for the number of lattice points contained in each ball-like shaped set. The estimates are in terms of the successive minima of the transformed lattices. Hence, a crucial part of the proof is finding lower bounds for the first successive minima of these lattices. Each lattice is transformed in a different way, and we need the transformed lattices to be somehow "controllable". This is precisely the step where weak admissibility comes into play. For further details, the reader can check Section 2 in Chapter II.

3 Khintchine-type subspaces

In the following subsection we describe an application of Theorem 1.3.

Let $\psi : [1, +\infty) \to (0, 1]$, let $\mathcal{N} \in \mathbb{N}$, and let

$$\mathscr{S}_{\mathcal{N}}(\psi) := \left\{ \boldsymbol{x} \in [0,1]^{\mathcal{N}} : \exists \text{ i.m. } q \in \mathbb{N} \text{ such that } \max_{i=1}^{\mathcal{N}} \|qx_i\| < \psi(q) \right\},$$

where i.m. stands for infinitely many. The set $\mathscr{S}_{\mathcal{N}}(\psi)$ is said to be the set of ψ-well approximable points. Khintchine [21] showed that a precise 0-1 law holds for the set $\mathscr{S}_{\mathcal{N}}(\psi)$ when the function ψ is non-increasing. Namely, he showed that

$$\mathscr{L}(\mathscr{S}_{\mathcal{N}}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{+\infty} \psi(q)^{\mathcal{N}} < +\infty \\ 1 & \text{if } \sum_{q=1}^{+\infty} \psi(q)^{\mathcal{N}} = +\infty, \end{cases}$$

where $\mathscr{L}$ stands for Lebesgue measure.

It is well-known this law is no longer valid on arbitrary embedded manifolds of $\mathbb{R}^{\mathcal{N}}$ such as proper rational affine subspaces [20, p.1]. Hence, the notion of Khintchine-type manifold becomes natural, i.e., a manifold for which Khintchine's law is still valid with respect to the "local" measure on the manifold. If the assumption that the function ψ is non-increasing can be dropped, we say that the manifold is of strong Khintchine type.

The theory of Khintchine-type manifolds has been developed by multiple authors. The case of non-degenerate manifolds (non-zero curvature at each point) is pretty well understood in the convergence case (see [6],[19],[35]). However, the non-degeneracy condition is not strictly necessary for a manifold to be of Khintchine type. Indeed, certain proper vector subspaces of $\mathbb{R}^{\mathcal{N}}$ have also been shown to be of Khintchine type for convergence [25],[33].

In [20], Huang and Liu investigate precisely which proper affine subspaces of $\mathbb{R}^{\mathcal{N}}$ are of (strong) Khintchine type for convergence. They consider a general affine subspace $\mathscr{H}(\boldsymbol{A}, \boldsymbol{b})$ of $\mathbb{R}^{\mathcal{N}}$ of dimension d, defined[4] by the matrix $\boldsymbol{A} \in \mathbb{R}^{d \times (\mathcal{N}-d)}$ and the vector $\boldsymbol{b} \in \mathbb{R}^{\mathcal{N}}$, and they draw a

[4]More precisely, $\mathscr{H}(\boldsymbol{A}, \boldsymbol{b}) := \left\{ \boldsymbol{x} \in \mathbb{R}^{\mathcal{N}} : (\boldsymbol{A}, \boldsymbol{b})(\boldsymbol{x}, 1) = 0 \right\}$, where $(\boldsymbol{A}, \boldsymbol{b}) \in \mathbb{R}^{\mathcal{M} \times (\mathcal{N}+1)}$ and $(\boldsymbol{x}, 1) \in \mathbb{R}^{\mathcal{N}+1}$.

connection between the Khintchine-type properties of $\mathscr{H}$ in the convergence case and certain Diophantine exponents of the matrices $\boldsymbol{A}$ and $(\boldsymbol{A}, \boldsymbol{b})$ [20, Theorem 1].

This connection is not new. For instance, Kleinbock [22],[23] had already studied extremal subspaces of $\mathbb{R}^{\mathcal{N}}$ (i.e., subspaces $\mathscr{H}$ such that $\mathrm{Vol}_d(\mathscr{H} \cap \mathscr{S}_{\mathcal{N}}(q^{-\nu})) = 0$ for specified values of ν), in terms of certain Diophantine exponents related to $\mathscr{H}$. Huang and Liu were however the first authors to produce examples of Khintchine-type subspaces for the convergence case in full generality (the difference here being that the function ψ is not necessarily of the form $q^{-\nu}$).

In Chapter II we improve on certain results of Huang and Liu by using Theorem 1.3. Huang and Liu [20, Theorem 1] show that

i) if $\omega_{sm}(\boldsymbol{A}, \boldsymbol{b}) < \mathcal{N}(d+1)$, the subspace $\mathscr{H}(\boldsymbol{A}, \boldsymbol{b})$ is of Khintchine type for convergence;

ii) if $\omega_{sm}(\boldsymbol{A}) < \mathcal{N}d$, the subspace $\mathscr{H}(\boldsymbol{A}, \boldsymbol{b})$ is of strong Khintchine type for convergence.

Here $\omega_{sm}(\boldsymbol{A})$ is the semimultiplicative exponent of the matrix $\boldsymbol{A}$ (see Definition 1.10 in Chapter II).

By means of Theorem 1.3 we can include the limit cases in their result, provided some additional restrictions hold. To do so, we need to introduce a refined semimultiplicative exponent that takes into account logarithmic factors. We denote this exponent by $\omega'_{sm}(\boldsymbol{A}, \omega_0)$ and we call it the semimultiplicative logarithmic exponent of the matrix $\boldsymbol{A}$ at ω_0.

Then, the following is a corollary of Theorem 1.3.

Corollary 3.1. *Let $\boldsymbol{A}$ and $\boldsymbol{b}$ be as above. Then,*

i) *if $\omega_{sm}(\boldsymbol{A}, \boldsymbol{b}) = \mathcal{N}(d+1)$ and $\omega'_{sm}((\boldsymbol{A}, \boldsymbol{b}), \mathcal{N}(d+1)) < 1 - 2(d+1)$, the subspace $\mathscr{H}(\boldsymbol{A}, \boldsymbol{b})$ is of Khintchine type for convergence;*

ii) *if $\omega_{sm}(\boldsymbol{A}) = \mathcal{N}d$ and $\omega'_{sm}(\boldsymbol{A}, \mathcal{N}d) < 1 - 2d$, the subspace $\mathscr{H}(\boldsymbol{A}, \boldsymbol{b})$ is of strong Khintchine type for convergence.*

Unfortunately, not much is known about matrices of prescribed semimultiplicative logarithmic order. In the very special case $d = 1$, we can deduce from [4, Theorem 1] that the set of matrices $\boldsymbol{A} \in \mathbb{R}^{1\times(\mathcal{N}-1)}$ with $\omega'_{sm}(\boldsymbol{A}, \mathcal{N}) = \omega_1$ for a fixed ω_1, has full Hausdorff dimension. This improves on [25, Theorem 1]. However, the techniques currently available are not powerful enough to prove the existence of matrices with prescribed semimultiplicative logarithmic order for $d > 1$. Very little is known also for higher values of ω_{sm} in terms of Khintchine-type properties.

We recall that Huang and Liu also investigate Khintchine-type conditions for subspaces with respect to the Hausdorff dimension (Baker-Schmidt problem) [20, Theorems 2 and 3]. In Chapter II we extend these results to the limit cases as for [20, Theorem 1]. For more details, we refer the reader to the introduction in Chapter II.

It is also worth noting that the divergence case in Khintchine theory for subspaces has equally widely been studied. For a deeper insight, the reader can consult [34].

4 More on Cantor-like sets and applications to exponential sums

As previously mentioned, Badziahin and Velani [2] studied two-dimensional badly approximable vectors in the multiplicative setting, both in the classical and mixed senses. We drew inspiration from their techniques to generalise a result of Moshchevitin (Corollary 1.5).

Their approach is essentially based on Cantor-like sets constructions. Badziahin and Velani also proved some measure-theoretic results for Cantor-like sets [2, Theorem 4]. They used methods from [11] to give lower bounds for the Hausdorff dimension of these sets, such as the mass distribution principle. We use similar methods to address a problem on bounded exponential sums recently raised by Lambert A'Campo.

In the sequel of this section we briefly recap the main results that we obtained in Chapter IV. We start off by introducing the main problem. Let $A \subset \mathbb{N}$ and let $\alpha \in (0, 1)$. For $N \in \mathbb{N}$ we consider the sum

$$S_A(\alpha, N) := \sum_{\substack{n\in A \\ n\leq N}} e(n\alpha),$$

where $e(x) := e^{2\pi i x} \in \mathbb{C}$ for $x \in \mathbb{R}$. If $A = \mathbb{N}$, we have

$$|S_A(\alpha, N)| = \left|\sum_{n\leq N} e(n\alpha)\right| \leq \left|\frac{e((N+1)\alpha) - e(\alpha)}{e(\alpha)-1}\right| \leq \frac{2}{|e(\alpha)-1|}$$

for all $N \in \mathbb{N}$. Hence, if the set A is finite or cofinite[5], for each choice of $\alpha \in (0,1)$ there exists a constant $C_{A,\alpha} > 0$ such that $|S_A(\alpha, N)| \leq C_{A,\alpha}$ independently of N. In a summer project on the circle method [37], Lambert A'Campo asked the following question.

Question 4.1 (L. A'Campo)**.** Are there infinite non-cofinite sets $A \subset \mathbb{N}$ such that for each $\alpha \in (0,1)$ there exists a constant $C_{A,\alpha} > 0$ for which $|S_A(\alpha, N)| \leq C_{A,\alpha}$ independently of $N \in \mathbb{N}$?

This question was later presented by Philipp Habegger as an open problem at the "Diophantine Approximation and Transcendence" conference held in Luminy in September 2018.

To get a first intuition on this problem, we confine ourselves to rational numbers in the unit interval. Suppose that $A \subset \mathbb{N}$ is such that $|S_A(\alpha, N)| \leq C_{A,\alpha}$ for all N and all $\alpha \in (0,1)$. Then, in particular, $S_A(\alpha, N)$ is bounded for all $\alpha \in \mathbb{Q} \cap (0,1)$. Take $\alpha = p/q$. The direction of the unitary vector $e(np/q)$ depends on the remainder of $n \pmod q$. Since the sum of such vectors, $S_A(N, p/q)$, must be bounded for all N, it is sensible to assume that each direction is hit a "fair" number of times as N grows, or that, in other words, the set A hits all remainders modulo q a "fair" number of times. This is clearly the case for the set $A = \mathbb{N}$. However, it is not immediately clear how to choose an infinite and non-cofinite set $A \subset \mathbb{N}$ that satisfies this condition for all $q \in \mathbb{N}$.

By asking that the sum $S_A(\alpha, N)$ is bounded in modulus for all $\alpha \in (0,1)$, we are, in principle, requiring that the set A is equally "equidistributed modulo irrational numbers". The rational and the irrational conditions are however incompatible if the set A is infinite and non-cofinite. Indeed, in Chapter IV we show that the answer to Question 4.1 is negative. More generally we prove the following result.

Proposition 4.2. *Let $\boldsymbol{a} := (a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers taking only finitely many values, and let*

$$E(\boldsymbol{a}) := \left\{ \alpha \in (0,1) : \sup_{N\in\mathbb{N}} \left|\sum_{n\leq N} a_n e(n\alpha)\right| < +\infty \right\}.$$

Assume that:

[5] A set $A \subset \mathbb{N}$ is cofinite if $A = \mathbb{N} \setminus B$, with $B \subset \mathbb{N}$ finite.

i) the set $E(\boldsymbol{a})$ contains an open non-empty interval;

ii) the set $E(\boldsymbol{a})$ contains $\mathbb{Q} \cap (0, 1/2]$.

Then the sequence $\boldsymbol{a}$ is ultimately constant.

An answer to Question 4.1 is provided by the case $a_n = \chi_A(n)$ and $E(\boldsymbol{a}) = (0, 1)$. In particular, condition $ii)$ is the rational "equidistribution" condition mentioned above, whereas condition $i)$ can be regarded as the analogous "equidistribution" condition for irrational numbers.

To prove Proposition 4.2 we use a result on complex power series by Duffin and Schaeffer from the early forties [10, Part II, Theorem I]. For a complete proof the reader can see Section 2 in Chapter IV.

It is worth noting that a very special case of Proposition 4.2 was proved by Lesigne and Petersen in 1989 [30], by using more involved machinery such as the Spectral Theorem for Hilbert spaces.

Of course, Proposition 4.2 is not the end of the story. One might ask how "large" a set $\mathbb{Q} \cap (0, 1/2] \subset E \subset (0, 1)$ has to be so that if $|S_A(\alpha, N)| \leq C_{A,\alpha}$ for all $\alpha \in E$, then A is either finite or cofinite. We show the following.

Proposition 4.3. *There exist full Hausdorff dimension sets E, with $\mathbb{Q} \cap (0, 1) \subset E \subset (0, 1)$, and infinite non-cofinite sets $A \subset \mathbb{N}$ such that $|S_A(\alpha, N)| \leq C_{A,\alpha}$ for all $\alpha \in E$.*

Proposition 4.3 shows that some full Hausdorff dimension sets E containing $\mathbb{Q} \cap (0, 1)$ admit infinite non-cofinite sets $A \subset \mathbb{N}$ with bounded sums for all $\alpha \in E$. This raises the question whether a full Hausdorff dimension set E can ever be enough to exclude the existence of an infinite non-cofinite set A. On the other hand, one might also ask whether containing an interval is actually a necessary condition. The tools that we use to prove Proposition 4.3 are not powerful enough to answer these questions.

Among such tools is the so-called factoradic representation of a real number, which we combine with [27, Example 2.3] to prove Proposition 4.3. Given a real number $\alpha \in [0, 1)$, it can be shown that there exists an "almost unique" sequence $\boldsymbol{a} = (a_n)_{n \in \mathbb{N}}$ with $a_n \in \{0, \ldots, n-1\}$ such that

$$\alpha = \sum_{n \geq 1} \frac{a_n}{n!}.$$

4 More on Cantor-like sets and applications to exponential sums

We show that all real numbers for which certain subsequences of factoradic digits a_{n_k} are null, give bounded exponential sums for an appropriate choice of the set A. The set of such real numbers has full Hausdorff dimension and zero Lebesgue measure. This can be proved by using Cantor-like sets constructions. For more details we refer the reader to Section 3 in Chapter IV.

We conclude the introduction, by listing the four research papers that constitute this thesis.

- **Sums of Reciprocals of Fractional Parts** [13], Chapter I:
  published in International Journal of Number Theory (https://doi.org/10.1142/S1793042119500416). This paper contains Theorem 1.2.
- **On a Counting Theorem for Weakly Admissible Lattices** [15], Chapter II:
  to appear on International Mathematical Research Notices (https://doi.org/10.1093/imrn/rnaa102). This paper contains Proposition 2.1, Theorem 1.3, and Corollary 3.1.
- **Multiplicatively Badly Approximable Matrices up to Logarithmic factors** [14], Chapter III: under review. This paper contains Corollary 1.5, and Corollary 1.6.
- **A Note on Bounded Exponential Sums** [12], Chapter IV:
  under review. This paper contains Proposition 4.2, and Proposition 4.3.

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Sums of Reciprocals of Fractional Parts

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Sums of Reciprocals of Fractional Parts

Let $\boldsymbol{\alpha} \in \mathbb{R}^N$ and $Q \geq 1$. We consider the sum $\sum_{\boldsymbol{q} \in [-Q,Q]^N \cap \mathbb{Z}^N \setminus \{\mathbf{0}\}} \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\|^{-1}$. Sharp upper bounds are known when $N = 1$, using continued fractions or the three distance theorem. However, these techniques do not seem to apply in higher dimension. We introduce a different approach, based on a general counting result of Widmer for weakly admissible lattices, to establish sharp upper bounds for arbitrary N. Our result also sheds light on a question raised by Lê and Vaaler in 2013 on the sharpness of their lower bound $\gg Q^N \log Q$.

1 Introduction

Sums of reciprocals of fractional parts have been studied by many authors in the light of their tight connections to, e.g., uniform distribution theory, metric Diophantine approximation, and lattice point counting (see [1], [6], [7], [8], [9], [12] and [14]). In this note, we establish a new, sharp upper bound for such sums.

We denote by $\|x\|$ the distance to the nearest integer of a real x. We write $A \ll B$ to mean that there exists a constant $c > 0$ (absolute or depending only on the parameters indicated) such that $A \leq cB$. We use $|\cdot|_2$ to denote the Euclidean norm on $\mathbb{R}^n$ and $|\cdot|_\infty$ to denote the maximum norm.

Let $N \in \mathbb{N} := \{1, 2, 3, \ldots\}$, and let $\boldsymbol{x} \cdot \boldsymbol{y} := \sum_{i=1}^{N} x_i y_i$ be the standard inner product on $\mathbb{R}^N$. Let $\boldsymbol{\alpha}$ be in $\mathbb{R}^N$ with $1, \alpha_1, \ldots, \alpha_N$ linearly independent over $\mathbb{Q}$, and suppose $Q_1, \ldots, Q_N \in (0, +\infty)$. Lê and Vaaler [11, Corollary 1.2] proved that for $X := [-Q_1, Q_1] \times \cdots \times [-Q_N, Q_N]$ and $Q :=$

$(Q_1 \cdots Q_N)^{1/N} \geq 1$ it holds

$$Q^N \log Q \ll \sum_{\boldsymbol{q} \in X \cap \mathbb{Z}^N \setminus \{\boldsymbol{0}\}} \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\|^{-1}. \tag{10}$$

They also showed that, whenever $\boldsymbol{\alpha}$ is multiplicatively badly approximable (see [3] and [11, (2.5)]), inequality (10) is sharp[6], i.e.,

$$\sum_{\boldsymbol{q} \in X \cap \mathbb{Z}^N \setminus \{\boldsymbol{0}\}} \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\|^{-1} \ll_{\boldsymbol{\alpha}} Q^N \log Q \tag{11}$$

for all $Q \geq 2$. However, if $N \geq 2$, every such $\boldsymbol{\alpha}$ yields a counterexample to the Littlewood conjecture. This follows from two facts:

- a matrix is multiplicatively badly approximable if and only if its transpose is multiplicatively badly approximable (see [5, Corollary 1] and [11, Theorem 2.2]);
- every submatrix of a multiplicatively badly approximable matrix is itself multiplicatively badly approximable (see [11, (2.6)]).

To date, Littlewood's conjecture is still open and the set of counterexamples to it, if not empty, has been shown to be extremely sparse. Indeed, Einsiedler, Katok, and Lindenstrauss [4] have proved that its Hausdorff dimension is zero. Despite this, it can be shown that for some vectors in $\mathbb{R}^N$ we can reverse inequality (10).

Definition 1.1. Let $N \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{R}^N$. Let $\phi : (0, +\infty) \to (0, 1]$ be a non-increasing[7], real-valued function. We say that $\boldsymbol{\alpha}$ is a ϕ-badly approximable vector if

$$|\boldsymbol{q}|_\infty^N \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\| \geq \phi(|\boldsymbol{q}|_\infty)$$

for all $\boldsymbol{q} \in \mathbb{Z}^N \setminus \{\boldsymbol{0}\}$. If ϕ can be chosen constant, we simply say that $\boldsymbol{\alpha}$ is badly approximable.

It was kindly pointed out to me by Victor Beresnevich that a simple gap-principle, already known in the literature (see [9, Proof of Lemma 3.3, p.123]), proves (11) for all badly approximable vectors $\boldsymbol{\alpha}$ in the special case $X = [-Q, Q]^N$. Note that the set of such vectors has full Hausdorff

[6] Note that Lê and Vaaler assumed $X := [0, Q_1] \times \cdots \times [0, Q_N]$, but this makes no difference. Suppose indeed that the sum taken over, e.g., the set $[-Q_1, 0] \times [0, Q_2] \times \cdots \times [0, Q_N]$ is big, then, we can multiply the first coordinate of $\boldsymbol{\alpha}$ by -1, obtaining a multiplicatively badly approximable vector such that the sum over X is now big.

[7] We say that ϕ is non-increasing if $\phi(x) \geq \phi(y)$, whenever $x < y$.

dimension in $\mathbb{R}^N$ (see [13]). We recall this proof here. Let $Q \geq 2$ and suppose that $\boldsymbol{\alpha} \in \mathbb{R}^N$ is ϕ-badly approximable. Then, for all distinct $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{Z}^N \cap X$, we have

$$\|\boldsymbol{\alpha} \cdot \boldsymbol{q}_1 \pm \boldsymbol{\alpha} \cdot \boldsymbol{q}_2\| = \|\boldsymbol{\alpha} \cdot (\boldsymbol{q}_1 \pm \boldsymbol{q}_2)\| \geq \frac{\phi(2Q)}{(2Q)^N}. \tag{12}$$

It follows that

$$|\|\boldsymbol{\alpha} \cdot \boldsymbol{q}_1\| - \|\boldsymbol{\alpha} \cdot \boldsymbol{q}_2\|| \geq \frac{\phi(2Q)}{(2Q)^N}.$$

Therefore, none of the intervals

$$\left[0, \frac{\phi(2Q)}{(2Q)^N}\right), \left[\frac{\phi(2Q)}{(2Q)^N}, \frac{2\phi(2Q)}{(2Q)^N}\right), \left[\frac{2\phi(2Q)}{(2Q)^N}, \frac{3\phi(2Q)}{(2Q)^N}\right), \ldots$$

contains more than one number of the form $\|\boldsymbol{\alpha} \cdot \boldsymbol{q}\|$ ($\boldsymbol{q} \in X \cap \mathbb{Z}^N \setminus \{\mathbf{0}\}$), and no such number lies in the first interval. Hence,

$$\sum_{\boldsymbol{q} \in X \cap \mathbb{Z}^N \setminus \{\mathbf{0}\}} \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\|^{-1} \leq \sum_{j=1}^{|X \cap \mathbb{Z}^N \setminus \{\mathbf{0}\}|} \frac{(2Q)^N}{j\phi(2Q)} \ll_N \frac{Q^N \log Q}{\phi(2Q)}. \tag{13}$$

For $N = 1$, the above upper bound can be improved on to

$$\ll Q \log Q + \frac{Q}{\phi(Q)} \tag{14}$$

using the theory of continued fractions (see [10, Theorem 2, p.37-40]). The bound given in (14) is best possible. However, if one allows the upper bound to be expressed in terms of the (least) denominator q_K of the K-th convergent of α, an even more precise result can be obtained via the three distance theorem, as shown by Beresnevich and Leong [2, Corollary 1].

Nevertheless, neither the techniques based on continued fractions nor those of Beresnevich and Leong using the three distance theorem seem to generalise in an obvious way to higher dimension. In this note we introduce yet another method, based on a recent counting result for weakly admissible lattices, which allows us to extend (14) to arbitrary dimension N.

Theorem 1.2. *Let $X := [-Q, Q]^N$ and let $\boldsymbol{\alpha} \in \mathbb{R}^N$ be a ϕ-badly approximable vector. Then, we have*

$$Q^N \log(Q\phi(Q)) \ll_N \sum_{\mathbf{q} \in X \cap \mathbb{Z}^N \setminus \{\mathbf{0}\}} \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\|^{-1} \ll_N Q^N \log Q + \frac{Q^N}{\phi(Q)} \tag{15}$$

for all $Q \geq 1$.

Theorem 1.2 implies immediately that (11) is satisfied whenever

$$|\boldsymbol{q}|_\infty^N \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\| \gg_{\boldsymbol{\alpha}} \frac{1}{\log(|\boldsymbol{q}|_\infty)},$$

and not just when $\boldsymbol{\alpha}$ is a badly approximable vector. Clearly, the lower bound is superseded by (10) (and non trivial only if $\log(Q\phi(Q)) \geq 1$), but we have decided to include it in Theorem 1.2 nonetheless. This is because both bounds in (15) can be proved almost at once, and the method we use is substantially different from Lê and Vaaler's. Note that the upper bound, in conjunction with the Khintchine-Groshev Theorem, implies that for every $\varepsilon > 0$ the set of $\boldsymbol{\alpha} \in \mathbb{R}^N$ for which

$$\sum_{\mathbf{q} \in X \cap \mathbb{Z}^N \setminus \{\mathbf{0}\}} \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\|^{-1} \ll_{\boldsymbol{\alpha}} Q^N (\log Q)^{1+\varepsilon},$$

has full Lebesgue measure. Moreover, the upper bound in (15) is sharp, in the sense that for all $N \geq 1$ there exists a sequence of positive integers $Q_i \to +\infty$ such that

$$\sum_{\mathbf{q} \in X_i \cap \mathbb{Z}^N \setminus \{\mathbf{0}\}} \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\|^{-1} \gg Q_i^N \log Q_i + \frac{Q_i^N}{\phi(Q_i)}, \tag{16}$$

where $X_i := [-Q_i, Q_i]^N$. To show this, let ϕ be chosen maximal, i.e., such that

$$\phi(x) = \min\{|\boldsymbol{q}|_\infty^N \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\| \mid 0 < |\boldsymbol{q}|_\infty \leq x,\ \boldsymbol{q} \in \mathbb{Z}^N\}$$

for $x \geq 1$ and $\phi(x) = 1$ for $x < 1$. Let $\boldsymbol{q}_i \in \mathbb{Z}^N$ be a sequence of pairwise distinct vectors such that $\phi(|\boldsymbol{q}_i|_\infty) = |\boldsymbol{q}_i|_\infty^N \|\boldsymbol{\alpha} \cdot \boldsymbol{q}_i\|$, and set $Q_i := |\boldsymbol{q}_i|_\infty$. Then, $\|\boldsymbol{\alpha} \cdot \boldsymbol{q}_i\|^{-1} = Q_i^N / \phi(Q_i)$. This implies that

$$\sum_{\mathbf{q} \in X_i \cap \mathbb{Z}^N \setminus \{\mathbf{0}\}} \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\|^{-1} \geq \frac{Q_i^N}{\phi(Q_i)}. \tag{17}$$

Hence, (16) follows from (10) and (17).

The main tool to prove Theorem 1.2 is Proposition 1.3, which gives a precise estimate for the size of the set

$$M(\boldsymbol{\alpha}, \varepsilon, Q) := \left\{(p, \boldsymbol{q}) \in \mathbb{Z}^{1+N} \setminus \{\mathbf{0}\} \;\middle|\; |\boldsymbol{\alpha} \cdot \boldsymbol{q} + p| \leq \varepsilon,\ |\boldsymbol{q}|_\infty \leq Q\right\}, \tag{18}$$

where $0 < \varepsilon \leq 1/2$ and $Q \geq 1$.

Proposition 1.3. *Let $\boldsymbol{\alpha} \in \mathbb{R}^N$ be a ϕ-badly approximable vector. Let $Q \geq 1$ and $0 < \varepsilon \leq 1/2$. Then, we have*

$$\Big| |M(\boldsymbol{\alpha}, \varepsilon, Q)| - 2^{N+1} \varepsilon Q^N \Big| \ll_N \left(\frac{\varepsilon Q^N}{\phi(Q)}\right)^{\frac{N}{N+1}}.$$

With Proposition 1.3 at hand, Theorem 1.2 can be derived from a simple dyadic summation. Proposition 1.3 itself is a straightforward consequence of a recent, general lattice point counting result, due to Widmer [16, Theorem 2.1], which we recall in the next section. Unfortunately, it seems unlikely that Theorem 1.2 could be generalised to the case of a non symmetric counting domain, i.e., of the form $X := [-Q_1, Q_1] \times \cdots \times [-Q_N, Q_N]$ with $Q_i \neq Q_j$, by using weak admissibility. Indeed, in this case, a different type of weak admissibility would be required to prove the analogue of Proposition 1.3. We are currently working on this problem. It is also worth mentioning that Theorem 1.2 is essentially a dual version of [15, Corollary 1.2], where Widmer provides similar estimates for sums of products of fractional parts of two linear forms in one variable.

2 Weakly admissible lattices and counting lattice points

To state [16, Theorem 2.1], we need to introduce some notation. We follow the notation used in [16], except that we write $\mathcal{N}$ (instead of N) for $\sum_{i=1}^{n} m_i$. We assume throughout that $\mathcal{N} \geq 2$.

Let n be a positive integer and let $\mathcal{S} := (\boldsymbol{m}, \boldsymbol{\beta})$, where $\boldsymbol{m} := (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_n) \in (0, +\infty)^n$. Let

$$\mathrm{Nm}_{\beta}(\underline{\boldsymbol{x}}) := \prod_{i=1}^{n} |\boldsymbol{x}_i|_2^{\beta_i}$$

be the multiplicative norm induced by $\boldsymbol{\beta}$ on $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n} \ni \underline{\boldsymbol{x}} := (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$. Let I be a non-empty subset of $\{1, \ldots, n\}$ and let also

$$C = C_I := \{\underline{\boldsymbol{x}} \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n} \mid \boldsymbol{x}_i = \boldsymbol{0} \ \ \forall i \in I\}.$$

We fix the couple $(\mathcal{S}, C)$ and for any $\Gamma \subset \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$ we consider the quantity

$$\nu(\Gamma, \varrho) := \inf \left\{ \mathrm{Nm}_{\beta}(\underline{\boldsymbol{x}})^{\frac{1}{t}} \;\middle|\; \underline{\boldsymbol{x}} \in \Gamma \setminus C,\ |\underline{\boldsymbol{x}}|_2 < \varrho \right\},$$

where $t := \beta_1 + \cdots + \beta_n$. We observe that $\nu(\Gamma, \cdot)$ is a decreasing function of ϱ, bounded by below from 0. Hence, we have the following definition from [16].

Definition 2.1. A full rank lattice Λ in $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$ is said to be weakly admissible for the couple $(\mathcal{S}, C)$, if $\nu(\Lambda, \varrho) > 0$ for all $\varrho \in (0, +\infty)$.

Before stating the counting theorem, we require some more notation. For any $\Gamma \subset \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$ we define

$$\lambda_1(\Gamma) := \inf \{ |\boldsymbol{x}|_2 \mid \boldsymbol{x} \in \Gamma \setminus \{\boldsymbol{0}\}\},$$

and we set

$$\mu(\Gamma, \varrho) := \min\{\lambda_1(\Gamma \cap C), \nu(\Gamma, \varrho)\}.$$

We can now state a simplified version of [16, Theorem 2.1], streamlined for our application.

Theorem 2.2 (Widmer)**.** *Let $n \in \mathbb{N}$ and let $(\mathcal{S}, C)$ be a couple as in Definition (2.1). For $\boldsymbol{Q} := (Q_1, \ldots, Q_n) \in (0, +\infty)^n$ we set*

$$\overline{Q} := \left(Q_1^{\beta_1} \cdots Q_n^{\beta_n}\right)^{\frac{1}{t}}$$

and $Q_{\max} := \max\{Q_1, \ldots, Q_n\}$. Let

$$Z_{\boldsymbol{Q}} := \prod_{i=1}^{n} [-Q_i, Q_i]^{m_i} \subset \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$$

and let Λ be a weakly admissible lattice for the couple $(\mathcal{S}, C)$. Then, there exists a real constant $c = c(\mathcal{N}) > 0$, only depending on the quantity $\mathcal{N} := \sum_{i=1}^{n} m_i$, such that

$$\left| |Z_{\boldsymbol{Q}} \cap \Lambda| - \frac{\mathrm{Vol}(Z_{\boldsymbol{Q}})}{\det \Lambda} \right| \leq c \inf_{0 < B \leq Q_{\max}} \left(\frac{\overline{Q}}{\mu(\Lambda, B)} + \frac{Q_{\max}}{B} \right)^{\mathcal{N}-1},$$

where $\mathrm{Vol}(Z_{\boldsymbol{Q}})$ denotes the volume of the set $Z_{\boldsymbol{Q}}$ and $\det \Lambda$ denotes the determinant of the lattice Λ.

3 Proofs

3.1 Proof of Proposition 1.3

First we note that if $(p, \boldsymbol{q}) \in M(\boldsymbol{\alpha}, \varepsilon, Q)$, then

$$\varepsilon Q^N \geq \|\boldsymbol{q} \cdot \boldsymbol{\alpha}\| \, |\boldsymbol{q}|_\infty^N \geq \phi(Q). \tag{19}$$

Suppose that $\varepsilon Q^N / \phi(Q) < 1$. Then, $M(\boldsymbol{\alpha}, \varepsilon, Q) = \emptyset$, by (19). Moreover,

$$2^{N+1} \varepsilon Q^N \ll_N \varepsilon Q^N \leq \frac{\varepsilon Q^N}{\phi(Q)} \leq \left(\frac{\varepsilon Q^N}{\phi(Q)}\right)^{\frac{N}{N+1}}.$$

Hence, Proposition 1.3 holds true whenever $\varepsilon Q^N / \phi(Q) < 1$. From now on, we can assume that

$$\frac{\varepsilon Q^N}{\phi(Q)} \geq 1. \tag{20}$$

Let

$$A_{\boldsymbol{\alpha}} := \left(\begin{array}{c|ccc} 1 & \alpha_1 & \dots & \alpha_N \\ \hline 0 & & & \\ \vdots & & \mathrm{I}_N & \\ 0 & & & \end{array}\right).$$

Define $\Lambda_{\boldsymbol{\alpha}} := A_{\boldsymbol{\alpha}}\mathbb{Z}^{N+1} \subset \mathbb{R}^{N+1}$ and $Z_{\varepsilon,Q} := [-\varepsilon, \varepsilon] \times [-Q, Q]^N \subset \mathbb{R}^{N+1}$. Then,

$$|M(\boldsymbol{\alpha}, \varepsilon, Q)| = |\Lambda_{\boldsymbol{\alpha}} \cap Z_{\varepsilon,Q}| - 1, \tag{21}$$

since $\mathbf{0} \in \Lambda_{\boldsymbol{\alpha}} \cap Z_{\varepsilon,Q}$. Therefore, to prove Proposition 1.3, it suffices to estimate the quantity $|\Lambda_{\boldsymbol{\alpha}} \cap Z_{\varepsilon,Q}|$. To this end, we use Theorem 2.2.

Let $n = 2$ and let $\boldsymbol{m} = \boldsymbol{\beta} := (1, N)$. Let $C := C_I$, with $I := \{2\}$. Then, all vectors $\boldsymbol{v} \in \Lambda \setminus C$ have the form

$$\boldsymbol{v} = A_{\boldsymbol{\alpha}} \begin{pmatrix} p \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha} \cdot \boldsymbol{q} + p \\ \boldsymbol{q} \end{pmatrix},$$

where $\boldsymbol{q} \in \mathbb{Z}^N \setminus \{\mathbf{0}\}$ and $p \in \mathbb{Z}$. Recall now that $\boldsymbol{\alpha}$ is a ϕ-badly approximable vector. Hence, for all $\boldsymbol{v} \in \Lambda_{\boldsymbol{\alpha}} \setminus C$ it holds

$$\mathrm{Nm}_{\beta}(\boldsymbol{v}) = |\boldsymbol{\alpha} \cdot \boldsymbol{q} + p||\boldsymbol{q}|_2^N \geq \|\boldsymbol{\alpha} \cdot \boldsymbol{q}\||\boldsymbol{q}|_\infty^N \geq \phi\left(|\boldsymbol{q}|_\infty\right) \geq \phi(|\boldsymbol{v}|_2).$$

Since ϕ is non-increasing, we can conclude that

$$\nu\left(\Lambda_{\boldsymbol{\alpha}}, \varrho\right) \geq \phi(\varrho)^{\frac{1}{N+1}} > 0 \tag{22}$$

for all real $\varrho \geq \lambda_1(\Lambda_{\boldsymbol{\alpha}} \setminus C)$. However, if $\varrho < \lambda_1\left(\Lambda_{\boldsymbol{\alpha}} \setminus C\right)$, then $\nu(\Lambda_{\boldsymbol{\alpha}}, \varrho) = +\infty$ and (22) trivially holds true. This shows that $\Lambda_{\boldsymbol{\alpha}} \subset \mathbb{R} \times \mathbb{R}^N$ is weakly admissible for the couple $((\boldsymbol{m}, \boldsymbol{\beta}), C)$. We can thus apply Theorem 2.2, with $\Lambda = \Lambda_{\boldsymbol{\alpha}}$ and $Z_{\boldsymbol{Q}} = Z_{\varepsilon,Q}$. By choosing $B := Q_{\max} = Q$, we get

$$\left| |Z_{\varepsilon,Q} \cap \Lambda_{\boldsymbol{\alpha}}| - \frac{\mathrm{Vol}\left(Z_{\varepsilon,Q}\right)}{\det \Lambda_{\boldsymbol{\alpha}}} \right| \leq c \left(\frac{\left(\varepsilon Q^N\right)^{\frac{1}{N+1}}}{\mu(\Lambda, Q)} + 1 \right)^N. \tag{23}$$

Since $\det \Lambda_{\boldsymbol{\alpha}} = 1$ and $\mathrm{Vol}\left(Z_{\varepsilon,Q}\right) = 2^{N+1}\varepsilon Q^N$, to conclude the proof, we just need to estimate the right-hand side of (23). We observe that $\lambda_1(\Lambda_{\boldsymbol{\alpha}} \cap C) = \lambda_1\left(\mathbb{Z} \times \{\mathbf{0}\}\right) = 1$. Hence,

$$\mu(\Lambda, Q) \geq \min\left\{1, \phi(Q)^{\frac{1}{N+1}}\right\} = \phi(Q)^{\frac{1}{N+1}},$$

by (22). Combining (21) and (23), and using (20), we find

$$\left| |M(\boldsymbol{\alpha}, \varepsilon, Q)| - 2^{N+1}\varepsilon Q^N \right| \ll_N \left(\left(\frac{\varepsilon Q^N}{\phi(Q)} \right)^{\frac{1}{N+1}} + 1 \right)^N \ll_N \left(\frac{\varepsilon Q^N}{\phi(Q)} \right)^{\frac{N}{N+1}}.$$

This completes the proof.

3.2 Proof of Theorem 1.2

We start by observing that

$$\sum_{\substack{\mathbf{q}\in[-Q,Q]^N \\ \cap\, \mathbb{Z}^N\setminus\{\mathbf{0}\}}} \|\boldsymbol{\alpha}\cdot\boldsymbol{q}\|^{-1} \leq \sum_{k=1}^{\infty} 2^{k+1}\left|\left\{ (p,\boldsymbol{q})\in\mathbb{Z}^{N+1}\setminus\{\mathbf{0}\} \;\middle|\; 2^{-k-1} < |\boldsymbol{\alpha}\cdot\boldsymbol{q}+p| \leq 2^{-k},\ |\boldsymbol{q}|_\infty \leq Q\right\}\right|$$

$$\leq \sum_{k=1}^{\infty} 2^{k+1}\left|M(\boldsymbol{\alpha},2^{-k},Q)\right| = \sum_{k=1}^{\lfloor \log_2(Q^N/\phi(Q))\rfloor} 2^{k+1}\left|M(\boldsymbol{\alpha},2^{-k},Q)\right|, \tag{24}$$

where the last equation is due to (19). Now, by Proposition 1.3, we know that

$$\left|\left|M(\boldsymbol{\alpha},2^{-k},Q)\right| - 2^{N+1-k}Q^N\right| \ll_N \left(\frac{2^{-k}Q^N}{\phi(Q)}\right)^{\frac{N}{N+1}}. \tag{25}$$

Hence, (24) yields

$$\sum_{\substack{\mathbf{q}\in[-Q,Q]^N \\ \cap\, \mathbb{Z}^N\setminus\{\mathbf{0}\}}} \|\boldsymbol{\alpha}\cdot\boldsymbol{q}\|^{-1} \ll_N \sum_{k=1}^{\lfloor \log_2(Q^N/\phi(Q))\rfloor} 2^{k+1}\left(2^{N+1-k}Q^N + \left(\frac{2^{-k}Q^N}{\phi(Q)}\right)^{\frac{N}{N+1}}\right)$$

$$\ll_N \sum_{k=1}^{\lfloor \log_2(Q^N/\phi(Q))\rfloor} \left(Q^N + 2^{\frac{k}{N+1}}\left(\frac{Q^N}{\phi(Q)}\right)^{\frac{N}{N+1}}\right)$$

$$\ll_N Q^N\log_2\left(\frac{Q^N}{\phi(Q)}\right) + \left(\frac{Q^N}{\phi(Q)}\right)^{\frac{1}{N+1}}\left(\frac{Q^N}{\phi(Q)}\right)^{\frac{N}{N+1}} \tag{26}$$

$$\ll_N Q^N\log Q + \frac{Q^N}{\phi(Q)}, \tag{27}$$

where (26) follows from the trivial estimate $\sum_{k=1}^{K} 2^{k/(N+1)} \leq 2^{K/(N+1)+1}$ and (27) is due to the fact that $1/\phi(Q) \geq \log(1/\phi(Q))$. This proves the upper bound.

To prove the lower bound, we notice that

$$\sum_{\substack{\mathbf{q}\in[-Q,Q]^N \\ \cap\, \mathbb{Z}^N\setminus\{\mathbf{0}\}}} \|\boldsymbol{\alpha}\cdot\boldsymbol{q}\|^{-1} \geq \sum_{k=1}^{\infty} 2^{k}\left|\left\{ (p,\boldsymbol{q})\in\mathbb{Z}^{N+1}\setminus\{\mathbf{0}\} \;\middle|\; 2^{-k-1} < |\boldsymbol{\alpha}\cdot\boldsymbol{q}+p| \leq 2^{-k},\ |\boldsymbol{q}|_\infty \leq Q\right\}\right|$$

$$\geq \sum_{k=1}^{\infty} 2^{k}\left(\left|M(\boldsymbol{\alpha},2^{-k},Q)\right| - \left|M(\boldsymbol{\alpha},2^{-k-1},Q)\right|\right). \tag{28}$$

From Proposition 1.3, we also know that for all $k \geq 1$ and $Q \geq 1$

$$\left|\left|M(\boldsymbol{\alpha},2^{-k},Q)\right| - 2^{N+1-k}Q^N\right| \leq c_N\left(\frac{2^{-k}Q^N}{\phi(Q)}\right)^{\frac{N}{N+1}}, \tag{29}$$

where c_N is a positive constant. Hence, whenever $1 \leq k \leq \log_2\left(Q^N\phi(Q)^N/c_N^{N+1}\right) =: K$, we have

$$\left(2^{N+1}-1\right)2^{-k}Q^N \leq \left|M(\boldsymbol{\alpha},2^{-k},Q)\right| \leq \left(2^{N+1}+1\right)2^{-k}Q^N.$$

This, in turn, shows that

$$\left|M(\boldsymbol{\alpha},2^{-k},Q)\right| - \left|M(\boldsymbol{\alpha},2^{-k-1},Q)\right| \geq 2^{-k}Q^N, \tag{30}$$

when $1 \leq k \leq K-1$. Therefore, provided $K-1 \geq 1$, we can plug (30) into (28) and restrict the sum to $k \leq \lfloor K-1 \rfloor$. This yields the lower bound

$$\sum_{\substack{\mathbf{q}\in[-Q,Q]^N \\ \cap\ \mathbb{Z}^N\setminus\{\mathbf{0}\}}} \|\boldsymbol{\alpha}\cdot\boldsymbol{q}\|^{-1} \geq \lfloor K-1 \rfloor Q^N \geq (K-2)Q^N, \tag{31}$$

which, of course, remains true for $K-1<1$. Now, a trivial lower bound for the sum of the reciprocals is

$$\sum_{\substack{\mathbf{q}\in[-Q,Q]^N \\ \cap\ \mathbb{Z}^N\setminus\{\mathbf{0}\}}} \|\boldsymbol{\alpha}\cdot\boldsymbol{q}\|^{-1} \gg_N Q^N,$$

since $\|\boldsymbol{\alpha}\cdot\boldsymbol{q}\|^{-1} \geq 2$ for all $\boldsymbol{q} \neq 0$. Hence, recalling that $K = \log_2\left(Q^N\phi(Q)^N/c_N^{N+1}\right)$, we conclude that

$$\sum_{\substack{\mathbf{q}\in[-Q,Q]^N \\ \cap\ \mathbb{Z}^N\setminus\{\mathbf{0}\}}} \|\boldsymbol{\alpha}\cdot\boldsymbol{q}\|^{-1} \gg_N (K-2)\,Q^N + \log_2\left(4c_N^{N+1}\right)Q^N \gg_N Q^N \log\left(Q\phi(Q)\right).$$

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On a Counting Theorem for Weakly Admissible Lattices

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On a counting theorem for weakly admissible lattices

We give a precise estimate for the number of lattice points in certain bounded subsets of $\mathbb{R}^n$ that involve "hyperbolic spikes" and occur naturally in multiplicative Diophantine approximation. We use Wilkie's o-minimal structure $\mathbb{R}_{\exp}$ and expansions thereof to formulate our counting result in a general setting. We give two different applications of our counting result. The first one establishes nearly sharp upper bounds for sums of reciprocals of fractional parts, and thereby sheds light on a question raised by Lê and Vaaler, extending previous work of Widmer and of the author. The second application establishes new examples of linear subspaces of Khintchine type thereby refining a theorem by Huang and Liu. For the proof of our counting result we develop a sophisticated partition method which is crucial for further upcoming work on sums of reciprocals of fractional parts over distorted boxes.

1 Introduction

1.1 Notation

Let X be a set. For any pair of functions $f, g : X \to \mathbb{R}$, we write $f \ll g$ ($f \gg g$) to mean that there exists a real number $c > 0$ such that $f(x) \leq cg(x)$ ($f(x) \geq cg(x)$) for all $x \in X$. If the constant c depends on any parameters, we write them under the symbol $\ll$ ($\gg$). We write $O_c(f)$ to indicate a function g such that $g \ll_c f$. We use $|\cdot|_2$ to denote the Euclidean norm on $\mathbb{R}^n$ and $|\cdot|_\infty$ to denote the maximum norm. We write $\mathbb{N}$ for the set $\{1, 2, 3, \dots\}$ of positive integers. We indicate by $\|x\|$ the distance from $x \in \mathbb{R}$ to the nearest integer, i.e., $\min\{|x - n| : n \in \mathbb{Z}\}$. We denote by diam$X$ the diameter (i.e., $\sup\{|x - y| : x, y \in X\}$) of a set $X \subset \mathbb{R}^n$, and we use

$\mathrm{Vol}_d(X)$ to indicate its d-dimensional Hausdorff measure ($d \in \mathbb{N}$). When the dimension d is not specified, we assume $d = n$.

1.2 Main result

In this paper we prove a general counting result for weakly admissible lattices. More specifically, we estimate the number of lattice points lying in the area bounded by a certain compact hypersurface defined in terms of the lattice structure. We generalise this result to any definable set in Wilkie's o-minimal structure $\mathbb{R}_{\exp}$ lying within the above mentioned hypersurface, and we derive an asymptotic formula for the number of lattice points contained in any such set. Our counting principle allows us to shed light on a question raised by Lê and Vaaler on the behaviour of certain sums of reciprocals of fractional parts. It also yields a refinement of a theorem proved by Huang and Liu on linear subspaces of Khintchine type.

Before stating the main result, we look at a special case that already captures the main features of our counting principle. Let $\mathcal{M}, \mathcal{N} \in \mathbb{N}$ and let $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M} \times \mathcal{N}}$. We denote by $L_1, \ldots, L_{\mathcal{M}}$ the rows of the matrix $\boldsymbol{L}$, and for any $\boldsymbol{x} \in \mathbb{R}^{\mathcal{N}}$ we set

$$L_i \boldsymbol{x} := \sum_{j=1}^{\mathcal{N}} L_{ij} x_j$$

for $i = 1, \ldots, \mathcal{M}$. We assume throughout the paper that 1 along with the coefficients $L_{i1}, \ldots, L_{i\mathcal{N}}$ of each of these linear forms are linearly independent over $\mathbb{Q}$. Let $\varepsilon, T \in (0, +\infty)$ and let $Q \in [1, +\infty)$. We consider the set

$$M(\boldsymbol{L}, \varepsilon, T, Q) := \Bigg\{ (\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^{\mathcal{M}} \times \left(\mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\} \right) : \prod_{i=1}^{\mathcal{M}} |L_i \boldsymbol{q} + p_i| < \varepsilon, \\ |L_i \boldsymbol{q} + p_i| \leq T, \; i = 1, \ldots, \mathcal{M}, \; |q_j| \leq Q, \; j = 1, \ldots, \mathcal{N} \Bigg\}.$$

Our goal is to estimate the cardinality of $M(\boldsymbol{L}, \varepsilon, T, Q)$. To this end, we let

$$\boldsymbol{A_L} := \left(\begin{array}{c|c} \boldsymbol{I}_{\mathcal{M}} & \boldsymbol{L} \\ \hline \boldsymbol{0} & \boldsymbol{I}_{\mathcal{N}} \end{array} \right) \in \mathbb{R}^{(\mathcal{M}+\mathcal{N}) \times (\mathcal{M}+\mathcal{N})},$$

where $\boldsymbol{I}_{\mathcal{M}}$ and $\boldsymbol{I}_{\mathcal{N}}$ are identity matrices of size $\mathcal{M}$ and $\mathcal{N}$ respectively, and we let $\Lambda_{\boldsymbol{L}} := \boldsymbol{A_L} \mathbb{Z}^{\mathcal{M}+\mathcal{N}}$. We also set

$$Z := \left\{ \boldsymbol{x} \in \mathbb{R}^{\mathcal{M}} : \prod_{i=1}^{\mathcal{M}} |x_i| < \varepsilon, \; |x_i| \leq T, \; i = 1, \ldots, \mathcal{M} \right\} \times [-Q, Q]^{\mathcal{N}}.$$

Then, we have

$$\#M(\boldsymbol{L},\varepsilon,T,Q) = \#\left((\Lambda_{\boldsymbol{L}} \cap Z) \setminus C\right),$$

where $C := \left\{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^{\mathcal{M}+\mathcal{N}} : \boldsymbol{y} = \boldsymbol{0}\right\}$. Therefore, estimating $\#M(\boldsymbol{L},\varepsilon,T,Q)$ is equivalent to estimating $\#(\Lambda_{\boldsymbol{L}} \cap Z)$, if we exclude the points of $\Lambda_{\boldsymbol{L}}$ that lie in C.

We now make some assumptions on the lattice $\Lambda_{\boldsymbol{L}}$. First, we assume that the distance between the points in $\Lambda_{\boldsymbol{L}} \setminus C$ and the coordinate subspaces of $\mathbb{R}^{\mathcal{M}+\mathcal{N}}$ orthogonal to C is always positive. In the worst case, this distance will be decaying as we move away from the origin. We want to control its decay rate in terms of the distance from the origin. Hence, we additionally assume that the distance between the points of $\Lambda_{\boldsymbol{L}} \setminus C$ and the coordinate subspaces orthogonal to C is bounded from below by a certain non-increasing function. To make this precise, we give the following definition.

Definition 1.1. Let $\phi : [1,+\infty) \to (0,1]$ be a non-increasing[8] function. We say that a matrix $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ is ϕ-semimultiplicatively badly approximable if

$$|\boldsymbol{q}|_\infty^{\mathcal{N}} \prod_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\| \geq \phi(|\boldsymbol{q}|_\infty)$$

for all $\boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}$. If the function ϕ can be chosen constant, we say that $\boldsymbol{L}$ is semimultiplicatively badly approximable.

For the lattice $\Lambda_{\boldsymbol{L}}$, the purely arithmetic property introduced in Definition 1.1 yields the geometric condition described above. Provided this geometric condition is fulfilled, we can derive an asymptotic estimate for $\#\left((\Lambda_{\boldsymbol{L}} \cap Z) \setminus C\right)$.

Proposition 1.2. *Let $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ be a ϕ-semimultiplicatively badly approximable matrix and suppose that $T^{\mathcal{M}}/\varepsilon \geq \mathrm{e}^{\mathcal{M}}$, where $\mathrm{e} = 2.71828\ldots$ is the base of the natural logarithm. Then, we have*

$$\left|\#\left((\Lambda_{\boldsymbol{L}} \cap Z) \setminus C\right) - \mathrm{Vol}Z\right| \ll_{\mathcal{M},\mathcal{N}} (1+T)^{\mathcal{M}+\mathcal{N}-1} \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} \left(\frac{\varepsilon Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{\mathcal{M}+\mathcal{N}-1}{\mathcal{M}+\mathcal{N}}},$$

where

$$\mathrm{Vol}Z = 2^{\mathcal{M}+\mathcal{N}} Q^{\mathcal{N}} \left[\varepsilon \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} + T^{\mathcal{M}}\left(1 - \left(1 - \frac{\varepsilon}{T^{\mathcal{M}}}\right)^{\mathcal{M}-1}\right)\right].$$

[8] We say that ϕ is non-increasing if $\phi(x) \geq \phi(y)$ for all $x < y$.

1 Introduction

We briefly explain how the proof is structured, so that we can highlight the main difficulties. The key idea is to decompose the set Z into approximately $\log{(T/\varepsilon)}^{\mathcal{M}-1}$ subsets. To each such subset we apply a different diagonal linear map and obtain a ball-like shaped set. We then count the points of the corresponding transformation of $\Lambda_{\boldsymbol{L}}$ lying in each of these sets. Note that for $\mathcal{M} = 10$, $T = 1$, and $\varepsilon = 1/10$, we already find more than 1800 different subsets and linear maps. The finer the subdivision, the more precise is each single estimate. However, when the subdivision is too fine, we end up summing too many error terms, and controlling the minima of the transformed lattices becomes rather difficult. The geometric condition introduced in Definition 1.1 allows us to give sufficiently good bounds on the first successive minima of these lattices to control the total error term. The essence of this partition method is summarised in Proposition 2.1, which is itself a crucial ingredient in our upcoming work on sums of reciprocals of fractional parts over general boxes.

Similar techniques, but in a much more specific setting, were used by Widmer in [20, Corollary 1.1] (case $\mathcal{M} = 2$, $\mathcal{N} = 1$ of Proposition 1.2) and [21, Theorem 2.1] (case $\mathcal{M} = 1$ of Theorem 1.5). Here, we extend Widmer's ideas to prove a highly general counting principle, that is the central object of this paper. In lieu of the lattice $\Lambda_{\boldsymbol{L}}$ and the set Z, we consider a general weakly admissible lattice in $\mathbb{R}^{\mathcal{M}+\mathcal{N}}$ and a general definable set contained in Z.

A rather different approach was used by Skriganov [18] to establish counting results for polyhedra. He used analytic (e.g., Poisson summation formula) and dynamical methods, and hence, required the dual lattice to be weakly admissible (see [19] for a comparison between weak admissibility for a lattice and its dual).

Before stating the main result, we introduce some notation and we recall the main definitions. Let $L \in \mathbb{N}$ and let $\boldsymbol{l} \in \mathbb{N}^L$. We set $V_{\boldsymbol{l}} := \prod_{h=1}^{L} \mathbb{R}^{l_h}$ and we write $\underline{\boldsymbol{v}} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_L)$ for any vector $\underline{\boldsymbol{v}}$ in $V_{\boldsymbol{l}}$. Note that each $\boldsymbol{\alpha} \in (0, +\infty)^L$ induces a multiplicative norm on the space $V_{\boldsymbol{l}}$, given by $\prod_{h=1}^{L} |\boldsymbol{v}_h|_2^{\alpha_i}$. We indicate this norm by $\mathrm{Nm}_{\boldsymbol{\alpha}}(\underline{\boldsymbol{v}})$. The following definition is a generalisation of the property considered in Definition 1.1, due to Widmer [21].

Definition 1.3. Let

$$C = C(I) := \{\underline{\boldsymbol{v}} \in V_{\boldsymbol{l}} : \boldsymbol{v}_h = \boldsymbol{0},\ h \in I\}$$

for some $\emptyset \neq I \subset \{1, \ldots, L\}$, and let $\mathcal{A} := \alpha_1 + \cdots + \alpha_L$. We say that a full rank lattice $\Lambda \subset V_{\boldsymbol{l}}$ is weakly admissible for the couple $((\boldsymbol{l}, \boldsymbol{\alpha}), C)$ if

$$\nu(\Lambda, \varrho) := \inf \left\{ \mathrm{Nm}_{\boldsymbol{\alpha}}(\underline{\boldsymbol{v}})^{1/\mathcal{A}} : \underline{\boldsymbol{v}} \in \Lambda \setminus C,\ |\underline{\boldsymbol{v}}|_2 < \varrho \right\} > 0$$

for all $\varrho > 0$, where we interpret $\inf \emptyset$ as $+\infty > 0$.

Note that weakly admissible lattices also appear in Skriganov's work [18, p.17]. However, his definition of weak admissibility assumes a stronger hypothesis on the lattice, that does not leave room for exceptional subspaces like C in Definition 1.3.

For our purpose, it is convenient to work with the product of two spaces of the form $V_{\boldsymbol{l}}$. We therefore adopt a double index notation. Let $M, N \in \mathbb{N}$ and let $\mathcal{S} := ((\boldsymbol{m}, \boldsymbol{n}), (\boldsymbol{\beta}, \boldsymbol{\gamma}))$, where $\boldsymbol{m} \in \mathbb{N}^M$, $\boldsymbol{n} \in \mathbb{N}^N$, $\boldsymbol{\beta} \in (0, +\infty)^M$, and $\boldsymbol{\gamma} \in (0, +\infty)^N$. Let $\mathcal{M} := \sum_{i=1}^M m_i$ and let $\mathcal{N} := \sum_{j=1}^N n_j$. Let also $\mathcal{B} := \sum_{i=1}^M \beta_i$ and let $\mathcal{C} := \sum_{j=1}^N \gamma_j$. We consider the vector space $V := V_{\boldsymbol{m}} \times V_{\boldsymbol{n}} := \prod_{i=1}^M \mathbb{R}^{m_i} \times \prod_{j=1}^N \mathbb{R}^{n_j}$ and we denote its vectors by $(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_N)$. As mentioned above, the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ induce a multiplicative norm on V, given by $\mathrm{Nm}_{(\boldsymbol{\beta},\boldsymbol{\gamma})}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) := \mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\boldsymbol{x}}) \mathrm{Nm}_{\boldsymbol{\gamma}}(\underline{\boldsymbol{y}})$, where $\mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\boldsymbol{x}}) := \prod_{i=1}^M |\boldsymbol{x}_i|_2^{\beta_i}$ and $\mathrm{Nm}_{\boldsymbol{\gamma}}(\underline{\boldsymbol{y}}) := \prod_{j=1}^N |\boldsymbol{y}_j|_2^{\gamma_j}$. Throughout this section, we fix a subspace $C \subset V$ of the form

$$C = C(I, J) := \left\{ (\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) \in V : \boldsymbol{x}_i = \boldsymbol{0},\ i \in I, \boldsymbol{y}_j = \boldsymbol{0},\ j \in J \right\},$$

where $I \subseteq \{1, \ldots, M\}$, $J \subseteq \{1, \ldots, N\}$, and $I \cup J \neq \emptyset$.

Now, we introduce a generalisation of the set Z appearing in Proposition 1.2. We let

$$\mathcal{H} := \left\{ (\underline{\boldsymbol{x}}, \varepsilon, T) \in V_{\boldsymbol{m}} \times (0, +\infty)^2 : \mathrm{Nm}_{\boldsymbol{\beta}}(\underline{\boldsymbol{x}})^{\frac{1}{\mathcal{B}}} < \varepsilon,\ |\boldsymbol{x}_i|_2 \leq T,\ i = 1, \ldots, M \right\},$$

and

$$\mathcal{R} := \left\{ (\underline{\boldsymbol{y}}, \boldsymbol{Q}) \in V_{\boldsymbol{n}} \times \mathbb{R}^N : |\boldsymbol{y}_j|_2 \leq Q_j,\ j = 1, \ldots, N \right\}.$$

Then, we set $\mathcal{Z} := \mathcal{H} \times \mathcal{R}$. Finally, for $\boldsymbol{Q} \in (0, +\infty)^N$ we define

$$Q := \left(\prod_{j=1}^N Q_j^{\gamma_j} \right)^{1/\mathcal{C}},$$

and for any $\Gamma \subset V_{\boldsymbol{l}}$ we define $\lambda_1(\Gamma) := \inf\{|\boldsymbol{v}|_2 : \boldsymbol{v} \in \Gamma \setminus \{\boldsymbol{0}\}\}$.

To make our counting result applicable to a large class of sets we use o-minimal structures.

Definition 1.4. A structure over $\mathbb{R}$ is a sequence $\mathfrak{S} = (\mathfrak{S}_n)_{n \in \mathbb{N}}$ of families of subsets of $\mathbb{R}^n$ such that for each n:

i) $\mathfrak{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$ (under the usual set-theoretic operations);

ii) $\mathfrak{S}_n$ contains every semi-algebraic subset of $\mathbb{R}^n$;

iii) if $A \in \mathfrak{S}_n$ and $B \in \mathfrak{S}_m$, then $A \times B \in \mathfrak{S}_{n+m}$;

iv) if $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the projection onto the first n coordinates and $A \in \mathfrak{S}_{n+m}$, then $\pi(A) \in \mathfrak{S}_n$.

A structure $\mathfrak{S}$ over $\mathbb{R}$ is said to be *o*-minimal if additionally:

v) the boundary of any set in $\mathfrak{S}_1$ is finite.

We would like to point out that the key result of this paper (Theorem 1.5) relies on the definition of *o*-minimal structure given in [2], which is more general than Definition 1.4 (e.g., the structure of semi-linear sets is *o*-minimal according to the definition in [2] but not *o*-minimal according to Definition 1.4). Despite this, we decided to use Definition 1.4 since it is enough for the purposes of this paper.

A set $S \subset \mathbb{R}^n$ is definable in the structure $\mathfrak{S}$ if $S \in \mathfrak{S}_n$. A map $f : A \to B$ is definable if its graph $\Gamma(f) := \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in A\} \subset A \times B$ is a definable set.

Let $t \in \mathbb{N}$ and let $\mathcal{W} \subset V_{\boldsymbol{l}} \times \mathbb{R}^t$ be a definable set. We call $\mathcal{W}$ a definable family in $V_{\boldsymbol{l}}$, and we call the variables $\boldsymbol{\tau} \in \mathbb{R}^t$ parameters of $\mathcal{W}$. For $\boldsymbol{\tau} \in \mathbb{R}^t$ we call the set

$$W_{\boldsymbol{\tau}} := \{\underline{\boldsymbol{v}} \in V_{\boldsymbol{l}} : (\underline{\boldsymbol{v}}, \boldsymbol{\tau}) \in \mathcal{W}\}$$

the fibre of $\mathcal{W}$ above $\boldsymbol{\tau}$. In our setting, the functions $f(x) = x^r = \exp(r \log x)$ (for any real r>0) and $\log x$, defined on $(0, +\infty)$, need to be definable. Therefore, we require that the o-minimal structure $\mathfrak{S}$ we are working with extends Wilkie's o-minimal structure $\mathbb{R}_{\exp}$ [22], i.e., we require that each set definable in $\mathbb{R}_{\exp}$ is also definable in $\mathfrak{S}$.

We recall that every subset of $V_{\boldsymbol{l}} \times \mathbb{R}^t$ of the form

$$\mathcal{W} = \left\{ (\underline{\boldsymbol{v}}, \boldsymbol{\tau}) \in V_{\boldsymbol{l}} \times \mathbb{R}^t : \mathfrak{F}(\underline{\boldsymbol{v}}, \boldsymbol{\tau}) \geq \mathbf{0} \ (> \mathbf{0}) \right\},$$

where $\mathfrak{F}$ is a finite system of functions in the variables $\underline{\boldsymbol{v}}$ and $\boldsymbol{\tau}$, obtained by the (suitably interpreted) composition of polynomials, exponential functions $\exp : \mathbb{R} \to \mathbb{R}$, and logarithms $\log : (0, +\infty) \to \mathbb{R}$, is definable in $\mathbb{R}_{\exp}$.

From now on, we see the set $\mathcal{Z}$ as a definable family in $\mathbb{R}_{\exp}$, with parameters $\boldsymbol{\eta} := (\varepsilon, T, \boldsymbol{Q}) \in (0, +\infty)^{2+N}$. In analogy with the above, we indicate its fibres by $Z_{\boldsymbol{\eta}}$. We can now state our main theorem.

Theorem 1.5. *Let $\Lambda \subset V$ be a weakly admissible lattice for the couple $(\mathcal{S}, C)$ and let $\mathcal{W} \subset V \times \mathbb{R}^t$ be definable in an o-minimal structure expanding $\mathbb{R}_{\exp}$. Suppose that for all $\boldsymbol{\tau} \in \mathbb{R}^t$ there exists $\boldsymbol{\eta}(\boldsymbol{\tau}) = (\varepsilon, T, \boldsymbol{Q}) \in (0, +\infty)^{2+N}$ such that $W_{\boldsymbol{\tau}} \subset Z_{\boldsymbol{\eta}(\boldsymbol{\tau})}$. Then, for all $\boldsymbol{\tau} \in \mathbb{R}^t$ and all choices of $\boldsymbol{\eta}(\boldsymbol{\tau})$ with $T/\varepsilon > e$ (where $e = 2.71828\dots$ is the base of the natural logarithm) we have*

$$\left| \#(\Lambda \cap W_{\boldsymbol{\tau}}) - \frac{\mathrm{Vol} W_{\boldsymbol{\tau}}}{\det \Lambda} \right| \ll_{\mathcal{W}, \beta, \gamma}$$

$$\inf_{0 < r \leq \mathrm{diam}(Z_{\boldsymbol{\eta}(\boldsymbol{\tau})})} \log\left(\frac{T}{\varepsilon}\right)^{M-1} \left(\frac{\left(\varepsilon^{\mathcal{B}} Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}}{\nu(\Lambda, r)} + \frac{\mathrm{diam}\left(Z_{\boldsymbol{\eta}(\boldsymbol{\tau})}\right)}{r} + \frac{\mathrm{diam}\left(Z_{\boldsymbol{\eta}(\boldsymbol{\tau})} \cap C\right)}{\lambda_1(\Lambda \cap C)} \right)^{\mathcal{M}+\mathcal{N}-1}.$$

Note the importance of the definability condition on the family $\mathcal{W}$. Indeed, if, e.g., $W_\tau = \Lambda \cap Z_{\eta(\tau)}$ for all $\boldsymbol{\tau} \in \mathbb{R}^t$, Theorem 1.5 obviously fails. The family $(\Lambda \times \mathbb{R}^t) \cap \mathcal{Z}$ is however not definable.

1.3 Applications I

Theorem 1.5 has some interesting applications in multiplicative Diophantine approximation. Let $\boldsymbol{Q} \in (0, +\infty)^{\mathcal{N}}$ and let $X := \prod_{j=1}^{\mathcal{N}} [-Q_j, Q_j]$. We set

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) := \sum_{\boldsymbol{q} \in X \cap \mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}} \prod_{i=1}^{\mathcal{M}} \|L_i \boldsymbol{q}\|^{-1}.$$

The function $S_{\boldsymbol{L}}(\boldsymbol{Q})$ is of major importance in several branches of Diophantine Approximation and Geometry of Numbers. For instance, Kuipers and Niederreiter use it to control the discrepancy of the sequence $\{q\boldsymbol{L}\}_{q \in \mathbb{Z}}$ via the Erdős-Turan inequality [13, p.122,129,131]. Hardy and Littlewood use it to count the number of lattice points contained in certain polygons [7],[8]. Beresnevich, Haynes, and Velani, work out very precise estimates for $S_{\boldsymbol{L}}(\boldsymbol{Q})$ in the one dimensional inhomogeneous setting [4]. Huang and Liu show how estimates of $S_{\boldsymbol{L}}(\boldsymbol{Q})$ can be used to solve the convergence case of the generalised Baker-Schmidt problem for simultaneous approximation on certain affine subspaces [10].

In this section, we focus on a question raised by Lê and Vaaler [14]. Lê and Vaaler show that for $Q := (Q_1 \cdots Q_{\mathcal{N}})^{1/\mathcal{N}} \geq 1$ it holds

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \gg_{\mathcal{M}, \mathcal{N}} Q^{\mathcal{N}} (\log Q)^{\mathcal{M}}, \tag{32}$$

independently of the choice of the matrix $\boldsymbol{L}$ [14, Corollary 1.2]. They also ask whether the estimate in (32) is sharp, i.e., whether there exists a matrix $\boldsymbol{L}$ such that

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll Q^{\mathcal{N}}(\log Q)^{\mathcal{M}}. \tag{33}$$

Lê and Vaaler themselves show that (33) holds true whenever the matrix $\boldsymbol{L}$ is multiplicatively badly approximable [14, Theorem 2.1]. However, multiplicatively badly approximable matrices seem unlikely to exist for $\mathcal{M}+\mathcal{N} \geq 3$, since each of them would provide a counterexample to the Littlewood conjecture. In the present section we prove a general estimate for the function $S_{\boldsymbol{L}}(\boldsymbol{Q})$, when $Q_1 = Q_2 = \cdots = Q_{\mathcal{N}}$. This estimate shows that, in the special case $Q_1 = Q_2 = \cdots = Q_{\mathcal{N}}$, Lê and Vaaler's hypothesis can be significantly weakened. Let us first recap some definitions (see [5][16] for a deeper insight).

Definition 1.6. Let $\phi : [1, +\infty) \to (0, 1]$ be non-increasing. We say that a matrix $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ is ϕ-additively badly approximable if

$$|\boldsymbol{q}|_\infty^{\mathcal{N}} \left(\max_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\|\right)^{\mathcal{M}} \geq \phi(|\boldsymbol{q}|_\infty) \tag{34}$$

for all $\boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}$. We say that $\boldsymbol{L}$ is ϕ-multiplicatively badly approximable if

$$\prod_{j=1}^{\mathcal{N}} \max\{1, |q_j|\} \prod_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\| \geq \phi\left(\prod_{j=1}^{\mathcal{N}} \max\{1, |q_j|\}^{\frac{1}{\mathcal{N}}}\right) \tag{35}$$

for all $\boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}$. If ϕ can be chosen constant in either case, we say that $\boldsymbol{L}$ is additively or multiplicatively badly approximable.

The additive and multiplicative conditions are very different. Schmidt [16] showed that the set of additively badly approximable matrices in $\mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ has full Hausdorff dimension. On the other hand, as mentioned above, multiplicatively badly approximable matrices are unlikely to exist for $\mathcal{M}+\mathcal{N} \geq 3$.

We introduce a new condition, which is hybrid between (34) and (35).

Definition 1.7. Let ϕ be as in Definition 1.6. We say that a matrix $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ is ϕ-semimultiplicatively badly approximable if

$$|\boldsymbol{q}|_\infty^{\mathcal{N}} \prod_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\| \geq \phi(|\boldsymbol{q}|_\infty) \tag{36}$$

for all $\boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}$. If the function ϕ can be chosen constant, we say that $\boldsymbol{L}$ is semimultiplicatively badly approximable.

Note that (35)⇒(36)⇒(34). Now, under the additional hypothesis $Q_1 = \cdots = Q_{\mathcal{N}}$, we have the following estimate for $S_{\boldsymbol{L}}(\boldsymbol{Q})$.

Corollary 1.8. *Let $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ be a ϕ-semimultiplicatively badly approximable matrix. Then, for $Q \geq 2$ we have*

$$\sum_{\substack{\boldsymbol{q}\in[-Q,Q]^{\mathcal{N}} \\ \cap\ \mathbb{Z}^{\mathcal{N}}\setminus\{\boldsymbol{0}\}}} \prod_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\|^{-1} \ll_{\mathcal{M},\mathcal{N}} Q^{\mathcal{N}} \log\left(\frac{Q^{\mathcal{N}}}{\phi(Q)}\right)^{\mathcal{M}} + \frac{Q^{\mathcal{N}}}{\phi(Q)} \log\left(\frac{Q^{\mathcal{N}}}{\phi(Q)}\right)^{\mathcal{M}-1}.$$

Corollary 1.8 immediately implies that, in the special case $Q_1 = \cdots = Q_{\mathcal{N}}$, all matrices $\boldsymbol{L}$ that are ϕ-semimultiplicatively badly approximable with $\phi(x) \gg_{\mathcal{M},\mathcal{N}} (\log x)^{-1}$ satisfy (33).

The case $\mathcal{M} = 2$, $\mathcal{N} = 1$ of Corollary 1.8 was proved by Widmer [20], whereas the analogous result for $\mathcal{M} = 1$ was proved by the author [6], by using Widmer's $\mathcal{M} = 1$ case of Theorem 1.5 [21, Theorem 2.1]. Huang and Liu addressed the general case [10, Theorem 6] by using the well-known gap principle [13, Proof of Lemma 3.3 p.123]. However, Huang and Liu's Theorem 6 contains an extra factor of $1/\phi(Q)$ in the error term, which we could get rid of in Corollary 1.8.

Unfortunately, the existence of ϕ-semimultiplicatively badly approximable matrices with $\phi(x) \gg_{\mathcal{M},\mathcal{N}} (\log x)^{-1}$ has not yet been established, except for $\mathcal{M} = \mathcal{N} = 1$. Despite this, at least in dimension 2, there is some heuristic evidence for their existence. For $\mathcal{M} = 2, \mathcal{N} = 1$ Badziahin showed that condition (35), with $\phi(x) = c_{\boldsymbol{L}}(\log x \log\log x)^{-1}$ ($c_{\boldsymbol{L}} > 0$ sufficiently small), holds true for a set of vectors of full Hausdorff dimension [1]. It follows from Corollary 1.8 that for $\mathcal{M} = 2, \mathcal{N} = 1$ the set of matrices $\boldsymbol{L}$ such that $S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{\boldsymbol{L}} Q(\log Q)^2 \log\log Q$ has full Hausdorff dimension. Badziahin and Velani also conjectured that the set of 2×1 ϕ-multiplicatively badly approximable matrices, with $\phi(x) = c_{\boldsymbol{L}}(\log x)^{-1}$ ($c_{\boldsymbol{L}} > 0$ sufficiently small), has full Hausdorff dimension [1, Conjecture 1]. To the best of our knowledge, nothing is known in higher dimension.

1.4 Applications II

Let $\psi : [1, +\infty) \to (0, 1]$. We consider the set

$$\mathscr{S}_{\mathcal{N}}(\psi) := \left\{\boldsymbol{x} \in \mathbb{R}^{\mathcal{N}} : \exists \text{ i.m. } q \in \mathbb{N} \text{ such that } \max_{i=1}^{\mathcal{N}} \|qx_i\| < \psi(q)\right\},$$

where i.m. stands for infinitely many. The set $\mathscr{S}_{\mathcal{N}}$ is said to be the set of simultaneously ψ-approximable points. A well-known theorem of Khintchine [11] relates the Lebesgue measure of the set $\mathscr{S}_{\mathcal{N}}(\psi)$ to the convergence of the sum $\sum_{q=1}^{+\infty} \psi(q)^{\mathcal{N}}$.

Khintchine showed that if ψ is non-increasing and $\sum_{q=1}^{+\infty} \psi(q)^{\mathcal{N}}$ converges, we have $\mathrm{Vol}\left(\mathscr{S}_{\mathcal{N}}(\psi)\right) = 0$, whereas if ψ is non-increasing and $\sum_{q=1}^{+\infty} \psi(q)^{\mathcal{N}}$ diverges, we have $\mathrm{Vol}\left(\mathscr{S}_{\mathcal{N}}(\psi)\right) = +\infty$. It is well known that this theorem becomes false when we restrict to certain submanifolds of $\mathbb{R}^{\mathcal{N}}$, such as proper rational affine subspaces. This leads naturally to the following definition.

Definition 1.9. Let $\mathcal{M} \subset \mathbb{R}^{\mathcal{N}}$ be a submanifold of dimension d. We say that $\mathcal{M}$ is of Khintchine type for convergence if for all non-increasing functions $\psi : [1, +\infty) \to (0, 1]$ we have

$$\sum_{q=1}^{+\infty} \psi(q)^{\mathcal{N}} < +\infty \Rightarrow \mathrm{Vol}_d\left(\mathscr{S}_{\mathcal{N}}(\psi) \cap \mathcal{M}\right) = 0.$$

We say that $\mathcal{M}$ is of Khintchine type for divergence if for all non-increasing functions $\psi : [1, +\infty) \to (0, 1]$ we have

$$\sum_{q=1}^{+\infty} \psi(q)^{\mathcal{N}} = +\infty \Rightarrow \mathrm{Vol}_d\left(\mathscr{S}_{\mathcal{N}}(\psi) \cap \mathcal{M}\right) = \mathrm{Vol}_d(\mathcal{M}).$$

If both conditions hold, we simply say that $\mathcal{M}$ is of Khintchine type.

We recall that there is also a notion of strong Khintchine type submanifold in $\mathbb{R}^{\mathcal{N}}$, i.e., a submanifold for which Definition 1.9 holds without the assumption that the function ψ is non-increasing.

It has been shown that many non-degenerate submanifolds (i.e., those that in some sense deviate from a hyperplane at each point) are of strong Khintchine type for convergence [9],[17]. It seems natural then, to ask whether the non-degeneracy condition is necessary for a submanifold to be of (strong) Khintchine type. The answer to this question is no, and indeed it turns out that even some proper affine subspaces of $\mathbb{R}^{\mathcal{N}}$ are of strong Khintchine type [12],[15]. So, what makes an affine subspace of (strong) Khintchine type? Since each affine subspace is defined by a real matrix, it appears interesting to try and establish a link between the Diophantine type of this matrix and the properties of the subspace in terms of the validity of the Khintchine Theorem. In a very recent paper [10] Huang and Liu made some progress in this direction.

Definition 1.10. Let $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$. We set

$$\omega_{sm}(\boldsymbol{L}) := \sup\left\{\gamma \in \mathbb{R} : \prod_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\| \leq |\boldsymbol{q}|_\infty^{-\gamma} \text{ has i.m. solutions } \boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}\right\}.$$

We call $\omega_{sm}(\boldsymbol{L})$ the semimultiplicative exponent of the matrix $\boldsymbol{L}$.

Observe that for $\phi_\gamma(x) := x^{-\gamma}$ ($\gamma \in \mathbb{R}$) we have

$\omega_{sm}(\boldsymbol{L}) = \inf\{\gamma \in \mathbb{R} : \exists c > 0$ such that $\boldsymbol{L}$ is $c\phi_{(\gamma-\mathcal{N})}$-semimultiplicatively badly approximable$\}$.

Let $d \geq 1$ be an integer, and let $\boldsymbol{A} \in \mathbb{R}^{d\times(\mathcal{N}-d)}$. Let also $\boldsymbol{\alpha}_0 \in \mathbb{R}^{\mathcal{N}-d}$. We define

$$\tilde{\boldsymbol{A}} := \begin{pmatrix} \boldsymbol{\alpha}_0 \\ \boldsymbol{A} \end{pmatrix} \in \mathbb{R}^{(d+1)\times(\mathcal{N}-d)} \quad \text{and} \quad \tilde{\boldsymbol{x}} := (1, \boldsymbol{x}) \in \mathbb{R}^{d+1} \text{ for } \boldsymbol{x} \in \mathbb{R}^d. \tag{37}$$

Then, we consider the following submanifold of $\mathbb{R}^{\mathcal{N}}$.

$$\mathscr{H} := \{(\boldsymbol{x}, \tilde{\boldsymbol{x}}\tilde{\boldsymbol{A}}) : \boldsymbol{x} \in [0,1]^d\}.$$

Huang and Liu [10, Theorem 1] proved that if $\omega_{sm}\left(\tilde{\boldsymbol{A}}\right) < \mathcal{N}(d+1)$, the submanifold $\mathscr{H}$ is of Khintchine type for convergence, whereas if $\omega_{sm}(\boldsymbol{A}) < \mathcal{N}d$, the submanifold $\mathscr{H}$ is of strong Khintchine type for convergence. We recall that, when $\mathcal{N} - d = 1$, for any $\omega_0 \geq 1$ there exist matrices $\boldsymbol{A} \in \mathbb{R}^{d\times 1}$ such that $\omega_{sm}(\boldsymbol{A}) = \omega_0$. This is a consequence of [5, Theorem 1]. More precisely, if $\mathcal{N} - d = 1$, we have

$$\dim\left\{\boldsymbol{A} \in \mathbb{R}^{d\times 1} : \omega_{sm}(\boldsymbol{A}) = \omega_0\right\} = d - 1 + \frac{2}{1+\omega_0},$$

where dim denotes the Hausdorff dimension. For $\mathcal{N} - d > 1$ the spectrum of the exponent $\omega_{sm}(\boldsymbol{A})$ is known to lie in the extended interval $[\mathcal{N} - d, +\infty]$. Heuristic evidence suggests that most likely it coincides with the whole interval $[\mathcal{N} - d, +\infty]$ (this is indeed true for the additive and multiplicative exponents, see [5, Theorem D and Theorem 1]), however, this fact hasn't been established. It can also be shown that for all $\varepsilon > 0$ the set of matrices $\boldsymbol{A} \in \mathbb{R}^{d\times(\mathcal{N}-d)}$ such that $\omega_{sm}(\boldsymbol{A}) \leq \mathcal{N} - d + \varepsilon$ has actually full Lebesgue measure (follows from the main result in [23]). Hence, Huang and Liu's theorem holds for generic matrices $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$.

One could ask if anything can be said about the limit cases, i.e., $\omega_{sm}(\tilde{\boldsymbol{A}}) = \mathcal{N}(d+1)$ and $\omega_{sm}(\boldsymbol{A}) = \mathcal{N}d$. We show that that, up to a logarithmic factor, these cases yield Khintchine type subspaces.

Definition 1.11. Let $\boldsymbol{L} \in \mathbb{R}^{\mathcal{M}\times\mathcal{N}}$ and let $\omega_0 \in \mathbb{R}$. We set

$$\omega'_{sm}(\boldsymbol{L},\omega_0) := \sup\left\{\gamma \in \mathbb{R} : \prod_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\| \leq |\boldsymbol{q}|_\infty^{-\omega_0} \log(|\boldsymbol{q}|_\infty)^{-\gamma} \text{ has i.m. solutions } \boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}} \setminus [-1,1]^{\mathcal{N}}\right\}.$$

We call $\omega_{sm}(\boldsymbol{L},\omega_0)$ the semimultiplicative logarithmic exponent of the matrix $\boldsymbol{L}$ at ω_0.

Corollary 1.12. *Let $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$ be as above. Then,*

i) if $\omega_{sm}\left(\tilde{\boldsymbol{A}}\right) = \mathcal{N}(d+1)$ and $\omega'_{sm}\left(\tilde{\boldsymbol{A}},\mathcal{N}(d+1)\right) < 1-2(d+1)$, the submanifold $\mathscr{H}$ is of Khintchine type for convergence;

ii) if $\omega_{sm}(\boldsymbol{A}) = \mathcal{N}d$ and $\omega'_{sm}(\boldsymbol{A},\mathcal{N}d) < 1-2d$, the submanifold $\mathscr{H}$ is of strong Khintchine type for convergence.

Unfortunately, not much is known about the existence of matrices with prescribed multiplicative logarithmic order. However, their existence is established in the additive setting [3]. We can therefore say something about the case $d = 1$. From [3, Theorem 1] we can easily deduce that if $d = 1$, there always exist matrices $\boldsymbol{A} \in \mathbb{R}^{1\times(\mathcal{N}-1)}$ such that $\omega'_{sm}(\boldsymbol{A},\mathcal{N}) = \omega_1$ for any given $\omega_1 \in \mathbb{R}$ (this implies $\omega_{sm}(\boldsymbol{A}) = \mathcal{N}$). More precisely, we have

$$\dim\left\{\boldsymbol{A} \in \mathbb{R}^{1\times(\mathcal{N}-1)} : \omega'_{sm}(\boldsymbol{A},\mathcal{N}) = \omega_1\right\} = \mathcal{N} - 2 + \frac{\mathcal{N}}{1+\mathcal{N}},$$

independently of the choice of ω_1 (here dim denotes the Hausdorff dimension). It follows from Corollary 1.12 that there exist strong Khintchine type lines in $\mathbb{R}^{\mathcal{N}}$ with exponent $\omega_{sm} = \mathcal{N}$, improving on [10, Theorem 1]. For higher values of ω_{sm} very little is known.

Now, [10, Theorem 1] follows in turn from [10, Theorems 2 and 3]. These results establish some Khintchine type conditions for the submanifold $\mathscr{H}$ with respect to general s-dimensional Hausdorff measures (i.e., s need not coincide with the dimension of the submanifold). The problem of establishing Khintchine type conditions with respect to general Hausdorff measures is widely known as the generalised Baker-Schmidt problem. Huang and Liu prove that such conditions hold for the convergence case, when $\omega_{sm}(\tilde{\boldsymbol{A}}) < (d+1)(\mathcal{N}-d+s)/(d+1-s)$ or $\omega_{sm}(\boldsymbol{A}) < d(\mathcal{N}-d+s)/(d+1-s)$. We show that [10, Theorems 2 and 3] can be refined to include the limit cases.

Proposition 1.13. *Let* $\boldsymbol{A} \in \mathbb{R}^{d\times(\mathcal{N}-d)}$ *and let* $\boldsymbol{\alpha}_0 \in \mathbb{R}^{\mathcal{N}-d}$. *Let* $\tilde{\boldsymbol{A}}$ *be the matrix defined in (37), and let* $s \in [0,+\infty)$. *Assume that* $\tilde{\boldsymbol{A}}$ *is* $\tilde{\phi}$*-semimultiplicatively badly approximable, where* $\tilde{\phi} : [1,+\infty) \to (0,1]$ *is a non-increasing function with the following properties:*

i) $\tilde{\phi}(\lambda x) \gg_\lambda \tilde{\phi}(x)$ *for all* $\lambda \gg 1$*;*

ii) $x^{-\gamma} \ll \tilde{\phi}(x) \ll 1/\log(x)$ *for some* $\gamma > 0$*;*

iii) there exists a non-increasing function $\hat{\psi} : [1,+\infty) \to (0,1]$ *such that*

iiia) $\sum_{q=1}^{+\infty} \hat{\psi}(q)^{\mathcal{N}-d+s} q^{d-s} < +\infty$*;*

iiib) $\tilde{\phi}\left(1/\hat{\psi}(x)\right) \hat{\psi}(x)^{\mathcal{N}-d} \gg_{\mathcal{N},d,s} \log(x)^d/x^{d+1}$.

Then, for all non-increasing approximating functions $\psi : [1,+\infty) \to (0,1]$ *such that* $\sum_{q=1}^{+\infty} \psi(q)^{\mathcal{N}-d+s} q^{d-s} < +\infty$*, we have* $\mathrm{Vol}_s\left(\mathscr{S}_{\mathcal{N}}(\psi) \cap \mathscr{H}\right) = 0$.

Note that when $\hat{\psi}(x)$ is of the form $\hat{\psi}(x) = x^{-\gamma'}$, with $\gamma' > 0$, condition *iii)* implies

$$\omega_m\left(\tilde{\boldsymbol{A}}\right) \le \frac{(d+1)(\mathcal{N}-d+s)}{d+1-s},$$

i.e., the hypothesis in Huang and Liu's theorem along with the limit case.

Proposition 1.14. *Let* $s \in [0,+\infty)$ *and let* $\boldsymbol{A} \in \mathbb{R}^{d\times(\mathcal{N}-d)}$ *be a* ϕ*-semimultiplicatively badly approximable matrix, where* $\phi : [1,+\infty) \to (0,1]$ *is a non-increasing function with the following properties:*

i) $\phi(\lambda x) \gg_\lambda \phi(x)$ *for all* $\lambda \gg 1$*;*

ii) $x^{-\gamma} \ll \phi(x) \ll 1/\log(x)$ *for some* $\gamma > 0$*;*

iii) there exists a function $\hat{\psi} : [1,+\infty) \to (0,1]$ *such that*

iiia) $\sum_{q=1}^{+\infty} \hat{\psi}(q)^{\mathcal{N}-d+s} q^{d-s} < +\infty$*;*

iiib) $\phi\left(1/\hat{\psi}(x)\right) \hat{\psi}(x)^{\mathcal{N}-d} \gg_{\mathcal{N},d,s} \log(x)^{d-1}/x^d$.

Then, for all approximating functions $\psi : [1, +\infty) \to (0, 1]$ *such that* $\sum_{q=1}^{+\infty} \psi(q)^{\mathcal{N}-d+s} q^{d-s} < +\infty$*, we have* $\mathrm{Vol}_s\left(\mathscr{S}_{\mathcal{N}}(\psi) \cap \mathscr{H}\right) = 0$.

With Propositions 1.13 and 1.14 at hand, the proof of Corollary 1.12 is straightforward. We sketch it below.

Proof. Let $\log^*(x) := \max\{1, \log(x)\}$ for all $x \in (0, +\infty)$. The proof follows from taking $\tilde{\phi}(x) = \tilde{c}x^{\mathcal{N}-d-\omega_{sm}(\tilde{\boldsymbol{A}})} \log^*(x)^{-\omega'_{sm}(\tilde{\boldsymbol{A}})-\tilde{\varepsilon}}$ $(\tilde{c} > 0)$ and $\phi(x) = cx^{\mathcal{N}-d-\omega_{sm}(\boldsymbol{A})} \log^*(x)^{-\omega'_{sm}(\boldsymbol{A})-\varepsilon}$ $(c > 0)$, and applying the case $s = d$ of Propositions 1.14 and 1.13 with $\hat{\psi}(x) := x^{-1/\mathcal{N}} \log^*(x)^{(-1-\varepsilon')/\mathcal{N}}$, where $\varepsilon, \tilde{\varepsilon}, \varepsilon'$ are small constants. $\square$

Note that we intentionally chose not to specify the function $\hat{\psi}$ in Propositions 1.13 and 1.14, since these results could be used to derive even finer Diophantine conditions on subspaces, involving, e.g, iterated logarithms.

2 Proof of Theorem 1.5

From now on, we fix an o-minimal structure $\mathfrak{S}$ extending $\mathbb{R}_{\exp}$, and we say that a set $S \subset \mathbb{R}^n$ is definable if it is definable in $\mathfrak{S}$. We fix the parameters $\boldsymbol{\tau}$ and $\boldsymbol{\eta}(\boldsymbol{\tau}) = (\varepsilon, T, \boldsymbol{Q}) \in (0, +\infty)^{2+N}$ such that $W_{\boldsymbol{\tau}} \subset Z_{\boldsymbol{\eta}(\boldsymbol{\tau})}$. For simplicity, we set $W := W_{\boldsymbol{\tau}}$ and $Z := Z_{\boldsymbol{\eta}(\boldsymbol{\tau})}$. We also write H for $H_{\varepsilon,T}$ and R for $R_{\boldsymbol{Q}}$.

To prove our estimate, we partition the set Z and we consider the induced partition on W. We then count the lattice points contained in each subset of this partition. Let

$$H_+ := H \cap \{\underline{\boldsymbol{x}} \in V_{\boldsymbol{m}} : \boldsymbol{x}_i \neq \boldsymbol{0}\}$$

and let $Z_+ := H_+ \times R$. Let also

$$H^i := H \cap \{\underline{\boldsymbol{x}} \in V_{\boldsymbol{m}} : \boldsymbol{x}_i = \boldsymbol{0}\}$$

and $Z^i := H^i \times R$ for $i = 1, \ldots, M$. We set $W_+ := W \cap Z_+$ and $W^i := W \cap Z^i$ for $i = 1, \ldots, M$. Then, we have

$$W = W_+ \cup \bigcup_{i=1}^{M} W^i.$$

Hence,

$$\left|\#(\Lambda \cap W) - \frac{\mathrm{Vol}W}{\det \Lambda}\right| \leq \left|\#(\Lambda \cap W_+) - \frac{\mathrm{Vol}W_+}{\det \Lambda}\right| + \sum_{i=1}^{M} \# \left(\Lambda \cap W^i\right),$$

To decompose the sets H_+ and W_+ we use the following crucial decomposition result. In the following proposition and throughout the paper the notation $\{a_i^k\}_{i=1,\dots,M}^{k\in\mathcal{K}}$ will stand for a double-index sequence and not for the k-th power of the real number a_i.

Proposition 2.1. *Assume that $T/\varepsilon > e$ (where $e = 2.71828\dots$ is the base of the natural logarithm). Then, there exists a partition of the set H_+ of the form $H_+ = \bigcup_{k\in\mathcal{K}} X_k$, and there exists a collection of linear maps $\{\varphi_k\}_{k\in\mathcal{K}}$, defined on the space $V_{\boldsymbol{m}}$, such that*

i) $\#\mathcal{K} \ll_{\boldsymbol{m},\boldsymbol{\beta}} \log\left(T/\varepsilon\right)^{M-1}$*;*

ii) each of the sets X_k for $k \in \mathcal{K}$ is definable;

iii) the maps φ_k for $k \in \mathcal{K}$ are defined by $\varphi_k(\underline{\boldsymbol{x}})_i = \exp\left(a_i^k - c\right)\boldsymbol{x}_i$ for $i = 1,\dots,M$, where $c \in \mathbb{R}$ is a constant only depending on $\boldsymbol{m}$ and the coefficients $a_i^k \in \mathbb{R}$ satisfy

iiia) $\exp\left(a_i^k - c\right) \gg_{\boldsymbol{m},\boldsymbol{\beta}} \varepsilon/T$ *for $i = 1,\dots,M$;*

iiib) $\sum_{i=1}^{M} \beta_i a_i^k = 0$*;*

iv) $\varphi_k\left(X_k\right) \subset \{|\boldsymbol{x}_i|_2 \leq \varepsilon,\ i = 1,\dots,M\}$ *for $k \in \mathcal{K}$.*

We prove Proposition 2.1 in Section 3. The following corollary is an immediate consequence of Proposition 2.1.

Corollary 2.2. *Let $\hat{X}_k := X_k \times R \subset V$ and let $\hat{\varphi}_k := (\varphi_k, \mathrm{id}) : V \to V$ for all $k \in \mathcal{K}$. Then,*

i) $Z_+ = \bigcup_{k\in\mathcal{K}} \hat{X}_k$ *is a partition of the set Z_+;*

ii) each of the sets $\hat{X}_k$ for $k \in \mathcal{K}$ is definable;

iii) $\hat{\varphi}_k(\hat{X}_k) \subset \{|\boldsymbol{x}_i|_2 \leq \varepsilon,\ i = 1,\dots,M\} \times R$ *for $k \in \mathcal{K}$.*

Corollary 2.2 yields the following partition of the set W_+.

$$W_+ = \bigcup_{k\in\mathcal{K}} W \cap \hat{X}_k.$$

Hence, we can write

$$\begin{aligned}
&\left|\#\left(\Lambda\cap W\right) - \frac{\mathrm{Vol}W}{\det\Lambda}\right| \le \left|\#(\Lambda\cap W_+) - \frac{\mathrm{Vol}W_+}{\det\Lambda}\right| + \sum_{i=1}^{M}\#(\Lambda\cap W^i)\\
&\le \sum_{k\in\mathcal{K}}\left|\#\left(\Lambda\cap W\cap\hat{X}_k\right) - \frac{\mathrm{Vol}(W\cap\hat{X}_k)}{\det\Lambda}\right| + \sum_{i=1}^{M}\#(\Lambda\cap W^i)\\
&= \sum_{k\in\mathcal{K}}\left|\#\left(\hat{\varphi}_k(\Lambda)\cap\hat{\varphi}_k\left(W\cap\hat{X}_k\right)\right) - \frac{\mathrm{Vol}\hat{\varphi}_k\left(W\cap\hat{X}_k\right)}{\det\hat{\varphi}_k(\Lambda)}\right| + \sum_{i=1}^{M}\#(\Lambda\cap W^i). \qquad (38)
\end{aligned}$$

Lemma 2.3. *Let $c := \dim C$. Then, for $i = 1,\ldots,M$ we have*

$$\#\left(\Lambda\cap W^i\right) \ll_{\boldsymbol{m},\boldsymbol{n}} 1 + \left(\frac{\mathrm{diam}(W\cap C)}{\lambda_1(\Lambda\cap C)}\right)^c.$$

Proof. By weak admissibility, we have $\Lambda\cap W_i \subset \Lambda\cap W\cap C$. Therefore, it is enough to estimate $\#(\Lambda\cap W\cap C)$. Now, $\Lambda\cap C$ is either $\{(\mathbf{0},\mathbf{0})\}$ or a full rank lattice in some subspace $C'\subset C$ with $\dim(C') = c' > 0$. To prove the claim, it suffices to show that for any bounded set $S\subset\mathbb{R}^n$ and any full rank lattice $\Gamma\subset\mathbb{R}^n$ we have

$$\#(\Gamma\cap S) \ll_n 1 + \left(\frac{\mathrm{diam}S}{\lambda_1(\Gamma)}\right)^n. \qquad (39)$$

This follows easily from [2, Lemmas 2.1 and 2.2]. Applying (39) to $(W\cap C')\cap(\Lambda\cap C')$, and noting that $c'\le c$ and $\lambda_1(\Lambda\cap C) = \lambda_1(\Lambda\cap C')$ yields

$$\begin{aligned}
\#\left(\Lambda\cap W^i\right) \le \#\left((W\cap C')\cap(\Lambda\cap C')\right) &\ll_{c'} 1 + \left(\frac{\mathrm{diam}(W\cap C)}{\lambda_1(\Lambda\cap C)}\right)^{c'}\\
&\ll_{\boldsymbol{m},\boldsymbol{n}} 1 + \left(\frac{\mathrm{diam}(W\cap C)}{\lambda_1(\Lambda\cap C)}\right)^{c}.
\end{aligned}$$

Note that in the last inequality we can replace c' by a bigger integer, due to the definition of the constant in (39) (see again [2, Lemmas 2.1 and 2.2]). □

We are left to estimate the quantity $\#\left(\hat{\varphi}_k(\Lambda)\cap\hat{\varphi}_k\left(W\cap\hat{X}_k\right)\right) - \mathrm{Vol}\hat{\varphi}_k\left(W\cap\hat{X}_k\right)/\det\hat{\varphi}_k(\Lambda)$ for $k\in\mathcal{K}$. By Corollary 2.2, we know that

$$\hat{\varphi}_k\left(W\cap\hat{X}_k\right) \subset \hat{\varphi}_k\left(\hat{X}_k\right) \subset \{|\boldsymbol{x}_i|_2 \le \varepsilon,\ i=1,\ldots,M\}\times R. \qquad (40)$$

To make the counting more effective, we reshape the set on the right-hand side of (40) into a ball-like shaped set. Let $\omega_1 : V \to V$ be the map

$$\omega_1(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) := \left(\underline{\boldsymbol{x}}, \frac{Q}{Q_1}\boldsymbol{y}_1, \dots, \frac{Q}{Q_N}\boldsymbol{y}_N\right),$$

and let $\omega_2 : V \to V$ be the map

$$\left(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}\right) \mapsto \left(\theta\underline{\boldsymbol{x}}, \theta^{-\frac{\mathcal{B}}{\mathcal{C}}}\underline{\boldsymbol{y}}\right),$$

where

$$\theta := \frac{\left(\varepsilon^{\mathcal{B}} Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}}{\varepsilon}.$$

Then, we have

$$\begin{aligned} &\omega_2 \circ \omega_1(\{|\boldsymbol{x}_i|_2 \le \varepsilon,\ i = 1, \dots, M\} \times R) \\ &\quad = \left\{|\boldsymbol{x}_i|_2 \le \left(\varepsilon^{\mathcal{B}} Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}},\ i = 1, \dots, M\right\} \times \left\{|\boldsymbol{y}_j|_2 \le \left(\varepsilon^{\mathcal{B}} Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}},\ j = 1, \dots, N\right\}. \end{aligned} \tag{41}$$

Now, to complete the estimate we use the following general counting result [2, Theorem 1.3], which we state for a vector space of the form $V_{\boldsymbol{l}}$ and a definable family.

Theorem 2.4 (Barroero-Widmer)**.** *Let $\boldsymbol{l} \in \mathbb{N}^L$ and let $\mathcal{L} := \sum_{h=1}^{L} l_h$. Let also $t \in \mathbb{N}$. Consider a full rank lattice $\Lambda \subset V_{\boldsymbol{l}}$ and a definable family $\mathcal{W}' \subseteq V_{\boldsymbol{l}} \times \mathbb{R}^t$. Suppose that each fibre $W'_{\boldsymbol{\tau}}$ of $\mathcal{W}'$ is bounded. Then, there exists a constant $c_{\mathcal{W}'} \in \mathbb{R}$, only depending on $\mathcal{W}'$, such that*

$$\left|\#(\Lambda \cap W'_{\boldsymbol{\tau}}) - \frac{\mathrm{Vol} W'_{\boldsymbol{\tau}}}{\det \Lambda}\right| \le c_{\mathcal{W}'} \sum_{s=0}^{\mathcal{L}-1} \frac{V_s(W'_{\boldsymbol{\tau}})}{\lambda_1 \cdots \lambda_s},$$

where $V_s\left(W'_{\boldsymbol{\tau}}\right)$ is the sum of the s-dimensional volumes of the orthogonal projections of $W'_{\boldsymbol{\tau}}$ onto every s-dimensional coordinate subspace of $V_{\boldsymbol{l}}$, and λ_s is the s-th successive minimum of the lattice Λ with respect to the Euclidean unit ball. By convention, $V_0(W'_{\boldsymbol{\tau}}) = \lambda_0 = 1$.

We fix $k \in \mathcal{K}$, and we apply Theorem 2.4 to the family

$$\mathcal{S}_k := \left\{(\omega_2 \circ \omega_1 \circ \hat{\varphi}_k(\underline{\boldsymbol{v}}), \boldsymbol{\tau}) : (\underline{\boldsymbol{v}}, \boldsymbol{\tau}) \in \mathcal{W} \cap \left(\hat{X}_k \times \mathbb{R}^t\right)\right\} \subset V \times \mathbb{R}^t.$$

This family is definable in view of Definition 1.4 and part *ii*) of Corollary 2.2 (note that $\omega_2 \circ \omega_1 \circ \hat{\varphi}_k$ is a definable map). Moreover, since the fibres of $\mathcal{Z}$ are bounded, the same holds true for the fibres of $\mathcal{S}_k$. Hence, by Theorem 2.4, Lemma 2.3, and Equations (38) and (41), we have

$$\left|\#(\Lambda\cap W)-\frac{\mathrm{Vol}W}{\det\Lambda}\right|\leq$$

$$\sum_{k\in\mathcal{K}}\left|\#\left(\omega_2\circ\omega_1\circ\hat{\varphi}_k(\Lambda)\cap\omega_2\circ\omega_1\circ\hat{\varphi}_k\left(W\cap\hat{X}_k\right)\right)-\frac{\mathrm{Vol}\omega_2\circ\omega_1\circ\hat{\varphi}_k\left(W\cap\hat{X}_k\right)}{\det\omega_2\circ\omega_1\circ\hat{\varphi}_k(\Lambda)}\right|$$

$$+\sum_{i=1}^{M}\#(\Lambda\cap W^i)\ll_{\mathcal{W},\boldsymbol{\beta},\boldsymbol{\gamma}}\left(\sum_{k\in\mathcal{K}}\sum_{s=0}^{\mathcal{M}+\mathcal{N}-1}\frac{\left(\varepsilon^{\mathcal{B}}Q^{\mathcal{C}}\right)^{\frac{s}{\mathcal{B}+\mathcal{C}}}}{\lambda_1\left(\omega_2\circ\omega_1\circ\hat{\varphi}_k(\Lambda)\right)^s}\right)+\left(\frac{\mathrm{diam}(W\cap C)}{\lambda_1(\Lambda\cap C)}\right)^c, \tag{42}$$

where $\lambda_1\left(\omega_2\circ\omega_1\circ\hat{\varphi}_k(\Lambda)\right)$ is the first successive minimum of the lattice $\omega_2\circ\omega_1\circ\hat{\varphi}_k(\Lambda)$.

Proposition 2.5. *Let $k\in\mathcal{K}$ and let λ_1 be the first successive minimum of the lattice $\omega_2\circ\omega_1\circ\hat{\varphi}_k(\Lambda)$. Then,*

$$\lambda_1\gg_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{\beta},\boldsymbol{\gamma}}\min\left\{\nu(\Lambda,r),\ \left(\varepsilon^{\mathcal{B}}Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}\frac{r}{\mathrm{diam}Z},\ \left(\varepsilon^{\mathcal{B}}Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}\frac{\lambda_1(\Lambda\cap C)}{\mathrm{diam}(Z\cap C)}\right\}$$

for all $r>0$. By convention, the last term is $+\infty$ if $C=\{(\mathbf{0},\mathbf{0})\}$.

We prove Proposition 2.5 in Section 4. Note that $C\subsetneq V$ implies $c=\dim(C)\leq\mathcal{M}+\mathcal{N}-1$. Hence, combining (42) and Proposition 2.5, we get that for all $r>0$

$$\left|\#(\Lambda\cap W)-\frac{\mathrm{Vol}W}{\det\Lambda}\right|\ll_{\mathcal{W},\boldsymbol{\beta},\boldsymbol{\gamma}}\#\mathcal{K}\left(1+\left(\frac{\left(\varepsilon^{\mathcal{B}}Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}}{\nu(\Lambda,r)}+\frac{\mathrm{diam}Z}{r}+\frac{\mathrm{diam}(Z\cap C)}{\lambda_1(\Lambda\cap C)}\right)^{\mathcal{M}+\mathcal{N}-1}\right), \tag{43}$$

where the last term is null if $C=\{(\mathbf{0},\mathbf{0})\}$. It follows that

$$\left|\#(\Lambda\cap W)-\frac{\mathrm{Vol}W}{\det\Lambda}\right|\ll_{\mathcal{W},\boldsymbol{\beta},\boldsymbol{\gamma}}$$

$$\inf_{0<r\leq\mathrm{diam}Z}\#\mathcal{K}\left(\frac{\left(\varepsilon^{\mathcal{B}}Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}}{\nu(\Lambda,r)}+\frac{\mathrm{diam}Z}{r}+\frac{\mathrm{diam}(Z\cap C)}{\lambda_1(\Lambda\cap C)}\right)^{\mathcal{M}+\mathcal{N}-1},$$

where we restrict to $r\leq\mathrm{diam}Z$ to get rid of the 1 at the right-hand side of (43).

By Proposition 2.1, we have $\#\mathcal{K}\ll_{\boldsymbol{m},\boldsymbol{\beta}}\log\left(T/\varepsilon\right)^{M-1}$ and thus, the proof is complete.

3 Proof of Proposition 2.1

In this section, we use the notation S_+ to indicate the set $S\setminus\bigcup_{i=1}^{M}\{\boldsymbol{x}_i=\mathbf{0}\}$ for any subset S of $V_{\boldsymbol{m}}$.

3.1 A partition of the boundary

We construct a partition of the set $(\partial H)_+$ and a collection of linear maps defined on $V_{\boldsymbol{m}}$, that satisfy parts $i) - iv)$ for the set $(\partial H)_+$. Then, we extend this partition to H_+ by taking cones, and we prove that the same maps work for the whole set. To this end, we consider the sets

$$(\partial H)_{\text{hyp}} := \left\{ \underline{\boldsymbol{x}} \in V_{\boldsymbol{m}} : \prod_{i=1}^{M} |\boldsymbol{x}_i|_2^{\beta_i} = \varepsilon^{\mathcal{B}}, |\boldsymbol{x}_1|_2, \ldots, |\boldsymbol{x}_M|_2 \le T \right\}$$

and $(\partial H)_{\text{non-hyp}} := (\partial H)_+ \setminus (\partial H)_{\text{hyp}}$.

3.2 The hyperbolic part

We start by proving parts $i), iii)$, and $iv)$ for the set $(\partial H)_{\text{hyp}}$. Let $\xi : \{\underline{\boldsymbol{x}} \in V_{\boldsymbol{m}} : \boldsymbol{x}_i \ne 0,\ i = 1, \ldots, M\} \to \mathbb{R}^M$ with $\xi(\underline{\boldsymbol{x}})_i := \log |\boldsymbol{x}_i|_2$ for $i = 1, \ldots, M$. We denote by z_i the coordinates of the codomain of ξ. We also introduce the sets

$$\pi := \left\{ \boldsymbol{z} \in \mathbb{R}^M : \sum_{i=1}^{M} \beta_i z_i = \mathcal{B} \log \varepsilon \right\}$$

and

$$S := (-\infty, \log T]^M .$$

Lemma 3.1. *There exists a partition of $(\partial H)_{\text{hyp}}$ of the form $(\partial H)_{\text{hyp}} = \bigcup_{k \in \mathcal{K}_{\text{hyp}}} \tilde{X}_k$, and there exists a collection of linear maps $\varphi_k : V_{\boldsymbol{m}} \to V_{\boldsymbol{m}}$ for $k \in \mathcal{K}_{\text{hyp}}$ that satisfy parts $i), iii)$, and $iv)$ of Proposition 2.1.*

Proof. First, we observe that

$$(\partial H)_{\text{hyp}} = \xi^{-1}(\pi \cap S).$$

Let P be any point on the hyperplane π and let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{M-1}\}$ be an orthonormal basis of $\text{lin}(\pi)$ (i.e., the only linear subspace associated to π). We consider a tiling of π given by the sets

$$T_{\boldsymbol{k}} := \{P + \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_{M-1} \boldsymbol{v}_{M-1} : k_i \le \lambda_i < k_i + 1 \text{ for } i = 1, \ldots, M-1\}.$$

for $\boldsymbol{k} \in \mathbb{Z}^{M-1}$. Since $\pi \cap S$ is bounded and $\text{diam}(T_{\boldsymbol{k}}) \gg_{\boldsymbol{m}} 1$, we trivially have

$$\#\{\boldsymbol{k} : T_{\boldsymbol{k}} \cap \pi \cap S \ne \emptyset\} \ll_{\boldsymbol{m}} (2 + \text{diam}(\pi \cap S))^{\mathcal{M}-1} . \tag{44}$$

Now, the set $\pi \cap S$ is a $(M-1)$-dimensional simplex, whose vertices V^i $(i = 1, \dots, M)$ have coordinates

$$V^i_h := \begin{cases} \log T & \text{if } h \neq i \\ \dfrac{1}{\beta_i} \log\left(\dfrac{\varepsilon^{\mathcal{B}}}{T^{\mathcal{B}-\beta_i}}\right) & \text{if } h = i \end{cases}$$

for $h = 1, \dots, M$. We define $V^0 := (\log T, \dots, \log T)$, and we consider the only aligned box whose vertices include the points $V^0, V^1, \dots, V^M$. We let $\boldsymbol{\mu}$ be its centre. Since the side length of this box is

$$\left| \frac{1}{\beta_i} \log\left(\frac{\varepsilon^{\mathcal{B}}}{T^{\mathcal{B}-\beta_i}}\right) - \log T \right| = \frac{\mathcal{B}}{\beta_i} \log\left(\frac{T}{\varepsilon}\right),$$

and since it contains $\pi \cap S$, by (44) we have

$$\#\{\boldsymbol{k} : T_{\boldsymbol{k}} \cap \pi \cap S \neq \emptyset\} \ll_{\boldsymbol{m},\boldsymbol{\beta}} \log\left(\frac{T}{\varepsilon}\right)^{M-1}. \tag{45}$$

Now, we set

$$\mathcal{K}_{\text{hyp}} := \left\{\boldsymbol{k} \in \mathbb{Z}^M : T_{\boldsymbol{k}} \cap S \neq \emptyset\right\}.$$

Note that $T_{\boldsymbol{k}} \cap \pi \cap S = T_{\boldsymbol{k}} \cap S$. Then, the sets $\{T_{\boldsymbol{k}} \cap S\}_{\boldsymbol{k} \in \mathcal{K}_{\text{hyp}}}$ form a partition of $\pi \cap S$, and part *i)* follows directly from (45). We associate with each of these sets a translation $\tau_{\boldsymbol{k}}$ of the form $\tau_{\boldsymbol{k}}(\boldsymbol{z}) := \boldsymbol{z} + \boldsymbol{a}^{\boldsymbol{k}}$, where $\boldsymbol{a}^{\boldsymbol{k}} \in \mathbb{R}^M$. In particular, we choose $\boldsymbol{a}^{\boldsymbol{k}}$ to be the distance vector from the centre of the tile $T_{\boldsymbol{k}}$ to the point $C := (\log \varepsilon, \dots, \log \varepsilon) \in \pi \cap S$. Given that $\boldsymbol{a}^{\boldsymbol{k}} \in \text{lin}(\pi)$, we have

$$\sum_{i=1}^{M} \beta_i a_i^{\boldsymbol{k}} = 0,$$

proving part *iiib)*. Now, C lies in the box of vertices $V_0, \dots, V_M$, whereas the centre of the tile $T_{\boldsymbol{k}}$ lies in a box of centre $\boldsymbol{\mu}$ and side length at most $\max_i \frac{\mathcal{B}}{\beta_i} \log\left(\frac{T}{\varepsilon}\right) + \text{diam}(T_{\boldsymbol{k}})$. Since $\text{diam}(T_{\boldsymbol{k}}) \ll_{\boldsymbol{m}} 1$, we have

$$\left|\boldsymbol{a}^{\boldsymbol{k}}\right| \ll_{\boldsymbol{m},\boldsymbol{\beta}} \log\left(\frac{T}{\varepsilon}\right).$$

Hence, part *iiia)* is proved (modulo the fact that the constant c depends uniquely on $\boldsymbol{m}$). Let $\tilde{X}_{\boldsymbol{k}} := \xi^{-1}(T_{\boldsymbol{k}} \cap S)$ for $\boldsymbol{k} \in \mathcal{K}_{\text{hyp}}$ and let $\varphi_{\boldsymbol{k}} : V_{\boldsymbol{m}} \to V_{\boldsymbol{m}}$ be the linear transformation defined by

$$\varphi_{\boldsymbol{k}}(\underline{\boldsymbol{x}})_i := e^{a_i^{\boldsymbol{k}}} \boldsymbol{x}$$

for $i = 1, \dots, M$. Since the following diagram commutes

$$\begin{array}{ccc} \{\boldsymbol{x}_i \neq 0,\ i = 1, \dots, M\} & \xrightarrow{\varphi_{\boldsymbol{k}}} & \{\boldsymbol{x}_i \neq 0,\ i = 1, \dots, M\}, \\ \downarrow \xi & & \downarrow \xi \\ \mathbb{R}^M & \xrightarrow{\tau_{\boldsymbol{k}}} & \mathbb{R}^M \end{array}$$

we have

$$\varphi_{\boldsymbol{k}}(X_{\boldsymbol{k}}) \subset \xi^{-1}\left\{|z_i - \log\varepsilon| \le \operatorname{diam}(T_{\boldsymbol{k}}) \text{ for } i = 1,\ldots,M\right\}$$
$$\subset \left\{|\boldsymbol{x}_i|_2 \le \varepsilon e^{\operatorname{diam}(T_{\boldsymbol{k}})} \text{ for } i = 1,\ldots,M\right\}.$$

To conclude the proof of part $iv)$ it suffices to rescale all coordinates by $e^c \ll_{\boldsymbol{m}} 1$, where $c := \operatorname{diam}(T_{\boldsymbol{k}})$. □

Lemma 3.2. *Each of the sets $\tilde{X}_k$ defined in Lemma 3.1 is definable.*

Proof. Let $\operatorname{pr}_\pi : \mathbb{R}^M \to \pi$ be the orthogonal projection onto the hyperplane π, and let

$$\tilde{T}_{\boldsymbol{k}} := \operatorname{pr}_\pi^{-1}(T_{\boldsymbol{k}})$$

for $\boldsymbol{k} \in \mathbb{Z}^M$. Each of the sets $\tilde{T}_{\boldsymbol{k}}$ is semi-linear, since it is bounded by a finite number of hyperplanes. Moreover, we have

$$T_{\boldsymbol{k}} \cap S = \tilde{T}_{\boldsymbol{k}} \cap \pi \cap S.$$

It follows that all the inequalities defining the set $T_{\boldsymbol{k}} \cap S$ are semi-linear in $\boldsymbol{z}$. Let $\mathfrak{L}(\boldsymbol{z})$ be the system defining $T_{\boldsymbol{k}} \cap S$. Then, $\tilde{X}_{\boldsymbol{k}}$ is defined by the system $\mathfrak{L}(\xi(\underline{\boldsymbol{x}}))$, which is a system of inequalities of generalised polynomials[9] in the variables $\boldsymbol{x}_i$. Hence, the set $\tilde{X}_k$ is definable. □

3.3 The non-hyperbolic part

Now, we prove parts $i), iii)$, and $iv)$ for the set $(\partial H)_{\text{non-hyp}}$.

Lemma 3.3. *There exists a partition of the set $(\partial H)_{\text{non-hyp}}$ of the form $(\partial H)_{\text{non-hyp}} = \bigcup_{k\in\mathcal{K}_{\text{non-hyp}}} \tilde{X}'_k$, and there exists a collection of linear maps $\varphi_k : V_{\boldsymbol{m}} \to V_{\boldsymbol{m}}$ for $k \in \mathcal{K}_{\text{non-hyp}}$ that satisfy parts $i), iii)$, and $iv)$ of Proposition 2.1.*

Proof. Let $\boldsymbol{z} \in \xi((\partial H)_{\text{non-hyp}})$. We define a unique point $\boldsymbol{z}^* \in \pi \cap S$ associated to $\boldsymbol{z}$ by the following procedure. By definition of $(\partial H)_{\text{non-hyp}}$, we have

$$\sum_{i=1}^{M} \beta_i z_i < \mathcal{B}\log\varepsilon.$$

[9] finite sums of monomials with non-negative real exponents. Note that the function $f(x) = x^r = \exp(r\log x)$ on $(0,+\infty)$ with real $r > 0$ is definable in $\mathbb{R}_{\exp}$.

We increase the first coordinate z_1 of $\boldsymbol{z}$ until either $\sum_{i=1}^{M} \beta_i z_i = \mathcal{B} \log \varepsilon$ or $z_1 = \log T$. We call the increased coordinate z_1^*. If

$$\beta_1 z_1^* + \sum_{i=2}^{M} \beta_i z_i = \mathcal{B} \log \varepsilon,$$

we stop and we set $\boldsymbol{z}^* := (z_1^*, z_2, \dots, z_M)$. Otherwise, we increase the second coordinate z_2 until either $\beta_1 z_1^* + \beta_2 z_2 + \sum_{i=3}^{M} \beta_i z_i = \mathcal{B} \log \varepsilon$, or $z_2 = \log T$. We call the increased coordinate z_2^*. If

$$\beta_1 z_1^* + \beta_2 z_2^* + \sum_{i=3}^{M} \beta_i z_i = \mathcal{B} \log \varepsilon,$$

we stop and we set $\boldsymbol{z}^* := (z_1^*, z_2^*, z_3, \dots, z_M)$. Otherwise, we repeat the same steps for the remaining coordinates. This procedure terminates, since $\mathcal{B} \log T \geq \mathcal{B} \log \varepsilon$. Moreover, we have that $\boldsymbol{z}^* \in (\partial H)_{\text{hyp}}$. Now, we set $\mathcal{K}_{\text{non-hyp}} := \mathcal{K}_{\text{hyp}}$, and for each $k \in \mathcal{K}_{\text{non-hyp}}$ we define

$$\tilde{X}_k' := \left\{ \underline{\boldsymbol{x}} \in (\partial H)_{\text{non-hyp}} : \xi(\underline{\boldsymbol{x}})^* \in \xi\left(\tilde{X}_k\right) = T_k \cap S \right\}.$$

Then, we have

$$(\partial H)_{\text{non-hyp}} = \bigcup_{k \in \mathcal{K}_{\text{hyp}}} \tilde{X}_k',$$

and this is a partition of $(\partial H)_{\text{non-hyp}}$ since the sets $T_k \cap S$ form a partition of $\pi \cap S$. We show that the sets $\tilde{X}_k'$ and the maps φ_k for $k \in \mathcal{K}_{\text{non-hyp}}$ (i.e., the maps introduced in Lemma 3.1) have the required properties. The proof of parts *i)* and *iii)* is trivial. To prove part *iv)* we observe that, by construction, for each point $\underline{\boldsymbol{x}} \in \tilde{X}_k'$ there are points $\underline{\boldsymbol{y}} \in \tilde{X}_k$ such that $|\boldsymbol{x}_i|_2 \leq |\boldsymbol{y}_i|_2$ for $i = 1, \dots, M$ (e.g., any point $\underline{\boldsymbol{y}} \in \xi^{-1}(\xi(\underline{\boldsymbol{x}})^*)$). Therefore, since

$$\varphi_k\left(\tilde{X}_k\right) \subset \{|\boldsymbol{x}_i|_2 \leq \varepsilon,\ i = 1, \dots, M\},$$

we have

$$\varphi_k\left(\tilde{X}_k'\right) \subset \{|\boldsymbol{x}_i|_2 \leq \varepsilon,\ i = 1, \dots, M\},$$

by the definition of the maps φ_k. □

Lemma 3.4. *Each of the sets $\tilde{X}_k'$ defined in Lemma 3.3 is definable.*

Proof. We have

$$\tilde{X}_k' = \Big\{ \underline{\boldsymbol{x}} \in (\partial H)_{\text{non-hyp}} : \exists \underline{\boldsymbol{x}}^* \in \tilde{X}_k \text{ such that } |\boldsymbol{x}_i^*| \geq |\boldsymbol{x}_i| \\ \text{and } |\boldsymbol{x}_i^*| > |\boldsymbol{x}_i| \Rightarrow (|\boldsymbol{x}_h^*| = T \text{ for } h < i) \Big\}.$$

Now, we consider the set

$$\tilde{X}_k'' := \left\{(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}^*) \in (\partial H)_{\text{non-hyp}} \times \tilde{X}_k : |\boldsymbol{x}_i^*| \geq |\boldsymbol{x}_i| \text{ and } |\boldsymbol{x}_i^*| > |\boldsymbol{x}_i| \Rightarrow (|\boldsymbol{x}_h^*| = T \text{ for } h < i)\right\},$$

and we let $\mathrm{pr} : V_{\boldsymbol{m}} \times V_{\boldsymbol{m}} \to V_{\boldsymbol{m}}$ be the projection onto the first cartesian factor. Then, $\tilde{X}_k' = \mathrm{pr}\left(\tilde{X}_k''\right)$. The set $\tilde{X}_k''$ is clearly definable since it is defined by algebraic inequalities and boolean operators, and since $\tilde{X}_k$ is definable. Hence, by the properties of *o*-minimal structures (see Definition 1.4), $\mathrm{pr}\left(\tilde{X}_k''\right)$ is a definable set. □

3.4 From the boundary to the whole set

Given a set $A \subset V_{\boldsymbol{m}}$, we denote by $\mathcal{C}(A)$ the cone generated by the set A, i.e., the set

$$\{t\underline{\boldsymbol{x}} : t \in (0, +\infty), \underline{\boldsymbol{x}} \in A\}.$$

Let $\mathcal{K} := \mathcal{K}_{\text{hyp}} \sqcup \mathcal{K}_{\text{non-hyp}}$, and let

$$X_k := \mathcal{C}\left(\tilde{X}_k\right) \cap H_+$$

for $k \in \mathcal{K}$ (where we drop the apex $'$ for the sets $\tilde{X}_k'$ with $k \in \mathcal{K}_{\text{non-hyp}}$). Then, clearly

$$H_+ = \bigcup_{k \in \mathcal{K}} X_k,$$

and this is a partition of the set H_+ (each line through the origin intersects the boundary at at most one point). We prove that the sets X_k and the maps φ_k satisfy parts $i) - iv)$ of Proposition 2.1. From Lemmas 3.1 and 3.3, we easily deduce

$$\#\mathcal{K} \ll_{\boldsymbol{m},\boldsymbol{\beta}} \log\left(\frac{T}{\varepsilon}\right)^{M-1},$$

proving part $i)$. To prove part $ii)$, we need the following lemma.

Lemma 3.5. *Let $D \subset V_{\boldsymbol{m}}$ be a definable set. Then, the set $\mathcal{C}(D) \subset V_{\boldsymbol{m}}$ is also definable.*

Proof. We have

$$\mathcal{C}(D) = \{\underline{\boldsymbol{x}} \in V_{\boldsymbol{m}} : \exists t \in (0, +\infty) \text{ such that } t\underline{\boldsymbol{x}} \in D\}.$$

We consider the set

$$\tilde{D} = \{(\underline{\boldsymbol{x}}, t) \in V_{\boldsymbol{m}} \times \mathbb{R} : t\underline{\boldsymbol{x}} \in D, t > 0\},$$

and we let $\mathrm{pr} : V_{\boldsymbol{m}} \times \mathbb{R} \to V_{\boldsymbol{m}}$ be the natural projection. Then,

$$\mathcal{C}(D) = \mathrm{pr}\left(\tilde{D}\right).$$

Now, the set $\tilde{D}$ is clearly definable, since D is definable. Hence, by the properties of o-minimal structures (see Definition 1.4), $\mathrm{pr}\left(\tilde{D}\right)$ is a definable set. □

From Lemmas 3.2, 3.4, and 3.5 it follows that $\mathcal{C}(\tilde{X}_k)$ is a definable set for each k, proving part *ii*). Part *iii*) is a straightforward consequence of Lemmas 3.1 and 3.3. To prove part *iv*), it suffices to note that for each point $\underline{\boldsymbol{x}} \in X_k$ there is a point $\underline{\boldsymbol{y}} \in \tilde{X}_k$ or $\underline{\boldsymbol{y}} \in \tilde{X}'_k$ such that $|\boldsymbol{x}_i|_2 \leq |\boldsymbol{y}_i|_2$ for $i = 1, \ldots, M$ (namely $\{\underline{\boldsymbol{y}}\} = \{t\underline{\boldsymbol{x}} : t \in (0, +\infty)\} \cap (\partial H)_+$). Hence, part *iv*) follows again from Lemmas 3.1 and 3.3, and by the definition of the maps φ_k.

4 Proof of Proposition 2.5

Let $\underline{\boldsymbol{v}} \neq \boldsymbol{0}$ be a vector of shortest length in the lattice $\omega_2 \circ \omega_1 \circ \hat{\varphi}_k(\Lambda)$. Then, $\boldsymbol{v}$ has the form

$$\underline{\boldsymbol{v}} = \left(\theta \exp\left(a_1^k - c\right)\boldsymbol{x}_1, \ldots, \theta \exp\left(a_M^k - c\right)\boldsymbol{x}_M, \theta^{-\frac{\mathcal{B}}{\mathcal{C}}}\frac{Q}{Q_1}\boldsymbol{y}_1, \ldots, \theta^{-\frac{\mathcal{B}}{\mathcal{C}}}\frac{Q}{Q_N}\boldsymbol{y}_N\right)$$

for some point $(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) \in \Lambda$. It follows that

$$|\underline{\boldsymbol{v}}|_2 = \left(\theta^2 \exp\left(2a_1^k - 2c\right)|\boldsymbol{x}_1|_2^2 + \cdots + \theta^2 \exp\left(2a_M^k - 2c\right)|\boldsymbol{x}_M|_2^2 + \theta^{-\frac{2\mathcal{B}}{\mathcal{C}}}\frac{Q^2}{Q_1^2}|\boldsymbol{y}_1|_2^2 + \cdots + \theta^{-\frac{2\mathcal{B}}{\mathcal{C}}}\frac{Q^2}{Q_N^2}|\boldsymbol{y}_N|_2^2\right)^{\frac{1}{2}}. \quad (46)$$

Fix $r > 0$. We consider three cases. Case 1:

- $\boldsymbol{x}_i \neq \boldsymbol{0}$ for $i = 1, \ldots, M$ and $\boldsymbol{y}_j \neq \boldsymbol{0}$ for $j = 1, \ldots, N$;
- $|(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}})|_2 < r$.

By applying the weighted arithmetic-geometric mean inequality to (46), with weights $\beta_1, \ldots, \beta_M$ and $\gamma_1, \ldots, \gamma_N$, we get

$$|\underline{\boldsymbol{v}}|_2 \gg_{\boldsymbol{m},\boldsymbol{\beta},\boldsymbol{\gamma}} \left(\mathrm{Nm}_{(\boldsymbol{\beta},\boldsymbol{\gamma})}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}})\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}} \geq \nu(\Lambda, r),$$

where we used the fact that $\sum_{i=1}^{M} \beta_i a_i^k = 0$ (see Proposition 2.1, part *iiib*)).

Case 2:

- $|(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}})|_2 \geq r$.

In this case it must be either $|\boldsymbol{x}_{i_0}|_2 \geq r/\sqrt{M+N}$ for some $1 \leq i_0 \leq M$ or $|\boldsymbol{y}_{j_0}|_2 \geq r/\sqrt{M+N}$ for some $1 \leq j_0 \leq N$.

Case 2a:

- there exists $1 \leq i_0 \leq M$ such that $|\boldsymbol{x}_{i_0}|_2 \geq r/\sqrt{M+N}$.

By ignoring all the terms but $\boldsymbol{x}_{i_0}$, we get

$$|\underline{\boldsymbol{v}}|_2 \gg_{\boldsymbol{m}} \theta e^{a_{i_0}^{\boldsymbol{k}}} |\boldsymbol{x}_{i_0}|_2 \gg_{\boldsymbol{m},\boldsymbol{n}} \theta e^{a_{i_0}^{\boldsymbol{k}}} r.$$

It follows from Proposition 2.1 part *iiia*) that

$$|\underline{\boldsymbol{v}}|_2 \gg_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{\beta}} \frac{\left(\varepsilon^{\mathcal{B}} Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}}{\varepsilon} \frac{\varepsilon}{T} r = \left(\varepsilon^{\mathcal{B}} Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}} \frac{r}{T} \geq \left(\varepsilon^{\mathcal{B}} Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}} \frac{r}{\operatorname{diam} Z}.$$

Case 2b:

- there exists $1 \leq j_0 \leq N$ such that $|\boldsymbol{y}_{j_0}|_2 \geq r/\sqrt{M+N}$.

By ignoring all the terms but $\boldsymbol{y}_{j_0}$, we get

$$|\underline{\boldsymbol{v}}|_2 \gg_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{\beta}} \theta^{-\frac{\mathcal{B}}{\mathcal{C}}} \frac{Q}{Q_{j_0}} r \geq \left(\varepsilon^{\mathcal{B}} Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}} \frac{r}{Q_{\max}} \geq \left(\varepsilon^{\mathcal{B}} Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}} \frac{r}{\operatorname{diam} Z},$$

where $Q_{\max} := \max\{Q_j : j = 1, \ldots, N\}$.

Case 3:

- $\boldsymbol{x}_{i_0} = \boldsymbol{0}$ for some $1 \leq i_0 \leq M$ or $\boldsymbol{y}_{j_0} = \boldsymbol{0}$ for some $1 \leq j_0 \leq N$.

We can suppose $C \neq \{(\mathbf{0}, \mathbf{0})\}$, otherwise this case does not occur. Since Λ is weakly admissible for (S, C) we have that $(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) \in C$. Now, let

$$\delta_{\underline{\boldsymbol{x}}} := \begin{cases} +\infty & \text{if } I = \{1, \dots, M\} \\ 1 & \text{otherwise} \end{cases},$$

$$\delta_{\underline{\boldsymbol{y}}} := \begin{cases} +\infty & \text{if } J = \{1, \dots, N\} \\ 1 & \text{otherwise} \end{cases},$$

and let $Q_{C,\max} := \max\{Q_j : j \notin J\}$ (if $J = \{1, \dots, N\}$, we set $Q_{C,\max} := 1$). Then, by Proposition 2.1 part *iiia*), we have

$$\begin{aligned} |\underline{\boldsymbol{v}}|_2 &\geq \min\left\{\delta_{\underline{\boldsymbol{x}}}\theta \min_i \exp\left(a_i^{\boldsymbol{k}} - c\right),\ \delta_{\underline{\boldsymbol{y}}}\theta^{-\frac{\mathcal{B}}{\mathcal{C}}}\frac{Q}{Q_{C,\max}}\right\}|(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}})|_2 \\ &\gg_{\boldsymbol{m},\boldsymbol{\beta}} \min\left\{\delta_{\underline{\boldsymbol{x}}}\frac{\left(\varepsilon^{\mathcal{B}}Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}}{\varepsilon}\frac{\varepsilon}{T},\ \delta_{\underline{\boldsymbol{y}}}\frac{\left(\varepsilon^{\mathcal{B}}Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}}{Q_{C,\max}}\right\}\lambda_1(\Lambda \cap C) \\ &\geq \left(\varepsilon^{\mathcal{B}}Q^{\mathcal{C}}\right)^{\frac{1}{\mathcal{B}+\mathcal{C}}}\frac{\lambda_1(\Lambda \cap C)}{\operatorname{diam}(Z \cap C)}. \end{aligned}$$

This concludes the proof.

5 Proof of Proposition 1.2

The set Z that we consider in Proposition 1.2 has a slightly different structure from the fibres of the family $\mathcal{Z}$ appearing in Theorem 1.5. In particular, it involves the maximum norm $|\cdot|_\infty$ instead of the Euclidean norm $|\cdot|_2$. Therefore, in order to apply Theorem 1.5 to the set Z, we need to introduce a new family $\mathcal{W}$ and see Z as a fibre of $\mathcal{W}$. Let $\boldsymbol{m} = \boldsymbol{\beta} := (1, \dots, 1) \in \mathbb{R}^{\mathcal{M}}$ and let $\boldsymbol{n} = \boldsymbol{\gamma} := \mathcal{N}$ (which implies $M = \mathcal{M}$ and $N = 1$ according to the notation described in the Introduction). We set $\mathcal{W} := \mathcal{H} \times \mathcal{R}^\infty$, where

$$\mathcal{H} := \left\{(\underline{\boldsymbol{x}}, \varepsilon', T') \in V_{\boldsymbol{m}} \times (0, +\infty)^2 : \operatorname{Nm}_{\boldsymbol{m}}(\underline{\boldsymbol{x}})^{\frac{1}{\mathcal{M}}} < \varepsilon',\ |x_i| \leq T',\ i = 1, \dots, M\right\},$$

and

$$\mathcal{R}^\infty := \{(\boldsymbol{y}, Q') \in V_{\boldsymbol{n}} \times \mathbb{R} : |\boldsymbol{y}|_\infty \leq Q'\}$$

(note that the definition of $\mathcal{H}$ hasn't changed). Then, $Z = W_{\boldsymbol{\tau}}$, where

$$\boldsymbol{\tau} := (\varepsilon', T', Q') = \left(\varepsilon^{\frac{1}{\mathcal{M}}}, T, Q\right).$$

5 Proof of Proposition 1.2

To prove proposition 1.2, we need to estimate

$$\#\left(M(\boldsymbol{L},\varepsilon,T,Q)\right)=\#\left((\Lambda_{\boldsymbol{L}}\cap W_{\boldsymbol{\tau}})\setminus C\right), \tag{47}$$

where $C := \{\boldsymbol{y}=\boldsymbol{0}\}\subset V$. We consider two different cases. First, we assume

$$\varepsilon Q^{\mathcal{N}}/\phi(Q)\geq 1. \tag{48}$$

In this case, we use Theorem 1.5 to estimate $\#(\Lambda_{\boldsymbol{L}}\cap W_{\boldsymbol{\tau}})$. A suitable choice for the parameter $\boldsymbol{\eta}(\boldsymbol{\tau})$ in order to have $W_{\boldsymbol{\tau}}\subset Z_{\boldsymbol{\eta}(\boldsymbol{\tau})}$ is $\boldsymbol{\eta}(\boldsymbol{\tau})=\left(\varepsilon^{\frac{1}{\mathcal{M}}},T,\sqrt{\mathcal{N}}Q\right)$. However, we first need to show that the lattice $\Lambda_{\boldsymbol{L}}$ is weakly admissible for the couple $(\mathcal{S},C)$, where $\mathcal{S}:=((\boldsymbol{m},\boldsymbol{n}),(\boldsymbol{\beta},\boldsymbol{\gamma}))$. We do this in the following lemma.

Lemma 5.1. *Let $\boldsymbol{m}=\boldsymbol{\beta}:=(1,\dots,1)\in\mathbb{R}^{\mathcal{M}}$ and let $\boldsymbol{n}=\boldsymbol{\gamma}:=\mathcal{N}$. Let also $\mathcal{S}:=((\boldsymbol{m},\boldsymbol{n}),(\boldsymbol{\beta},\boldsymbol{\gamma}))$ and let $C:=\{\boldsymbol{y}=\boldsymbol{0}\}$. Then,*

$$\nu(\Lambda_{\boldsymbol{L}},\varrho)\geq\phi(\varrho)^{\frac{1}{\mathcal{M}+\mathcal{N}}} \tag{49}$$

for all $\varrho>0$. Therefore, the lattice $\Lambda_{\boldsymbol{L}}$ is weakly admissible for the couple $(\mathcal{S},C)$ (see Definition 1.3).

Proof. Let $\varrho\in(0,+\infty)$. If $\varrho\leq\lambda_1(\Lambda_{\boldsymbol{L}}\setminus C)$, then $\nu(\Lambda_{\boldsymbol{L}},\varrho)=+\infty$ and (49) holds true. We can thus suppose that $\varrho>\lambda_1(\Lambda_{\boldsymbol{L}}\setminus C)$. Let $\boldsymbol{v}\in\Lambda_{\boldsymbol{L}}\setminus C$ with $|\boldsymbol{v}|_2<\varrho$. Then,

$$\boldsymbol{v}=(L_1(\boldsymbol{q})+p_1,\dots,L_{\mathcal{M}}(\boldsymbol{q})+p_{\mathcal{M}},\boldsymbol{q})$$

for some $\boldsymbol{p}\in\mathbb{Z}^{\mathcal{M}}$ and $\boldsymbol{q}\in\mathbb{Z}^{\mathcal{N}}\setminus\{\boldsymbol{0}\}$. It follows from the hypothesis that

$$\mathrm{Nm}_{\beta,\gamma}(\boldsymbol{v})=|\boldsymbol{q}|_2^{\mathcal{N}}\prod_{i=1}^{\mathcal{M}}|L_i\boldsymbol{q}+p_i|\geq|\boldsymbol{q}|_\infty^{\mathcal{N}}\prod_{i=1}^{\mathcal{M}}\|L_i\boldsymbol{q}\|\geq\phi(|\boldsymbol{q}|_\infty)\geq\phi(\varrho),$$

where we used the fact that ϕ is non-increasing. Hence, $\nu(\Lambda_{\boldsymbol{L}},\varrho)\geq\phi(\varrho)^{1/(\mathcal{M}+\mathcal{N})}$. □

By applying Theorem 1.5 to $W_{\boldsymbol{\tau}}\subset Z_{\boldsymbol{\eta}(\boldsymbol{\tau})}$, we find

$$|\#(\Lambda_{\boldsymbol{L}}\cap Z)-\mathrm{Vol}Z|\ll_{\mathcal{M},\mathcal{N}}$$
$$\inf_{0<r\leq\mathrm{diam}\left(Z_{\boldsymbol{\eta}(\boldsymbol{\tau})}\right)}\log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1}\left(\left(\frac{\varepsilon Q^{\mathcal{N}}}{\phi(r)}\right)^{\frac{1}{\mathcal{M}+\mathcal{N}}}+\frac{T+Q}{r}+\frac{T}{\lambda_1(\Lambda_{\boldsymbol{L}}\cap C)}\right)^{\mathcal{M}+\mathcal{N}-1}. \tag{50}$$

Now, since $\Lambda_{\boldsymbol{L}} \cap C = \mathbb{Z}^{\mathcal{M}} \times \{\boldsymbol{0}\}$, we have $\lambda_1(\Lambda_{\boldsymbol{L}} \cap C) = 1$. Hence, by choosing $r = Q$ in (50), we deduce

$$|\#(\Lambda_{\boldsymbol{L}} \cap Z) - \mathrm{Vol}Z| \ll_{\mathcal{M},\mathcal{N}} \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} \left(\left(\frac{\varepsilon Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{1}{\mathcal{M}+\mathcal{N}}} + 1 + T\right)^{\mathcal{M}+\mathcal{N}-1}. \quad (51)$$

An easy integration shows that

$$\mathrm{Vol}Z = 2^{\mathcal{M}+\mathcal{N}} Q^{\mathcal{N}} \left[\varepsilon \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} + T^{\mathcal{M}}\left(1 - \left(1 - \frac{\varepsilon}{T^{\mathcal{M}}}\right)^{\mathcal{M}-1}\right)\right].$$

Thus, (47) and (51) imply

$$\begin{aligned}&\left|\#M(\boldsymbol{L}, \varepsilon, T, Q) - 2^{\mathcal{M}+\mathcal{N}} Q^{\mathcal{N}} \left[\varepsilon \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} + T^{\mathcal{M}}\left(1 - \left(1 - \frac{\varepsilon}{T^{\mathcal{M}}}\right)^{\mathcal{M}-1}\right)\right]\right| \\ &\qquad \ll_{\mathcal{M},\mathcal{N}} \#(\Lambda_{\boldsymbol{L}} \cap Z \cap C) + \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} \left(\left(\frac{\varepsilon Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{1}{\mathcal{M}+\mathcal{N}}} + 1 + T\right)^{\mathcal{M}+\mathcal{N}-1}. \quad (52)\end{aligned}$$

Since $\#(\Lambda_{\boldsymbol{L}} \cap Z \cap C) \leq (2T+1)^{\mathcal{M}}$ and by (48)

$$\left(\frac{\varepsilon Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{1}{\mathcal{M}+\mathcal{N}}} + 1 + T \leq \left(\frac{\varepsilon Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{1}{\mathcal{M}+\mathcal{N}}} (1+T),$$

the required estimate is a straightforward consequence of (52).

We are now left to prove the claim for $\varepsilon Q^{\mathcal{N}}/\phi(Q) < 1$.

Lemma 5.2. *Suppose that $\varepsilon Q^{\mathcal{N}}/\phi(Q) < 1$. Then, $\Lambda_{\boldsymbol{L}} \cap Z \subset C$.*

Proof. Assume by contradiction that there exists $\underline{\boldsymbol{v}} \in (\Lambda_{\boldsymbol{L}} \cap Z) \setminus C$. Then,

$$\underline{\boldsymbol{v}} = (L_1(\boldsymbol{q}) + p_1, \ldots, L_{\mathcal{M}}(\boldsymbol{q}) + p_{\mathcal{M}}, \boldsymbol{q})$$

for some $\boldsymbol{p} \in \mathbb{Z}^{\mathcal{M}}$ and $\boldsymbol{q} \in \mathbb{Z}^{\mathcal{N}} \setminus \{\boldsymbol{0}\}$. Since $\underline{\boldsymbol{v}} \in Z$, we have

$$|\boldsymbol{q}|_\infty^{\mathcal{N}} \prod_{i=1}^{\mathcal{M}} \|L_i \boldsymbol{q}\| \leq |\boldsymbol{q}|_\infty^{\mathcal{N}} \prod_{i=1}^{\mathcal{M}} |L_i \boldsymbol{q} + p_i| \leq Q^{\mathcal{N}} \varepsilon < \phi(Q),$$

and this contradicts (32). $\square$

If $\varepsilon Q^{\mathcal{N}}/\phi(Q) < 1$, it follows from Lemma 5.2 and (47) that $M(\boldsymbol{L},\varepsilon,T,Q) = \emptyset$. Hence, to prove Proposition 1.2, it suffices to show that

$$2^{\mathcal{M}+\mathcal{N}} \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} \varepsilon Q^{\mathcal{N}} \ll_{\mathcal{M},\mathcal{N}} (1+T)^{\mathcal{M}+\mathcal{N}-1} \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} \left(\frac{\varepsilon Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{\mathcal{M}+\mathcal{N}-1}{\mathcal{M}+\mathcal{N}}}, \tag{53}$$

and that

$$2^{\mathcal{M}+\mathcal{N}} Q^{\mathcal{N}} T^{\mathcal{M}} \left(1-\left(1-\frac{\varepsilon}{T^{\mathcal{M}}}\right)^{\mathcal{M}-1}\right) \ll_{\mathcal{M},\mathcal{N}} (1+T)^{\mathcal{M}+\mathcal{N}-1} \log\left(\frac{T^{\mathcal{M}}}{\varepsilon}\right)^{\mathcal{M}-1} \left(\frac{\varepsilon Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{\mathcal{M}+\mathcal{N}-1}{\mathcal{M}+\mathcal{N}}}. \tag{54}$$

Inequality (53) follows immediately from the assumption $\varepsilon Q^{\mathcal{N}}/\phi(Q) < 1$. To prove (54), we notice that

$$1-\left(1-\frac{\varepsilon}{T^{\mathcal{M}}}\right)^{\mathcal{M}-1} \ll_{\mathcal{M}} \frac{\varepsilon}{T^{\mathcal{M}}}, \tag{55}$$

and again we use the fact that $\varepsilon Q^{\mathcal{N}}/\phi(Q) < 1$. The proof is hence complete.

6 Proof of Corollary 1.8

We notice that

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{q}\in[-Q,Q]^{\mathcal{N}}\\ \cap\,\mathbb{Z}^{\mathcal{N}}\setminus\{\boldsymbol{0}\}}} \prod_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\|^{-1} \\
&\le \sum_{k=0}^{\infty} 2^{k+1} \# \left\{\boldsymbol{q}\in[-Q,Q]^{\mathcal{N}}\cap\mathbb{Z}^{\mathcal{N}}\setminus\{\boldsymbol{0}\} : 2^{-k-1} \le \prod_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\| < 2^{-k}\right\} \\
&\le \sum_{k=0}^{\infty} 2^{k+1} \# \left\{\boldsymbol{q}\in[-Q,Q]^{\mathcal{N}}\cap\mathbb{Z}^{\mathcal{N}}\setminus\{\boldsymbol{0}\} : \prod_{i=1}^{\mathcal{M}} \|L_i\boldsymbol{q}\| < 2^{-k}\right\} \\
&= \sum_{k=0}^{\infty} 2^{k+1} \# M\left(\boldsymbol{L}, 2^{-k}, \frac{1}{2}, Q\right) \\
&= \sum_{k=0}^{\left\lfloor \log_2\left(\frac{Q^{\mathcal{N}}}{\phi(Q)}\right)\right\rfloor} 2^{k+1} \# M\left(\boldsymbol{L}, 2^{-k}, \frac{1}{2}, Q\right),
\end{aligned} \tag{56}$$

where the last equality follows from (47) and Lemma 5.2 ($\varepsilon = 2^{-k}$).

We use Proposition 1.2 to estimate the right-hand side of (56). Since we need $T^{\mathcal{M}}/\varepsilon \ge e^{\mathcal{M}}$, i.e., $2^{k-\mathcal{M}} \ge e^{\mathcal{M}}$, we split the sum into two parts, one for $2^{k-\mathcal{M}} < e^{\mathcal{M}}$ and one for $2^{k-\mathcal{M}} \ge e^{\mathcal{M}}$. We find

$$\sum_{\substack{\boldsymbol{q}\in[-Q,Q]^{\mathcal{N}}\\ \cap\,\mathbb{Z}^{\mathcal{N}}\setminus\{\boldsymbol{0}\}}}\prod_{i=1}^{\mathcal{M}}\|L_i\boldsymbol{q}\|^{-1}\leq\sum_{k=0}^{\lfloor\mathcal{M}(1+1/\log 2)\rfloor}2^{k+1}\#M\left(\boldsymbol{L},2^{-k},\frac{1}{2},Q\right)$$

$$+\sum_{k=\lceil\mathcal{M}(1+1/\log 2)\rceil}^{\left\lfloor\log_2\left(\frac{Q^{\mathcal{N}}}{\phi(Q)}\right)\right\rfloor}2^{k+1}\#M\left(\boldsymbol{L},2^{-k},\frac{1}{2},Q\right)$$

$$\ll_{\mathcal{M},\mathcal{N}}Q^{\mathcal{N}}+\sum_{k=\lceil\mathcal{M}(1+1/\log 2)\rceil}^{\left\lfloor\log_2\left(\frac{Q^{\mathcal{N}}}{\phi(Q)}\right)\right\rfloor}2^{k+1}(k-\mathcal{M})^{\mathcal{M}-1}\left(2^{-k}Q^{\mathcal{N}}+\left(\frac{2^{-k}Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{\mathcal{M}+\mathcal{N}-1}{\mathcal{M}+\mathcal{N}}}\right)\tag{57}$$

$$\ll_{\mathcal{M},\mathcal{N}}\sum_{k=0}^{\left\lfloor\log_2\left(\frac{Q^{\mathcal{N}}}{\phi(Q)}\right)\right\rfloor}k^{\mathcal{M}-1}\left(Q^{\mathcal{N}}+2^{\frac{k}{\mathcal{M}+\mathcal{N}}}\left(\frac{Q^{\mathcal{N}}}{\phi(Q)}\right)^{\frac{\mathcal{M}+\mathcal{N}-1}{\mathcal{M}+\mathcal{N}}}\right),\tag{58}$$

where we estimate $\#M\left(\boldsymbol{L},2^{-k},1/2,Q\right)$ by $Q^{\mathcal{N}}$ for $k\leq\lfloor\mathcal{M}\left(1+1/\log 2\right)\rfloor$. Note that in (57) we used (55) to bound the volume term in Proposition 1.2 by $2^{-k}Q^{\mathcal{N}}$. The required result follows from (58) combined with the trivial estimates $\sum_{k=0}^{K}k^{\mathcal{M}-1}\leq K^{\mathcal{M}}$ and $\sum_{k=0}^{K}k^{\mathcal{M}-1}2^{\frac{k}{\mathcal{M}+\mathcal{N}}}\ll_{\mathcal{M},\mathcal{N}}K^{\mathcal{M}-1}2^{\frac{K}{\mathcal{M}+\mathcal{N}}}$.

7 Proof of Propositions 1.13 and 1.14

The proofs of Theorems [10, Theorems 2 and 3] rely on [10, Theorems 7 and 8]. We state here the refined versions of Theorems 7 and 8. Let $\tilde{\boldsymbol{a}}=(q,\boldsymbol{a})$, where $q\in\mathbb{Z}$ and $\boldsymbol{a}\in\mathbb{Z}^d$. For $0<\delta<1/2$ we set

$$\mathcal{A}(q,\delta):=\#\left\{\boldsymbol{a}\in\mathbb{Z}^d:\boldsymbol{a}\in(0,q]^d,\ \left\|\tilde{\boldsymbol{a}}\tilde{\boldsymbol{A}}\right\|<\delta\right\},$$

and

$$\mathcal{N}(Q,\delta):=\sum_{q=1}^{Q}\mathcal{A}(q,\delta),$$

where $Q\in\mathbb{N}$.

Lemma 7.1. *Let $\boldsymbol{A}\in\mathbb{R}^{d\times(\mathcal{N}-d)}$ and let $\boldsymbol{\alpha}_0\in\mathbb{R}^{\mathcal{N}-d}$. Let $\tilde{\boldsymbol{A}}$ be the matrix defined in (37). Assume that $\tilde{\boldsymbol{A}}$ is $\tilde{\phi}$-semimultiplicatively badly approximable, where $\tilde{\phi}:[1,+\infty)\to(0,1]$ is such* *that $\tilde{\phi}(\lambda x) \gg_\lambda \tilde{\phi}(x)$ for all $\lambda \gg 1$. Then, for all $\varepsilon' > 0$ we have*

$$\left|\mathcal{N}(Q,\delta) - 2^{\mathcal{N}-d}\delta^{\mathcal{N}-d}\sum_{q=1}^{Q} q^d\right| \le \varepsilon'\delta^{\mathcal{N}-d}Q^{d+1} + O_{\varepsilon',\mathcal{N},d}\left(\log\left(\frac{1/\delta^{\mathcal{N}-d}}{\tilde{\phi}(1/\delta)}\right)^{d+1} + \frac{1}{\tilde{\phi}(1/\delta)}\log\left(\frac{1/\delta^{\mathcal{N}-d}}{\tilde{\phi}(1/\delta)}\right)^{d}\right).$$

Lemma 7.2. *Let $\boldsymbol{A} \in \mathbb{R}^{d\times(\mathcal{N}-d)}$ be a ϕ-semimultiplicatively badly approximable matrix, where $\phi : [1,+\infty) \to (0,1]$ is such that $\phi(\lambda x) \gg_\lambda \phi(x)$ for all $\lambda \gg 1$. Then, for all $\varepsilon' > 0$ we have*

$$\left|\mathcal{A}(q,\delta) - 2^{\mathcal{N}-d}\delta^{\mathcal{N}-d}q^d\right| \le \varepsilon'\delta^{\mathcal{N}-d}q^d + O_{\varepsilon',\mathcal{N},d}\left(\log\left(\frac{1/\delta^{\mathcal{N}-d}}{\phi(1/\delta)}\right)^{d} + \frac{1}{\phi(1/\delta)}\log\left(\frac{1/\delta^{\mathcal{N}-d}}{\phi(1/\delta)}\right)^{d-1}\right).$$

For simplicity, we prove Lemma 7.2 first.

Proof. From Huang and Liu's proof of [10, Theorem 7], we have

$$\mathcal{A}(q,\delta) \le \left(2\delta + \frac{1}{J+1}\right)^{\mathcal{N}-d}\left(q^d + \sum_{0<|\boldsymbol{j}|_\infty\le J}\prod_{u=1}^{d}\|A_u(\boldsymbol{j})\|^{-1}\right)$$

for any $J \in \mathbb{N}$ (recall that A_u denotes the linear form induced by the u-th row of the matrix $\boldsymbol{A}$). We apply Corollary 1.8 to estimate the right-hand side. We conclude the proof as in [10], by setting $J = \kappa/\delta$ for some large enough κ and by using the fact that $\phi(\kappa/\delta) \gg_\kappa \phi(1/\delta)$. □

The proof of Lemma 7.1 is along the same lines.

Now, we show how to prove Proposition 1.14. We follow [10]. First, we note that without loss of generality we can assume $\psi(x) \ge \hat{\psi}(x)$ for all x, since otherwise we replace ψ with $\max\left\{\hat{\psi}(x),\psi(x)\right\}$, and we prove that the Hausdorff dimension of the set $\mathscr{S}_{\mathcal{N}}\left(\max\left\{\hat{\psi}(x),\psi(x)\right\}\right) \supset \mathscr{S}_{\mathcal{N}}(\psi)$ is zero. Note that by condition *iiia*), the function $\max\left\{\hat{\psi}(x),\psi(x)\right\}$ satisfies

$$\sum_{q=1}^{+\infty}\max\left\{\hat{\psi}(x),\psi(x)\right\}^{\mathcal{N}-d+s}q^{d-s} < +\infty.$$

In view of this, in condition *iiib*) we can replace $\hat{\psi}$ with ψ. To prove the Proposition 1.14, we need to estimate $\mathcal{A}(q, C\psi(q))$, where C is some large constant depending on $\boldsymbol{A}$ (see [10, Proof of Thm. 2]). By applying Lemma 7.2 with $\varepsilon' = 1$, we find

$$\left|\mathcal{A}(q, C\psi(q)) - (2C)^{\mathcal{N}-d}\psi(q)^{\mathcal{N}-d}q^d\right| \\ \ll_{C,\mathcal{N},d} \psi(q)^{\mathcal{N}-d}q^d + \log\left(\frac{1/\psi(q)^{\mathcal{N}-d}}{\phi(1/\psi(q))}\right)^d + \frac{1}{\phi(1/\psi(q))}\log\left(\frac{1/\psi(q)^{\mathcal{N}-d}}{\phi(1/\psi(q))}\right)^{d-1}. \quad (59)$$

Then, from *ii*) we deduce

$$\log\left(\frac{1/\psi(q)^{\mathcal{N}-d}}{\phi(1/\psi(q))}\right)^d \ll_{\mathcal{N},d,\gamma} \log\left(\frac{1}{\psi(q)}\right)^d \ll_{\mathcal{N},d} \frac{1}{\phi(1/\psi(q))}\log\left(\frac{1/\psi(q)^{\mathcal{N}-d}}{\phi(1/\psi(q))}\right)^{d-1}. \quad (60)$$

Finally, condition *iiib*) with ψ in lieu of $\hat{\psi}$ implies

$$\frac{1}{\phi(1/\psi(q))}\log\left(\frac{1/\psi(q)^{\mathcal{N}-d}}{\phi(1/\psi(q))}\right)^{d-1} \ll_{\mathcal{N},d,s} \frac{\psi(q)^{\mathcal{N}-d}q^d}{\log(q)^{d-1}}\log\left(\frac{q^d}{\log(q)^{d-1}}\right)^{d-1} \ll_d \psi(q)^{\mathcal{N}-d}q^d. \quad (61)$$

Hence, from (59), (60), and (61) we deduce $\mathcal{A}(q, C\psi(q)) \ll_{C,\mathcal{N},d,s,\gamma} \psi(q)^{\mathcal{N}-d}q^d$, and we can conclude just as in [10].

To prove Proposition 1.13, we use Lemma 7.1, and parts *ii*) and *iii*) to obtain an estimate of $\mathcal{N}(Q, C\psi(Q))$.

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Multiplicatively Badly Approximable Matrices up to Logarithmic Factors

under review, 17 pages

Multiplicatively badly approximable matrices up to logarithmic factors

Let $\|x\|$ denote the distance from $x \in \mathbb{R}$ to the nearest integer. In this paper, we prove an existence and density statement for matrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ satisfying

$$\liminf_{|\boldsymbol{q}|_\infty \to +\infty} \prod_{j=1}^{n} \max\{1, |q_j|\} \log\left(\prod_{j=1}^{n} \max\{1, |q_j|\}\right)^{m+n-1} \prod_{i=1}^{m} \|A_i \boldsymbol{q}\| > 0,$$

where the vector $\boldsymbol{q}$ ranges in $\mathbb{Z}^n$ and A_i are the rows of the matrix $\boldsymbol{A}$. This result extends a previous result of Moshchevitin for 2-dimensional vectors to arbitrary dimension. The estimates needed to apply Moshchevitin's method to the case $m > 2$ are not currently available. We therefore develop a substantially different method, that allows us to overcome this issue. We also generalise this existence result to the inhomogeneous setting. Matrices with the above property appear to have a very small sum of reciprocals of fractional parts. This fact helps us to shed light on a question raised by Lê and Vaaler, thereby proving some new estimates for such sums in higher dimension.

1 Introduction

1.1 Notation

For $x \in \mathbb{R}$ we denote by $\|x\|$ the distance from x to the nearest integer. For a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ we denote by $A_i \in \mathbb{R}^n$ the rows of $\boldsymbol{A}$ ($i = 1, \ldots, m$). If $\boldsymbol{x} \in \mathbb{R}^n$, we denote by $A_i\boldsymbol{x}$ the sum $\sum_{j=1}^{n} A_{ij}x_j$. Given a set X and a pair of functions $f, g : X \to \mathbb{R}$, we write $f \ll g$ (or $f \gg g$) if there exists a constant $c > 0$ such that $f(x) \leq cg(x)$ (or $f(x) \geq cg(x)$) for all $x \in X$. If the constant c depends on some parameters, we write them under the symbol $\ll$ (or $\gg$). We denote by $|\cdot|_2$ the Euclidean norm and by $|\cdot|_\infty$ the supremum norm in $\mathbb{R}^n$. We also denote by dist_2 and dist_∞ the Euclidean and supremum distances respectively. For a set $X \subset \mathbb{R}^n$ we denote by $\mathrm{diam}(X)$ its diameter and by $\mathrm{Vol}(X)$ its n-dimensional Hausdorff measure. If X is a (hyper)cube, we denote by $\mathrm{edge}(X)$ the length of its edges, i.e., its 1-dimensional faces. If $f : \mathbb{Z}^n \to [0, +\infty)$ is a function, we denote by $\liminf_{|\boldsymbol{q}|_\infty \to +\infty} f(\boldsymbol{q})$ the number $\liminf_{q\to+\infty} \min\{f(\boldsymbol{q}) : |\boldsymbol{q}|_\infty = q\}$. Finally, we intend any product $\prod_a^b$ where $b < a$ as 1.

1.2 Background

It is well known that the set of real numbers $\alpha \in \mathbb{R}$ such that

$$\liminf_{q\to\infty} q\|q\alpha\| > 0 \tag{62}$$

is non empty and has full Hausdorff dimension. Such numbers are called badly approximable. The notion of bad approximablity can be extended to a higher-dimensional setting. In particular, a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ (where m, n are positive integers) is said to be badly approximable if, in analogy to (62),

$$\liminf_{|\boldsymbol{q}|_\infty\to+\infty} |\boldsymbol{q}|_\infty^n \max_{i=1}^{m}\{\|A_i\boldsymbol{q}\|\}^m > 0.$$

Schmidt [14] showed that the set of such matrices has full Hausdorff dimension in $\mathbb{R}^{m\times n}$.

A multiplicative generalisation of (62) has also extensively been studied [4]. A matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is said to be multiplicatively badly approximable if

$$\liminf_{|\boldsymbol{q}|_\infty\to+\infty} \prod_{j=1}^{n} \max\{1, |q_j|\} \prod_{i=1}^{m} \|A_i\boldsymbol{q}\| > 0. \tag{63}$$

To simplify the notation, throughout this paper we write

$$\prod(\boldsymbol{q}) := \prod_{j=1}^{n} \max\{1, |q_j|\}$$

for all $\boldsymbol{q} \in \mathbb{Z}^n$.

Proving the existence of multiplicatively badly approximable matrices is a major problem in Diophantine approximation. The famous Littlewood conjecture states that for any pair of real numbers $\alpha, \beta \in \mathbb{R}$ it holds

$$\liminf_{q\to\infty} q\|q\alpha\|\|q\beta\| = 0, \tag{64}$$

or that, in other words, there are no 2×1 multiplicatively badly approximable matrices. If the Littlewood conjecture were true, there would be no multiplicatively badly approximable matrices (except for $n = m = 1$). This follows from the fact that every submatrix of a multiplicatively badly approximable matrix is itself multiplicatively badly approximable and from the transference principle (see [10, Theorem 2.2]).

Since the set of multiplicatively badly approximable matrices with this definition could be empty, some authors (Badziahin, Velani, etc.) started working on a different definition of multiplicative bad approximability. Their idea was to weaken the Diophantine condition in (63) to allow for more flexibility. One possible way of doing this could be increasing the exponent of the factor $\prod(\boldsymbol{q})$. This modification however, adds too many matrices to the set in consequence of the two following $0-1$ results.

Theorem 1.1 (Gallagher)**.** *Let m be a positive integer and let $\psi : \mathbb{N} \to (0,1]$ be a non-increasing*[10] *function. Let also*

$$W^{\times}(m,1,\psi) := \left\{ \boldsymbol{A} \in [0,1]^{m\times 1} : \prod_{i=1}^{m} \|A_i q\| < \psi\left(|q|\right) \text{ for infinitely many } q \in \mathbb{Z} \right\}.$$

Then,

$$\mathscr{L}(W^{\times}(m,1,\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{+\infty} \psi\left(q\right) \log\left(\psi(q)^{-1}\right)^{m-1} < +\infty \\ 1 & \text{if } \sum_{q=1}^{+\infty} \psi\left(q\right) \log\left(\psi(q)^{-1}\right)^{m-1} = +\infty \end{cases},$$

where $\mathscr{L}$ stands for the m-dimensional Lebesgue measure.

Theorem 1.2 (Sprindžuk)**.** *Let m, n be positive integers and let $\psi : \mathbb{N} \to (0,1]$ be any function. Let also*

$$W^{\times}(m,n,\psi) := \left\{ \boldsymbol{A} \in [0,1]^{m\times n} : \prod_{i=1}^{m} \|A_i \boldsymbol{q}\| < \psi\left(\prod(\boldsymbol{q})\right) \text{ for infinitely many } \boldsymbol{q} \in \mathbb{Z}^n \right\}.$$

Then,

$$\mathscr{L}(W^{\times}(m,n,\psi)) = \begin{cases} 0 & \text{if } \sum_{\boldsymbol{q}\in\mathbb{Z}^n} \psi\left(\prod(\boldsymbol{q})\right) \log\left(\psi\left(\prod(\boldsymbol{q})\right)^{-1}\right)^{m-1} < +\infty \\ 1 & \text{if } \sum_{\boldsymbol{q}\in S} \psi\left(\prod(\boldsymbol{q})\right) \log\left(\psi\left(\prod(\boldsymbol{q})\right)^{-1}\right)^{m-1} = +\infty \end{cases},$$

where $\mathscr{L}$ stands for the mn-dimensional Lebesgue measure, and S is any infinite set of pairwise linearly independent vectors in $\mathbb{Z}^n$.

Theorem 1.1 follows from the more general [8, Theorem 1], whereas Theorem 1.2 follows from its multi-dimensional analogue [15, Chapter 1, Theorem 13]. Note that there is a discrepancy

[10] We say that a function $f : A \to B$ with $A, B \subset \mathbb{R}$ is non-increasing if $f(x) \geq f(y)$ for all $x < y$.

between the cases $n = 1$ and $n > 1$. In particular, Theorem 1.2 does not imply Theorem 1.1, since for $n = 1$ there are no infinite subsets of pairwise linearly independent vectors in $\mathbb{Z}$.

Gallagher and Sprindžuk's Theorems both imply that the set of matrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ such that

$$\liminf_{|\boldsymbol{q}|_\infty\to+\infty} \prod(\boldsymbol{q})^{1+\varepsilon} \prod_{i=1}^{m} \|A_i\boldsymbol{q}\| > 0,$$

has full Lebesgue measure in $\mathbb{R}^{m\times n}$ for all $\varepsilon > 0$. Therefore, a finer indicator (in comparison with the exponent of $\prod(\boldsymbol{q})$) is required to find a non-empty set that would not coincide with almost all the space $\mathbb{R}^{m\times n}$. A natural approach is to allow for logarithmic factors, i.e., to consider the set

$$\mathrm{Mad}^{\lambda}(m,n) := \left\{\boldsymbol{A} \in \mathbb{R}^{m\times n} : \liminf_{|\boldsymbol{q}|_\infty\to+\infty} \prod(\boldsymbol{q}) \left(\log \prod(\boldsymbol{q})\right)^{\lambda} \prod_{i=1}^{m} \|A_i\boldsymbol{q}\| > 0\right\}. \qquad (65)$$

It follows from Theorems 1.1 and 1.2 that $\mathrm{Mad}^{\lambda}(m,n)$ has full Lebesgue measure for $\lambda > m+n-1$ and zero Lebesgue measure for $\lambda \leq m+n-1$. However, it could happen, for example, that $\mathrm{Mad}^{\lambda}(m,n)$ is empty for $\lambda \leq m+n-1$, and this is the case that we treat in this paper.

To have a better understanding of the situation, we look at the analogue of the set $\mathrm{Mad}^{\lambda}(m,n)$ in the standard setting, i.e., the set

$$\mathrm{Bad}^{\lambda}(m,n) := \left\{\boldsymbol{A} \in \mathbb{R}^{m\times n} : \liminf_{|\boldsymbol{q}|_\infty\to+\infty} |\boldsymbol{q}|_\infty^{n} (\log|\boldsymbol{q}|_\infty)^{\lambda} \max\{\|A_1\boldsymbol{q}\|,\dots,\|A_m\boldsymbol{q}\|\}^m > 0\right\}. \qquad (66)$$

The analogue of Theorems 1.1 and 1.2 in the additive setting is the Khintchine-Groshev Theorem (see references in [3]), which we report for the convenience of the reader.

Theorem 1.3 (Khintchine-Groshev)**.** *Let m, n be positive integers and let $\psi : \mathbb{N} \to (0,1]$ be a non-increasing function. Let also*

$$W^{+}(m,n,\psi) := \left\{\boldsymbol{A} \in [0,1]^{m\times n} : \max_{i=1}^{m}\{\|A_i\boldsymbol{q}\|\}^m < \psi(|\boldsymbol{q}|_\infty) \text{ for infinitely many } \boldsymbol{q} \in \mathbb{Z}^n\right\}.$$

Then,

$$\mathscr{L}(W^{+}(m,n,\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{+\infty} \psi(q)q^{n-1} < +\infty \\ 1 & \text{if } \sum_{q=1}^{+\infty} \psi(q)q^{n-1} = +\infty \end{cases},$$

where $\mathscr{L}$ stands for the mn-dimensional Lebesgue measure.

This theorem, in combination with Schmidt's dimensional result for badly approximable matrices [14] and Dirichlet's Theorem, implies that

$$\mathrm{Bad}^{\lambda}(m,n)=\begin{cases}\emptyset & \text{if } \lambda<0\\ \text{full Hausdorff dimension but zero Lebesgue measure set} & \text{if } 0\leq\lambda\leq 1\\ \text{full Lebesgue measure set} & \text{if } \lambda>1\end{cases}. \quad (67)$$

In particular, we observe that "shaving off" a logarithm factor from the Lebesgue $0-1$ "switch over" gives the set of badly approximable matrices.

Let us move back to the multiplicative framework and draw a comparison. To keep things simple we set $m=2$, $n=1$. We note that Theorem 1.3 for $m=2$, $n=1$ and Theorem 1.1 for $m=2$ differ only by the presence of a logarithmic factor in the sum. In particular, Gallagher's Theorem implies that

$$\mathscr{L}\left(\mathrm{Mad}^{\lambda}(2,1)\right)=\begin{cases}0 & \text{if } \lambda\leq 2\\ +\infty & \text{if } \lambda>2\end{cases}.$$

Drawing inspiration from (67) and from the "shaving off" phenomenon, Badziahin and Velani [2, Statements L1-L3] made the following conjecture.

Conjecture 1.4 (Badziahin-Velani)**.**

$$\mathrm{Mad}^{\lambda}(2,1)=\begin{cases}\emptyset & \text{if } \lambda<1\\ \text{full Hausdorff dimension but zero Lebesgue measure set} & \text{if } 1\leq\lambda\leq 2\\ \text{full Lebesgue measure set} & \text{if } \lambda>2\end{cases}.$$

This is also supported by heuristic volume arguments of Peck [12], and Pollington and Velani [13] (see references in [2]). If this conjecture were true, the set $\mathrm{Mad}^1(2,1)$ would be rightfully regarded as the multiplicative analogue of the set $\mathrm{Bad}^0(2,1)$ (i.e., the set of badly approximable vectors in $\mathbb{R}^2$). Note that Conjecture 1.4 implies the Littlewood Conjecture.

Multiple authors have contributed towards a partial solution of 1.4. Moshchevitin [11] was the first to show that $\mathrm{Mad}^2(2,1)\neq\emptyset$ by using the so-called Peres-Schlag method. Subsequently, Bugeaud and Moschevitin [5] showed that $\dim\mathrm{Mad}^2(2,1)=2$, where dim denotes the Hausdorff dimension. Finally, Badziahin [1] showed that $\dim\mathrm{Mad}^{\lambda}(2,1)=2$ for all $\lambda>1$. The case $\lambda=1$ of this conjecture is still unsolved.

1.3 Main result

In analogy with Moshchevitin's result [11], we show in this paper that the set $\mathrm{Mad}^{m+n-1}(m,n)$ is dense and uncountable, in particular, it is non-empty for all $m,n \in \mathbb{N}$. We furthermore generalise this result to the inhomogeneous setting.

Let $C \subset \mathbb{R}^{m\times n}$ be a cube of edge ℓ. For $f : [0,+\infty) \to [1,+\infty)$ non-decreasing[11], $\boldsymbol{\gamma} \in \mathbb{R}^m$, and $c > 0$ we consider the set

$$\mathrm{Mad}_{m,n}(C,\boldsymbol{\gamma},f,c) := \Bigg\{ \boldsymbol{A} \in C : \prod(\boldsymbol{q}) \|A_1\boldsymbol{q}+\gamma_1\| \cdots \|A_m\boldsymbol{q}+\gamma_m\| > \frac{c}{f(\prod(\boldsymbol{q}))} \\ \text{for all } \boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \Bigg\}.$$

For $x \in [0,+\infty)$ we set $\log^*(x) := \log(\max\{e,x\})$, where $e = 2.71828\ldots$ is the base of the natural logarithm. Then, the following holds.

Proposition 1.5. *Let $m,n \in \mathbb{N}$ with $m+n \geq 3$, let C be a cube in $\mathbb{R}^{m\times n}$, and let $\boldsymbol{\gamma} \in \mathbb{R}^m$. Then, there exists a constant $c = c(m,n,\ell) > 0$, only depending on the integers m and n, and the edge of the cube C, such that for any countable (possibly finite) family of hyperplanes $\mathscr{H}$ lying in $\mathbb{R}^{m\times n}$ we have*

$$\mathrm{Mad}_{m,n}\Big(C,\boldsymbol{\gamma},\log^*(x)^{m+n-1},c\Big) \setminus \bigcup_{H\in\mathscr{H}} H \neq \emptyset.$$

Proposition 1.5 immediately implies the following corollary.

Corollary 1.6. *Let $m+n \geq 3$. Then, for all $\boldsymbol{\gamma} \in \mathbb{R}^m$ the set*

$$\mathrm{Mad}^{m+n-1}(m,n,\boldsymbol{\gamma}) := \left\{ \boldsymbol{A} \in \mathbb{R}^{m\times n} : \liminf_{|\boldsymbol{q}|_\infty \to +\infty} \prod(\boldsymbol{q}) \Big(\log \prod(\boldsymbol{q})\Big)^{m+n-1} \prod_{i=1}^{m} \|A_i\boldsymbol{q}+\gamma_i\| > 0 \right\}$$

is everywhere dense in $\mathbb{R}^{m\times n}$ and does not lie on a countable union of hyperplanes.

Note that, for certain values of $\boldsymbol{\gamma} \in \mathbb{R}^m$ the fact that the set $\mathrm{Mad}^{m+n-1}(m,n,\boldsymbol{\gamma})$ is uncountable is trivial (e.g., take $n=1$, $\gamma_1,\ldots,\gamma_{m-1} \notin \mathbb{Q}$, $\gamma_m = 0$, $A_1,\ldots,A_{m-1} \in \mathbb{Z}$, and A_m badly approximable). However, the fact that $\mathrm{Mad}^{m+n-1}(m,n,\boldsymbol{\gamma})$ does not lie on a countable union of hyperplanes implies that there exist matrices $\boldsymbol{A} \in \mathrm{Mad}^{m+n-1}(m,n,\boldsymbol{\gamma})$ whose entries, along with 1 and the entries of $\boldsymbol{\gamma}$, are linearly independent over $\mathbb{Q}$. This excludes many of the most trivial

[11] We say that a function $f : A \to B$ with $A, B \subset \mathbb{R}$ is non-decreasing if $f(x) \leq f(y)$ for all $x < y$.

examples. That said, for $\boldsymbol{\gamma} = \mathbf{0}$ even the non-emptiness of the set $\mathrm{Mad}^{m+n-1}(m, n, \boldsymbol{\gamma})$ appears to be non-trivial.

It is worth observing that to prove Proposition 1.5 we do not follow the Peres-Schlag method, i.e., the method used by Moshchevitin to show that $\mathrm{Mad}^2(2, 1) \neq \emptyset$ (see [11]). Moshchevitin's proof relies both on the one dimensional case ($m = n = 1$), and on estimates for the sum

$$\sum_{q=1}^{Q} \frac{1}{q\|q\alpha\|}.$$

This sum is known to grow like $O(\log(Q)^2)$ for almost all $\alpha \in \mathbb{R}$ [9, Theorem 6 b)]. However, to apply inductively Moshchevitin's argument in dimension, e.g., $m \times 1$, one would require an estimate of the form

$$\sum_{q=1}^{Q} \frac{1}{q\|q\alpha_1\| \ldots \|q\alpha_m\|} \ll_m (\log Q)^{m+1}$$

for at least some vectors $(\alpha_1, \ldots, \alpha_m)$. At present, such an estimate is only known to hold for multiplicatively badly approximable vectors (to see this, it suffices to apply Abel's summation formula and [10, Theorem 2.1]). Hence, a different method is required.

To prove Proposition 1.5, we work directly in a higher-dimensional setting without relying on induction. We generalise a construction introduced by Badziahin and Velani [2], in order to produce a multi-dimensional Cantor-like set contained in $\mathrm{Mad}_{m,n}(C, \boldsymbol{\gamma}, f, c)$. Such construction requires to count lattice points lying in sets with "hyperbolic spikes". To accomplish this, we use an elementary geometric argument that is the key to the whole proof. The core of this argument can be found in Lemma 4.1. We remark that Badziahin's proof [1] of the fact that $\dim \mathrm{Mad}^{\lambda}(2, 1) = 1$ for $\lambda > 1$ also relies on induction.

Proposition 1.5 is one log factor off from the conjecturally optimal result (i.e., the extension of Conjecture 1.4 to higher dimension). Specifically, we could not prove that $\mathrm{Mad}^{\lambda}(m, n) \neq \emptyset$ for $m + n - 2 \leq \lambda < m + n - 1$ in consequence of some overcounting issues araising in the proof of Proposition 1.5. Badziahin [1] used a rather convoluted strategy to overcome such issues, thus improving on the estimates of Moshchevitin. However, his methods appear quite hard to generalise to a higher-dimensional and/or inhomogeneous setting.

We conclude by saying that it would be equally desirable to prove a dimensional result for the set $\mathrm{Mad}^{m+n-1}(m, n)$. Unfortunately, the methods used in this paper do not seem powerful enough to obtain such result. Indeed, the (suitably generalised) hypothesis in Badziahin and Velani's [2, Theorem 4] does not hold for our construction. An adaptation of [2, Theorem 4] to our setting appears equally challenging, due to an obstruction in [2, Lemma 2].

1.4 Applications

Let $m, n \in \mathbb{N}$, let $\boldsymbol{Q} \in (0, +\infty)^n$, and let $X := \prod_{j=1}^{n}[-Q_j, Q_j]$. Let also $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ such that the entries $L_{i1}, \ldots, L_{in} \in \mathbb{R}$ together with 1 are linearly independent over $\mathbb{Z}$ for $i = 1, \ldots, m$. We consider the sum

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) := \sum_{\boldsymbol{q}\in X\cap\mathbb{Z}^n\setminus\{\boldsymbol{0}\}} \prod_{i=1}^{m} \|L_i\boldsymbol{q}\|^{-1}.$$

Sums of this shape are of major importance in Diophantine approximation and have extensively been studied (see [7] for a brief summary). Lê and Vaaler [10, Corollary 1.2] showed that for $Q := (Q_1 \cdots Q_n)^{1/n} \geq 1$ it holds

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \gg_{m,n} Q^n(\log Q)^m$$

independently of the choice of the matrix $\boldsymbol{L}$. They also asked whether this estimate is sharp, i.e., whether there exist matrices $\boldsymbol{L}$ such that

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{m,n} Q^n(\log Q)^m.$$

In [10, Theorem 2], they showed that this holds true for multiplicatively badly approximable matrices, but since these matrices are not known to exist, the question remains open. Proposition 1.5 allows us to find matrices with "relatively small" (even though not optimal) upper bounds.

Let $\phi : [1, +\infty) \to (0, 1]$ be a non-increasing function. In [7, Corollary 1.8] the author proved that if a matrix $\boldsymbol{L}$ is ϕ-semimultiplicatively badly approximable, i.e., if

$$|\boldsymbol{q}|_\infty^n \prod_{i=1}^{m} \|L_i\boldsymbol{q}\| \geq \phi(|\boldsymbol{q}|_\infty)$$

for all $\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$, then we have

$$\sum_{\substack{\boldsymbol{q}\in[-Q,Q]^n\\ \cap\, \mathbb{Z}^n\setminus\{\boldsymbol{0}\}}} \prod_{i=1}^{m} \|L_i\boldsymbol{q}\|^{-1} \ll_{m,n} Q^n \log\left(\frac{Q^n}{\phi(Q)}\right)^m + \frac{Q^n}{\phi(Q)} \log\left(\frac{Q^n}{\phi(Q)}\right)^{m-1} \tag{68}$$

for $Q \geq 2$. Since $\prod_{j=1}^{n} \max\{1, |q_j|\} \leq |\boldsymbol{q}|_\infty^n$ for all $\boldsymbol{q} \in \mathbb{Z}^n$, from Proposition 1.5 we easily deduce the following.

Corollary 1.7. *Let $m, n \in \mathbb{N}$. Then, there exist uncountably many matrices $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ such that*

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{m,n} Q^n (\log Q)^{2m+n-2} \tag{69}$$

for all $\boldsymbol{Q} = (Q, \ldots, Q)$ with $Q \geq 2$.

Note that the linear independence of the row entries of $\boldsymbol{L}$ together with 1 over $\mathbb{Z}$ follows from the definition of $\mathrm{Mad}^{m+n-1}(m,n)$.

This result is not best possible. In particular, by (68), we have that for $\boldsymbol{L} \in \mathrm{Mad}^{1+\varepsilon}(2,1)$ it holds

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_{\varepsilon} Q(\log Q)^{1+\varepsilon}$$

for any $\varepsilon > 0$ (such matrices $\boldsymbol{L}$ exist since $\dim \mathrm{Mad}^{\lambda}(2,1) = 2$ for $\lambda > 1$). Hence, for $m = 2,\ n = 1$ inequality (69) is not sharp. It is also well-known (see [6]) that set of $1 \times n$ matrices $\boldsymbol{L}$ such that

$$S_{\boldsymbol{L}}(\boldsymbol{Q}) \ll_n Q^n \log Q$$

has full Hausdorff dimension in $\mathbb{R}^{1\times n}$. Thus, (69) is again not sharp for $m = 1$. However, to the best of our knowledge, for $m \geq 3$ or $m = 2,\ n \geq 2$ the existence of matrices satisfying (69) was not previously known.

2 Generalised Cantor sets in higher dimension

In this section we introduce a simple generalisation[12] of the one-dimensional construction used by Badziahin and Velani in [2]. This generalisation will be useful in the proof of Proposition 1.5. From now on the word cube will stand for ball in the supremum norm.

Let $l \in \mathbb{N}$ and let C be a closed cube in $\mathbb{R}^l$. For $k \geq 0$ let $\boldsymbol{R} := (R_k)$ be a sequence of natural numbers, and let $\boldsymbol{r} := (r_k)$ and $\boldsymbol{h} := (h_k)$ be sequences of non-negative integers with $0 \leq h_k \leq k$.

Our goal is to construct a Cantor-like set contained in C depending on the sequences $\boldsymbol{R}$, $\boldsymbol{r}$, and $\boldsymbol{h}$. We denote such set by $\boldsymbol{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$. To this end, we introduce two sequences $\mathcal{I}_k$ and $\mathcal{J}_k$ of

[12] To be precise, our construction is simplified. In [2], Badziahin and Velani consider a double-index sequence $\boldsymbol{r} = (r_{h,k})$, whereas we consider two one-index sequences $\boldsymbol{r} = (r_k)$ and $\boldsymbol{h} = (h_k)$, since this is enough for our application.

cube collections such that each cube in these collections lies in C $(k \geq 0)$. We set $\mathcal{I}_0 = \mathcal{J}_0 := \{C\}$ and we define $\mathcal{I}_k$ and $\mathcal{J}_k$ by recursion on k. We do this in two steps. Suppose that we have constructed $\mathcal{I}_h$ and $\mathcal{J}_h$ for $h = 0, \ldots, k$. Then,

STEP 1 we split each cube $J \in \mathcal{J}_k$ into R_k^l cubes of equal volume. We call $\mathcal{I}_{k+1}$ the family of all the cubes obtained via this splitting procedure for J ranging in $\mathcal{J}_k$; note that for $I \in \mathcal{I}_{k+1}$

$$\mathrm{edge}(I) = R_k^{-l}\mathrm{edge}(J) \quad \text{and} \quad \#\mathcal{I}_{k+1} = R_k^l \#\mathcal{J}_k;$$

STEP 2 for each $J \in \mathcal{J}_{h_k}$ we remove from $\mathcal{I}_{k+1}$ at most r_k cubes $I \in \mathcal{I}_{k+1}$ such that $I \subset J$. We call $\mathcal{J}_{k+1}$ the family given by the remaining cubes in $\mathcal{I}_{k+1}$.

Finally, we set

$$\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r}) := \bigcap_{k=1}^{\infty} \bigcup_{J \in \mathcal{J}_k} J.$$

Note that the sequences $\boldsymbol{R}$, $\boldsymbol{r}$, and $\boldsymbol{h}$ do not determine a unique set, but a number of different sets obtained via the procedure described above. This follows from the fact that we did not specify which cubes we remove in the second step (we only gave a bound on their number). We call every set constructed by using the sequences $\boldsymbol{R}$, $\boldsymbol{r}$, and $\boldsymbol{h}$, in the cube C, a $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set. We also observe that, by construction,

$$\#\mathcal{J}_{k+1} \geq R_k^l \#\mathcal{J}_k - r_k \#\mathcal{J}_{h_k} \tag{70}$$

for all $k \geq 0$.

Now, the following proposition extends [2, Theorem 3].

Proposition 2.1 (multidimensional Baziahin-Velani)**.** *Let $\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$ be a $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set, where $C \subset \mathbb{R}^l$ is a cube, and let*

$$t_k := R_k^l - \frac{r_k}{\prod_{i=h_k}^{k-1} t_i}$$

for $k \geq 1$. If $t_k > 0$ for all $k \geq 0$, then we have $\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r}) \neq \emptyset$.

The proof is almost straightforward and we give it directly in this section.

Proof. We shall prove by induction on k that for $k \geq 1$

$$\#\mathcal{J}_k \geq t_{k-1}\#\mathcal{J}_{k-1}. \tag{71}$$

The fact that $t_k > 0$ for all k, along with (71), implies

$$\#\mathcal{J}_k \geq \left(\prod_{h=0}^{k-1} t_h\right)\#\mathcal{J}_0 > 0.$$

Hence, every $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set is the intersection of a family of nested compact non-empty sets, and therefore non-empty.

We are left to prove that $\#\mathcal{J}_k \geq t_{k-1}\#\mathcal{J}_{k-1}$ for all $k \geq 1$. By (70), we have $\#\mathcal{J}_1 \geq R_0^l\#\mathcal{J}_0 - r_0\#\mathcal{J}_0 = t_0\#\mathcal{J}_0$, and this proves the case $k = 1$. Now, let us assume that for all $1 \leq h \leq k$ it holds $\#\mathcal{J}_h \geq t_{h-1}\#\mathcal{J}_{h-1}$. Then, in particular, we have

$$\#\mathcal{J}_k \geq \left(\prod_{i=h_k}^{k-1} t_i\right)\#\mathcal{J}_{h_k}.$$

This, combined with (70), gives

$$\#\mathcal{J}_{k+1} \geq R_k^l\#\mathcal{J}_k - r_k\#\mathcal{J}_{h_k} \geq \left(R_k^l - \frac{r_k}{\prod_{i=h_k}^{k-1} t_i}\right)\#\mathcal{J}_k = t_k\#\mathcal{J}_k,$$

whence the claim. □

3 Proof of Proposition 1.5

The strategy is simple enough: by picking suitable parameters, we construct a non-empty $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set $\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$ lying in $\mathrm{Mad}_{m,n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H \in \mathscr{H}} H$.

To do so, we fix a non-decreasing[13] sequence of integers $\boldsymbol{R} = (R_k)$ with $R_k \geq 1$, a sequence of non-negative integers $\boldsymbol{h}$, with $0 \leq h_k \leq k$, and a strictly increasing unbounded function $F : \{0\} \cup \mathbb{N} \to [1, +\infty)$. In the following technical lemma we specify the values of a sequence $\boldsymbol{r}$ (in terms of c, ℓ, F, $\boldsymbol{R}$, and $\boldsymbol{h}$) for which there exists a (possibly empty) $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set lying in $\mathrm{Mad}_{m,n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H \in \mathscr{H}} H$.

Lemma 3.1. *Assume that*

[13] We say that a sequence $\{x_i\}_{i \in \{0\} \cup \mathbb{N}}$ of real numbers is non-decreasing if $x_i \leq x_{i+1}$ for all i.

i) $2^m c < e^{-1}$;

ii) $F(0) = 1$ *and* $F(k+1)/F(k) \geq e$ *for all* $k \geq 0$;

iii) $F(k+1)^2(\log^* F(k+1))^{m+n-1} \leq c\ell^{-1} \prod_{h=0}^{k} R_h$ *for all* $k \geq 0$.

Then, there is a $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$*-Cantor set contained in* $\mathrm{Mad}_{m+n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H \in \mathscr{H}} H$ *with* $\boldsymbol{r}$ *given by*

$$r_k := c(m,n) \left[\mathfrak{f}(c, \ell, \boldsymbol{R}, \boldsymbol{h}, k) \prod_{h=h_k}^{k} R_h^{mn} + \prod_{h=h_k}^{k} R_h^{mn-1} \right], \tag{72}$$

where the factor $\mathfrak{f}(c, \ell, \boldsymbol{R}, \boldsymbol{h}, k)$ *has the form*

$$\mathfrak{f}(c, \ell, \boldsymbol{R}, \boldsymbol{h}, k) := c \log\left(\frac{1}{2^m c}\right)^{m-1} \frac{1}{\log^* F(k)} \log\left(\frac{F(k+1)}{F(k)}\right)^{n-1} \left(\log\left(\frac{F(k+1)}{F(k)}\right) + \ell^{-m}\left(2F(k)^{-m/n} - F(k+1)^{-m/n}\right) \prod_{h=0}^{h_k-1} R_h^m \right), \tag{73}$$

and $c(m,n) > 0$ *is a constant only depending on* m *and* n.

Lemma 3.1 is a key result in our method. Its proof, although quite technical, is essentially based on elementary geometric considerations. We prove Lemma 3.1 in Section 4.

Now, we need to show that the $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$-Cantor set constructed in Lemma 3.1 is non-empty. To do so, we use a non-emptiness condition involving the values of the sequence $\boldsymbol{r}$.

Lemma 3.2. *Let* $\boldsymbol{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$ *be a* $(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$*-Cantor set. If for all* $k \geq 0$ *we have*

$$r_k \leq \frac{g_k}{\max\{2, k\}} \prod_{h=h_k}^{k} R_h^{mn}, \tag{74}$$

where $g_k := \max\{2, h_k\}/(8\max\{2, k-1\})$*, then the set* $\boldsymbol{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$ *is non-empty.*

We prove this lemma in Section 5.

To conclude the proof of Proposition 1.5, it is enough to show that both the hypotheses of Lemma 3.1 and Lemma 3.2 simultaneously hold for an appropriate choice of the parameters $c, F, \boldsymbol{R}$, and $\boldsymbol{h}$. With this in mind, we fix a constant $R > 0$, and we set $R_k := R$, $F(k) := R^{k/3}$, and $h_k := \lfloor k/(3n) \rfloor$ for all $k \geq 0$. Then, we prove that, provided R is large enough, the constant c has enough room to satisfy both the hypotheses of Lemma 3.1 and Lemma 3.2.

With our choice of $\boldsymbol{R}, F$, and $\boldsymbol{h}$, condition $ii)$ in Lemma 3.1 becomes $R \geq e^3$, whereas condition $iii)$ becomes

$$R^{\frac{2(k+1)}{3}} \log^* \left(R^{\frac{k+1}{3}}\right)^{m+n-1} \leq c\ell^{-1} R^{k+1},$$

whence

$$\ell R^{-\frac{k+1}{3}} \log^* \left(R^{\frac{k+1}{3}}\right)^{m+n-1} \leq c. \tag{75}$$

On the other hand, by substituting (72) into (74), we obtain

$$c(m,n)\left[\mathfrak{f}(c,\ell,\boldsymbol{R},\boldsymbol{h},k)\prod_{h=h_k}^{k} R_h^{mn} + \prod_{h=h_k}^{k} R_h^{mn-1}\right] \leq \frac{g_k}{\max\{2,k\}} \prod_{h=h_k}^{k} R_h^{mn},$$

which, with our choice of $\boldsymbol{R}, F$, and $\boldsymbol{h}$, is equivalent to

$$\mathfrak{f}(c,\ell,R,k) + \frac{1}{R^{\left(k-\lfloor\frac{k}{3n}\rfloor+1\right)}} \leq \frac{g_k c(m,n)^{-1}}{\max\{2,k\}}. \tag{76}$$

Since g_k is bounded away from 0 for all k, by choosing R suitably large in terms of m and n, we can ignore the second term at the left-hand side of (76). Hence, we are just left to prove

$$\mathfrak{f}(c,\ell,R,k) \leq \frac{c'(m,n)}{\max\{2,k\}},$$

where $c'(m,n)$ is a constant only depending on m and n. By using (73), this can be written as

$$c\log^*\left(\frac{1}{2^m c}\right)^{m-1} \frac{1}{\max\{1,k\}} \log^*\left(R^{1/3}\right)^{n-1} \left(\log^*\left(R^{1/3}\right) + \ell^{-m} R^{\frac{-mk}{3n}} \left(2 - R^{-\frac{m}{3n}}\right) R^{\lfloor\frac{k}{3n}\rfloor m}\right) \leq \frac{c'(m,n)}{\max\{2,k\}}, \tag{77}$$

where we ignored a factor of $\log\left(R^{1/3}\right)$ at the denominator, coming from $\log^*(F(k))$ for $k \geq 1$. Assuming $\ell < 1$, condition (77) holds if we have

$$c\log^*\left(\frac{1}{2^m c}\right)^{m-1} \leq \mathrm{const}''(m,n)\ell^m \log^*\left(R^{1/3}\right)^{-n}, \tag{78}$$

where $\mathrm{const}''(m,n)$ is some other positive constant only depending on m and n.

To conclude the proof, we pick a small real number $\varepsilon > 0$. Since $\log^*(1/2^m c)^{m-1} \ll_{m,\varepsilon} c^{-\varepsilon}$, condition (78) is in turn implied by

$$c \leq \mathrm{const}'''(m,n,\varepsilon)\ell^{\frac{m}{1-\varepsilon}} \log^*\left(R^{1/3}\right)^{-\frac{n}{1-\varepsilon}}, \tag{79}$$

where $\mathrm{const}'''(m,n,\varepsilon)$ is a suitably chosen positive constant only depending on m, n, and ε. The claim is then proved on noting that (75) and (79) can simultaneously hold for a sufficiently large value of R.

4 Proof of Lemma 3.1

4.1 Construction of the Cantor-like set

For each $P := (\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\boldsymbol{0}\})$, with $gcd(p_1, \ldots, p_m, q_1, \ldots, q_n) = 1$, we introduce the following "bad" set.

$$\Delta(P) := \Bigg\{ \boldsymbol{X} \in \mathbb{R}^{m\times n} : \prod_{i=1}^{m} |X_i\boldsymbol{q} + \gamma_i + p_i| \le \frac{c}{\prod(\boldsymbol{q}) \log^*(\prod(\boldsymbol{q}))^{m+n-1}}, \\ |X_i\boldsymbol{q} + \gamma_i + p_i| \le \frac{1}{2}\ i = 1, \ldots, m \Bigg\}, \quad (80)$$

where we ignore the dependence on $\boldsymbol{\gamma}$ and c for simplicity. We also enumerate the hyperplanes in $\mathscr{H}$, indexing them for $k \in \{0\} \cup \mathbb{N}$. Then, we define the families $\mathcal{J}_k$ of our Cantor-like set so that the intersection of their cubes avoids all the "bad" sets $\Delta(P)$ and the hyperplanes H_k for $k \in \mathbb{N}$. More precisely, for each $J \in \mathcal{J}_k$ we require that $J \cap (\Delta(P) \cup H_h) = \emptyset$ for all the points P with $\prod(\boldsymbol{q}) < F(k)$ and all hyperplanes H_h with $h \le k$ (where we assume $H_0 = \emptyset$). If this condition is satisfied, we have

$$\begin{aligned}\mathbf{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r}) &\subset \bigcap_{k=0}^{+\infty} \bigcap_{\prod(\boldsymbol{q})<F(k)} \bigcap_{h\le k} C \setminus (\Delta(P) \cup H_h) \\ &= C \setminus \left(\bigcup_{P} \Delta(P) \cup \bigcup_{H\in\mathscr{H}} H \right) = \mathrm{Mad}_{m,n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H\in\mathscr{H}} H, \quad (81)\end{aligned}$$

thus showing the claim. Note that (81) holds because the function $F(k)$ is unbounded.

We construct the families $\mathcal{J}_k$ by recursion on $k \ge 0$. For each k we need to ensure

$$J \in \mathcal{J}_k \Rightarrow J \cap \left(\bigcup_{\prod(\boldsymbol{q})<F(k)} \Delta(P) \cup \bigcup_{h\le k} H_h \right) = \emptyset. \quad (82)$$

If $k = 0$, we have $\mathcal{J}_0 = \{C\}$ and $F(0) = 1$. Therefore, by definition,

$$\bigcup_{\prod(\boldsymbol{q})<1} \Delta(P) \cup \bigcup_{h\le 0} H_h = \emptyset.$$

This shows that $\mathcal{J}_0$ satisfies (82). For $k \ge 1$ we subdivide the points $P \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\boldsymbol{0}\})$ into "workable" families. Namely, we define

$$C(k) := \left\{ P \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\boldsymbol{0}\}) : F(k-1) \le \prod(\boldsymbol{q}) < F(k) \right\}.$$

Suppose that we have constructed the family $\mathcal{J}_k$ in such a way that for $J \in \mathcal{J}_k$ (82) holds (note that $\mathcal{J}_k$ can be empty). If $\mathcal{J}_k = \emptyset$, we set $\mathcal{J}_{k+1} := \emptyset$, if $\mathcal{J}_k \neq \emptyset$, we proceed as follows. Since any cube in $\mathcal{I}_{k+1}$ lies within some cube in $\mathcal{J}_k$, it is enough to construct $\mathcal{J}_{k+1}$ in such a way that if $J \in \mathcal{J}_{k+1}$ then $J \cap (\Delta(P) \cup H_{k+1}) = \emptyset$ for all $P \in C(k+1)$. To define $\mathcal{J}_{k+1}$, we therefore remove from $\mathcal{I}_{k+1}$ all the cubes I such that $I \cap (\Delta(P) \cup H_{k+1}) \neq \emptyset$ for some $P \in C(k+1)$. This procedure yields a possibly empty Cantor-like set $\boldsymbol{K}(C, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{r})$, contained in $\mathrm{Mad}_{m,n}\left(C, \boldsymbol{\gamma}, \log^*(x)^{m+n-1}, c\right) \setminus \bigcup_{H \in \mathscr{H}} H$.

To conclude the proof, we just need to estimate the number of "small" cubes $I \in \mathcal{I}_{k+1}$ that need to be removed from each "big" cube $J \in \mathcal{J}_{h_k}$ to avoid the sets $\Delta(P)$ for $P \in C(k+1)$ and the hyperplane H_{k+1}, and show that such number is smaller than r_k defined in (72).

We start by counting the cubes intersecting the sets $\Delta(P)$ for $P \in C(k+1)$. In particular, for a fixed $J \in \mathcal{J}_{h_k}$ it is enough to estimate

$$\#\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\}.$$

If $\mathcal{I}_{k+1} = \emptyset$, there is nothing to prove. Otherwise, we write

$$\begin{aligned}\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\}\\ = \bigcup_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} \bigcup_{\substack{P \in C(k+1) \\ \boldsymbol{q}(P) = \boldsymbol{q}}} \{I \in \mathcal{I}_{k+1} : I \cap J \cap \Delta(P) \neq \emptyset\},\end{aligned}$$

whence we deduce

$$\#\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\} \leq \sum_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} A(\boldsymbol{q}) B(\boldsymbol{q}), \tag{83}$$

where

$$A(\boldsymbol{q}) := \max_{\substack{P \in C(k+1) \\ \boldsymbol{q}(P) = \boldsymbol{q}}} \#\{I \in \mathcal{I}_{k+1} : I \cap J \cap \Delta(P) \neq \emptyset\} \tag{84}$$

and

$$B(\boldsymbol{q}) := \#\left\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J \cap \Delta(P) \neq \emptyset\right\}. \tag{85}$$

We estimate the factors $A(\boldsymbol{q})$ and $B(\boldsymbol{q})$ separately.

4.2 Estimate of $A(\boldsymbol{q})$

To estimate $A(\boldsymbol{q})$, we need the following counting result.

4 Proof of Lemma 3.1

Lemma 4.1. *Let* $\boldsymbol{\gamma}' \in \mathbb{R}^m$, $\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$, *and* $\varepsilon, T \in (0, +\infty)$, *with* $\varepsilon/T^m < e^{-1}$ *(where* $e = 2.71828\dots$ *is the base of the natural logarithm). Let also*

$$\mathscr{C} := \left\{ \boldsymbol{X} \in \mathbb{R}^{m\times n} : \prod_{i=1}^{m} |X_i\boldsymbol{q} + \gamma_i'| \leq \varepsilon,\ |X_i\boldsymbol{q} + \gamma_i'| \leq T,\ i = 1, \dots, m \right\},$$

and let $\mathscr{D} \subset \mathbb{R}^{m\times n}$ *be a cube such that* $\mathscr{D} \cap \mathscr{C} \neq \emptyset$. *Finally, let* $\delta > 0$, $\boldsymbol{V} \in \mathbb{R}^{m\times n}$, *and* Λ *be the grid* $\delta\mathbb{Z}^{m\times n} + \boldsymbol{V}$. *Then, we have*

$$\delta^{mn}\#\{\textit{tiles } \tau \textit{ of the grid } \Lambda : \tau \cap \mathscr{D} \cap \mathscr{C} \neq \emptyset\} \leq 2^{2m-1}\frac{\varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m}{|\boldsymbol{q}|_\infty^m} \log^*\left(\frac{(T + n|\boldsymbol{q}|_\infty\delta)^m}{\varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m}\right)^{m-1} (\mathrm{edge}(\mathscr{D}) + 2\delta)^{m(n-1)}, \quad (86)$$

where a tile is any set of the form $\{\boldsymbol{X} \in \mathbb{R}^{m\times n} : \delta S_{ij} + V_{ij} \leq X_{ij} \leq \delta(S_{ij} + 1) + V_{ij},\ i = 1, \dots, m,\ j = 1, \dots, n\}$ *for some* $\boldsymbol{S} \in \mathbb{Z}^{m\times n}$.

We prove this result in Section 6.

Now, we note that if

a) $\varepsilon \gg_{m,n} (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m$;

b) $T \gg_{m,n} |\boldsymbol{q}|_\infty\delta$;

c) $\mathrm{edge}(\mathscr{D}) \gg_{m,n} \delta$;

(86) implies

$$\#\{\text{tiles } \tau \text{ of the grid } \Lambda : \tau \cap \mathscr{D} \cap \mathscr{C} \neq \emptyset\} \ll_{m,n} \frac{\varepsilon}{\delta^{mn}|\boldsymbol{q}|_\infty^m} \log^*\left(\frac{T^m}{\varepsilon}\right)^{m-1} \mathrm{edge}(\mathscr{D})^{m(n-1)}. \quad (87)$$

This is precisely the assertion that we need to prove the claim. We fix a point $P \in C(k+1)$ and a cube $J \in \mathscr{J}_{h_k}$, and we apply (87) to $\mathscr{C} = \Delta(P)$, $\mathscr{D} = J$, and to the grid Λ formed by the cubes $I \in \mathcal{I}_{k+1}$. We have $\varepsilon = c\left(\prod(\boldsymbol{q})\left(\log^*\prod(\boldsymbol{q})\right)^{m+n-1}\right)^{-1}$, $T = 1/2$, and $\delta = \ell\prod_{h=0}^{k} R_h^{-1}$ (note that by hypothesis $\varepsilon/T^m < e^{-1}$).

We show that conditions $a)$, $b)$, and $c)$ hold in this specific case. If condition $b)$ is satisfied, then to prove condition $a)$, it is enough to show that $\varepsilon \gg_{m,n} T^{m-1}|\boldsymbol{q}|_\infty \delta$. By definition of $C(k+1)$ and part $iii)$ in the hypotheses of Lemma 3.1, we have

$$\frac{\varepsilon}{|\boldsymbol{q}|_\infty} = \frac{c}{|\boldsymbol{q}|_\infty \prod(\boldsymbol{q}) \left(\log^* \prod(\boldsymbol{q})\right)^{m+n-1}} \geq \frac{c}{F(k+1)^2 (\log^* F(k+1))^{m+n-1}} \geq \ell \prod_{h=0}^{k} R_h^{-1} = \delta. \quad (88)$$

Hence, $\varepsilon \gg_{m,n} T^{m-1}|\boldsymbol{q}|_\infty \delta$, and we have $a)$. Condition $b)$ is equivalent to $1/|\boldsymbol{q}|_\infty \gg_{m,n} \delta$, which is again implied by (88). Finally, condition $c)$ is clearly satisfied since $\mathrm{edge}(J) \geq \mathrm{edge}(I)$ for any $I \in \mathcal{I}_{k+1}$.

Thus, we can apply (87) to obtain

$$\begin{aligned} A(\boldsymbol{q}) = \max_{\substack{P \in C(k+1) \\ \boldsymbol{q}(P)=\boldsymbol{q}}} \#\{I \in \mathcal{I}_{k+1} : I \cap J \cap \Delta(P) \neq \emptyset\} \ll_{m,n} \frac{c}{|\boldsymbol{q}|_\infty^m \prod(\boldsymbol{q}) \left(\log^* \prod(\boldsymbol{q})\right)^{m+n-1}} \\ \log^* \left(\frac{\prod(\boldsymbol{q}) \log^* \left(\prod(\boldsymbol{q})\right)^{m+n-1}}{2^m c} \right)^{m-1} \ell^{-m} \prod_{h=0}^{h_k - 1} R_h^{-m(n-1)} \prod_{h=0}^{k} R_h^{mn} \\ \ll_{m,n} \frac{c \log^*(1/(2^m c))^{m-1}}{|\boldsymbol{q}|_\infty^m \prod(\boldsymbol{q}) \left(\log^* \prod(\boldsymbol{q})\right)^n} \ell^{-m} \prod_{h=0}^{h_k-1} R_h^{-m(n-1)} \prod_{h=0}^{k} R_h^{mn}. \quad (89) \end{aligned}$$

4.3 Estimate of $B(\boldsymbol{q})$

We are now left to estimate $\#\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J \cap \Delta(P) \neq \emptyset\}$ for each given $\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$ such that $F(k) \leq \prod(\boldsymbol{q}) < F(k+1)$. For $P \in C(k+1)$ we consider the hyperspace

$$\pi(P) := \{\boldsymbol{X} \in \mathbb{R}^{m \times n} : X_i \boldsymbol{q} + \gamma_i + p_i = 0,\ i = 1, \ldots, m\},$$

i.e., the "core" of the set $\Delta(P)$. We show that to count the number of points $P \in C(k+1)$ such that $\Delta(P)$ intersects J, it is enough to count the number of points $P \in C(k+1)$ such that the thinner set $\pi(P)$ intersects an "inflation" of J. In particular, we claim that

$$\begin{aligned} &\#\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J \cap \Delta(P) \neq \emptyset\} \\ &\qquad \leq \#\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset\}, \quad (90) \end{aligned}$$

where $J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$ is the "inflation" of the cube J by the quantity $\sqrt{m}/|\boldsymbol{q}|_\infty$, i.e., the set $\{\boldsymbol{X} \in \mathbb{R}^{m \times n} : \mathrm{dist}_\infty(\boldsymbol{X}, J) \leq \sqrt{m}/|\boldsymbol{q}|_\infty\}$. To prove (90) we show that for any fixed P in the left-hand side of (90) we have

$$J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset.$$

Indeed, for any $\boldsymbol{Y} \in \Delta(P)$ we have

$$|Y_i\boldsymbol{q} + \gamma_i + p_i| \leq 1/2 \text{ for } i = 1, \dots, m.$$

Moreover, for $i = 1, \dots, m$ the Euclidean distance in $\mathbb{R}^m$ between the vector Y_i and the hyperplane $\{X_i\boldsymbol{q} + \gamma_i + p_i = 0\}$ is given by $|Y_i\boldsymbol{q} + \gamma_i + p_i|/|\boldsymbol{q}|_2$. Hence, the Euclidean distance in $\mathbb{R}^{m\times n}$ between the vector $\boldsymbol{Y}$ and $\pi(P)$ is at most $\sqrt{m}/(2|\boldsymbol{q}|_2)$. This shows that for any point $\boldsymbol{Y} \in \Delta(P)$, we have

$$\mathrm{dist}_2(\boldsymbol{Y}, \pi(P)) \leq \frac{\sqrt{m}}{2|\boldsymbol{q}|_2}.$$

Since $J \cap \Delta(P) \neq \emptyset$, we deduce

$$\mathrm{dist}_\infty(J, \pi(P)) \leq \mathrm{dist}_2(J, \pi(P)) \leq \mathrm{dist}_2(J \cap \Delta(P), \pi(P)) \leq \frac{\sqrt{m}}{2|\boldsymbol{q}|_2} \leq \frac{\sqrt{m}}{2|\boldsymbol{q}|_\infty}.$$

Hence, by definition of distance, $J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset$, whence the claim.

We are now left to bound the right-hand side in (90). To do this, we note that the distance between two hyperspaces $\pi(P)$ and $\pi(P')$ with same $\boldsymbol{q}$ is at least $1/(n|\boldsymbol{q}|_\infty)$. Indeed, assume that $X_i, X_i' \in \mathbb{R}^n$ satisfy $X_i\boldsymbol{q} + \gamma_i + p_i = 0$ and $X_i'\boldsymbol{q} + \gamma_i + p_i' = 0$, with $p_i \neq p_i'$. Then, by the Cauchy-Schwartz inequality, we have

$$\mathrm{dist}_\infty(X_i, X_i') \geq \frac{\mathrm{dist}_2(X_i, X_i')}{\sqrt{n}} \geq \frac{|(X_i - X_i')\boldsymbol{q}|}{\sqrt{n}|\boldsymbol{q}|_2} \geq \frac{|p_i - p_i'|}{\sqrt{n}|\boldsymbol{q}|_2} \geq \frac{1}{n|\boldsymbol{q}|_\infty}.$$

This shows that for all $\boldsymbol{X} \in \pi(P)$ and $\boldsymbol{X}' \in \pi(P')$ we have

$$\mathrm{dist}_\infty(\boldsymbol{X}, \boldsymbol{X}') \geq \frac{1}{n|\boldsymbol{q}|_\infty}. \tag{91}$$

Now, if $J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset$ for some P, by a dimensional argument[14], the hyperspace $\pi(P)$ must intersect at least one m-dimensional face of the cube $J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$. For each P such that $J_{\sqrt{m}/|\boldsymbol{q}|_\infty} \cap \pi(P) \neq \emptyset$ we select a point $Q(P)$ on an m-dimensional face of $J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$ lying in $\pi(P)$. We know, by (91), that all such points are at least at a distance of $1/(n|\boldsymbol{q}|_\infty)$ away from each other in the supremum distance. To evaluate their number, we fix any m-dimensional face E of $J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$ and we enlarge it by $1/(2n|\boldsymbol{q}|_\infty)$ in all directions, i.e., we consider the set $E_{1/(2n|\boldsymbol{q}|_\infty)}$. Then, for each intersection point $Q(P) \in \pi(P) \cap J_{\sqrt{m}/|\boldsymbol{q}|_\infty}$ we take an mn-dimensional cube of edge $1/(n|\boldsymbol{q}|_\infty)$ centred at $Q(P)$. All these cubes are contained in the "inflated" face $E_{1/(2n|\boldsymbol{q}|_\infty)}$

[14] Observe that $\pi(P)$ must intersect the boundary of J, hence some $(mn-1)$-dimensional face F of J. The intersection of $\pi(P)$ with the hyperspace generated by F has dimension at least $\dim(F) + \dim(\pi(P)) - \dim(F + \pi(P)) \geq \dim(\pi(P)) - 1$. Hence, we have a hyperspace of dimension $\pi(P) - 1$ intersecting a cube (F) of dimension $mn - 1$. The argument can be run inductively.

and they all have disjoint interiors. Comparing the volume of these cubes and the volume of the inflated face, we find

$$\begin{aligned}
&\#\{P \in C(k+1) : E \cap \pi(P) \neq \emptyset\} \left(\frac{1}{n|\boldsymbol{q}|_\infty}\right)^{mn} \\
&\qquad \leq \operatorname{Vol}\left(E_{1/(2n|\boldsymbol{q}|_\infty)}\right) = \left(\operatorname{edge}\left(J_{\sqrt{m}/|\boldsymbol{q}|_\infty}\right) + \frac{1}{n|\boldsymbol{q}|_\infty}\right)^m \left(\frac{1}{n|\boldsymbol{q}|_\infty}\right)^{m(n-1)} \\
&\qquad = \left(\operatorname{edge}(J) + \frac{1+2n\sqrt{m}}{n|\boldsymbol{q}|_\infty}\right)^m \left(\frac{1}{n|\boldsymbol{q}|_\infty}\right)^{m(n-1)}. \qquad (92)
\end{aligned}$$

Since the number of m-dimensional faces of a cube only depends on m, from (90) and (92) we deduce

$$\begin{aligned}
&B(\boldsymbol{q}) = \#\{P \in C(k+1) : \boldsymbol{q}(P) = \boldsymbol{q},\ J \cap \Delta(P) \neq \emptyset\} \leq \#\{P \in C(k+1) : \\
&\boldsymbol{q}(P) = \boldsymbol{q},\ J_{\sqrt{m}/(2|\boldsymbol{q}|_\infty)} \cap \pi(P) \neq \emptyset\} \ll_{m,n} (|\boldsymbol{q}|_\infty \operatorname{edge}(J) + 1)^m \ll_{m,n} |\boldsymbol{q}|_\infty^m \ell^m \prod_{h=1}^{h_k-1} R_h^{-m} + 1. \qquad (93)
\end{aligned}$$

4.4 Conclusion

To conclude the proof of Lemma 3.1, we combine (83), (89), and (93) to obtain

$$\begin{aligned}
&\#\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\} \\
&\ll_{m,n} \sum_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} \left(\frac{c \log^*(1/(2^m c))^{m-1}}{|\boldsymbol{q}|_\infty^m \prod(\boldsymbol{q}) \left(\log^* \prod(\boldsymbol{q})\right)^n} \ell^{-m} \prod_{h=0}^{h_k-1} R_h^{-m(n-1)} \prod_{h=0}^{k} R_h^{mn}\right) \\
&\qquad \left(|\boldsymbol{q}|_\infty^m \ell^m \prod_{h=1}^{h_k-1} R_h^{-m} + 1\right).
\end{aligned}$$

Hence, by using the fact that $|\boldsymbol{q}|_\infty^m \geq \prod(\boldsymbol{q})^{m/n}$, we find

$$\begin{aligned}
&\#\{I \in \mathcal{I}_{k+1} : \exists P \in C(k+1)\ I \cap J \cap \Delta(P) \neq \emptyset\} \ll_{m,n} c \log^* \left(\frac{1}{2^m c}\right)^{m-1} \prod_{h=h_k}^{k} R_h^{mn} \\
&\qquad \frac{1}{(\log^* F(k))^n} \sum_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} \frac{1}{\prod(\boldsymbol{q})} \left(1 + \frac{\ell^{-m}}{\prod(\boldsymbol{q})^{m/n}} \prod_{h=0}^{h_k-1} R_h^m\right). \qquad (94)
\end{aligned}$$

Now, a simple integration shows that

$$\sum_{\substack{\boldsymbol{q} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\} \\ F(k) \leq \prod(\boldsymbol{q}) < F(k+1)}} \prod(\boldsymbol{q})^{-1} \ll_n (\log^* F(k+1))^{n-1} \log^* \left(\frac{F(k+1)}{F(k)}\right),$$

and

$$\sum_{\substack{\boldsymbol{q}\in\mathbb{Z}^n\setminus\{\boldsymbol{0}\}\\ F(k)\le\prod(\boldsymbol{q})<F(k+1)}} \prod(\boldsymbol{q})^{-1-m/n} \ll_n (\log^* F(k+1))^{n-1}\left(2F(k)^{-m/n}-F(k+1)^{-m/n}\right).$$

Therefore, from (94) and from $(\log^* F(k+1))^{n-1} \le \log^*(F(k+1)/F(k))^{n-1}(\log^* F(k))^{n-1}$, we deduce

$$\begin{aligned}\#\{I\in\mathcal{I}_{k+1} : \exists P\in C(k+1)\ I\cap J\cap\Delta(P)\neq\emptyset\} \ll_{m,n}& \\ c\log^*\left(\frac{1}{2^m c}\right)^{m-1}\frac{1}{\log^* F(k)}\log^*\left(\frac{F(k+1)}{F(k)}\right)^{n-1}& \\ \left(\log^*\left(\frac{F(k+1)}{F(k)}\right)+\ell^{-m}\left(2F(k)^{-m/n}-F(k+1)^{-m/n}\right)\prod_{h=0}^{h_k-1}R_h^m\right)\prod_{h=h_k}^{k}R_h^{mn}.& \qquad (95)\end{aligned}$$

We are now left to count all the cubes in $\mathcal{I}_{k+1}$ lying in J and intersecting the hyperplane H_{k+1}. From the set of cubes $I\in\mathcal{I}_{k+1}$ such that $I\cap J\cap H_{k+1}\neq\emptyset$, we select a maximal subset S of pairwise disjoint cubes (with disjoint boundary). For each of these cubes I we pick a point lying in $I\cap H_{k+1}$. The points that we picked are, by construction, at least $\mathrm{edge}(I)$ distant from each other in the supremum norm. Then, we take $(mn-1)$-dimensional cubes in H_{k+1} of edge $\mathrm{edge}(I)$ around each such point. By construction, these cubes are disjoint. Comparing the volume of the union of the cubes with the volume of the set $(J\cap H_{k+1})$ inflated in the Euclidean distance by the quantity $\mathrm{diam}(I)$ in H_{k+1}, i.e., the set $\{\boldsymbol{X}\in H_{k+1} : \mathrm{dist}_2(\boldsymbol{X}, J\cap H_{k+1})\le\mathrm{diam}(I)\}$, we find

$$\#S\cdot\mathrm{edge}(I)^{mn-1} \ll_{m,n} (\mathrm{diam}(J\cap H_{k+1})+\mathrm{diam}(I))^{mn-1} \ll_{m,n} \mathrm{edge}(J)^{mn-1},$$

whence

$$\#\{I\in\mathcal{I}_{k+1} : I\cap J\cap H_{k+1}\neq\emptyset\} \ll_{m,n} \#S \ll_{m,n} \left(\frac{\mathrm{edge}(J)}{\mathrm{edge}(I)}\right)^{mn-1} = \prod_{h=h_k}^{k} R_h^{mn-1}. \qquad (96)$$

Combining (95) and (96), the proof of Lemma 3.1 is concluded.

5 Proof of Lemma 3.2

We show by induction on k that

$$t_k \ge R_k^{mn}\left(1-\frac{1}{\max\{2,k\}}\right) > 0 \qquad (97)$$

for all $k \geq 0$. By Proposition 2.1, this is enough to prove the claim. If $k = 0$ we have

$$t_0 = R_0^{mn} - r_0 \geq R_0^{mn} - \frac{g_0}{2} R_0^{mn}.$$

Hence, the base case is proved, given that $g_0 = 1/8$. Now, assume that (97) holds for $0 \leq h \leq k$. Then, we have

$$\begin{aligned} t_{k+1} &= R_{k+1}^{mn} - \frac{r_{k+1}}{\prod_{i=h_{k+1}}^{k} t_i} \\ &\geq R_{k+1}^{mn} - \frac{r_{k+1}}{\prod_{i=h_{k+1}}^{k} R_h^{mn} \left(1 - \max\{2, i\}^{-1}\right)}. \end{aligned} \tag{98}$$

Moreover, for $k \geq 0$

$$\prod_{i=h_{k+1}}^{k} \left(1 - \max\{2, i\}^{-1}\right) \geq \frac{\max\{2, h_{k+1}\} - 1}{4 \max\{2, k\}} \geq \frac{\max\{2, h_{k+1}\}}{8 \max\{2, k\}} = g_{k+1}.$$

Hence, by (98) and by the hypothesis, we deduce

$$t_{k+1} \geq R_{k+1}^{mn} - \frac{g_{k+1}^{-1} r_{k+1}}{\prod_{h=h_{k+1}}^{k} R_h^{mn}} \geq R_{k+1}^{mn} \left(1 - \frac{1}{\max\{2, k+1\}}\right).$$

6 Proof of Lemma 4.1

For a set $\mathscr{A} \subset \mathbb{R}^{m \times n}$ we denote by $\mathscr{A}_\delta$ the "inflation" of $\mathscr{A}$ by the quantity δ, i.e., the set $\{\boldsymbol{X} \in \mathbb{R}^{m \times n} : \mathrm{dist}_\infty(\boldsymbol{X}, \mathscr{A}) \leq \delta\}$. First, we show that

$$\delta^{mn} \#\{\text{tiles } \tau \text{ of the lattice } \Lambda : \tau \cap \mathscr{D} \cap \mathscr{C} \neq \emptyset\} \leq \mathrm{Vol}(\mathscr{D}_\delta \cap \mathscr{C}_\delta). \tag{99}$$

This follows from the fact that for any tile τ of Λ we have

$$\tau \cap \mathscr{D} \cap \mathscr{C} \neq \emptyset \Rightarrow \tau \subset \mathscr{D}_\delta \cap \mathscr{C}_\delta. \tag{100}$$

To see why (100) holds, it is enough to observe that for all points $P \in \tau \cap \mathscr{C}$ and all points $Q \in \tau$ we have $\mathrm{dist}_\infty(P, Q) \leq \delta$. Hence, $\tau \subset \mathscr{C}_\delta$. The same is true for $\mathscr{D}$, whence (100).

To conclude the proof, we need to estimate $\mathrm{Vol}(\mathscr{D}_\delta \cap \mathscr{C}_\delta)$. By definition, if $\boldsymbol{X} \in \mathscr{C}_\delta$, then there is some $\boldsymbol{X'} \in \mathscr{C}$ such that $\mathrm{dist}_\infty(\boldsymbol{X}, \boldsymbol{X'}) \leq \delta$. Hence,

$$\begin{aligned} &\prod_{i=1}^{m} |X_i \boldsymbol{q} + \gamma_i'| \leq \prod_{i=1}^{m} \left(|X_i' \boldsymbol{q} + \gamma_i'| + n|\boldsymbol{q}|_\infty \delta\right) \\ &\leq \prod_{i=1}^{m} |X_i' \boldsymbol{q} + \gamma_i'| + \sum_{I \subsetneq \{1, \ldots, m\}} \left(\prod_{i \in I} |X_i' \boldsymbol{q} + \gamma_i'|\right) (n|\boldsymbol{q}|_\infty \delta)^{m-|I|} \leq \varepsilon + (T + n|\boldsymbol{q}|_\infty \delta)^m - T^m. \end{aligned} \tag{101}$$

Now, let $\boldsymbol{\mu}$ be the centre of the cube $\mathscr{D}$. Then, by (101), we have

$$\mathscr{D}_\delta \cap \mathscr{C}_\delta \subset \left\{ \boldsymbol{X} \in \mathbb{R}^{m\times n} : \begin{cases} \prod_{i=1}^m |X_i\boldsymbol{q} + \gamma_i'| \leq \varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m & \\ |X_i\boldsymbol{q} + \gamma_i'| \leq T + n|\boldsymbol{q}|_\infty\delta & i = 1,\dots,m \\ |X_{ij} - \mu_{ij}| \leq \mathrm{edge}(\mathscr{D})/2 + \delta & i = 1,\dots,m,\ j = 1,\dots,n \end{cases} \right\}. \tag{102}$$

Without loss of generality, we can assume $|\boldsymbol{q}|_\infty = |q_n|$. We proceed by considering the linear transformation $\xi : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ defined by

$$\xi(\boldsymbol{X})_{ij} = \begin{cases} X_{ij} & \text{if } j \neq n \\ X_i\boldsymbol{q} & \text{if } j = n \end{cases}.$$

Under the action of ξ, the right-hand side of (102) is sent into a subset of

$$\left\{ \boldsymbol{X} \in \mathbb{R}^{m\times n} : \begin{cases} \prod_{i=1}^m |X_{in} + \gamma_i'| \leq \varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m & \\ |X_{in} + \gamma_i'| \leq T + n|\boldsymbol{q}|_\infty\delta & i = 1,\dots,m \\ |X_{ij} - \mu_{ij}| \leq \mathrm{edge}(\mathscr{D})/2 + \delta & i = 1,\dots,m\ j = 1,\dots,n-1 \end{cases} \right\}. \tag{103}$$

Now, the determinant of ξ is $|q_n|^m = |\boldsymbol{q}|_\infty^m \neq 0$, so ξ is a bijective linear transformation. Therefore, to obtain an estimate of $\mathrm{Vol}(\mathscr{D}_\delta \cap \mathscr{C}_\delta)$, it is enough to estimate the volume of the set in (103). A simple integration shows that the volume of this set is bounded from above by

$$2^{2m-1}(\varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m)\log^*\left(\frac{(T + n|\boldsymbol{q}|_\infty\delta)^m}{\varepsilon + (T + n|\boldsymbol{q}|_\infty\delta)^m - T^m}\right)^{m-1}(\mathrm{edge}(\mathscr{D}) + 2\delta)^{m(n-1)}.$$

Hence, $\mathrm{Vol}(\mathscr{D}_\delta \cap \mathscr{C}_\delta)$ is bounded from above by this quantity divided by the absolute value of the determinant of ξ, that is $|q_n|^m = |\boldsymbol{q}|_\infty^m$.

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A note on Bounded Exponential Sums

under review, 12 pages

A note on bounded exponential sums

Let $A \subset \mathbb{N}$, $\alpha \in (0,1)$, and for $x \in \mathbb{R}$ let $e(x) := e^{2\pi i x}$. We set

$$S_A(\alpha, N) := \sum_{\substack{n \in A \\ n \le N}} e(n\alpha).$$

Recently, Lambert A'Campo posed the following question: is there an infinite non-cofinite set $A \subset \mathbb{N}$ such that for all $\alpha \in (0,1)$ the sum $S_A(\alpha, N)$ has bounded modulus as $N \to +\infty$? In this note we show that such sets do not exist. To do so, we use a theorem by Duffin and Schaeffer on complex power series. We extend our result by proving that if the sum $S_A(\alpha, N)$ is bounded in modulus on an arbitrarily small interval and on the set of rational points, then the set A has to be either finite or cofinite. On the other hand, we show that there are infinite non-cofinite sets A such that $|S_A(\alpha, N)|$ is bounded for all $\alpha \in E \subset (0,1)$, where E has full Hausdorff dimension and $\mathbb{Q} \cap (0,1) \subset E$.

1 Introduction

We denote by $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of natural numbers and for $x \in \mathbb{R}$ we denote by $e(x)$ the complex number $e^{2\pi i x}$. Let $A \subset \mathbb{N}$, $\alpha \in (0,1)$, and $N \in \mathbb{N}$. We consider the sum

$$S_A(\alpha, N) := \sum_{\substack{n \in A \\ n \le N}} e(n\alpha).$$

We observe that for all $\alpha \in (0,1)$

$$\left| \sum_{n \le N} e(n\alpha) \right| = \left| \frac{e((N+1)\alpha) - e(\alpha)}{e(\alpha) - 1} \right| \le \frac{2}{|e(\alpha) - 1|}.$$

Hence, if the set $A \subset \mathbb{N}$ is finite or cofinite[15], the sum $S_A(\alpha, N)$ is bounded in modulus, i.e., for each $\alpha \in (0,1)$ there exists a constant $C_{A,\alpha} > 0$, only depending on the set A and the real α, such that $|S_A(\alpha, N)| \leq C_{A,\alpha}$ for all $N \in \mathbb{N}$.

Lambert A'Campo has raised the following question.

Question 1.1 (L. A'Campo)**.** Are there infinite non-cofinite sets $A \subset \mathbb{N}$ such that for each $\alpha \in (0,1)$ there exists a constant $C_{A,\alpha} > 0$ for which $|S_A(\alpha, N)| \leq C_{A,\alpha}$ for all $N \in \mathbb{N}$?

Question 1.1 was presented by Philipp Habegger during the problem session at the "Diophantine Approximation and Transcendence" conference held in Luminy from September 10th to 14th 2018. In this note we answer Question 1.1, by showing that such sets A do not exist. More generally, we prove the following.

Proposition 1.2. *Let $\boldsymbol{a} := (a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers taking only finitely many values, and let*

$$E(\boldsymbol{a}) := \left\{ \alpha \in (0,1) : \sup_{N\in\mathbb{N}} \left| \sum_{n\leq N} a_n e(n\alpha) \right| < +\infty \right\}.$$

Assume that

i) the set $E(\boldsymbol{a})$ contains an open non-empty interval;

ii) the set $E(\boldsymbol{a})$ contains $\mathbb{Q} \cap (0, 1/2]$.

Then, the sequence $\boldsymbol{a}$ is ultimately constant.

An answer to Question 1.1 is provided by the case $a_n = \chi_A(n)$ and $E(\boldsymbol{a}) = (0,1)$, where χ_A is the characteristic function of the set A, i.e., $\chi_A(n) = 1$ if $n \in A$ and 0 otherwise.

To make things easier, we give the following definition.

Definition 1.3. Let $E \subset (0,1)$. We say that a set $A \subset \mathbb{N}$ has BES (bounded exponential sums) over E if for each $\alpha \in E$ there exists a constant $C_{A,\alpha} > 0$ such that $|S_A(\alpha, N)| \leq C_{A,\alpha}$ for all $N \in \mathbb{N}$.

[15] A set $A \subset \mathbb{N}$ is cofinite if $A \setminus \mathbb{N}$ is finite.

We note that a set $A \subset \mathbb{N}$ has BES over $(0,1)$ if and only if it has BES over $(0,1/2]$. Indeed, for $A \subset \mathbb{N}$ and $\alpha \in (0,1)$ we have

$$\overline{S_A(\alpha, N)} = \sum_{\substack{n \le N \\ n \in A}} \overline{e(n\alpha)} = \sum_{\substack{n \le N \\ n \in A}} e(n(-\alpha)) = \sum_{\substack{n \le N \\ n \in A}} e(n(1-\alpha)) = S_A(1-\alpha, N), \tag{104}$$

proving that the function $S_A(\alpha, N)$ is bounded if and only if the function $S_A(1-\alpha, N)$ is bounded. This shows that condition $ii)$ in Proposition 1.2 is equivalent to $\mathbb{Q} \cap (0,1) \subset E(\boldsymbol{a})$.

Now, we analyse the two conditions appearing in Proposition 1.2. Condition $ii)$ is clearly necessary. Indeed, the series

$$\sum_{n=0}^{+\infty} e(-pn/q)e(n\alpha) = (1 - e(-p/q)e(\alpha))^{-1}$$

has finitely many complex coefficients and is only unbounded at the rational $\alpha = p/q$. If we assume $a_n \in \{0,1\}$, condition $ii)$ can be replaced by "for each $q \geq 2$ there exists $0 < p \leq q-1$ such that $(p,q) = 1$ and $p/q \in E(\boldsymbol{a})$"[16]. This is again necessary since, e.g., the set $A = \{qn\}_{n \in \mathbb{N}}$ has BES over $E = (0,1) \setminus \{p/q : p = 1, \dots, q-1\}$ for all integers $q \geq 2$. On the other hand, condition $i)$ in Proposition 1.2 is not strictly necessary. To see this, one can use a slightly modified version of Theorem 2.1 in Section 2 (see [4]) which shows that the result of Proposition 1.2 still holds if we remove from the interval contained in $E(\boldsymbol{a})$ a zero Lebesgue measure set. It is then natural to ask whether the presence of an interval (up to zero measure sets) in a subset $E \subset (0,1)$ is necessary to avoid the existence of an infinite non-cofinite set $A \subset \mathbb{N}$ with BES over E. In other words, is a purely measure-theoretic condition enough? Note that there are subsets $E \subset (0,1)$ such that $\mathscr{L}(E) = 1 - \varepsilon$ $(0 < \varepsilon < 1)$ and $\mathscr{L}(E \cap I) < \mathscr{L}(I)$ for any interval $I \subsetneq (0,1)$, here $\mathscr{L}$ stands for the Lebesgue measure[17]. In view of this, we study subsets $\mathbb{Q} \cap (0,1/2] \subset E \subset (0,1)$ that admit infinite non-cofinite sets $A \subset \mathbb{N}$ with BES over E. "How big" can such subsets E be? A partial answer is provided by the following.

Proposition 1.4. *There exist infinite non-cofinite sets $A \subset \mathbb{N}$ with* BES *over a subset $\mathbb{Q} \cap (0,1/2] \subset E \subset (0,1)$ of full Hausdorff dimension.*

Proposition 1.4 is a consequence of the following result.

[16]Note that if we do not assume $(p,q) = 1$, the result no longer holds. Consider, e.g., the rational function $(z+1)/(z^4-1)$. This function is unbounded only at 1, $e(1/4)$, and $e(3/4)$, and has a power series expansion whose coefficients are not ultimately constant. However, for all even q we could choose $p = q/2$, so that the hypotheisis still holds.

[17]An example of such a set could be the following. Assume that $\mathbb{Q} \cap (0,1) = \{q_n\}_{n \geq 1}$ is a numbering of the rational numbers and let $0 < \varepsilon < 1$. Consider the set $E = \bigcup_{n \geq 1} (q_n - \varepsilon 2^{-n}, q_n + \varepsilon 2^{-n})$. We have $\mathscr{L}((0,1) \setminus E) \geq 1 - \varepsilon$. Moreover, since every non-empty interval $I \subset (0,1)$ contains a rational, we have $\mathscr{L}(((0,1) \setminus E) \cap I) < \mathscr{L}(I)$.

Proposition 1.5. *Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Then, the set $A(f) := \{n + f(n)! : n \in \mathbb{N}\}$ has* BES *over $E = \mathbb{Q} \cap (0, 1/2]$. Moreover, any function f such that*

i) $\sum_{i \geq 1} 1/f(i) < +\infty$,

ii) $\sup_{i \in \mathbb{N}} (1/i!)^{\varepsilon} \prod_{f(j) \leq i} (f(j) + 1) < +\infty$ *for all* $0 < \varepsilon < 1$,

gives raise to a set $A(f)$ that has BES *over some subset $\mathbb{Q} \cap (0, 1/2] \subset E(f) \subset (0, 1)$ of full Hausdorff dimension.*

A function that satisfies both *i)* and *ii)* is $f(n) = n^2$. We give more details in Section 4.

In view of the above discussion, we are led to the following questions.

Question 1.6.

a) Are there any positive Lebesgue measure subsets $\mathbb{Q} \cap (0, 1/2] \subset E \subset (0, 1)$ that admit infinite non-cofinite sets $A \subset \mathbb{N}$ with BES over E?

b) Are there any zero Lebesgue measure subsets $\mathbb{Q} \cap (0, 1/2] \subset E \subset (0, 1)$ that admit no infinite non-cofinite sets $A \subset \mathbb{N}$ with BES over E?

The techniques used in this note do not seem powerful enough to tackle Question 1.6.

A closely related question to Question 1.1 was studied by Lesigne and Petersen in [5], where they prove the following result.

Theorem 1.7 (Lesigne-Petersen)**.** *There are no sequences $\boldsymbol{a} := \{a_k\}_{k \in \mathbb{Z}}$ with $a_k \in \{\pm 1\}$ such that*

$$\sup_{\substack{m,n \in \mathbb{Z} \\ n \geq 0}} \left| \sum_{k=m}^{m+n} a_k e^{-ik\theta} \right| \leq c(\theta) \tag{105}$$

for all $\theta \in [-\pi, \pi)$ ($c(\theta)$ being a positive real constant depending on θ).

To prove Theorem 1.7, Lesigne and Petersen consider the compact metric space $[-1, +1]^{\mathbb{Z}}$ (endowed with the product distance) and the shift endomorphism σ. They fix a sequence

$\boldsymbol{a} \in [-1,+1]^{\mathbb{Z}}$ satisfying (105), and they set X to be the topological closure of the orbit of $\boldsymbol{a}$ under σ. By using the spectral theorem for Hilbert spaces, they prove that any shift-invariant probability measure μ defined on X (whose existence is guaranteed by the Bogolyubov-Krylov Theorem [1, Theorem 1.1]) must be concentrated on the point $\mathbf{0}$, i.e., the sequence given by all zeroes. This is clearly never true when $\boldsymbol{a} \in \{\pm 1\}^{\mathbb{Z}}$.

We note that the hypothesis in Theorem 1.7 is slightly different from that of Question 1.1, the key difference being the fact that the sum in (105) is bounded also for $\theta = 0$. After carefully reading Lesigne and Petersen's proof, we believe that their argument can be applied to show that, once the constraint for $\theta = 0$ is removed, any shift invariant probability measure μ defined on the closure X of the σ-orbit of a sequence $\boldsymbol{a} \in \{0,1\}^{\mathbb{Z}}$ satisfying (105) must be concentrated either on the point $\mathbf{0}$ or on the point $\mathbf{1}$. In this case we say (using the terminology from [5]) that $\boldsymbol{a}$ is essentially 0 or essentially 1. This, however, does not imply that the sequence $\boldsymbol{a}$ is eventually constant. Indeed, it is easy to see that the set $A = \{n + n! : n \in \mathbb{N}\}$ has an essentially zero indicator function $\chi_A : \mathbb{Z} \to \{0,1\}$. To show this, it is sufficient to observe that the closure of the orbit of χ_A under σ is the set

$$X = \left\{\mathbf{0}, \sigma^n\left(\chi_A\right), \sigma^n\left(\chi_{\{0\}}\right) : n \in \mathbb{Z}\right\}.$$

We conclude the introduction by noting that Proposition 1.2 provides a simpler and more elementary proof of Theorem 1.7.

2 Proof of Proposition 1.2

To prove Proposition 1.2 we use the following powerful result by Duffin and Schaeffer [2, Part II, Theorem I].

Theorem 2.1 (Duffin-Schaeffer)**.** *Let*

$$u(z) := \sum_{n=0}^{+\infty} b_n z^n$$

be a power series defined over the unit disk $D := \{|z| < 1\} \subset \mathbb{C}$. Assume that the coefficients $b_n \in \mathbb{C}$ take only finitely many different values. Then, if the series $u(z)$ is bounded in a sector $S := \{\theta_1 \le \arg(z) \le \theta_2,\ |z| < 1\}$ of the disk D, where $0 \le \theta_1 < \theta_2 \le 2\pi$, the sequence $\{b_n\}$ is *ultimately periodic*[18].

Let I be an open interval contained in $E(\boldsymbol{a})$. We consider the function $f : I \to [0, +\infty)$ defined by

$$f(\alpha) := \sup_{N\in\mathbb{N}} \left| \sum_{n\leq N} a_n e(n\alpha) \right|.$$

This is a Baire class 1 function since it is the point-wise limit[19] of a sequence of continuous functions (see [6, Definition 11.1]).

By [6, Theorem 11.4], we know that the set of continuity points of such functions is dense in their domain. Hence, f has a continuity point P in I. This means that we can find an interval (α_1, α_2) around P such that the image $f((\alpha_1, \alpha_2))$ is contained in a small interval around $f(P)$. It follows that f is bounded in (α_1, α_2) by some constant $M > 0$.

For $z \in D := \{|z| < 1\}$ we let

$$u(z) := \sum_{n=0}^{+\infty} a_n z^n,$$

where $a_0 := 0$. By applying Abel's summation formula, we find that for all $\alpha \in (\alpha_1, \alpha_2)$, all $0 \leq r < 1$, and all integers $A \geq 1$ it holds

$$\begin{aligned} \left| \sum_{n=0}^{A} a_n r^n e(n\alpha) \right| &\leq \left| \sum_{n=0}^{A} a_n e(n\alpha) \right| r^A + \sum_{n=0}^{A-1} \left| \sum_{j=0}^{n} a_j e(j\alpha) \right| \left(r^n - r^{n+1} \right) \\ &\leq f(\alpha) r^A + f(\alpha) \left(1 - r^A \right) = f(\alpha) \leq M. \end{aligned} \tag{106}$$

Taking the limit for $A \to +\infty$, we obtain

$$|u(z)| \leq M$$

for $z \in S := \{2\pi\alpha_1 \leq \arg(z) \leq 2\pi\alpha_2,\ |z| < 1\}$. Hence, by Theorem 2.1, the sequence $\boldsymbol{a}$ is ultimately periodic.

To conclude the proof, we show that if $f(\alpha) < +\infty$ for all rational numbers $\alpha \in \mathbb{Q} \cap (0, 1)$ (or equivalently in $\mathbb{Q} \cap (0, 1/2]$ by (104)), the period of the sequence $\{a_n\}$ is 1. Suppose that

[18] This is not explicitly stated in the theorem, but it is stated at the end of the proof (see [2, Part II, Section 4]).

[19] Note that the supremum of a sequence of continuous functions $\{f_m\}$ can be turned into a limit by considering the continuous functions $f_M := \sup_{m\leq M} f_m$.

ultimately $\boldsymbol{a}$ has a period of length $q \geq 1$, i.e., $a_n = a_{n+q}$ for all $n \geq K$, where K is some large integer. Then, for $z \in D$ we have

$$u(z) = \sum_{n=0}^{K-1} a_n z^n + \sum_{n=0}^{+\infty} z^{qn+K} \left(\sum_{j=0}^{q-1} a_{K+j} z^j \right) = \sum_{n=0}^{K-1} a_n z^n + z^K \left(\sum_{j=0}^{q-1} a_{K+j} z^j \right) \frac{1}{1-z^q}.$$

Since $|u(re^{i\alpha})| \leq f(\alpha)$ for all $0 \leq r < 1$ and all $\alpha \in E(\boldsymbol{a})$ (to see this, use (106)), the function $u(z)$ cannot have a pole at a non trivial root of unity. Hence, the polynomial $1 + z + \cdots + z^{q-1}$ must divide $\sum_{j=0}^{q-1} a_{K+j} z^j$, thus showing that $a_{K+j} = a_{K+j'}$ for all $j \neq j'$.

3 Proof of Proposition 1.5

Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function and let $A(f) = \{n + f(n)! : n \in \mathbb{N}\}$. Clearly, $A(f)$ is neither finite nor cofinite. For $N \geq 0$ we estimate the sum

$$\sum_{n \leq N} e((n + f(n)!)\alpha).$$

First, we show that this sum is bounded for all $\alpha \in \mathbb{Q} \cap (0,1)$. Let $\alpha := p/q$, with $p, q \in \mathbb{N}$ and $q \geq 2$. Then, for all $n \geq q$ we have

$$n + f(n)! \equiv n \pmod{q}.$$

It follows that for $N \geq q$

$$\begin{aligned} \left| \sum_{n \leq N} e((n + f(n)!)\alpha) \right| &\leq \left| \sum_{n<q} e((n + f(n)!)\alpha) \right| + \left| \sum_{q \leq n \leq N} e(n\alpha) \right| \\ &\leq \left| \sum_{n<q} e((n + f(n)!)\alpha) \right| + \left| \sum_{n \leq q} e(n\alpha) \right| + \left| \sum_{n \leq N} e(n\alpha) \right|, \quad (107) \end{aligned}$$

and the right-hand side in (107) is bounded for $N \to +\infty$. To prove the second part of Proposition 1.5, we need the following auxilary result (see [7, Section 2]).

Lemma 3.1. *Let $\alpha \in [0,1)$. Then, there exists a sequence of integers $(s_n(\alpha))_{n \in \mathbb{N}}$ such that $0 \leq s_n(\alpha) \leq n-1$ and*

$$\alpha = \sum_{n \geq 1} \frac{s_n(\alpha)}{n!}.$$

The sequence $(s_n(\alpha))_{n \in \mathbb{N}}$ associated to α is unique, if we exclude all those sequences s_n such that $s_n = n-1$ for all sufficiently large n. Under this limitation, the sequence $s_n(\alpha)$ is eventually null if and only if $\alpha \in \mathbb{Q}$.

Remark 3.2. Let N be a fixed integer and let $(s_n)_{n>N}$ be a sequence of integers such that $0 \leq s_n \leq n-1$ for all $n > N$. Then, we have

$$\sum_{n>N} \frac{s_n}{n!} \leq \frac{1}{N!}.$$

This follows from the equality

$$\sum_{n=N+1}^{M} \frac{n-1}{n!} = \frac{1}{N!} - \frac{1}{M!},$$

valid for all $M \geq N+1$. This important fact will be used later on in the proof.

For a real number $\alpha \in [0,1)$ we call the unique sequence $(s_n(\alpha))_{n\in\mathbb{N}}$ given by Lemma 3.1 (that does not eventually coincide with $n-1$) the factoradic representation of α and we call the integer $s_n(\alpha)$ of such sequence the n-th factoradic digit of α.

Let $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ be another sequence of strictly positive integers and assume that

$$\sum_{n\geq 1} 1/a_n < +\infty. \tag{108}$$

We consider the set

$$E(f, \boldsymbol{a}) := \left\{ \alpha \in (0,1) : s_{f(i)+1}(\alpha) \leq \frac{f(i)+1}{a_i} \text{ for all } i \geq 1 \right\}.$$

Note that $E(f, \boldsymbol{a}) \neq \emptyset$ whenever f is not the identity function. We shall show that for any function f satisfying condition $i)$ and any sequence $\boldsymbol{a}$ satisfying (108) the set $A(f)$ has BES over $E(f, \boldsymbol{a})$, thereby proving that $A(f)$ has BES over $E = E(f, \boldsymbol{a}) \cup (\mathbb{Q} \cap (0,1))$.

By Abel's summation formula, we have

$$\left| \sum_{n\leq N} e((n + f(n)!)\alpha) \right|$$
$$\leq \left| \sum_{n\leq N} e(n\alpha) \right| |e(f(N)!\alpha)| + \sum_{n\leq N-1} \left| \sum_{i\leq n} e(i\alpha) \right| |e(f(n)!\alpha) - e(f(n+1)!\alpha)|. \tag{109}$$

Hence, to bound the left-hand side of (109) it is enough to bound the sum

$$\sum_{n\leq N-1} |e(f(n)!\alpha) - e(f(n+1)!\alpha)| \leq 2 \sum_{n\leq N} |e(f(n)!\alpha) - 1|. \tag{110}$$

Let $\{\theta\}$ denote the fractional part of any real number $\theta > 0$. By using the inequality $|e(\theta) - 1| \leq 2\pi\{\theta\}$ (valid for $\theta \in [0, +\infty)$), we obtain

$$\sum_{n\leq N} |e(f(n)!\alpha) - 1| \leq 2\pi \sum_{n\leq N} \{f(n)!\alpha\}. \tag{111}$$

Now, since $\alpha \in E(f, \boldsymbol{a})$ and $f(n) \geq n$, we have

$$\begin{aligned}\{f(n)!\alpha\} &= \left\{ f(n)! \sum_{i\geq 1} \frac{s_i(\alpha)}{i!} \right\} = \left\{ \frac{s_{f(n)+1}(\alpha)}{f(n)+1} + \frac{s_{f(n)+2}(\alpha)}{(f(n)+1)(f(n)+2)} + \cdots \right\} \\ &\leq \frac{s_{f(n)+1}(\alpha)}{f(n)+1} + \frac{s_{f(n)+2}(\alpha)}{(f(n)+1)(f(n)+2)} + \cdots \\ &\leq \frac{1}{a_n} + \frac{1}{f(n)+1} + \frac{1}{(f(n)+1)(f(n)+2)} + \cdots \\ &\leq \frac{1}{a_n} + \frac{e}{f(n)+1}.\end{aligned} \tag{112}$$

Thus, combining (109),(110),(111), and (112), we get

$$\left| \sum_{n\leq N} e((n+f(n)!)\alpha) \right| \leq \frac{2}{|e(\alpha)-1|} \left(1 + 4\pi \sum_{n\leq N} \left(\frac{1}{a_n} + \frac{e}{f(n)+1} \right) \right).$$

By (108) and condition i), the right hand side is bounded, proving the claim.

Now, we show that the set $E(f, \boldsymbol{a}) \cup (\mathbb{Q} \cap (0,1))$ has full Hausdorff dimension whenever the function f satisfies condition ii). To give a lower bound for the Hausdorff dimension of $E(f, \boldsymbol{a}) \cup (\mathbb{Q} \cap (0,1))$ we use the so called mass distribution principle (see [3, Principle 4.2]).

Lemma 3.3. *Let μ be a probability measure supported on a bounded subset X of $\mathbb{R}$. Suppose that there are strictly positive constants a, s and ℓ_0 such that*

$$\mu(B) \leq a|B|^s \tag{113}$$

for any interval[20] *B of length $|B| \leq \ell_0$. Then,* $\dim(X) \geq s$*, where* $\dim$ *denotes the Hausdorff dimension of a set.*

We can take X to be $E(f, \boldsymbol{a}) \cup (\mathbb{Q} \cap [0,1])$, since adding a finite number of points to a set does not change its Hausdorff dimension. To apply Lemma 3.3, we need to construct a probability measure μ whose support is contained in X. We use a standard limit procedure to define μ (see [3, Proposition 1.7]). For $i \in \mathbb{N}$ we let $\rho_i := 1/i!$ and

$$Z_i := \{\alpha \in [0,1) : s_j(\alpha) = 0 \text{ for } j > i\}.$$

First, we observe that

$$E(f, \boldsymbol{a}) \cup (\mathbb{Q} \cap [0,1]) \supset \bigcap_{i\in\mathbb{N}} \bigcup_{\alpha \in (E(f,\boldsymbol{a})\cup\{0\})\cap Z_i} [\alpha, \alpha+\rho_i]. \tag{114}$$

[20] Note that it is enough to consider intervals since the ball-defined Hausdorff dimension coincides with the classical Hausdorff dimension (see [3, Section 2.4])

Indeed, by definition, for each α lying in the right-hand side of (114) and each $i \in \mathbb{N}$ there exists $\alpha_i \in (E(f, \boldsymbol{a}) \cup \{0\}) \cap Z_i$ such that $\alpha \in [\alpha_i, \alpha_i + \rho_i]$. This means that either $\alpha = \alpha_i + \rho_i \in \mathbb{Q} \cap [0, 1]$ (by Remark 3.2) or $0 \leq \alpha - \alpha_i < \rho_i$, i.e., $\alpha - \alpha_i$ is a number between 0 and 1 such that $s_j(\alpha - \alpha_i) = 0$ for $j \leq i$. Thus, when we add $\alpha - \alpha_i$ to α_i the digits before the i-th do not change, showing that $s_j(\alpha) = s_j(\alpha_i)$ for $j \leq i$. It follows that $\alpha \in E(f, \boldsymbol{a}) \cup \{0\}$.

Now, for all $i \in \mathbb{N}$ and all $\alpha \in (E(f, \boldsymbol{a}) \cup \{0\}) \cap Z_i$ we define

$$\mu((\alpha, \alpha + \rho_i)) := \frac{1}{\#((E(f, \boldsymbol{a}) \cup \{0\}) \cap Z_i)}, \tag{115}$$

where we take open intervals to make sure that for any fixed i all the sets $(\alpha, \alpha + \rho_i)$ are disjoint.

Remark 3.4. Note that for $\alpha \in (E(f, \boldsymbol{a}) \cup \{0\}) \cap Z_i$ we have

$$\#\{\beta \in (E(f, \boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} : \beta_i = \alpha\} = \frac{\#((E(f, \boldsymbol{a}) \cup \{0\}) \cap Z_{i+1})}{\#((E(f, \boldsymbol{a}) \cup \{0\}) \cap Z_i)},$$

where β_i is the truncation of β at the i-th digit. Hence,

$$\begin{aligned} \mu((\alpha, \alpha + \rho_i)) = \frac{1}{\#((E(f, \boldsymbol{a}) \cup \{0\}) \cap Z_i)} &= \sum_{\substack{\beta \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} \\ \beta_i = \alpha}} \frac{1}{\#((E(f, \boldsymbol{a}) \cup \{0\}) \cap Z_{i+1})} \\ = \sum_{\substack{\beta \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} \\ \beta_i = \alpha}} \mu((\beta, \beta + \rho_{i+1})) &= \sum_{\substack{\beta \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} \\ (\beta, \beta + \rho_{i+1}) \subset (\alpha, \alpha + \rho_i)}} \mu((\beta, \beta + \rho_{i+1})). \end{aligned} \tag{116}$$

By [3, Proposition 1.7] and (116), Equation (115) induces a unique well-defined Borel measure μ on $\mathbb{R}$, with the property[21] that

$$\mu\left(\mathbb{R} \setminus \bigcup_{\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i} (\alpha, \alpha + \rho_i)\right) = 0 \tag{117}$$

for all $i \in \mathbb{N}$, and supported on the set

$$\bigcap_{i \in \mathbb{N}} \bigcup_{\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i} [\alpha, \alpha + \rho_i] \subset X. \tag{118}$$

To prove (113), we fix a number $i_0 \in \mathbb{N}$ and a real $0 < s < 1$. We consider an interval $B \subset [0, 1]$ of length $|B|$ less than ρ_{i_0}. Clearly, there exists an index $i \in \mathbb{N}$, $i \geq i_0$, such that

$$\rho_{i+1} < |B| \leq \rho_i. \tag{119}$$

[21] Stated a few lines above [3, Proposition 1.7]

By (117), we have

$$\mu(B) \leq \mu\left(\bigcup_{\substack{\alpha\in(E(f,\boldsymbol{a})\cup\{0\})\cap Z_{i+1}\\(\alpha,\alpha+\rho_{i+1})\cap B\neq\emptyset}} (\alpha,\alpha+\rho_{i+1})\right),$$

and it is straightforward to see that

$$\#\{\alpha\in Z_{i+1} : (\alpha,\alpha+\rho_{i+1})\cap B\neq\emptyset\} \leq \frac{|B|}{1/(i+1)!}+2,$$

since the intervals $(\alpha,\alpha+\rho_{i+1})$ are pairwise disjoint and each of them has length $1/(i+1)!$. Hence, we have

$$\begin{aligned}\mu(B) \leq &\sum_{\substack{\alpha\in(E(f,\boldsymbol{a})\cup\{0\})\cap Z_{i+1}\\(\alpha,\alpha+\rho_{i+1})\cap B\neq\emptyset}} \mu((\alpha,\alpha+\rho_{i+1}))\\ &\leq \left(\frac{|B|}{1/(i+1)!}+2\right)\frac{1}{\#((E(f,\boldsymbol{a})\cup\{0\})\cap Z_{i+1})}. \qquad (120)\end{aligned}$$

Now, we observe that for all $j\in\mathbb{N}$

$$\#((E(f,\boldsymbol{a})\cup\{0\})\cap Z_j) \geq \frac{j!}{\prod_{f(k)+1\leq j}(f(k)+1)}, \qquad (121)$$

since for $\alpha\in E(f,\boldsymbol{a})\cup\{0\}$ the 0 digit is always allowed in the $(f(k)+1)$-th position independently of $\boldsymbol{a}$. Thus, by (119), (120), and (121), we deduce

$$\begin{aligned}\mu(B) &\leq \left(\frac{|B|}{1/(i+1)!}+2\right)\frac{\prod_{f(j)+1\leq i+1}(f(j)+1)}{(i+1)!} \leq 3(i+1)!|B|\frac{\prod_{f(j)\leq i}(f(j)+1)}{(i+1)!}\\ &\leq 3|B|^{1-s}\prod_{f(j)\leq i}(f(j)+1)|B|^s \leq 3\left(\frac{1}{i!}\right)^{1-s}\prod_{f(j)\leq i}(f(j)+1)|B|^s.\end{aligned}$$

It then follows from Lemma 3.3 that any function f such that

$$\sup_{i\in\mathbb{N}}\left(\frac{1}{i!}\right)^{1-s}\prod_{f(j)\leq i}(f(j)+1) < +\infty$$

for all $0<s<1$ gives raise to a set $E(f,\boldsymbol{a})$ of full Hausdorff dimension.

4 An example

To conclude this note, we give an example of a function $f:\mathbb{N}\to\mathbb{N}$ satisfying both conditions $i)$ and $ii)$ of Proposition 1.5. We let $s:=1-\varepsilon$ and $f(i)=i^2$ for $i\in\mathbb{N}$. Condition $i)$ is clearly satisfied. Moreover, we have

$$\left(\frac{1}{i!}\right)^{1-s} \prod_{f(j)\leq i} (f(j)+1) = \frac{\prod_{j\leq\sqrt{i}}(j^2+1)}{(i!)^{\varepsilon}}. \tag{122}$$

Now, when $i > \lceil 3/\varepsilon \rceil \left\lfloor \sqrt{i} \right\rfloor$, we find

$$(i!)^{\varepsilon} \geq \left(\underbrace{1\cdots 1}_{\lceil 3/\varepsilon \rceil \text{ times}} \cdot \underbrace{2\cdots 2}_{\lceil 3/\varepsilon \rceil \text{ times}} \cdots\cdots \underbrace{\left\lfloor \sqrt{i} \right\rfloor \cdots \left\lfloor \sqrt{i} \right\rfloor}_{\lceil 3/\varepsilon \rceil \text{ times}} \right)^{\varepsilon} \geq \prod_{j\leq\sqrt{i}} j^3.$$

Hence, the right-hand side in (122) is always bounded, showing that ii) holds.

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