A NOTE ON BOUNDED EXPONENTIAL SUMS

REYNOLD FREGOLI

Abstract. Let $A \subset \mathbb{N}$, $\alpha \in (0,1)$, and $e(x) := e^{2\pi i x}$ for $x \in \mathbb{R}$. We set

$$S_A(\alpha, N) := \sum_{\substack{n \in A \\ n \leq N}} e(n\alpha).$$

Recently, Lambert A'Campo posed the following question: is there an infinite non-cofinite set $A \subset \mathbb{N}$ such that for all $\alpha \in (0,1)$ the sum $S_A(\alpha, N)$ has bounded modulus as $N \to +\infty$? In this note we show that such sets do not exist. To do so, we use a theorem by Duffin and Schaeffer on complex power series. We extend our result by proving that if the sum $S_A(\alpha, N)$ is bounded in modulus on an arbitrarily small interval and on the set of rational points, then the set A has to be either finite or cofinite. On the other hand, we show that there are infinite non-cofinite sets $A \subset \mathbb{N}$ such that $|S_A(\alpha, N)|$ is bounded independently of N for all $\alpha \in E \subset (0,1)$, where $\mathbb{Q} \cap (0,1) \subset E$ and E has full Hausdorff dimension.

1. Introduction

Let $\mathbb{N}$ denote the set $\{1, 2, 3, \dots\}$ of natural numbers and let $e(x)$ denote the complex number $e^{2\pi i x}$ for $x \in \mathbb{R}$. Let $A \subset \mathbb{N}$, $\alpha \in (0,1)$, and $N \in \mathbb{N}$. In this note, we study the sum

$$S_A(\alpha, N) := \sum_{\substack{n \in A \\ n \leq N}} e(n\alpha).$$

For all $\alpha \in (0,1)$ we have

$$\left| \sum_{n \leq N} e(n\alpha) \right| = \left| \frac{e((N+1)\alpha) - e(\alpha)}{e(\alpha) - 1} \right| \leq \frac{2}{|e(\alpha) - 1|}.$$

Hence, if the set $A \subset \mathbb{N}$ is finite or cofinite[1], the sum $S_A(\alpha, N)$ is bounded in modulus independently of the value assigned to N. By this, we mean that for each $\alpha \in (0,1)$ there exists a constant $C_{A,\alpha} > 0$, only depending on the set A and the real number α, such that $|S_A(\alpha, N)| \leq C_{A,\alpha}$ for all $N \in \mathbb{N}$.

In [7], Lambert A'Campo raised the following question.

Question 1.1 (L. A'Campo)**.** Are there infinite non-cofinite sets $A \subset \mathbb{N}$ such that for each $\alpha \in (0,1)$ we have $|S_A(\alpha, N)| \leq C_{A,\alpha}$ for all $N \in \mathbb{N}$?

Question 1.1 was presented by Philipp Habegger as an open problem at the "Diophantine Approximation and Transcendence" conference held in Luminy from September 10th to 14th 2018.

In this note, we answer Question 1.1, by showing that such sets A do not exist. More generally, we prove the following.

Date: October 27, 2020, and in revised form ....

1991 *Mathematics Subject Classification.* 11L03, 11L07 ; 28A78, 40C15.

[1] A set $A \subset \mathbb{N}$ is cofinite if $A \setminus \mathbb{N}$ is finite.

Proposition 1.2. *Let $\boldsymbol{a} := (a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers taking only finitely many values, and let*

$$E(\boldsymbol{a}) := \left\{ \alpha \in (0,1) : \sup_{N\in\mathbb{N}} \left| \sum_{n\leq N} a_n e(n\alpha) \right| < +\infty \right\}.$$

Assume that:

i) the set $E(\boldsymbol{a})$ contains an open non-empty interval;

ii) the set $E(\boldsymbol{a})$ contains $\mathbb{Q}\cap(0,1/2]$.

Then, the sequence $\boldsymbol{a}$ is ultimately constant.

An answer to Question 1.1 is provided by the case $a_n = \chi_A(n)$ and $E(\boldsymbol{a}) = (0,1)$, where χ_A denotes the characteristic function of the set A.

To help the reader, we give the following definition.

Definition 1.3. Let $E \subset (0,1)$. We say that a set $A \subset \mathbb{N}$ has BES (bounded exponential sums) over E if for each $\alpha \in E$ there exists a constant $C_{A,\alpha} > 0$ such that $|S_A(\alpha, N)| \leq C_{A,\alpha}$ for all $N \in \mathbb{N}$.

We note that a set $A \subset \mathbb{N}$ has BES over $(0,1)$ if and only if it has BES over $(0,1/2]$. Indeed, for $A \subset \mathbb{N}$ and $\alpha \in (0,1)$ we have

$$\overline{S_A(\alpha,N)} = \sum_{\substack{n\leq N\\ n\in A}} \overline{e(n\alpha)} = \sum_{\substack{n\leq N\\ n\in A}} e(n(-\alpha)) = \sum_{\substack{n\leq N\\ n\in A}} e(n(1-\alpha)) = S_A(1-\alpha,N), \tag{1}$$

proving that the function $S_A(\alpha,N)$ is bounded if and only if the function $S_A(1-\alpha,N)$ is bounded. This shows that part *ii)* in Proposition 1.2 is equivalent to $\mathbb{Q}\cap(0,1) \subset E(\boldsymbol{a})$.

Now, we analyse the two conditions appearing in Proposition 1.2. Condition *ii)* is clearly necessary. Indeed, the series

$$\sum_{n=0}^{+\infty} e(-pn/q)e(n\alpha) = (1 - e(-p/q)e(\alpha))^{-1}$$

has finitely many complex coefficients and diverges only at the rational $\alpha = p/q$. If we assume $a_n \in \{0,1\}$, condition *ii)* can be replaced by "for each $q \geq 2$ there exists $0 < p \leq q-1$ such that $(p,q) = 1$ and $p/q \in E(\boldsymbol{a})$"[2]. This is again necessary since, e.g., the set $A = \{qn\}_{n\in\mathbb{N}}$ has BES over $E = (0,1) \setminus \{p/q : p = 1,\dots,q-1\}$ for all integers $q \geq 2$.

On the other hand, condition *i)* in Proposition 1.2 is not strictly necessary. To see this, one may use a slightly modified version of Theorem 2.1 in Section 2 (see [5]) which shows that the result of Proposition 1.2 still holds if we remove from the interval contained in $E(\boldsymbol{a})$ a zero Lebesgue measure set.

It is then natural to ask whether the presence of an interval (up to zero measure sets) in a subset $E \subset (0,1)$ is necessary to avoid the existence of an infinite non-cofinite set $A \subset \mathbb{N}$ with BES over E. In other words, is a purely measure-theoretic condition enough? Note that there are subsets $E \subset (0,1)$ such that $\mathscr{L}(E)$ is arbitrarily close to 1 and $\mathscr{L}(E\cap I) < \mathscr{L}(I)$ for any

[2] Note that if we do not assume $(p,q) = 1$, the result no longer holds. Consider, e.g., the rational function $(z+1)/(z^4-1)$. This function is unbounded only at 1, $e(1/4)$, and $e(3/4)$, and has a power series expansion whose coefficients are not ultimately constant. However, for all even q we could choose $p = q/2$, so that the hypotheisis still holds.

interval $I \subsetneq (0,1)$, the symbol $\mathscr{L}$ denoting the Lebesgue measure[3]. It thus appears interesting to study subsets $\mathbb{Q} \cap (0,1/2] \subset E \subset (0,1)$ that admit infinite non-cofinite sets $A \subset \mathbb{N}$ with BES over E. "How large" can such subsets E of $(0,1)$ be? We partially answer this question.

Proposition 1.4. *There exist infinite non-cofinite sets $A \subset \mathbb{N}$ with* BES *over a subset $\mathbb{Q} \cap (0,1/2] \subset E \subset (0,1)$ of full Hausdorff dimension.*

Proposition 1.4 is a consequence of the following result.

Proposition 1.5. *Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Then,*

i) *the set $A(f) := \{n + f(n)! : n \in \mathbb{N}\}$ has* BES *over $\mathbb{Q} \cap (0,1/2]$;*

ii) *if the function f satisfies:*

a) $\sum_{i\geq 1} 1/f(i) < +\infty$,

b) $\sup_{i\in\mathbb{N}} (1/i!)^{\varepsilon} \prod_{f(j)\leq i}(f(j)+1) < +\infty$ *for all $0 < \varepsilon < 1$,*

the set $A(f)$ has BES *over some subset $E(f) \subset (0,1)$ of full Hausdorff dimension.*

A function that satisfies both $a)$ and $b)$ is $f(n) = n^2$. We give more details about this in Section 4.

In view of the above discussion, we ask two questions.

Question 1.6.

a) Are there any positive Lebesgue measure subsets $\mathbb{Q} \cap (0,1/2] \subset E \subset (0,1)$ that admit infinite non-cofinite sets $A \subset \mathbb{N}$ with BES over E?

b) Are there any zero Lebesgue measure subsets $\mathbb{Q} \cap (0,1/2] \subset E \subset (0,1)$ that admit no infinite non-cofinite sets $A \subset \mathbb{N}$ with BES over E?

The techniques used in this note do not seem powerful enough to tackle Question 1.6.

We conclude the introduction by remarking that a closely related question to Question 1.1 was studied by Lesigne and Petersen [6].

Theorem 1.7 (Lesigne-Petersen)**.** *There are no sequences $\boldsymbol{a} := (a_k)_{k\in\mathbb{Z}}$ with $a_k \in \{\pm 1\}$ such that*

$$\sup_{\substack{m,n\in\mathbb{Z}\\ n\geq 0}} \left| \sum_{k=m}^{m+n} a_k e(\alpha) \right| \leq c(\alpha) \tag{2}$$

for all $\alpha \in [0,1)$ ($c(\alpha)$ being a positive real constant depending on α).

To prove Theorem 1.7, Lesigne and Petersen consider the compact metric space $[-1,+1]^{\mathbb{Z}}$ (endowed with the product distance) and the shift endomorphism σ. They fix a sequence $\boldsymbol{a} \in [-1,+1]^{\mathbb{Z}}$ satisfying (2), and they set X to be the topological closure of the orbit of $\boldsymbol{a}$ under σ. By using the Spectral Theorem for Hilbert spaces, they prove that any shift-invariant probability measure μ defined on X (whose existence is guaranteed by the Bogolyubov-Krylov Theorem [2, Theorem 1.1]) must be concentrated on the point $\mathbf{0}$, i.e., the sequence given by all zeroes. This is clearly never true when $\boldsymbol{a} \in \{\pm 1\}^{\mathbb{Z}}$.

We note that the hypothesis in Theorem 1.7 is slightly different from that of Question 1.1, the key difference being the fact that the sum in (2) is bounded also for $\alpha = 0$. After carefully

[3]An example of such a set could be the following. Assume that $\mathbb{Q} \cap (0,1) = \{q_n\}_{n\geq 1}$ is a numbering of the rational numbers and let $0 < \varepsilon < 1$. Consider the set $E = \bigcup_{n\geq 1}(q_n - \varepsilon 2^{-n}, q_n + \varepsilon 2^{-n})$. We have $\mathscr{L}((0,1) \setminus E) \geq 1 - \varepsilon$, and for all intervals $I \subset (0,1)$ we have $\mathscr{L}(((0,1) \setminus E) \cap I) < \mathscr{L}(I)$, since every non-empty interval contains a rational.

reading Lesigne and Petersen's proof, we believe that their argument can be applied to show that, once the constraint for $\alpha = 0$ is removed, any shift invariant probability measure μ defined on the closure X of the σ-orbit of a sequence $\boldsymbol{a} \in \{0,1\}^{\mathbb{Z}}$ satisfying (2) must be concentrated either on the point $\mathbf{0}$ or on the point $\mathbf{1}$. In this case, we say (using the terminology from [6]) that $\boldsymbol{a}$ is essentially 0 or essentially 1. This, however, does not imply that the sequence $\boldsymbol{a}$ is eventually constant. Indeed, it is easy to see that the set $A = \{n + n! : n \in \mathbb{N}\}$ has an essentially zero indicator function $\chi_A : \mathbb{Z} \to \{0,1\}$. To show this, it is sufficient to observe that the closure of the orbit of χ_A under σ is the set

$$X = \left\{\mathbf{0}, \sigma^n\left(\chi_A\right), \sigma^n\left(\chi_{\{0\}}\right) : n \in \mathbb{Z}\right\}.$$

Note that Proposition 1.2 provides a simpler and more elementary proof of Theorem 1.7.

2. Proof of Proposition 1.2

To prove Proposition 1.2, we use a powerful result by Duffin and Schaeffer on complex power series [3, Part II, Theorem I]. We remark that the statement that we give below does not appear explicitly in [3, Part II, Theorem I], but it can be found at the end of the proof section [3, Part II, Section 4].

Theorem 2.1 (Duffin-Schaeffer)**.** *Let*

$$u(z) := \sum_{n=0}^{+\infty} b_n z^n$$

be a power series whose coefficients $b_n \in \mathbb{C}$ *take only finitely many different values. Assume that* $u(z)$ *is bounded in a sector* $S := \{\theta_1 \leq \arg(z) \leq \theta_2,\ |z| < 1\}$ *of the unit disk, where* $0 \leq \theta_1 < \theta_2 \leq 2\pi$*. Then, the sequence* $\{b_n\}$ *is ultimately periodic.*

Now, we show how to use Theorem 2.1 to prove the Proposition 1.2. Let I be an open interval contained in $E(\boldsymbol{a})$ and let $f : I \to [0, +\infty)$ be defined by

$$f(\alpha) := \sup_{N \in \mathbb{N}} \left| \sum_{n \leq N} a_n e(n\alpha) \right|.$$

The function f is a Baire class 1 function, since it is the point-wise limit[4] of a sequence of continuous functions (see [8, Definition 11.1]). From [8, Theorem 11.4], we know that the set of continuity points of such functions is dense in their domain. Hence, f has at least one continuity point P in I. We can therefore find an interval (α_1, α_2) around P whose image $f((\alpha_1, \alpha_2))$ is contained in a small interval around the point $f(P)$, deducing that f is bounded in (α_1, α_2) by some constant $M > 0$.

For $z \in D := \{|z| < 1\}$ we let

$$u(z) := \sum_{n=0}^{+\infty} a_n z^n,$$

[4]Note that the supremum of a sequence of continuous functions $\{f_m\}$ can be turned into a limit by considering the continuous functions $f_M := \sup_{m \leq M} f_m$.

where $a_0 := 0$. Summing by parts (see [1]), we find that for all $\alpha \in (\alpha_1, \alpha_2)$, all $0 \le r < 1$, and all integers $A \ge 1$ it holds

$$\left|\sum_{n=0}^{A} a_n r^n e(n\alpha)\right| \le \left|\sum_{n=0}^{A} a_n e(n\alpha)\right| r^A + \sum_{n=0}^{A-1}\left|\sum_{j=0}^{n} a_j e(j\alpha)\right| \left(r^n - r^{n+1}\right) \tag{3}$$
$$\le f(\alpha) r^A + f(\alpha)\left(1 - r^A\right) = f(\alpha) \le M.$$

Then, taking the limit for $A \to +\infty$, we obtain

$$|u(z)| \le M$$

for $z \in S := \{2\pi\alpha_1 \le \arg(z) \le 2\pi\alpha_2,\ |z| < 1\}$. Hence, by Theorem 2.1, the sequence $\boldsymbol{a}$ is ultimately periodic.

To conclude the proof, we show that if $f(\alpha) < +\infty$ for all rational numbers $\alpha \in \mathbb{Q} \cap (0,1)$ (or equivalently in $\mathbb{Q} \cap (0, 1/2]$ by (1)), the period of the sequence (a_n) is 1. Suppose that ultimately $\boldsymbol{a}$ has a period of length $q \ge 1$, i.e., $a_n = a_{n+q}$ for all $n \ge K$, where K is some large integer. Then, for $|z| < 1$ we have

$$u(z) = \sum_{n=0}^{K-1} a_n z^n + \sum_{n=0}^{+\infty} z^{qn+K}\left(\sum_{j=0}^{q-1} a_{K+j} z^j\right) = \sum_{n=0}^{K-1} a_n z^n + z^K \left(\sum_{j=0}^{q-1} a_{K+j} z^j\right)\frac{1}{1 - z^q}.$$

Since $|u(re^{i\alpha})| \le f(\alpha)$ for all $0 \le r < 1$ and all $\alpha \in E(\boldsymbol{a})$ (to see this, use (3)), the function $u(z)$ cannot have a pole at a non trivial root of unity. Hence, the polynomial $1 + z + \cdots + z^{q-1}$ must divide $\sum_{j=0}^{q-1} a_{K+j} z^j$, thus showing that $a_{K+j} = a_{K+j'}$ for all $j \ne j'$.

□

3. Proof of Proposition 1.5

Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function and let $A(f) := \{n + f(n)! : n \in \mathbb{N}\}$. To prove part *i*), it is sufficient to show that the sum

$$S_{A(f)}(\alpha, N + f(N)!) = \sum_{n \le N} e((n + f(n)!)\alpha)$$

is bounded for all $\alpha \in \mathbb{Q} \cap (0,1)$, independently of N.

Let $\alpha := p/q$, with $p, q \in \mathbb{N}$ and $q \ge 2$. Then, for all $n \ge q$ we have

$$n + f(n)! \equiv n \pmod{q}.$$

It follows that for $N \ge q$

$$\left|\sum_{n \le N} e((n + f(n)!)\alpha)\right| \le \left|\sum_{n < q} e((n + f(n)!)\alpha)\right| + \left|\sum_{q \le n \le N} e(n\alpha)\right| \tag{4}$$
$$\le \left|\sum_{n < q} e((n + f(n)!)\alpha)\right| + \left|\sum_{n \le q} e(n\alpha)\right| + \left|\sum_{n \le N} e(n\alpha)\right|,$$

and the right-hand side in (4) is bounded for $N \to +\infty$.

To prove part *ii*) of Proposition 1.5, we need the following auxilary result (see [9, Section 2]).

Lemma 3.1. *Let $\alpha \in [0,1)$. Then, there exists a sequence of integers $(s_n(\alpha))_{n \in \mathbb{N}}$ such that $0 \le s_n(\alpha) \le n - 1$ and*

$$\alpha = \sum_{n \ge 1} \frac{s_n(\alpha)}{n!}.$$

The sequence $(s_n(\alpha))_{n\in\mathbb{N}}$ associated to α is unique, if we exclude all those sequences s_n such that $s_n = n-1$ for all sufficiently large n. Under this limitation, the sequence $s_n(\alpha)$ is eventually null if and only if $\alpha \in \mathbb{Q}$.

For a real number $\alpha \in [0,1)$ we call the unique sequence $(s_n(\alpha))_{n\in\mathbb{N}}$ given by Lemma 3.1 (that does not eventually coincide with $n-1$) the factoradic representation of α, and we call the integer $s_n(\alpha)$ of such sequence the n-th factoradic digit of α.

Remark 3.2. Let N be a fixed integer and let $(s_n)_{n>N}$ be a sequence of integers such that $0 \leq s_n \leq n-1$ for all $n > N$, and such that s_n is not eventually equal to $n-1$. Then, we have

$$\sum_{n>N} \frac{s_n}{n!} < \frac{1}{N!}. \tag{5}$$

This follows from the equality

$$\sum_{n=N+1}^{M} \frac{n-1}{n!} = \frac{1}{N!} - \frac{1}{M!},$$

valid for all integers $M \geq N+1$. Equation (5) shows that the position of the digits in a factoradic representation is consistent with the order $<$ on the set of real numbers, as we have for decimal representations. This important fact will be used later on in the proof.

Let $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ be another sequence of strictly positive integers and assume that

$$\sum_{n\geq 1} 1/a_n < +\infty. \tag{6}$$

We consider the set

$$E(f,\boldsymbol{a}) := \left\{\alpha \in (0,1) : s_{f(i)+1}(\alpha) \leq \frac{f(i)+1}{a_i} \text{ for all } i \geq 1\right\}.$$

We shall show that for any function f satisfying condition $a)$ and any sequence $\boldsymbol{a}$ satisfying (6) the set $A(f)$ has BES over $E(f,\boldsymbol{a})$, thereby proving that $A(f)$ has BES over $E = E(f,\boldsymbol{a}) \cup (\mathbb{Q}\cap(0,1))$.

Summing by parts, we find

$$\begin{aligned} &\left|\sum_{n\leq N} e((n+f(n)!)\alpha)\right| \\ &\qquad \leq \left|\sum_{n\leq N} e(n\alpha)\right| |e(f(N)!\alpha)| + \sum_{n\leq N-1}\left|\sum_{i\leq n} e(i\alpha)\right| |e(f(n)!\alpha) - e(f(n+1)!\alpha)|. \end{aligned} \tag{7}$$

Hence, to bound the left-hand side of (7), it is enough to bound the sum

$$\sum_{n\leq N-1} |e(f(n)!\alpha) - e(f(n+1)!\alpha)| \leq 2\sum_{n\leq N} |e(f(n)!\alpha) - 1|. \tag{8}$$

Let $\{\theta\}$ denote the fractional part of any real number $\theta \geq 0$. By using the inequality $|e(\theta)-1| \leq 2\pi\{\theta\}$ (valid for $\theta \in [0,+\infty)$), we obtain

$$\sum_{n\leq N} |e(f(n)!\alpha) - 1| \leq 2\pi \sum_{n\leq N} \{f(n)!\alpha\}. \tag{9}$$

Now, since $\alpha \in E(f, \boldsymbol{a})$ and $f(n) \geq n$, we have

$$\begin{aligned} \{f(n)!\alpha\} &= \left\{ f(n)! \sum_{i\geq 1} \frac{s_i(\alpha)}{i!} \right\} = \left\{ \frac{s_{f(n)+1}(\alpha)}{f(n)+1} + \frac{s_{f(n)+2}(\alpha)}{(f(n)+1)(f(n)+2)} + \cdots \right\} \\ &\leq \frac{s_{f(n)+1}(\alpha)}{f(n)+1} + \frac{s_{f(n)+2}(\alpha)}{(f(n)+1)(f(n)+2)} + \cdots \\ &\leq \frac{1}{a_n} + \frac{1}{f(n)+1} + \frac{1}{(f(n)+1)(f(n)+2)} + \cdots \\ &\leq \frac{1}{a_n} + \frac{e}{f(n)+1}. \end{aligned} \tag{10}$$

Thus, combining (7),(8),(9), and (10), we obtain

$$\left| \sum_{n\leq N} e((n + f(n)!)\alpha) \right| \leq \frac{2}{|e(\alpha)-1|} \left(1 + 4\pi \sum_{n\leq N} \left(\frac{1}{a_n} + \frac{e}{f(n)+1} \right) \right).$$

By (6) and part *a*), the right hand side is bounded, proving the claim.

To conclude the proof, we show that the set $E(f, \boldsymbol{a})$ has full Hausdorff dimension whenever the function f satisfies condition *b*). To give a lower bound for the Hausdorff dimension of the set $E(f, \boldsymbol{a})$, we use the so called Mass-Distribution Principle (see [4, Principle 4.2]).

Lemma 3.3 (Mass-Distribution Principle)**.** *Let μ be a probability measure supported on a bounded subset X of $\mathbb{R}$. Suppose that there are strictly positive constants a, s, and ℓ_0 such that*

$$\mu(B) \leq a|B|^s \tag{11}$$

for any interval[5] $B \subset \mathbb{R}$ of length $|B| \leq \ell_0$. Then, $\dim(X) \geq s$*, where* dim *denotes the Hausdorff dimension of a set.*

We can take X to be the set $E(f, \boldsymbol{a}) \cup (\mathbb{Q} \cap [0,1])$, since adding a countable number of points to a set does not change its Hausdorff dimension. To apply Lemma 3.3, we need to construct a probability measure μ whose support is contained in X. To define μ we use [4, Proposition 1.7], which we state here as a lemma, for the convenience of the reader.

Lemma 3.4. *Let $n \geq 1$, let $E \subset \mathbb{R}^n$, and let $\{\mathcal{E}_k\}_{k\geq 0}$ be a sequence of set collections such that for all $k \geq 0$ the sets $U \in \mathcal{E}_k$ are pairwise disjoint Borel subsets of $\mathbb{R}^n$. Let $\mathcal{E}_0 := \{E\}$, and assume that for all $k \geq 1$ each set $U \in \mathcal{E}_k$ is contained in exactly one set $U_{k-1} \in \mathcal{E}_{k-1}$ and contains only a finite number of sets $U_{k+1} \in \mathcal{E}_{k+1}$. Assume also that the maximum diameter of the sets in $\mathcal{E}_k$ tends to 0 as k grows. Finally, let $\mu : \bigcup_{k\geq 0} \mathcal{E}_k \to (0, +\infty)$ be a function with the property that for all $k \geq 0$ and all $U \in \mathcal{E}_k$ it holds*

$$\mu(U) = \sum_{\substack{V \in \mathcal{E}_{k+1} \\ V \subset U}} \mu(V). \tag{12}$$

Then, μ can be extended to a unique Borel measure μ^* on $\mathbb{R}^n$ such that $\mu^*(A) = 0$ on all Borel sets $A \subset \mathbb{R}^n$ for which $A \cap \bigcup_{U\in\mathcal{E}_k} U = \emptyset$ for some $k \geq 0$. Moreover, the support of such measure μ^* is contained in the set

$$\bigcap_{k\geq 0} \overline{\bigcup_{U\in\mathcal{E}_k} U},$$

[5] Note that it is enough to consider intervals since the ball-defined Hausdorff dimension coincides with the classical Hausdorff dimension (see [4, Section 2.4]).

where $\overline{X}$ *denotes the topological closure of a set* $X \subset \mathbb{R}^n$.

Now, we show how to construct μ by using Lemma 3.4. For $i \in \mathbb{N}$ we let $\rho_i := 1/i!$ and

$$Z_i := \{\alpha \in [0,1) : s_j(\alpha) = 0 \text{ for } j > i\}.$$

Then, we set

$$\mathcal{E}_i := \{(\alpha, \alpha + \rho_i)\}_{\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i},$$

and for all $\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i$ we define

$$\mu((\alpha, \alpha + \rho_i)) := \frac{1}{\#\mathcal{E}_i} = \frac{1}{\#((E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i)}. \tag{13}$$

Clearly, the sets $\mathcal{E}_i$ have the properties required to apply Lemma 3.4. We just need to show that μ defined in (13) satisfies (12).

Remark 3.5. Let β_i denote the truncation of a number $\beta \in [0,1)$ at the i-th factoradic digit (so that $\beta_i \in Z_i$). We note that for $\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i$ it holds

$$\#\{\beta \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} : \beta_i = \alpha\} = \frac{\#((E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1})}{\#((E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i)} = \frac{\#\mathcal{E}_{i+1}}{\#\mathcal{E}_i}.$$

In view of Remark 3.2, we have

$$\begin{aligned}\mu((\alpha, \alpha + \rho_i)) = \frac{1}{\#\mathcal{E}_i} &= \frac{1}{\#\mathcal{E}_{i+1}} \sum_{\substack{\beta \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} \\ \beta_i = \alpha}} 1 = \sum_{\substack{\beta \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} \\ \beta_i = \alpha}} \frac{1}{\#\mathcal{E}_{i+1}} \\ &= \sum_{\substack{\beta \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} \\ \beta_i = \alpha}} \mu((\beta, \beta + \rho_{i+1})) = \sum_{\substack{\beta \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} \\ (\beta, \beta + \rho_{i+1}) \subset (\alpha, \alpha + \rho_i)}} \mu((\beta, \beta + \rho_{i+1})).\end{aligned} \tag{14}$$

This shows that μ satisfies (12).

By Lemma 3.4, Equation (13) induces a unique well-defined Borel measure μ on $\mathbb{R}$, with the property that

$$\mu\left(\mathbb{R} \setminus \bigcup_{\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i} (\alpha, \alpha + \rho_i)\right) = 0 \tag{15}$$

for all $i \in \mathbb{N}$, and supported on the set

$$\bigcap_{i \in \mathbb{N}} \bigcup_{\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i} [\alpha, \alpha + \rho_i].$$

Now, in order to apply Lemma 3.3 to the set X, we need to prove that

$$\bigcap_{i \in \mathbb{N}} \bigcup_{\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i} [\alpha, \alpha + \rho_i] \subset E(f,\boldsymbol{a}) \cup (\mathbb{Q} \cap [0,1]) = X. \tag{16}$$

This will show that the measure μ constructed above is supported on X. To see that (16) holds, we observe that, by definition, for each α lying in the left-hand side of (16) and each $i \in \mathbb{N}$ there exists $\alpha_i \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i$ such that $\alpha \in [\alpha_i, \alpha_i + \rho_i]$. This means that either $\alpha = \alpha_i + \rho_i \in \mathbb{Q} \cap [0,1]$ (by Remark (3.2)), or $0 \leq \alpha - \alpha_i < \rho_i$, i.e., $\alpha - \alpha_i$ is a number between 0 and 1 such that $s_j(\alpha - \alpha_i) = 0$ for $j \leq i$. Hence, when we add $\alpha - \alpha_i$ to α_i, the digits before the i-th do not change, showing that $s_j(\alpha) = s_j(\alpha_i)$ for $j \leq i$. Since this is true for all $i \in \mathbb{N}$, by the definition of $E(f,\boldsymbol{a})$, we have $\alpha \in E(f,\boldsymbol{a}) \cup \{0\}$.

We are now left to prove (11). Let $B \subset [0,1]$ be an interval. Clearly, there exists an index $i \in \mathbb{N}$ such that

$$\rho_{i+1} < |B| \leq \rho_i. \tag{17}$$

Then, by (15), we have

$$\mu(B) \leq \mu\left(\bigcup_{\substack{\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} \\ (\alpha, \alpha+\rho_{i+1}) \cap B \neq \emptyset}} (\alpha, \alpha + \rho_{i+1}) \right),$$

and it is straightforward to see that

$$\#\{\alpha \in Z_{i+1} : (\alpha, \alpha + \rho_{i+1}) \cap B \neq \emptyset\} \leq \frac{|B|}{1/(i+1)!} + 2,$$

since the intervals $(\alpha, \alpha + \rho_{i+1})$ are pairwise disjoint and each of them has length $1/(i+1)!$. Hence, we deduce

$$\mu(B) \leq \sum_{\substack{\alpha \in (E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1} \\ (\alpha, \alpha+\rho_{i+1}) \cap B \neq \emptyset}} \mu((\alpha, \alpha + \rho_{i+1})) \leq \left(\frac{|B|}{1/(i+1)!} + 2 \right) \frac{1}{\#((E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_{i+1})}. \tag{18}$$

Now, we observe that for all $i \in \mathbb{N}$ it holds

$$\#((E(f,\boldsymbol{a}) \cup \{0\}) \cap Z_i) \geq \frac{i!}{\prod_{f(j)+1 \leq i} (f(j)+1)}, \tag{19}$$

since for $\alpha \in E(f,\boldsymbol{a}) \cup \{0\}$ the 0 digit is always allowed in the $(f(j)+1)$-th position, independently of $\boldsymbol{a}$. Thus, by (17), (18), and (19), we conclude that

$$\begin{aligned} \mu(B) &\leq \left(\frac{|B|}{1/(i+1)!} + 2 \right) \frac{\prod_{f(j)+1 \leq i+1} (f(j)+1)}{(i+1)!} \leq 3(i+1)!|B| \frac{\prod_{f(j) \leq i} (f(j)+1)}{(i+1)!} \\ &= 3|B|^{1-s} \left(\prod_{f(j) \leq i} (f(j)+1) \right) |B|^s \leq 3 \left(\frac{1}{i!} \right)^{1-s} \left(\prod_{f(j) \leq i} (f(j)+1) \right) |B|^s. \end{aligned}$$

It then follows from Lemma 3.3 that any function f such that

$$\sup_{i \in \mathbb{N}} \left(\frac{1}{i!} \right)^{1-s} \prod_{f(j) \leq i} (f(j)+1) < +\infty$$

for all $0 < s < 1$ gives raise to a set $E(f,\boldsymbol{a})$ of full Hausdorff dimension.

4. An example

To conclude this note, we give an example of a function $f : \mathbb{N} \to \mathbb{N}$ satisfying both conditions *a*) and *b*) in Proposition 1.5 part *ii*). We let $s := 1 - \varepsilon$ and $f(i) = i^2$ for $i \in \mathbb{N}$. Condition *a*) is clearly satisfied. Moreover, we have

$$\left(\frac{1}{i!} \right)^{1-s} \prod_{f(j) \leq i} (f(j)+1) = \frac{\prod_{j \leq \sqrt{i}} (j^2+1)}{(i!)^{\varepsilon}}. \tag{20}$$

Now, when $i > \lceil 3/\varepsilon \rceil \lfloor \sqrt{i} \rfloor$, we find

$$(i!)^\varepsilon \geq \left(\underbrace{1 \cdots 1}_{\lceil 3/\varepsilon \rceil \text{ times}} \cdot \underbrace{2 \cdots 2}_{\lceil 3/\varepsilon \rceil \text{ times}} \cdot \cdots \cdot \underbrace{\lfloor \sqrt{i} \rfloor \cdots \lfloor \sqrt{i} \rfloor}_{\lceil 3/\varepsilon \rceil \text{ times}} \right)^\varepsilon \geq \prod_{j \leq \sqrt{i}} j^3.$$

Hence, the right-hand side in (20) remains bounded as $i \to +\infty$.

Acknowledgements

I am extremely grateful to Jeffrey Vaaler for pointing me to Helson's paper [5] and for the very fruitful discussions that we had during his visit at Royal Holloway. I would like thank my supervisor, Martin Widmer, who attended the conference in Luminy and introduced me to this problem, for his encouragement and precious advice. I would also like to thank Lambert A'Campo and Cedric Pilatte for spotting a mistake in an early version of this manuscript. Finally, I am thankful to Royal Holloway University of London, for funding my position here, and to the referee for his thorough checking and good advice.

References

[1] N. H. Abel. Untersuchungen über die Reihe $1 + \frac{m}{1} \cdot x + \frac{m \cdot (m-1)}{1 \cdot 2} \cdot x^2 + \cdots$. *J. Reine Angew. Math.*, 1:311–339, 1826.

[2] I. P. Cornfeld and Ya. G. Sinai. *Dynamical Systems II.* Springer-Verlag, 1989.

[3] R. J. Duffin and A. C. Schaeffer. Power series with bounded coefficients. *Am. J. Math.*, 67(No. 1):141–154, 1945.

[4] K. Falconer. *Fractal Geometry.* John Wiley & Sons Ltd., second edition, 2003.

[5] H. Helson. On a theorem of Szegő. *Proc. Am. Math. Soc.*, 6(No. 2):235–242, 1955.

[6] E. Lesigne and K. Petersen. Bounded exponential sums. *Osaka J. Math.*, 27:277–280, 1990.

[7] L. A'Campo (under the supervision of Philipp Habegger). The circle method, applications to the partition function and beyond, 2017.

[8] A. C. M. van Rooij and W. H. Schikhof. *A second course on real functions.* Cambridge University Press, 1982.

[9] J. F. Weisz. Real numbers in factorial representation. *Int. J. Math. Educ. Sci. Technol.*, 25(No. 1):25–29, 1994.

Department of Mathematics, Royal Holloway, University of London, TW20 0EX Egham, UK
E-mail address: `Reynold.Fregoli.2017@live.rhul.ac.uk`