Ordinal Allocation

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Abstract

A generalization of the well-known Vickrey auctions are *lottery qualification auctions* – where the m highest bidders win the good with uniform probability, and pay the $m+1$st highest bid upon winning. A *random lottery qualification mechanism* decides the integer m randomly. We characterize the class of mechanisms which are payoff equivalent to the random lottery qualification auctions. The key property characterizing this class of mechanisms is one which states that only the ordinal comparison of willingness-to-pay across individuals is relevant in determining the allocation. The mechanisms can be seen as compromising between ex-post utility efficiency and monetary efficiency.

1 Introduction

A mechanism designer seeks to allocate an object. She wants to base this allocation on the participants' valuations, but she is not necessarily concerned with revenue maximization. For example, a government may worry that allocating the object to the participant with the highest valuation may tend to induce a monopoly. Or, there may be legal constraints involved with using money in certain ways; as in kidney exchange.

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To this end, we discuss, in a private-value setting, a class of mechanisms which naturally generalize the Vickrey mechanism. These mechanisms do not guarantee *ex-post efficiency* (Green and Laffont (1977)), the property that the good goes to a participant with the highest valuation. The building blocks of the mechanisms we study here are not new: they generalize the Vickrey (Vickrey (1961); Clarke (1971); Groves (1973)) auction in a natural way. We imagine a single-unit auction with transfers and quasilinear preferences. The class of *lottery qualification auctions*, introduced by Harstad and Bordley (1996), is parametrized by an integer. The participants bid, and the m highest bidders each have a (uniform) chance of winning the object. The winner gets the object, and pays the $m+1$st highest price.

This primitive class generalizes both the second-price Vickrey auction (when $m = 1$), and a straight uniform lottery with no transfers (as in models of kidney exchange (Roth, Sönmez, and Ünver, 2004)).

Ex-post efficiency (Green and Laffont (1977)) usually has the interpretation that, after all transfers have been paid (the money presumably burned or given to the mechanism designer), no reallocation of the goods (or money) results in Pareto dominance. However, in a well-known phenomenon in mechanism design (Green and Laffont (1977); Holmström (1979); Laffont and Maskin (1980)), the allocations recommended by these mechanisms are not in general efficient. Money must be burned or injected. This is because only the VCG mechanism is ex-post incentive compatible and efficient (Green and Laffont (1977); Holmström (1979); Laffont and Maskin (1980)), and it is not ex-post budget-balanced. From the point of view of a mechanism designer who operates outside of the mechanism (and does not enjoy any profits), this is pure waste as much as "misallocating" the good is.

On the other hand, ex-post efficiency can be viewed as preventing a kind of aftermarket from occurring, after any money is burned. By preventing ex-post trade of objects (or ex-post trade of objects for money), the mechanism designer can ensure that the recommended allocation is actually implemented. This would not necessarily be so were ex-post efficiency violated. So, the mechanisms we consider are probably best used when consumption of recommended objects can be credibly enforced (such as with auctioning bandwidth, or perhaps exchange of kidneys).

So, ex-post efficency and incentive compatibility imply that some money must be burned. However; observe that the absence of budget-balance is not inevitable unless we require both of these properties. For example, we may simply take the object and randomly allocate it without even asking the agents' valuations. In fact, some variant of this takes place in kidney exchange, where for ethical and legal reasons, monetary valuation of kidneys are not even considered and money is not permitted (Roth, Sönmez, and Ünver (2004)). The important takeaway here is that money need not be burned. And, in fact, in our general class, there is a kind of tradeoff. The more we seek to satisfy ex-post efficiency, in equilibrium, the less "monetary efficiency" will be satisfied.

Our main result characterizes the class of lottery qualification auctions as (the extreme points of) the class of mechanisms satisfying four basic properties. The first three are standard: incentive compatibility, anonymity, and a property stating that those who bid nothing pay nothing. The key property here is a property we term *ordinality.* Ordinality requires that the probabilistic allocation of the object be determined solely by the ordinal ranking of the individuals' willingness to pay for the object. The chances of winning the object are completely determined by who values the object more (or less). Ex-post efficiency implies ordinality: it forces the participant with the highest valuation to necessarily obtain the good. As far as we know, ordinality is novel to this context. But axioms along these lines are frequently posited in the theory of matching with risk; see, *e.g.* Bogomolnaia and Moulin (2001).

We observe a critical difference between classical notions of ordinality (as in Bogomolnaia and Moulin (2001) and our notion. In the work of Bogomolnaia and Moulin (2001), the ordinality represents a preference structure. Their goal is to allocate objects probabilistically, and they assume preferences are only observed over nonrandom alternatives. So, in their case, this ordinal structure can easily be elicited without asking an individual about any kind of cardinal information. In contrast, our axiom is, in a way, less compelling as it works *across* individuals. Viewing the amount an individual is willing to pay for an object as a utility, our notion gives a natural interpersonally comparable notion of utility. It is the ordinal structure of this utility across agents which we study. We see no obvious method of eliciting this structure without first eliciting the cardinal utility (value of the object). Thus, our axiom should be viewed as more of an axiom which simplifies the structure of the problem, rather than one that has some underlying economic content.

In order to justify our claims about Pareto efficiency, we demonstrate a profile of valuations for which no mechanism in our class Pareto dominates any other mechanism. In other words, for this particular profile, *all* such mechanisms are Pareto efficient among the class of achievable allocations. In fact, there is a "large" set of such valuations, but we do not demonstrate this formally.

Section 2 describes the model and the main result. Section 3 demonstrates the profile with the Pareto non-dominance result. Finally, Section 4 concludes.

2 Model and Characterization

Let $N \equiv \{1, \ldots, n\}$ be a finite set of agents, with generic element $i \in N$. The set of *valuations* is $\Theta = [0, +\infty)$. Agents are risk-neutral and the utility an agent $i \in N$ with valuation $\theta_i \in \Theta$ gathers from receiving the object with probability $p \in [0, 1]$ less transfer t is $u^i(p, t : \theta_i) = \theta_i p - t$.

Though the interpretation of our mechanisms involves random payments, we consider only the relevant welfare of individuals. To this end, the set of *allocations* is $Y = \Delta(N) \times \Re^N$, with typical element (p, t). Let us comment on this definition. First, a more general type of allocation would take as a primitive a *deterministic allocation*, which would be an element of $N \times \Re^N$. Then, a random allocation would be a simple lottery over these objects. However, any such "generalized allocation" would induce an allocation of the form we consider, merely by taking appropriate expectations. So long as our primary axioms are about welfare, and our characterization speaks only to the welfare of the agents, it is without loss to consider this type of allocation.

A *valuation profile* is an element $\theta \in \Theta^N$, written $\theta = \{\theta_i\}_{i \in N}$. A *valuation profile without ties* is an element $\theta \in \Theta^N$ for which for all $i, j \in N$ with $i \neq j$, $\theta_i \neq \theta_j$. The set of valuation profiles without ties is written $\overline{\Theta^N}$. For ease of exposition, we will primarily concern ourselves with valuation profiles that have no ties.

To this end, define an *allocation rule* to be a function $f : \overline{\Theta^N} \to Y$. We often write $f(\theta) = (p(\theta), t(\theta))$. The notation $p_i(\theta)$ refers to the probability that agent i receives the good with profile θ, and $t_i(\theta)$ refers to the transfer for agent i with profiles θ

First, an allocation rule is *incentive compatible* if for all $\theta \in \overline{\Theta^N}$, all $i \in N$, and all $\theta' \in \Theta$ for which $(\theta', \theta_{-i}) \in \overline{\Theta^N}$, it satisfies

$$\theta_i p_i(\theta) - t_i(\theta) \geq \theta_i p_i(\theta', \theta_{-i}) - t_i(\theta', \theta_{-i}).$$

An allocation rule is *anonymous* if for all $\theta \in \overline{\Theta^N}$, any permutation $\sigma : N \to N$, and any $i \in N$, we have $p_i(\theta \circ \sigma) = p_{\sigma(i)}(\theta)$ and $t_i(\theta \circ \sigma) = t_{\sigma(i)}(\theta)$.

The allocation and transfer rules are *normalized* so that for all $\theta \in \overline{\Theta^N}$, if $\theta_i = 0$, then $\theta_i p_i(\theta) - t_i(\theta) = 0$. Observe that normalization and incentive compatibility jointly imply *individual rationality*, which means $\theta_i p_i(\theta) - t_i(\theta) \geq 0$. Another consequence is that when $\theta_i = 0$, $t_i(\theta) = 0$.

Finally, the allocation rule is *ordinal* if for all $\theta \in \overline{\Theta^N}$ and any strictly increasing function $\varphi : \Theta \to \Theta$, we have $p(\theta) = p(\varphi \circ \theta)$.

Ordinality is the substantive axiom introduced in our work. Observe that the classical axiom of *ex-post* efficiency implies ordinality. Our framework allows interpersonal comparison of utility. In other words, we can meaningfully talk of one individual as obtaining a higher utility from the good than another (their willingness to pay). Ordinality is the statement that only the ordinal content of the interpersonal comparison of utility is relevant for determining the random allocation.

Let $\theta_{[m]}$ be the m-th order statistic, so that on $\overline{\Theta^N}$, $\theta_{[1]} > \theta_{[2]} > \ldots > \theta[n]$. Because we consider $\theta \in \overline{\Theta^N}$, the n order statistics are distinct.

When $1 \leq m \leq n-1$, we define a *m-lottery qualification auction* (Harstad and Bordley (1996)) to be the mechanism on $\overline{\Theta^N}$ taking the form:

$$p_i(\theta) = \begin{cases} \frac{1}{m} \text{ if } \theta_i > \theta_{[m+1]} \\ 0 \text{ otherwise} \end{cases}$$

and

$$t_i(\theta) = \begin{cases} \frac{\theta_{[m+1]}}{m} \text{ if } \theta_i > \theta_{[m+1]} \\ 0 \text{ otherwise} \end{cases} ,$$

We define the *n-lottery qualification auction* to consist of uniform randomization across all agents, without any payments. We write p^m, and t^m, for the m-lottery qualification auction, and transfer function respectively. The rule itself is written f^m. [1]

Observe that this mechanism is a "reduced form" of the mechanism which awards the object to one of the m highest bidders with uniform probability, asking them to pay the $m+1$st highest price. Underlying uncertainty about the payoff is collapsed via expectation.

A rule is a *random lottery qualification auction* if there is $q \in \Delta(n)$ such that for all $i \in N$, $p_i(\theta) = \sum_{m=1}^{n} q^m p_i^m(\theta)$ and $f_i(\theta) = \sum_{m=1}^{n} q^m t_i^m(\theta)$.

Theorem 1. *An allocation rule is a random lottery qualification auction if and only if it satisfies ordinality, incentive compatibility, anonymity, and normalization.*

Proof. Let q be a random lottery qualification auction. Anonymity, ordinality, and normalization are all immediate. We establish that incentive compatibility is satisfied.

We first verify incentive compatibility for lottery qualification auctions. The result is trivial for the n-lottery qualification auction.

So, consider a lottery qualification auction, f^m. For any i, given our domain assumption, there are two possibilities: either $\theta_i > \theta_{[m+1]}$ or $\theta_i < \theta_{[m]}$. Suppose that $\theta_i > \theta_{[m+1]}$. Then for any $\theta' > \theta_{[m+1]}$, $f^m(\theta', \theta_{-i}) = f^m(\theta)$, so that $u^i_{\theta_i}(f^m(\theta)) = u^i_{\theta_i}(f^m(\theta', \theta_{-i}))$. For any $\theta' < \theta_{[m+1]}$ (recall our allocation rule is defined on $\overline{\Theta^N}$), observe that $u^i_{\theta_i}(f^m(\theta)) = \frac{\theta_i}{m} - \frac{\theta_{[m+1]}}{m} > 0 = u^i(f^m(\theta', \theta_{-i}))$.

Now suppose that $\theta_i < \theta_{[m]}$. For any $\theta' < \theta_{[m]}$, $f(\theta) = f(\theta', \theta_{-i})$. Suppose instead $\theta' > \theta_{[m]}$. Then $u_{\theta_i}(f^m(\theta)) = 0$, and $u_{\theta_i}(f^m(\theta', \theta_{-i})) = \frac{\theta_i}{m} - \frac{\theta_{[m]}}{m} < 0$.

We have shown that for any $m = 1, \ldots, n$, and all $\theta \in \overline{\Theta^N}$, all $i \in N$ and all $\theta' \in \Theta$ for which $(\theta', \theta_{-i}) \in \overline{\Theta^N}$, we have $u^i_{\theta_i}(f^m(\theta)) \geq u^i_{\theta_i}(f^m(\theta', \theta_{-i}))$. Now observe that for any $\tau \in \overline{\Theta^N}$, $u^i(\theta_i)(f^q(\tau)) = \sum_{k=1}^{n} q(k) u(f^k(\tau))$. Incentive compatibility therefore follows directly by linearity.

Conversely, we will show that an allocation rule satisfying the axioms is a random lottery qualification auction.

As a first step, fix $1 \in N$. Observe that if $\theta_2^* > \ldots > \theta_n^* > 0$, then by incentive compatibility $p_1(\theta_1, \theta_2^*, \ldots, \theta_n^*)$ weakly increases in θ_1. The argument is standard.

[1] This mechanism is usually understood as a hybrid of a "rationing" mechanism and an auction; see, *e.g.*, Parlour, Prasnikar, and Rajan (2007).

Secondly, by ordinality, it follows that $p_1(\cdot, \theta_2^*, \ldots, \theta_n^*)$ is constant on the intervals: $(\theta_2^*, +\infty)$, $(\theta_{k+1}^*, \theta_k^*)$ for $k = 1, \ldots, n-1$, and $[0, \theta_n^*)$. Define $p(1) \equiv p_1(\theta_1, \theta_2^*, \ldots, \theta_n^*)$ for any $\theta_1 > \theta_2^*$, $p(k) \equiv p_1(\theta_1, \theta_2^*, \ldots, \theta_n^*)$ for any $\theta_1 \in (\theta_k, \theta_{k-1})$, and finally $p(n) \equiv p_1(\theta_1, \theta_2^*, \ldots, \theta_n^*)$ when $0 \leq \theta_1 < \theta_n^*$.

Observe that by ordinality and symmetry, it follows that for any $i \in N$ and any $\theta \in \overline{\Theta^N}$, we have $p_i(\theta) \equiv p(k)$ where $p(k)$ is such that $\theta_i = \theta_{[k]}$. It therefore also follows by the definition of allocation rule that $\sum_{k=1}^n p(k) = 1$.

Now, define $q(n) = np(n)$, and observe that $0 \leq q(n)$. For $m < n$, define $q(m) = m(p(m) - p(m+1))$. By monotonicity of p, $0 \leq q(m)$. Finally, observe that $\sum_m q(m) = np(n) + \sum_{m<n} m(p(m) - p(m+1)) = \sum_m p(m) = 1$. We claim that the q-lottery qualification auction returns the probabilities p specified by the mechanism. Let us write $p(n+1) = 0$ to simplify notation. Let $m \in \{1, \ldots, n\}$; the q-lottery qualification auction awards the individual whose valuation is in the m-th position with probability $\sum_{k \geq m} q(k)\frac{1}{k}$. But $\sum_{k \geq m} q(k)\frac{1}{k} = \sum_{k \geq m} k(p(k) - p(k+1))\frac{1}{k} = \sum_{k \geq m}(p(k) - p(k+1)) = p(m)$.

Finally, the transfer function t is uniquely defined by incentive compatibility and normalization.

There is nothing at all novel in this simple revenue equivalence argument (see *e.g.* Myerson (1981)), but we replicate it here simply because our domain is $\overline{\Theta^N}$ rather than Θ^N. Recall that normalization implies that $t_i(\theta) = 0$ whenever $\theta_i = 0$. Let $v_i(\theta_i; \theta_{-i}) \equiv \theta_i p_i(\theta) - t_i(\theta)$, and observe that by incentive compatibility, $v_i(\theta_i; \theta_{-i}) = \sup_{\theta'} \theta_i p_i(\theta', \theta_{-i}) - t_i(\theta', \theta_{-i})$; so that v_i is the supremum of a collection of affine functions (in θ_i) defined on $\Theta \setminus \{\theta_2, \ldots, \theta_n\}$. In particular, $p(k)$ form the ordered collection of subgradients of $v_i(\theta_1; \theta_{-i})$. This set of subgradients is monotone, and hence cyclically monotone; they can clearly be extended (in a set valued-sense) uniquely to the points $\{\theta_2, \ldots, \theta_n\}$ to preserve monotonicity. Theorem 24.9 of Rockafellar (1970) demonstrates that $v_i(\theta_1; \theta_{-i})$ is therefore defined uniquely by the $p(i)$ and the fact that $v_i(0; \theta_{-i}) = 0$.

Observe then that $t_i(\theta_1; \theta_{-i}) = \theta_i p_i(\theta_i; \theta_{-i}) - v_i(\theta_i; \theta_{-i}) \geq \theta_i p_i(\theta'; \theta_{-i}) - v_i(\theta'; \theta_{-i})$. That is, $t_i(\theta_1; \theta_{-i}) = \sup_{\theta'} \theta_i p_i(\theta'; \theta_{-i}) - v_i(\theta'; \theta_{-i})$; t_i is a Fenchel conjugate of v_i (see, *e.g.* Rockafellar (1970), Section 12). Since v_i is uniquely determined, so is t_i. □

3 On the Lack of Pareto Dominance across q-Lottery Qualification Auctions

Here we show that there is a profile $\theta \in \overline{\Theta^N}$ which has the property that, for any $q, r \in \Delta(n)$ where $q \neq r$, the induced payoffs from q and r are not Pareto ranked. That is, given any two lottery qualification auctions, some agents will strictly prefer one, and others will strictly prefer the other. In terms of welfare of the agents participating in the allocation rule, therefore, ex-post efficiency cannot be used as an efficiency argument. In other words, absent revenue considerations, there are no efficiency arguments dictating which of the allocation rules to use. Naturally, a mechanism designer who also accounts for the seller's revenue will find the lottery qualification auctions to be Pareto-unranked.

Proposition 1. *There exists $\theta^* \in \overline{\Theta^N}$ such that for any $q, r \in \Delta(n)$ for which $q \neq r$, there exists $i \in N$ for which $u^i_{\theta^*_i}(f^r_i(\theta^*)) > u^i_{\theta^*_i}(f^q_i(\theta^*))$.*

Proof. We demonstrate a particular profile $\theta^* \in \overline{\Theta^N}$ where the payoffs $u^i_{\theta^*_i}(f^m(\theta^*))$ as $m \in \{1, \ldots, n\}$ are Pareto unranked and form the vertices of a simplex. This is enough to prove the Proposition.

First, observe that in the following matrix, the rows and columns are each linearly independent. The columns are intended to index individuals $N \equiv \{1, \ldots, n\}$ and the rows the k-lottery qualification auction. We plan to construct a profile $\theta \in \overline{\Theta^N}$ for which, up to individual scale transformations, entry (l, k) in this matrix approximates the utility agent k gets from the l-lottery qualification allocation rule.

$$A \equiv \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & k & \cdots & n \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \end{array}$$

To this end, consider a profile $\theta^* \in \overline{\Theta^N}$ whereby $\theta_1^* > \theta_2^* > \ldots \theta_n^*$. A corresponding matrix of utilities from participating in each of the allocation rules is as follows:

$$G(\theta^*) \equiv \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & k & \cdots & n \end{array} \\ \begin{bmatrix} \theta_1^* - \theta_2^* & 0 & \cdots & 0 & \cdots & 0 \\ \frac{\theta_1^*-\theta_3^*}{2} & \frac{\theta_2^*-\theta_3^*}{2} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{\theta_1^*-\theta_{k+1}^*}{k} & \frac{\theta_2^*-\theta_{k+1}}{k} & \cdots & \frac{\theta_k^*-\theta_{k+1}^*}{k} & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{\theta_1^*}{n} & \frac{\theta_2^*}{n} & \cdots & \frac{\theta_k^*}{n} & \cdots & \frac{\theta_n^*}{n} \end{bmatrix} \end{array}$$

Here, $G_{i,j}(\theta^*)$ (the i-th row and j-th column) is the payoff of the i-lottery qualification allocation rule to individual j.

Now, let us multiply each column k by $\frac{1}{\theta_k^*}$, resulting in a rescaling of utility for each individual, but no change in the ranking.

$$B(\theta^*) \equiv \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & k & \cdots & n \end{array} \\ \begin{bmatrix} 1 - \frac{\theta_2^*}{\theta_1^*} & 0 & \cdots & 0 & \cdots & 0 \\ \frac{1}{2} - \frac{\theta_3^*}{2\theta_1^*} & \frac{1}{2} - \frac{\theta_3^*}{2\theta_2^*} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{1}{k} - \frac{\theta_{k+1}^*}{k\theta_1^*} & \frac{1}{k} - \frac{\theta_{k+1}}{k\theta_2^*} & \cdots & \frac{1}{k} - \frac{\theta_{k+1}^*}{k\theta_k^*} & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \end{array}$$

Evidently, for any $\epsilon > 0$, we can choose $\sup_{k=1,\ldots,n-1} \frac{\theta_{k+1}^*}{\theta_k^*} < \epsilon$, rendering $B(\theta^*)$ arbitrarily close to A. Choose θ^* so that the rows of $B(\theta^*)$ are linearly independent, such that for each $k = 1, \ldots, n$, $\arg\max_l B_{k,l}(\theta^*) = k$, and finally such that for each column k, $B_{i,k}(\theta^*)$ strictly decreases as i increases (which is the case when $\epsilon < 1/n^2$).

Obviously, the rows (and columns) of $B(\theta^*)$ are linearly independent.

Now, we claim that for each $p, q \in \Delta(n)$ with $p \neq q$, it follows that neither $p^T B(\theta^*) \geq q^T B(\theta^*)$, or $q^T B(\theta^*) \geq p^T B(\theta^*)$. Here, the inequalities are vector inequalities.

So, suppose without loss that there are $p, q \in \Delta(n)$ for which $p \neq q$ and $p^T B(\theta^*) \geq q^T B(\theta^*)$. Clearly, this is true if and only if there is $x \in \Re^n \setminus \{0\}$ for which $\sum_i x_i = 0$ and $x^T B(\theta^*) \geq 0$, where by linear independence, $x^T B(\theta^*) \neq 0$. So, suppose by means of contradiction that there is such an x. Let $B^k(\theta^*)$ denote the k-th column of $B(\theta^*)$. It is clear that there are weights $\lambda_1, \ldots, \lambda_n > 0$ such that $\sum_{i=1}^n \lambda_i B^i(\theta^*) = \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$.

Thus, if there were such an x, we would obtain:

$$x^T(\sum_{i=1}^{n} \lambda_i B^i(\theta^*)) = 0,$$

and, since $x^T B(\theta^*) \geq 0$, but $x^T B(\theta^*) \neq 0$, there is some k for which $x^T B^k(\theta^*) > 0$. Consequently, since each $\lambda_i > 0$, there would also exist k' for which $x^T B^{k'}(\theta^*) < 0$, contradicting the fact that $x^T B(\theta^*) \geq 0$. □

Often in quasilinear allocation, the sum of utilities is a relevant object of study. The reason for this is that when utility is freely transferable, an allocation is Pareto optimal exactly when it maximizes a sum of utilities. When looking for efficient, or ex-post efficient allocations, maximization of the sum of utilities is therefore the correct criterion. In contrast, it is not the case that a higher sum of utilities necessarily leads to Pareto dominance. Rather, a higher sum of utilities for one allocation over another simply means that there exist additional transfers which render the allocation with the higher sum Pareto dominant to the one with the lower sum. In our context, we are comparing a given set of allocations *without additional transfers*. Thus, we are not claiming that all of the allocations we study are efficient amongst the class of all allocations, but rather only amongst the class of allocations which can arise from our class of mechanisms.

4 Related Literature and Conclusion

An obvious point is that our allocation rules focus on valuation profiles without ties. In particular, we do not even allow individuals to announce ties in deviations. This is in the interest of simplicity: when there are no ties, all valuation profiles are ordinally equivalent to each other. When ties are possible, there are as many ordinal equivalence classes as there are ordered number partitions of n (see Aigner (2012)). The situation becomes cumbersome and not particularly interesting formally, though still amenable to analysis. We demonstrate an example here with the case of $n = 3$.

Example 1. *An* ordered number partition of n *is a finite sequence of positive integers summing to* n*. The following are the ordered number partitions of* 3*:*

1. *1,1,1*
2. *1,2*
3. *2,1*
4. *3*

We associate each of these with a certain type of valuation profile. The ordered partition $1, 1, 1$ *is associated with the type of valuation profile considered in the body of this paper—one in which all valuations are distinct. The partition* $1, 2$ *is associated with a single individual with the highest valuation, and a tie for the lowest. Likewise,* $2, 1$ *represents a tie for the highest valuation, and a single bidder with the lowest. Finally,* 3 *is the situation in which all bids are tied.*

An ordinal allocation rule in this context associates with each ordered partition a profile of probabilities. Let us demonstrate by example. Suppose we associate with the ordered partition $1, 1, 1$*, the profile* $(.6, .3, .1)$*. This profile of probabilities means that the highest bidder gets the good with* $.6$ *chance, the middle with* $.2$ *and the lowest with* $.1$*. Importantly, this profile is weakly decreasing and sums to one.*

Now, let us associate with the ordered partition $1, 2$ *the profile of probabilities given by* $(.5, .25)$*. We interpret this as stating that the highest bidder achieves the good with probability* $.5$*, and the lowest two with probability* $.25$ *each. Critically,* $.25 \in [.1, .3]$*. This constraint ensures that a bidder's probability of winning the good is weakly increasing in their announcement (As they cross from the lowest bidder, receiving the good with probability* $.1$*, to tying, they receive a probability of* $.25$*. In becoming the middle bidder, they receive a probability of* $.3$*).*

Likewise, we can associate with the ordered partition $2, 1$ *the profile of probabilities* $(.45, .1)$*. Thus, the highest two bidders each have a probability of* $.45$ *of winning the good, and the lowest bidder a probability of* $.1$*. Here, it is critical that* $.45 \in [.3, .6]$*, so that a bidder raising their valuation from the middle valuation to the highest maintains the appropriate monotonicity constraint.*

Finally, of necessity, the ordered partition 3 *is associated with the profile* $(1/3)$*, where each individual gets a* $1/3$ *chance of obtaining the good. Again, observe that* $1/3 \in [.1, .45]$*, and* $1/3 \in [.25, .5]$*, ensuring that the monotonicity implied by incentive compatibility is satisfied.*

In general, these allocation rules are characterized by a collection of linear inequalities of this type (to ensure weak monotonicity). Transfers are always then uniquely determined via the normalization constraint. Unfortunately, we do not have a characterization of the type of Theorem 1, though one is probably possible. We do mention that the random tie-breaking rule, whereby when a set S *of agents tie, then they each receive the good with the probability* $avg_{i \in S}(p_i)$*, is characterized by a strengthening of the ordinality condition:* $p_i(\theta) = p_i(\phi(\theta))$ *for all values* θ *where* ϕ *is strictly increasing.*

A classical characterization of incentive compatible allocation rules absent efficiency is due to Roberts (1979) (there are corrections due to Carbajal, McLennan, and Tourky (2013); Vohra (2011). These results do not apply to our case as our space of types is one-dimensional (and thus lacks the necessary richness conditions to apply those results).

Other related characterizations of Vickrey style rules (for multiple units) are given by Ashlagi and Serizawa (2012); Green and Laffont (1977); Chew and Serizawa (2007); Atlamaz and Yengin (2008); Yengin (2012).

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