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DOCTORAL THESIS

Essays in Theoretical Political Economy

Author:
Mark WINTER

Supervisor:
Dr. Sophie BADE
Dr. Michael NAEF
Dr. Philip NEARY

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Declaration of Authorship

I, Mark WINTER, declare that this thesis titled, "Essays in Theoretical Political Economy " and the work presented in it are my own. I confirm that:

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""Democracy is the worst form of government, except for all those other forms that have been tried from time to time...." "

Winston Churchill

ROYAL HOLLOWAY UNIVERSITY OF LONDON

Abstract

Faculty of Management and Economics

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Essays in Theoretical Political Economy

by Mark WINTER

Chapter One

This chapter reports the results of an in-lab experiment to test two design features of two round elections. (1) uncertainty about round two participants and (2) the flexible threshold for first round victory. The chapter finds experimental evidence of the existence of Duverger's Law equilibrium (predominance of two candidates) and the sincere equilibrium in two round divided majority election games with uncertainty and flexible thresholds. The effect of flexible thresholds and uncertainty on the occurrence of these two equilibrium are tested. The chapter finds that the flexible threshold has a greater effect than uncertainty when it comes to Duverger's Law type effects. It is found that in general the effects of threshold and uncertainty are complementary but with a large interaction effect. The Duverger Law equilibrium exists and is more stable than alternative equilibria including the sincere equilibrium. Elections with a lower threshold and or uncertainty have higher levels of minority candidate victory. While coordination levels are in general high enough to mitigate these effects, the lower threshold and uncertainty reduce the room for error and magnify the effect of error. The secondary findings of this chapter are the existence of non-symmetric equilibrium in the experiment that voters do not take dominated strategies in the experiment under these new parameters of uncertainty and a lower threshold, therefore supporting the assumptions necessary for the other results. The second if these is not

claimed as a contribution to the literature as others have made the same findings though not under these specific election rules. The non-symmetric equilibrium suggest further study as it questions the assumption of symmetric voting in equilibrium.

Chapter Two

This chapter presents a three candidate election model with the key feature of two 'large' candidates and one 'small' candidate. This design mimics many political systems for example the US and Uk. with two large parties and a number of small parties. under these conditions two voting rules, Plurality and Instant run-off, are analysed. The analysis starts with the sincere equilibrium. It is found to exist for all voter distributions (under the basic conditions of two 'large' candidates) in this model for the instant run-off rule. The plurality rule is found to only have a sincere equilibrium under a set of restrictions. As such the instant run-off is a better system for those that want incentivise sincere voting. It is also found that when sincere voting gives different outcomes for the two rules the instant run-off is preferred by a strict majority of voters. The second part of this chapter is the main contribution to the literature. The application of level-k thinking to the model described above with the assumption of bounded rationality. These finds also support the argument that the instant run-off is a preferable system. It always gives and outcome that is weakly preferred by a strict majority of voters and when the two voting rules give the same outcome the instant run-off reaches this with a weakly lower level of necessary thinking. The sincere equilibrium is found to be the only equilibrium at higher levels of thinking for instant run-off while plurality has 3 possible equilibrium based on the voter distribution: the sincere equilibrium, the Duverger's Law equilibrium and the non-pivotal equilibrium. The chapter gives supporting evidence for 'wasted' votes that go to candidates that can not win the election coming from voters who are bounded by their rational and have not worked out that they should be strategic.

Chapter Three

The final chapter links the first two together somewhat but primarily aims to give supporting evidence for the assumptions made in the first two chapters. It presents a three candidate election with entry. Two candidates have a first move advantage and a third candidate enters after them. This simple candidate model is analysed for the 4 rules used in the first two chapters. The spatial equilibria that are found under the four rules fit to the assumptions about candidate policy position in the other two chapters with varying success. The plurality rule that is investigated in chapter two fits well and the entry model creates two 'large' candidates who get the first mover advantage. This suggest one rationale for the set up of chapter two is a first mover advantage that creates the 'large' candidates. This rationale is less clear for the instant run-off where the results of the entry game is for the entrant to copy one of the first movers and this creates two 'small' candidates. The two round elections with and without flexible threshold do not fit well into the equilibrium of the entry game. The interesting thing about the two round election with and without a flexible threshold though is that they fit very well with such a game where an entrant makes a small error in their strategy. So the entry game in disequilibrium could explain in part the divided majority set up. This raises questions in the divided majority especially with a flexible threshold; why does the minority candidate not get slightly more moderate to try to win enough votes to win as a minority?

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Dedicated to my Family and Friends that have supported me over eight years of study and work to get to this point.

Chapter 1

Does the Runoff Election Solve the Coordination Problem?

1.1 Introduction

Run-off elections with majority requirement are the most prevalent system of presidential elections in the world. Of the 111 countries that directly elect a president, 87 use a system with some form of run-off. Of these elections, 78 are absolute majority rules without additional conditions[1] Understanding how this election rule works and how voters respond to this rule is therefore essential to a wide range of countries and their people. Such voting rules are also applicable to committee decision making at board level, panel interviews, and any other situation where a group must make a collective decision.

The multi-candidate run-off election can require up to two rounds of voting. In round one, voters cast a vote for a single candidate. The candidate with the most votes wins if they pass a predefined share of the vote. This share is the victory threshold. When no candidate passes the victory threshold the election moves to round two. The two candidates with the most round one votes contest this run-off election. Due to the nature of a two-candidate election, the winner of round two will have a majority of the votes in round two[2].

With the range of applications and the prevalence at a presidential level, it is unsurprising that literature in this area has expanded in recent years. Theoretical work is the area that has

[1]Information compiled with additional research from IDEA electoral system design database. For a summary of these results and details on other systems worldwide, see appendix A.

[2]This ignores 50/50 ties, I discuss this in the model section.

expanded most with work by, Bouton (2013), Bouton and Gratton (2015), Bordignon, Nannicini, and Tabellini (2016, 2017), Callander (2005) Haan and Volkerink (2001) among others. I discuss these contributions and other related work in section 1.2.1 and 1.2.2. From an experimental perspective, this includes work by Morton and Rietz (2007), Blais, Laslier, Laurent, Sauger, and Van der Straeten (2007), and Van der Straeten, Laslier, Sauger, and Blais (2010). This literature is expanded on in 1.2.3.

The absolute majority run-off rule for better or worse stops the victory of minority candidates. As Morton and Rietz (2007) among others have argued the historic motivation of the run-off rule was in part to disenfranchise ethnic minorities in the southern states of the USA. When used instead to defeat political extremes it is framed more positively, for example in the French presidential elections since 1965[3]. This type of run-off election is also predicted to be more conducive to preference revelation compared to one round elections. Allowing for sincere revelation in both rounds without the risk of an upset victory where a Condorcet Loser[4] wins the election. Morton and Rietz (2007) finds that the run-off election does lead to higher sincere voting in round one followed by coordination to defeat the minority candidate in round two.

The analysis of the run-off election however is incomplete. This chapter will aim to address some of the unanswered questions in this area. Firstly, by running two election rounds uncertainty seeps in. Voting in round one does not necessitate voting in round two. Equally, a voter can vote in round two without voting in round one. There is then uncertainty over who votes in each round. If a Condorcet loser (minority candidate) can become a Condorcet winner in round two, how will players in round one act? Will they continue to treat the minority candidate as though it cannot win?

Secondly, the majority requirement is just one feasible threshold level. While previous analysis has compared plurality and the majority run-off, it has ignored the pros and cons of the other alternatives. Specifically a threshold between plurality and absolute majority runoff.

Bouton (2013) have modeled these questions theoretically but a gap exists regarding experimental evidence to support their findings, and other questions around uncertainty and threshold

[3] Specifically the second round defeats of Jean-Marie Le Pen and Marine Le Pen in 2002 and 2017 respectively.
[4] A candidate that would lose a head to head election against all other candidates

level in runoff elections.

In considering alternative thresholds, it is necessary to understand why an election designer might wish to change the threshold. At its heart, there is a trade-off when setting a threshold between cost and upset risk. The lowest possible threshold (plurality[5]) never requires a second round of voting and is therefore cheaper to run but has a large round one upset risk. A minority candidate must only exceed the vote tally of each of the majority candidates to win the election. Super majorities[6](any threshold above 50%) and the majority threshold often require a second round[7] and therefore cost more to run but eliminate round one upset risk. One objective of this chapter is to test if this trade-off exists for a threshold between plurality and majority. The uncertainty over populations is important not in understanding the actions/outcome in round two but in understanding how this affects decision making in round one.

The certainty in the existing analysis on run-off elections is pivotal to the results regarding sincere revelation in round one and coordination in round two. How voters respond to this uncertainty is important in understanding runoff elections. Will voters respond to uncertainty by entrenching their vote, aiming to remove uncertainty in the area they can control, or will they respond to uncertainty by trying to be strategic and insincere?

To answer these questions I present in section 1.3 a run-off election model with variable thresholds and uncertainty. This model follows the standard divided majority literature with three voter types and three candidates. Two types represent a divided majority while the third represents a unified minority. The majority both prefer candidate A and B to candidate C but are divided over their preference between A and B, while the minority prefer candidate C to A and B and are indifferent over these two. There is then a flexible threshold for round one.

Following the model section 1.5 details the experiment. The experiment will test the absolute majority threshold and a double threshold of 40% threshold and absolute majority threshold. In the second case a candidate wins if they get above 40% of the vote and no other candidate gets

[5]Under plurality, the most votes wins without considering a threshold. This is equivalent to there being no threshold. In practice with 3 candidates this is any threshold from 0 to 1/3

[6]There can only be a super majority in round one, in round two it must revert to plurality or the election result has an existence problem.

[7]Of the 10 French elections run with a majority threshold all have required the second round.

40% but if two candidates get above 40% one can still win if they get an absolute majority.

The main departure from Bouton (2013) is that where he uses a Poisson distribution to calculate group size in each round I fix the group size at 13 or 14 depending on treatment. This allows the experiment to run with a fixed number of real participants. The model then randomly selects the voters in round two to generate uncertainty. There will be five participants in round two under uncertainty.

A second change that needs to be discussed is the decision to exclude abstention as an option in the main election. As abstention is the primary motivator for uncertainty over who participates in each round it was necessary to control this to effectively measure the effect of uncertainty. The experiment exogenously creates uncertainty in specific treatments to compare them to treatments without uncertainty. If the experiment allowed abstention, this clear distinction would be removed. Section 1.2.4 includes a discussion of the possible causes of abstention.

I find that the changes to the generation of uncertainty and the exclusion of abstentions does not effect the predictions of Bouton (2013). The theoretical results for this model are presented with their proofs in section 1.4 and are in line with the findings of Bouton (2013).

The theoretical section of the chapter finds the same results as Bouton (2013). Both the Duverger Law equilibrium and the sincere equilibrium should occur in all 6 treatments in this experiment. This finding is supported by the results in the chapter. Using the same method as Forsythe, Myerson, Rietz, and Weber (1993) to compare a Duverger's Law type effect against random voting the chapter finds that a Duvergers Law type effect occurs in 32.10% of all elections in the experiment. This effect is found to occur most often in treatments $FC6$, $FU6$ and $FU5$. Therefore the flexible threshold seems to increase the occurrence of the Duverger's Law equilibrium. This supports the conjecture in this chapter that a lower threshold creates a greater cost to no reaching the Duverger's Law equilibrium and therefore incentivises participants to try to reach it.

When we test the effect of the flexible threshold on the occurrence of Duverger's Law equilibrium we find the effect to be a 27.8% increase in occurrence. This effect goes up when the interaction effect with uncertainty is included. Uncertainty also has a significant effect of the occurrence of Duverger's Law equilibrium but only when the interaction effect is taken into account. This suggest that of the two variables changing the threshold has a dominant effect. The effect of uncertainty, 17.3%, is smaller than the interaction effect, 33.3%, which is negative. So the flexible threshold and uncertainty effect the Duverger's Law equilibrium in the same direction. Both variables have a positive effect on the occurrence of the Duverger's Law equilibrium.

The second Symmetric Perfect Bayesian equilibrium that is predicted by the theory to exist is the sincere equilibrium. The findings of the experiment support this prediction. Using the same method as Forsythe, Myerson, Rietz, and Weber (1993) but now to compare random voting with sincere voting it is found that the sincere equilibrium occurs in 9.26% of elections. This is less than the Duverger's Law equilibrium but still a significant percentage of the time. The effect of threshold and uncertainty on the existence of the sincere equilibrium is also significant but smaller. The effect of the flexible threshold on the occurrence of the sincere equilibrium is 14.8% which rises to 23.5% when the interaction effect is considered. Unlike the Duverger's Law equilibrium uncertainty does has a significant effect, 11.1% without the interaction effect. The effect of uncertainty rises to 12% with the interaction effect, 17.3%, which again is negative. So the flexible threshold and uncertainty effect the sincere equilibrium in the same direction. Both variables have a negative effect on the sincere equilibrium

It is hard to test the third of these claims without data on plurality for compassion so I will not make any claims for that prediction. The findings of the chapter support both prediction one and two. The Duverger equilibria is the most prevalent outcome in the experiment and exists for all treatments. By design of the experiment, the sincere equilibrium is predicted to exist in all treatments. As predicted, the sincere equilibrium does always exist though it is less prevalent than the Duverger equilibria. Finally, the expectation that the sincere equilibrium is less likely with a lower threshold and uncertainty is found in the results of the experiment.

The benefit of testing the theory using an experiment is we can control for factors that it is impossible to control for in the real world including true voter preference, which is not fully reported by analysis of voting decisions. It is also possible to test multiple thresholds where in the real world any given election only has one. Finally, it is possible to decide to introduce uncertainty where in the real world data the level of uncertainty or its existence is not known and cannot be isolated.

The experiment has been able to report the effect of an intermediate threshold. This helps in analysing the trade-off inherent in selecting a threshold. Its limit is that there are still only three thresholds considered in the literature and this makes finding an 'ideal' threshold if one exists still very difficult. Additionally the experiment makes no claims to predict difference in populations between rounds just report the effect the existence of such difference can cause.

1.2 Literature Review

The review of the literature is broken into 4 parts.

1. An overview of key theoretical topics
2. The theoretical work on the run-off system
3. The experimental work on the divided majority
4. A brief review of the abstention literature

1.2.1 Theoretical Frame work

This chapter is focused on the voting decisions of a population of voters. One key theory in this literature is the Duverger law. Initially proposed by Duverger (1959) it states that plurality rule elections within single winner takes all elections tend to favour two party systems and the Duverger Law equilibrium is a voting equilibrium where only two candidates receive a positive vote share in the election. For a mathematical proof of Duverger's Law see Palfrey (1988). Riker (1982) extends the Duverger law and defines Duverger's hypothesis. Duverger's hypothesis states that double ballot majority systems (hereafter called the run-off system) and Proportional Representation tend to favour multiparty politics. This hypothesis has been called into question often, in the first instance by Riker himself who stated that in its deterministic form it can be abandoned. Callander (2005) considers two party and multiparty systems and finds that two party systems can be stable under run-off systems. This chapter fits into this literature by testing the Duverger Hypothesis equilibrium under an experimental framework and calling it into question in the run-off system under uncertainty and with lower thresholds.

In understanding the actions of voters, it is necessary to consider the equilibria that can exist within voting models. The first stage of such an analysis is the work by Nash (1950, 1951) and the formulation of the Nash equilibrium. A strategy profile is a Nash equilibrium if no player can do better by unilaterally changing his or her strategy fixing the strategies of other players. This equilibrium concept is insufficient for the work in this chapter as it is with other such group voting models. With just a small number of voters it is possible for any candidate winning to be a Nash equilibrium including Condorcet losers[8] and candidates that are every voter's strict worst option. Besley and Coate (1997) apply elimination of weakly dominated strategies to the plurality system. In doing so the strategy of voting for the least preferred candidate is eliminated. In two candidate elections this just leaves sincere voting. This chapter will use the same process. An extension of this work conducted by Dhillon and Lockwood (2004) gave the conditions under which a plurality voting game is dominance-solvable. This work refines the set of strategies in the plurality election system but leaves questions about the run-off election system and what these refinements mean for the run-off system. The next section explores the work done in the theoretical literature on the run-off system.

1.2.2 Run-off literature

In terms of this chapter the two most important papers in this literature are Bouton (2013) and Bouton and Gratton (2015). These papers present the two key theoretical questions that this chapter investigates in its experiment. They prove that the Duverger equilibrium must exist for any threshold level and with uncertainty and give the conditions for the existence of the sincere equilibrium. The hypotheses of Bouton (2013) and Bouton and Gratton (2015) are supported by the results of the experiment conducted under similar conditions in this chapter.

Haan and Volkerink (2001) shows that with run-off elections parties take median voter positions in a downs-Hotelling environment[9]. As such the run-off rule induces centrist politics. This supports the idea that the majority will win in the run-off election. It does not address the divided nature of the majority. Bordignon, Nannicini, and Tabellini (2016, 2017) support the Duverger hypothesis that there will be more viable candidates in a run-off election. They allow for party formation with many parties and find that while there are more parties in the run-off election the power of the extremes is reduced as they have less bargaining power in forming

[8] A candidate that would lose a head to head election against all other candidates
[9] for details on the downs-hotelling model see Downs (1957), Hotelling (1929)

coalitions. They also find that the moderate parties prefer to run alone. This supports the findings that run-off rules see coordination between voters instead of between parties. Additionally, the failure of the extremist supports the idea that the minority is less likely to win in run-off elections.

1.2.3 Divided Majority literature

There is limited real world capacity to test the concepts proposed in these papers as there is limited scope for changing election rules and too many unknowns. A solution to this is to run controlled experiments. This method has been applied to many of the theories above in different contexts. This section will focus solely on the work done to understand the divided majority problem.

Myerson and Weber (1993) presents the structure of the divided majority in the first example of their paper. This paper framed the questions regarding the divided majority and gave a clear structure to test the question experimentally though it did not run an experiment.

The first papers to use this theoretical work were Forsythe, Myerson, Rietz, and Weber (1993, 1996) in a set of experiments that formalized the divided majority experimental design. These papers gave evidence for pre-election polling and election histories as coordination methods for the divided majority. They showed that the divided majority problem can be reduced by polling and election histories but that the Condorcet loser still wins a significant number of elections. They find that the outcome of coordination is the emergence of the Duverger's Law equilibrium where the divided majority coordinate on a single candidate. Following this work Reitz, Myerson, and Weber (1998) used the same structure to test the signaling effect of financing. Their findings were similar, showing financing reducing the Condorcet loser winning occurrence by a similar amount to polls. Using the same design Gerber, Morton, and Rietz (1998) considers multimember districts. Their findings support the idea that proportional representation is preferable for minority candidates and that allowing for cumulative voting benefits the minority candidate. As this election is not a winner takes all election it applies less well to the work in this chapter. More recent work on the divided majority by Bouton, Castanheira, and Llorente-Saguer (2016,

2017) have also found a Duverger's Law type effect under the conditions of aggregate uncertainty for plurality and approval voting.

The first attempt to understand the run-off election using the divided majority was conducted by Morton and Rietz (2007). They showed that the run-off election system was very successful in coordinating the voters of the divided majority. The level of sincere voting increases by a large amount and the probability that the Condorcet loser wins is greatly reduced. The candidate with the most votes in round one, generally the minority candidate, rarely won in the second round. This contradicts real world evidence found in Bullock and Johnson (1992). They find that primary leaders usually win in run-offs and that larger primary leads usually lead to stronger wins in the run-off. They call this the "leader-loses" myth. This contradiction between the theory and experiment and the real world data can be explained in a number of ways. Most obviously that Bullock and Johnson (1992) elections are not divided majorities in the same way that Morton and Rietz (2007) experiments are. An alternative though is that there is some level of coordination in run-off elections. This chapter helps to explain the contradiction by considering the effect of uncertainty and suggests this could be one reason why there is this difference.

Alternative methods of studying the divided majority have been presented in papers by Blais, Laslier, Laurent, Sauger, and Van der Straeten (2007) Blais, Labbé-St-Vincent, Laslier, Sauger, and Van der Straeten (2011) Van der Straeten, Laslier, Sauger, and Blais (2010) using the downs-Hotelling model, with more than 3 candidates. In using alternative structures my work does not fit into these sections as clearly. Blais, Labbé-St-Vincent, Laslier, Sauger, and Van der Straeten (2011) and Blais, Laslier, Laurent, Sauger, and Van der Straeten (2007) find that there is less difference than expected between plurality and run-off. Though they do support the idea that run-off leads to 3 viable candidates it does not appear to help extremists. As they give symmetrical voter distribution they make no statement about a divided majority as there is no clear majority group. Van der Straeten, Laslier, Sauger, and Blais (2010) add to these findings to show that under run-off and instant run-off election rules voters take heuristic voting methods as the strategic decisions are found to be harder. Then Bouton, Castanheira, and Llorente-Saguer (2016) uses a state of the world signal to compare the two systems to compare information aggregation.

1.2.4 Abstention Literature

A central question in this chapter is how voters in round one respond to uncertainty over participation in round two. For such a question to arise it must be the case that voters will abstain in one or both elections. There are many reasons this happens and this is by no means an exhaustive list. The first reason to consider is an external shock that occurs on one of the election days. For example, Gomez, Hansford, and Krause (2007) investigate the effect of weather on voting level and find that rain reduces participation in presidential elections at 1% per inch of rain. Snow shows a smaller but equally significant effect. Gerber, Green, and Larimer (2008) investigated the importance of social pressure on voter turnout and found a profound effect on turnout when participation is public. This social pressure can change between the rounds and based on the perceived importance of the round. Myatt (2015) makes similar theoretical predictions about the effect of election importance on turn out. The importance of each round can vary based on perceived victory margin and the likelihood of selecting a winner. The importance of perceived victory margin has been well documented in among others Shachar and Nalebuff (1999). Finally, Valentino, Brader, Groenendyk, Gregorowicz, and Hutchings (2011) presents an analysis of the effect of emotion on voting decision using experiments, national surveys, and pooled data from 1980-2004 to argue that anger is a stronger motivator than anxiety or enthusiasm. Obviously, people's emotions can vary between election days.

This chapter does not make a claim to predict abstention or explain it but uses its existence, as seen in one way or another in these papers, to motivate uncertainty that is implied by abstention in run-off elections.

1.3 Model

The model in this paper is an altered version of the model in Bouton (2013) of two round elections with runoff. In Bouton (2013) the electorate,N, is distributed according to a Poisson distribution of mean $n : \ N \sim P(n)$. This distribution lends itself to large electorates as shown by Myerson (1998, 2000). These large electorates do not lend themselves to laboratory experiments. Instead the model will consider a fixed electorate. The details of this electorate are defined later in this section. The subsequent equilibria are solved in 1.4.

1.3.1 Players, Candidates, and Preferences

The model here is an extensive form game with two voting rounds. Voting in each round is simultaneous. The finite set N represent the players, hereafter called voters. The number of individual voters, i, will equal n.

$$i \in N = \{1, 2, ...n\}$$

The election has three candidates and the set of candidates is Ω. An individual candidate is ω.

$$\omega \in \Omega = \{A, B, C\}$$

There are a set of types, T[10]. A type defines the utility, U, a voter gets from the outcome, O, which is the candidate that wins. The focus of this chapter is the divided majority. As such, the voter's type , t, will be one of three specific utilities. $U(O|t)$ equals the utility a voter of type t gets from an outcome O. There are three outcomes, $O = (A),\ (B)\ or (C)$

$$t \in T = \{\alpha, \beta, \gamma\}$$
$$\alpha \ : \ U(A|\alpha) > U(B|\alpha) > U(C|\alpha)$$
$$\beta \ : \ U(B|\beta) > U(A|\beta) > U(C|\beta)$$
$$\gamma \ : \ U(C|\gamma) > U(A|\gamma) = U(B|\gamma)$$

Value t_i, is a player's type. The number η_t, represents the number of voters of a given type ,t, in the electorate. The types will be distributed in a similar way to the first example from Myerson and Weber (1993)[11]. The γ type voter will represent a strict minority of the electorate. The α and β type voters will represent an equal 50/50 split of the remaining voters and collectively represent a majority. Finally the γ type voters will represent a larger proportion of the electorate than α and β type voters do on their own. This information is common knowledge.

$$t_i \in T$$
$$\eta_t = (\#i|t_i = t)$$
$$\eta_\gamma < \tfrac{n}{2}$$
$$\eta_\alpha = \eta_\beta$$
$$\eta_\gamma > \tfrac{n}{3} > \eta_\alpha, \eta_\beta$$

[10] These are the same types as in the stylized model in Bouton and Gratton (2015)

[11] The example in their paper uses proportions 0.3,0.3,0.4, the theory here is more generalised

The election consists of two rounds $r \in R = \{1,2\}$.

1.3.2 Histories

H is a set of finite sequences. Each member of H is a history and is terminal if a candidate wins the election; the set of terminal histories is denoted Z. The set of actions available at some non-terminal history h is denoted $A(h)$.

The function P assigns to each non-terminal history a subset of players from $N \cup \{c\}$ and a subset of candidates from the set Ω. $P(h)$ is then the set of players who take an action after h. If $P(h) = c$ then chance determines the action.

All players participate in round one. All candidates are on the ballot. A voter then casts a first round vote, v_i^1, for one of the three candidates. The set of actions are a vector of first round votes,V^1.

$$P(\emptyset) = N\,, \quad A_i(\emptyset) = \Omega \;\forall\, i \in N$$
$$a_i(\emptyset) = v_i^1$$
$$V^1 = (v_1^1, v_2^1, ...v_n^1)$$

At any history $h = (V^1)$, there is a threshold $\tau \in [0,1]$. If in round one exactly one candidate gets more than $\tau * n$ votes, the outcome O of the election is victory for that candidate. The history is terminal. If two candidates are above the threshold and one candidate has a strict majority the outcome is victory for the candidate with a strict majority.

If no candidate wins in round one the game continues. For all histories $h = (V^1) \in H \backslash Z$; the player is $P(h) = c$ and a function f_c draws a set of voters K, of size $k \leq n$ and a linear order of set Ω from a uniform distribution on all such pairs.

For any non-terminal history, $h = (V^1, f_c)$ (after any first round vote count that does not lead to a terminal history and after the function f_c), $P(h)$ assigns the set K to vote and selects two candidates ω' and ω'' to be on the ballot.

$$P(V^1, f_c) = K\,, \quad A_i(V^1, f_c) = \{\omega', \omega''\} \;\forall\, i \in\; K$$

ω' and ω'' are chosen such that neither is the unique candidate with the lowest vote share in the first round. When two candidates have the equal lowest vote share, ω' will be neither of these candidates. Then ω'' is the higher ranked candidate, among the candidates tied with minimal vote share, according to the random draw generated by f_c[12].

In round two, the outcome, O, of the election is that the candidate with the most votes wins the election. If there is a tie, then the candidate ranked higher by f_c wins.

1.3.3 Strategy

A voter's mixed strategy, σ_i, is a probability measure over the set of player i's pure strategies, for each non-terminal history for voters when $i \in P(h)$.

$$\sigma_i : \; T \times \{H \backslash Z\} \rightarrow \Delta\{A, B, C\} \;\; s.t. \;\; \begin{cases} \sigma_i(t, (\emptyset)) \rightarrow \Delta\{A, B, C\} \\ \sigma_i(t, (V^1, f_c)) \rightarrow \Delta\{\omega', \omega''\} \end{cases} \tag{1.1}$$

1.3.4 Equilibrium Concept and Refinements

The equilibrium concept used is a symmetric Perfect Bayesian Equilibrium with elimination of weakly dominated strategies.

The beliefs of the voter here are over the probability attributed to each possible voting population in round two. Voters are selected at random as defined above so with a known n and k the probability of different voting populations can easily be calculated[13].

The strategies must be sequentially rational (optimal in expectations given the beliefs) and consistent (updated according to the strategies and Bayes' rule on any path that has a positive

[12] This chapter excludes a three-way tie in round one as in no treatment of the experiment is $\frac{n}{3}$ a whole number so a three-way tie is not possible. This condition is not necessary for the results.

[13] An example of this calculation for the values of n and k used in the experiment are given in appendix B

probability).

The symmetry refinement means that players of the same type will choose the same strategies in equilibrium. Then equation 1.1 is defined by the player's type:

$$\sigma : T \times (H \backslash Z) \rightarrow \Delta\{A, B, C\} \;\; s.t. \;\; \begin{cases} \sigma : T \times (\emptyset) \rightarrow \Delta\{A, B, C\} \\ \sigma : T \times (V^1, f_c) \rightarrow \Delta\{\omega', \omega''\} \end{cases}$$

The third refinement will be the indifference refinement: that when a voter is indifferent over two actions their strategy will be to mix these strategies equally e.g. if a voter is indifferent between candidates A and B their strategy will be $(0.5, 0.5, 0)$. From this sincere voting can be defined.

Definition 1.1. A voting strategy can be defined as sincere at any given non-terminal history. A strategy of sincere voting is pure strategy (e.g. $(1, 0, 0)$) voting for the candidate that, if they win, maximises utility. When a voter has multiple candidates that maximise their utility a sincere voting strategy is a mixed strategy where all candidates that maximise their utility have an equal probability of being voted for (e.g. $(0.5, 0.5, 0)$).

Two equilibria of specific interest in this chapter are the sincere voting equilibrium and the Duverger Law Equilibrium.

Definition 1.2. A sincere voting equilibrium round is a round of voting where the strategy of sincere voting is an equilibrium for all voters in that round.

Definition 1.3. The Duverger Law Equilibrium round is an equilibrium in which only two candidates get a positive proportion of the votes cast in a given round.

As weakly dominated strategies are eliminated, it is only necessary to consider cases when a voter can change the outcome of an election. This will be defined as a pivotal vote.

Definition 1.4. A voter is pivotal if changing the probability with which they vote for the set of candidates changes the probability of an outcome in the game. A voter can be pivotal in changing the candidate that wins the election in either round one or round two. A voter can also be pivotal if they change one or both candidates that progress to round two.

1.4 Theoretical Results

1.4.1 Second Round equilibrium

Theorem 1.1. *All subgames that start at history $h \in H \backslash (\emptyset)$ have a unique equilibrium*

In round two there are two candidates and the election is a winner takes all plurality. It has been shown in Besley and Coate (1997) that when weakly dominated strategies are eliminated the only strategy that remains for all players is the sincere voting strategy. This is a pure strategy for the candidate that maximises their utility of the remaining candidates. For voter γ when A and B remain due to the indifference refinement their strategy is $(0.5, 0.5, 0)$.

At the history $h = (V^1)$ where $h \in H \backslash Z$ the function f_c randomly picks the set of voters, K, to participate in the next round. These voters all vote sincerely therefore the winner depends only on $P(V^1, f_c)$. If the two candidate that remain are ω and ω' then ω wins if the function f_c selects more voters that prefer ω to ω' to be part of the set K than those who prefer ω' to ω. The probability that a given distribution of types is drawn is known. As such the probability that an outcome occurs is known.

After the voting in round one either one of three outcomes occur, $O = (A), (B), (C)$ or one of three second round sub games occur $(AB), (AC), (BC)$ and each has a unique equilibrium[14]. For the first three there is a known outcome and for the last three a lottery over two winners with known probabilities.

The table below gives the strategy taken by each type in the unique symmetric perfect Bayesian equilibrium with elimination of weakly dominated strategies for each possible round two.

TABLE 1.1: Voting strategy by type in the unique symmetrical Perfect Bayesian equilibrium in round two

Type, t,	Second round candidates		
	A against B (AB)	A against C (AC)	B against C (BC)
α	$(1, 0, 0)$	$(1, 0, 0)$	$(0, 1, 0)$
β	$(0, 1, 0)$	$(1, 0, 0)$	$(0, 1, 0)$
γ	$(0.5, 0.5, 0)$	$(0, 0, 1)$	$(0, 0, 1)$

[14] (A), (B) and (C) are all outcomes at terminal histories and have no subgame

As each subgame that follow on from the $(\emptyset)$ history has a unique equilibrium the rest of the results will focus on the normal form game that arises from the first round choices and the equilibrium that follow them.

1.4.2 The equilibrium strategy for γ type voters

Theorem 1.2. *At $h = (\emptyset)$ all γ type voter's have a unique equilibrium strategy of* $(0, 0, 1)$.

Proof: In all cases a γ type voter is pivotal; voting C is a best response. With elimination of weakly dominated strategies $(0, 0, 1)$ is the only strategy that remains. All pivotal cases are now presented[15].

Case 1: When two candidates, ω and ω', are above the threshold the election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega'\}$. Both candidates have a strictly positive probability of winning[16], $O = (\omega)$ or $O = (\omega')$. A voter is pivotal when a deviation takes a candidate ω' below the threshold leading to candidate ω winning, $O = (\omega)$ in round one.

1A) When $\omega = C$ and $\omega' = A, B$ the outcome with the deviation, $O = (\omega)$, gives a higher utility than without deviation where the game moves to the sub game $h = (V^1, f_c)$. In a positive proportion of the sub games the outcome is $O = (\omega')$ and this outcome is worse than the outcome $O = (\omega)$. When $\omega = C$ the best response of the γ type voter will be to vote for ω.

1B) When $\omega = A, B$ and $\omega' = C$ the outcome with the deviation, $O = (\omega)$, gives a lower utility than without deviation where the game moves to the sub game $h = (V^1, f_c)$. In a positive proportion of the sub games the outcome is $O = (\omega')$ and this outcome is better than the outcome $O = (\omega)$. When $\omega = A, B$ and $\omega' = C$ the best response of the γ type voter will be to not vote for ω.

1C) When $\omega, \omega' \neq C$. The outcome with deviation, $O = (\omega)$, gives the same utility as the outcome without the deviation, where the game moves to the sub game $h = (V^1, f_c)$. When $\omega = A, B$ and $\omega' \neq C$ the γ type voter is indifferent between all actions. In summary of case 1

[15] When pivotal a voter can choose to deviate. It is assumed that candidate ω' is the candidate the voter deviates away from for readability and ω is the candidate deviated to

[16] unless $K = N$ in round two then candidate A or candidate B win with certainty against candidate C who loses with certainty

the best response of the γ type voter is to change their vote to ω when $\omega = C$ and to not change their vote toω when $\omega = A, B$.

Case 2: When one candidate, ω', is above the threshold the outcome is victory for this candidate; $O = (\omega')$. A voter is pivotal when a deviation takes ω' below the threshold and the election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega'\}$.

2A) When $\omega = C$ and $\omega' = A, B$ deviation leads to subgame $h = (V^1, f_c)$, there are are a positive proportion of subgames where ω and ω' win the election. This gives a higher utility than the outcome without the deviation, $O = (\omega')$. When $\omega = C$ the best response of the γ type voter is to vote for ω.

2B) When $\omega = A, B$ and $\omega' = C$ deviation leads to subgame $h = (V^1, f_c)$, there are are a positive proportion of subgames where ω and ω' win the election. This gives a lower utility than the outcome without the deviation, $O = (\omega')$. When $\omega = A, B$ and $\omega = C$ the best response of the γ type voter will be to not vote for ω.

2C) When $\omega, \omega' \neq C$ deviation leads to subgame $h = (V^1, f_c)$, there are are positive a proportion of subgames where ω and ω' win the election. This gives the same utility as the outcome without deviation, (ω'). When $\omega = A, B$ and $\omega' \neq C$ the γ type voter is indifferent between all actions. In summary of case 2 the best response of the γ type voter is to change their vote to ω when $\omega = C$ and not change their vote to ω when $\omega = A, B$.

Case 3: As with case 2; one candidate ω' is above the threshold. A voter is pivotal if can take another candidate, ω, above the threshold. In such a case the deviation to ω takes that candidate above the threshold and the same subgame as case 2 occurs, $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega'\}$. Without deviation the same outcome as case 2 occurs therefore the same results as before follow[17].

Case 4: When no candidates are above the threshold the top two candidates, ω, ω' move to the second round and this leads to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega'\}$. A voter is pivotal when a deviation takes a candidate, ω, above the threshold leading to ω winning,

[17] A deviation away from ω' to ω can take ω' below the threshold and take ω above the threshold at the same time, in this case the outcome changes from (ω') to (ω) and if $(\omega\omega')$ is preferred to (ω') then (ω) is preferred to (ω'). The same results then follow.

$O = (\omega)$. This has the same results as case 1.

Case 5: When deviation away from ω' puts that candidate last the election leads to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega''\}$ where ω'' is the third candidate.

5A) When $\omega'' = C$, deviation leads to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega''\}$, there are a positive proportion of subgames where ω and ω'' win the election. Not deviating leads to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega'\}$, there are a positive proportion of subgames where ω and ω' win the election. As outcome ω'' is preferred to the outcome ω and ω' for the γ type voter then their best response is to vote for ω''.

5B) When $\omega'' = A, B$ and $\omega' = C$ deviation leads to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega''\}$. There are a positive proportion of subgames where ω and ω'' win the election. Not deviating leads to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega'\}$, there are are positive proportion of subgames where ω and ω' win the election. As outcome ω' is preferred to the outcome ω and ω'' for the γ type voter then their best response is to not change their vote.

5C) When $\omega'' = A, B$ and $\omega = C$ deviation leads to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega''\}$, there are a positive proportion of subgames where ω and ω'' win the election. There are a positive proportion of subgames where ω and ω'' win the election. Not deviating leads to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega'\}$, there are a positive proportion of subgames where ω and ω' win the election. As candidate ω wins in the same proportion of subgames with and without deviation and the utility from the outcomes ω' and ω'' are equal the γ type voter is indifferent between deviating and not deviating.

When a γ voter's best response is to deviate from a strategy it is away from voting for A or B to vote for C. With elimination of weakly dominated strategies the γ type voter's strategy at $h = (\emptyset)$ is $(0, 0, 1)$. ■

1.4.3 Dominated Strategies for the α and β type voters

Theorem 1.3. *At the empty set voting for candidate A weakly dominates voting for candidate C for all α type voters. At the empty set voting for candidate B weakly dominates voting for candidate C for all*

β type voters. In equilibrium all α type voters and β type voters will vote for candidate C with zero probability. $\sigma((\emptyset), \alpha, C) = 0$, $\sigma((\emptyset), \beta, C) = 0$.

Proof: In the proof to theorem 1.2 it was shown that the γ voter votes for C in all equilibrium as they strictly prefer candidate C to candidate A and candidate B. This proof did not consider cases where voting for C effected the probability that candidate A or candidate B win when the probability that candidate C winning did not change. e.g. When candidate C is one of two candidates in round two and a deviation can change if candidate A or candidate B are the other candidate. As in such cases the probability that candidate C wins is unchanged and the γ type voter is indifferent between candidate A and candidate B. The proof to theorem 1.2 proves that the γ type voter will vote for C as such a vote increases the probability that candidate C wins. The same logic can also be used to prove that the α and β type voters will prefer to vote for A and B respectively as in doing so they decrease the probability that candidate C wins. To compete the proof that voting C is weakly dominated by voting for A and B respectively it is necessary to consider the cases where deviation to or from C does not increase or decrease the probability that candidate C wins but does effect the probability that A or B win.

Firstly, when $\omega = (C)$, if a deviation to or from C changes candidate ω' in the subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega'\}$ this could improve the utility of the α or β type vote[18]. Secondly, if deviation to or from C changes from the outcome $O = (A)$ to the outcome $O = (B)$ or the sub game $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, B\}$, or from the outcome $O = (B)$ to the outcome $O = (A)$ or the sub game$h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, B\}$, or finally from the subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, B\}$ to the outcome $O = (A)$ or $O = (B)$. [19]. Using the fact that all γ voters vote C and that the γ voters represent strictly more than one third of the population; it is shown that these two additional cases either never occur or that deviating to C is weakly dominated by an alternative deviation.

As the γ voters represent over one third of the population and all γ voters vote C candidate C will get at least $\frac{1}{3}$ of the votes cast and they can never be last. The subgame $h = (V^1, f_c)$

[18]the γ voter is indifferent between the subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, C\}$ and the subgame between $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{B, C\}$ as the probability that the population drawn gives C a majority is the same in both cases. The α type voter prefers the sub game $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, C\}$ to the subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{B, C\}$ and the beta voter has the opposite preferences

[19] the γ type voter is indifferent between these two outcomes and the subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, B\}$ but the α and β type voters are not

such that $A_i(V^1, f_c) = \{A, B\}$ can not occur. This eliminates the second cases that needed to be considered.

When $\omega = (C)$ for a deviation to C to change ω' in the subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, \omega'\}$ it must change the candidate that comes second. When two candidates are above the threshold this is not possible as C is already above the threshold so deivation to C can not take another candidate above the threshold. Assume the subgame prior to deviation is $\omega' = A$. A deviation from A to C takes candidate A below the threshold but it will not take candidate B above the threshold and the outcome is $O = (C)$ as candidate C is the only candidate above the threshold. The outcome of the deviation, $O = (C)$, gives a lower utility than the outcome without the deviation for an α or β type voter.

Alternatively, no candidate is above the threshold[20]. Assume the outcome prior to deviation is that candidate A and C are the candidates with the most votes but neither passes the threshold this leads to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, C\}$. A deviation from A to C leads to candidate A having fewer votes than candidate B (or equal). The outcome is now candidate B and candidate C having the most votes, subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{B, C\}$. For a β type voter this is better than subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, C\}$. However, this outcome can also be achieved by deviating to B which is strictly better in one or more of the other cases discussed. Therefore voting for C is weakly dominated by voting for B for the β type voter. The same logic holds for the α voter.

Using elimination of weakly dominated strategies; the best response of the α and β type voter is to never vote C at $h = (\emptyset)$. ■

At this point the strategies of the γ type voter have been fully characterised; all γ type voters always vote for C. The α and β type voters will only choose between A and B. There are 4 feasible Symmetric Perfect Bayesian equilibria to analyse. 1) all α and β type voters vote for A, 2) all α and β type voters vote for B, 3) all α type voters vote for A and all β type voters vote for B and 4) all α type voters vote B and all β type voters vote B. The next three theorems cover these 4 Symmetric Perfect Bayesian equilibria.

The cases 1 and 2 are both Duverger Law equilibria. Case 3 is, under a set of necessary

[20]The cases where one candidate is above the threshold is not relevant as the starting outcome is $O = (A), (B), (C)$ not $O = (AC), (BC)$.

conditions which will be defined, a sincere equilibria. Case 4 is completely insincere voting where all α and β type voters are not sincere. This is an equilibrium under a set of necessary conditions which will be defined.

1.4.4 Existence of Duverger's Law equilibrium

Theorem 1.4. *The Duverger Law equilibrium exists for all threshold levels and any level of uncertainty.*

When all α and β type voters vote for A the outcome is $O = (A)$. Candidate A has all the votes from type α and β and this represents an absolute majority. Therefore regardless of the threshold level candidate A exceeds the threshold level. When candidate C also exceeds the threshold level, $\eta_\gamma > \tau * n$, candidate A will still win as they also pass the strict majority threshold and candidate C does not.

The outcome, $O = (A)$, is the best possible outcome for α type voters; there is no possible deviation that can increase their utility. The β type voters would prefer the outcome $O = (B)$. If a β type voter deviates to B candidate B will now have one vote. They remain the candidate with the least votes. The deviation will not lead to $O = (B)$. However if candidate A was less than one vote above the threshold[21] the election will move to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, C\}$. In this subgame candidate C has a positive probability of winning the election (except in the case where $P(V^1, f_c) = N$ and all voters vote in round two and candidate A still wins with certainty). The expected outcome of this subgame is therefore worse than the outcome without deviation $O = (A)$. So no voter benefits from deviation and the Duverger Law outcome is an equilibrium. This result holds under the same logic when candidate B receives all the α and β type voters votes. ■

1.4.5 Existence of Sincere equilibrium

Theorem 1.5. *The sincere equilibrium exists for any level of uncertainty and as long as $\eta_\alpha, \eta_\beta \leq \tau * n - 1$.*

[21] or one vote above a strict majority when candidate C is above the lower threshold

All α type voters vote for A and all β type voters vote for B. When $\eta_\gamma > \tau * n$ the outcome is $O = (C)$. When $\eta_\gamma \leq \tau * n$ as $\eta_\alpha = \eta_\beta$ the election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, C\}$ where ω is chosen by the random draw f_c and is A or B. Candidate A and B have an equal probability of reaching round two.

When $\eta_\gamma > \tau * n$ the outcome is $O = (C)$. This is the worst possible outcome for the α and β type voters. If a single α type voter deviates to B then candidate B now has a vote total of $\eta_\beta + 1$. If $\eta_\beta + 1 > \tau * n$ the election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{B, C\}$ and both candidates have a positive probability of winning. This outcome is preferable to $O = (C)$ and sincere voting is not an equilibrium. The same result holds if a β type voter deviates to A. If $\eta_\beta + 1 \leq \tau * n$ then the outcome of the election is still $O = (C)$ and deviation has no effect. In this case sincere voting is an equilibrium as both voting types cannot deviate and change the outcome of the election. Therefore when $\eta_\gamma > \tau * n$ there is a sincere equilibrium if and only if $\eta_\alpha, \eta_\beta \leq \tau * n - 1$.

When $\eta_\gamma \leq \tau * n$ the election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, C\}$ where ω is chosen by the random draw f_c and is A or B. Candidate A and B have an equal probability of reaching round two. As $\eta_\gamma \leq \tau * n$ and $\eta_\gamma > \eta_\alpha, \eta_\beta$ a single deviation by the α and β type voter cannot take A or B above the threshold. If an α type voter deviates to B candidate B has strictly more votes than candidate A. Candidate A therefore has strictly the least votes of the three candidates. The election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{B, C\}$. The α type voters strictly prefer the subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, C\}$ which occurred with 50% probability without deviation. The expected outcome of the election is now worse for the α type voter. Therefore they will not deviate to B. The same result holds if a β type voter deviates to A. So when $\eta_\gamma \leq \tau * n$ and by extension $\eta_\alpha, \eta_\beta \leq \tau * n - 1$ there is always a sincere equilibrium. ■

Theorem 1.6. *Complete insincere voting is an equilibrium if and only if* $\eta_\gamma > \tau * n$ *and* $\eta_\alpha, \eta_\beta \leq \tau * n - 1$

All α type voters vote B and all β type voters vote A. When $\eta_\gamma > \tau * n$ the outcome is $O = (C)$. When $\eta_\gamma \leq \tau * n$ as $\eta_\alpha = \eta_\beta$ the election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, C\}$ where ω is chosen by the random draw f_c and is A or B. Candidate A and

B have an equal probability of reaching round two.

When $\eta_\gamma > \tau * n$ the outcome is $O = (C)$. This is the worst possible outcome for the α and β type voters. If a single α type voter deviates to A then candidate A now has a vote total of $\eta_\beta + 1$. If $\eta_\beta + 1 > \tau * n$ the election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, C\}$ and both candidates have a positive probability of winning. This outcome is preferable to $O = (C)$ and completely insincere voting is not an equilibrium. If $\eta_\beta + 1 \leq \tau * n$ then the outcome of the election is still $O = (C)$ and deviation has no effect. The same result holds if a β type voter deviates to B. In this case sincere voting is an equilibrium as both voting types cannot deviate and change the outcome of the election. Therefore when $\eta_\gamma > \tau * n$ there is an equilibrium where all voters are insincere if and only if $\eta_\alpha, \eta_\beta \leq \tau * n - 1$.

When $\eta_\gamma \leq \tau * n$ the election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{\omega, C\}$ where ω is chosen by the random draw f_c and is A or B. Candidate A and B have an equal probability of reaching round two. As $\eta_\gamma \leq \tau * n$ and $\eta_\gamma > \eta_\alpha, \eta_\beta$ a single deviation by the α and β type voter can not take A or B above the threshold. If an α type voter deviates to A candidate A has strictly more votes than candidate B. Candidate B therefore has strictly the least votes of the three candidates. The election moves to subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{A, C\}$. The α type voters strictly prefer this subgame to the subgame $h = (V^1, f_c)$ such that $A_i(V^1, f_c) = \{B, C\}$ which occurred with 50% probability without deviation. The expected outcome of the election is now better for the α type voter. Therefore they will deviate to A. The same result holds if a β type voter deviates to B. So when $\eta_\gamma \leq \tau * n$ there is no equilibrium where all voters are insincere. ∎

One assumption in this chapter thus far has been that the equilibrium is symmetric. All voters of a given type vote the same way in equilibrium. If this condition is dropped the sincere equilibrium and the Duverger's Law equilibrium are no longer the only equilibrium that exist. A third type of non-symmetric equilibrium exist. Like the sincere equilibrium these equilibrium only exist under a specific set of conditions. However these conditions are not on the distribution of voter types but instead the distribution of votes. The necessary distribution of votes for a non-symmetric equilibrium to exist are defined in appendix C.

1.5 Experimental Procedure and Design

All sessions were run in the Economics computer lab at Royal Holloway University of London. The experiment was programmed and conducted by myself[22] using z-tree software Fischbacher (2007). Subjects were all students from the university from a range of academic disciplines as well as being both UG and PG students. A session was run over roughly 60 - 90 minutes depending on subject response speed. Subjects were paid an attendance fee of £6 and 6 payments selected at random from the session. The payment related to the outcome of elections as shown in table 1.2. Subject payments ranged from £10.20 to £15 with an average of £12.66.

The experiment was run over 12 sessions. 6 sessions with 16 subjects and 6 sessions with 8 subjects. A total of 144 subjects participated in the experiment and no individual participated multiple times. There were a total of 6 treatments with each subject participating in 3 of the treatments. Table 1.3 in section 1.5.2 summarises the breakdown of treatments. Within each treatment there were 9 periods. This gives a total of 27 periods per session. Each period is a separate election and an election consists of 8 subjects. In the large sessions with 16 subjects they were split in half at the start of the session and these groups remained the same throughout the session. The experiment had a total of 18 independent observations of voting groups.

Each period within a treatment is run identically and explained here. Prior to a treatment starting all rules are read out and subjects take a short quiz to check understanding.

1.5.1 Experimental Period

The basic design that tests majority requirement with full round two participation was similar to those previously run by Morton and Rietz (2007) which compared the majority requirement to plurality. As part of the literature on divided majority this experiment builds on the design from Forsythe, Myerson, Rietz, and Weber (1993, 1996). The number of subjects is the same as those used in these experiments[23].

A period starts with the allocation of type to each subject. A subject is either of type α or type β. Subjects will now be defined as type α voters or type β voters in a given period. The third type

[22] My thanks to Doctor Bjoern Hartig for his help and advice in setting up the experiment
[23] with the exception of the lower γ as the effect of this is what is of interest

of voter, type γ, have a dominant strategy and will be computer simulated similar to a number of previous studies, Bouton, Castanheira, and Llorente-Saguer (2016, 2017), Battaglini, Morton, and Palfrey (2008, 2010). Morton and Tyran (2012) have shown that one group's preferences are not affected by the preferences of another group. Simulating type γ votes should therefore not affect the behavior of type α or type β voters.

There are 8 subjects and 4 are allocated type α and 4 type β. These types are allocated every period. There will be 5/6 γ type voters depending on treatment. This gives an election population of 13/14 voters.

There are three candidates, A, B and C and each type of voter has a given preference relationship over these three types as shown in the set-up section. Table 1.2 gives the corresponding monetary payments the subjects see. All payments are in pounds. The numbers were selected to reflect similar relative payments from similar experiments and with the aim of an average hourly earnings of £10.

TABLE 1.2: Subject's payoff table from the experiment

Type	A	B	C	# of voters
α	1.50	1.20	0.20	4
β	1.20	1.50	0.20	4

Subjects are not given the payoffs for the simulated voters but are told that all γ type voters are indifferent between A and B and strictly prefer C. They are also told that these voters will always vote for C[24].

Of the nine periods per treatment two will be selected at random and the payment made in those two periods are what the subjects receive. This table is common knowledge and appears on each screen that subjects see.

Subjects will be asked to participate in a pre-election poll. The experiment asks whether subjects co-ordinate, the poll allows for a simple definition of co-ordination at the voter level. This is expanded on in 1.5.2.

In the poll they get 4 choices, Vote A, Vote B, Vote C, or abstain. Abstention is allowed in the poll to allow for an additional signal. If one voter abstains this leads to 7 votes (by subjects

[24]due to the number of γ type voters and the fact they all vote for C in round one there is no round two where C will not be an option, therefore the γ voter never has to chose between A and B

ignoring simulated votes). If all other voters vote for A or B this guarantees a poll leader. Therefore a voter who is unsure whether a poll will give a definitive leader can abstain. Including it here allows us to investigate if subjects consider this and if such a signal is successful.

A vote in the poll is not binding in regard to the round one vote. Finally, the result of the poll and their vote in the poll have no effect on the payments they receive. All of this is made clear to them prior to the experiment.

The result of the poll is then made public to everyone regardless of if they voted or not. Subjects see the total number of poll votes for each candidate but do not see who voted for each candidate or the types of those who voted for each candidate. All γ types will vote for C and this is known.

The election now begins. Voters are asked to vote in round one of the election. As in Guarnaschelli, McKelvey, and Palfrey (2000) voting is compulsory and abstention is not allowed [25]. After everyone has voted, subjects see a screen telling them the number of votes each candidate has received. If a candidate wins the election, depending on the threshold as explained in the setup, the period ends and subjects are told the result and move to the next period.

If the necessary conditions are not met the election moves to round two. The candidate with the least votes is eliminated and subjects are told this. At this stage subjects, depending on treatment, are either asked to vote in round two or are sent to a wait screen till round two is finished.

All those selected will vote again. A subject's vote in this round does not have to be the same as their vote in round one. The results of this vote are then shown to all voters regardless of if they were asked to vote or not. This round will, by definition, have a winner. All subjects are told the winner of the elections and the period is over. The screen goes to the next period wait screen.

This completes the description of a single period. There is no wait time between periods but at the end of a treatment (9 periods) the screen stops and the rules of the next treatment are outlined. At the end of the 3rd treatment a final screen appears telling subjects how much they

[25] Forsythe, Myerson, Rietz, and Weber (1993, 1996) allowed for abstention for a similar divided majority game under plurality and found abstention rate was only 0.65%

have made in each treatment and a total for the session.

The next section outlines the treatment design within subject and between subjects.

1.5.2 Between-subject and Within-subject Design

TABLE 1.3: Summary of Treatments based on the 3 dimensions that define their differences

Threshold Level		Within subject variables		
		$\gamma = 6$		$\gamma = 5$
		Certainty	Uncertainty	Uncertainty
Between Subject Treatment	50 %	MC6	MU6	MU5
	Flexible	FC6	FU6	FU5

Table 1.3 shows a subject will participate under either the 50% victory condition (absolute majority required to win) or the flexible victory condition with a 50% threshold as well as the lower 40% threshold.

Under one of these two victory conditions a subject will participate in 3 treatments within a session. The 3 treatments are:

1. There are 6 γ type voter and all voters participate in round two (when reached).
2. There are 6 γ type voters and 5 are randomly chosen to participate in round two (from the set of 14 voters). This creates uncertainty over who will participate in round two.
3. There are 5 γ type voters and 5 are randomly chosen to participate in round two (from the set of 13 voters). This creates uncertainty over who will participate in round two (this risk that the γ type voters are a round two majority is smaller)

The order that subjects participate in treatments was changed for each session such that each treatment came first, second and third within a session an equal amount of the times.

The number of subjects that participated under the majority rule was equal to the number that participated in the flexible rule. As such the number of groups that had each treatment first, second and third was equal [26].

[26]there were 18 groups and each treatment came 1st 2nd and 3rd in the order 3 times

1.5.3 Coordination

As discussed in section 1.4.4 there exist two equilibria where only two candidates get a positive share of the vote[27]. These two Duverger equilibria exist when all subjects (representing all α and β type voters) vote for A or B. These will be defined as the coordinated equilibria [28]. This defines coordination at the group level however the aim of the analysis in this chapter is to also consider coordination at the individual level. To do this two forms of individual level coordination are defined. A player can co-ordinate on the candidate that won the poll this period or the candidate that won the election in the last period.

Definition 1.5. Poll coordinated **(PC)** voting: An individual makes a coordinated vote when they vote for the candidate (between A and B) that has the most votes in the poll that occurred before the election. When both candidates A and B have the same number of votes there is no poll coordinated voting.

Definition 1.6. History coordinated **(HC)** voting: An individual makes a coordinated vote when they vote for the candidate (between A and B) that won the last period's election.

The results of the experiment will only present the Poll coordination. Having analyzed the results for both coordination methods it was found that polling is a more successful coordination mechanism. A candidate winning the poll prior to an election was a better predictor of voter action than the previous election result. Once irrelevant data is removed[29] the correlation between the winner of the last election and the candidate a player voted for was 0.2450 while the correlation between the poll leader and the candidate a player voted for was much higher at 0.7164. The results when using the **(HC)** definition of coordinated voting gives similar results to those presented but with lower significants.

1.6 Results

The results are broken down into the key theorems and the less important propositions that were proved in section 1.4, the set of conjectures that come from the results of the experiment and

[27] excludes the case where only A and B get a positive share of the vote as γ type voters will always vote for C

[28] These are specific cases of Duverger equilibria, they are defined to separate them from alternative Duverger equilibria

[29] For details about what data was removed and why see appendix

predictions based on the payoffs in each treatment. These predictions are laid out prior to each conjecture. Finally a set of propositions that test the less important theorems that were used to reach the key theorems and the assumptions made regarding the effect of polling are presented. Each theorem, conjecture or proposition is followed by the findings of the experiment that either support or oppose the theorem, conjecture or proposition. Prior to this I give a summary of the data.

1.6.1 Summary

As stated in 1.5.2 there were 6 treatments and each treatment represents a specific voting rule. For each treatment 72 voters participated in that voting rule and there were 9 independent sets of voters. Each independent set of voters voted 9 times under each election rule giving 81 elections in total for each treatment: a total of 486 elections.

Candidate A won in 51.4% of the elections. Candidate B won in 28.2% of elections and candidate C in 20.4% of elections. The outcomes for each treatment is given in table 1.4 in percentage terms and in brackets the number of elections this occurred.

TABLE 1.4: Summary of election winner by treatment

Outcome	MC6	MU6	MU5	FC6	FU6	FU5	Total
A wins	48.1% (39)	55.6% (45)	40.7% (33)	45.7% (37)	51.9% (42)	66.7% (54)	51.4% (250)
B wins	42.0% (34)	21.0% (17)	45.7% (37)	28.4% (23)	12.4% (10)	19.8% (16)	28.2% (137)
C wins	9.9% (8)	23.5% (19)	13.6% (11)	25.9% (21)	35.8% (29)	13.6% (11)	20.4% (99)

Part of these results come from random draws where candidates are tied so there is expected to be some variation. With that in mind, candidate A's odds of victory do not seem to be affected much by threshold and the only case where a large effect is seen, is between $MU5$ and $FU5$. This shows the effect of threshold with a small uncertainty. This rise is generated from a fall in candidate B's odds of victory. This suggests that uncertainty is having an effect on coordination, as players coordinate to mitigate the effect of a lower threshold. This effect is not prevalent when considering the effect of lower thresholds under large uncertainty, or no uncertainty. The effect seems to come from the fact that in $FU5$ the minority is not large enough to pass the variable threshold, but voters are coordinating as if they were. This decision at the voter level will be expanded on in the main results.

Threshold does seem to have an effect on candidate C's odds of victory, but only when candidate C can pass the variable threshold, suggesting this effect comes from round one upset risk not round two upset risk.

The second summary we give is the occurrence of round two. This is only needed when no candidate passes the variable threshold or two candidates pass the variable threshold and neither passes the majority threshold. The election reaches the second round in 45% of periods. The outcome for each treatment is given in table 1.5.

TABLE 1.5: Occurrence of the second round by treatment

Outcome	MC6	MU6	MU5	FC6	FU6	FU5	Total
2nd round needed	86%	68%	56%	17%	30%	19%	46%
2nd round not needed	14%	32%	44%	83%	70%	81%	54%

There is a clear difference in the need for round two between treatments. The starkest is the effect of the threshold under certainty (from $MC6$ to $FC6$). The need for round two falls by 69% points. A similar but smaller effect is seen under uncertainty (from $MU6toFU6$), of roughly 38% points. Uncertainty itself also has an effect but much smaller. It reduces the need for round two by 18% points with just the majority threshold (from $MC6$ to $MU6$). This effect is not seen with a lower threshold, suggesting that with a lower threshold, either coordination is insufficient to avoid the upset risk, or coordination is complete and therefore takes A or B above the majority threshold. We can see this from the fact that C does not win 83% of the elections but we know they always pass the variable threshold.

1.6.2 Duverger's Law Equilibria

Theorem 1.7. *Duverger's Law equilibria always exist under uncertainty and all thresholds. It exists in all 6 treatments*

The theoretical results here say that for any threshold and for any level of uncertainty there must exist a Duverger's Law equilibrium. It is predicted that such an equilibrium exists in all 6 treatments in this experiment[30]. It is important to class what results will be interpreted as a

[30]See Appendix G for the experimental example that shows Duverger's Law equilibrium existence

Duverger's Law equilibrium. This chapter uses the same definition of a Duverger's Law equilibrium as Forsythe, Myerson, Rietz, and Weber (1993). Forsythe, Myerson, Rietz, and Weber (1993) suppose that all voters randomly select a candidate such that all candidates are equally likely to be selected. The CDF for votes received by a single candidate is calculated and a one sided test asking what vote totals make a candidate's vote total significantly smaller than expected is carried out. With 14 voters and three candidates the random election gives an outcome where at least one candidate gets 0 or 1 votes with 2.7% probability. The probability that at least one candidate gets 0, 1 or 2 is 10.5%. Using these densities they reject the idea that a vote is random in favour of a Duverger's Law type effect if and only if at least one candidate gets 0 or 1 votes.

This chapter will use the same system but updated for our purposes. There are only 8 real voters in this experiment not 14 therefore the corresponding probabilities are not the same. Equally we have shown in the theory section that voters never select candidate C therefore we treat random voting as a random vote between candidate A and candidate B. Using these conditions the probability that candidate A or B gets 0 votes is 0.78% while the probability that candidate A or B get 0 or 1 is 7.03%. Using these numbers I reject random voting in favour of a Duverger's Law type effect if and only if candidate A or B gets 0 votes. In the cases when this happens we can be 99% confident that a Duverger's Law type effect did happen as opposed to a random effect.

Table 1.6 shows the percentage of elections where either candidate A or candidate B got zero votes. Therefore it shows the percentage of elections where random voting is rejected in favour of a Duverger's Law type effect. This is broken down by treatment to allow fo comparison for the prevalence of Duverger's equilibrium when the threshold or uncertainty is changed. They are also broken down by period to see if a Duverger's Law equilibrium exists more at the start or end of the treatment.

The first observation from the table is that a Duverger's Law equilibrium exists in at least 10% of elections in all six treatments supporting the theorem that a Duverger's Law equilibrium always exists. Learning seems to have an effect; the occurrence of a Duverger's Law equilibrium increases by 8% from the first 3 rounds to the last 3 rounds. This suggest that as the players learn the game better they start to coordinate on a single candidate and a Duverger's Law equilibrium occur more often. Breaking this down by treatment in five of the six treatments the occurrence of a Duverger's Law equilibrium increases in the later periods. In the case of the $(MU5)$, $(FU6)$ and $(FU5)$ it is by a substantial percentage, 14.8%, 25.9% and 11.1% respectively. Uncertainty seems to be something that takes voters longer to adapt to. The two treatments with certainty do not see large changes in the occurrence of Duverger's Law equilibria over time.

TABLE 1.6: The percentage of elections with Duverger's Law equilibrium in the first round of voting for each of the treatments

	Percentage of elections that ended in the Duverger equilibrium		
Treatment	All 9 Periods	Period 1-3	Period 7-9
MC6 (Majority Threshold, Certainty and 6 γ types)	13.58%	11.11%	14.81%
MU6 (Majority Threshold, Uncertainty and 6 γ types	30.86%	37.04%	25.93%
MU5 (Majority Threshold, Uncertainty and 5 γ types	13.58%	7.41%	22.22%
FC6 (Flexible Threshold, Certainty and 6 γ types)	58.02%	51.85%	55.56%
FU6 (Flexible Threshold, Uncertainty and 6 γ types	41.98%	33.33%	59.26%
FU5 (Flexible Threshold, Uncertainty and 5 γ types	34.57%	18.52%	29.63%
All treatments	32.10%	26.54%	34.57%

Notes: Each treatment has 81 elections and 9 of the 18 independent groups participated in each treatment. The value is the percentage of times that either Duverger's Law equilibrium exists, where candidate A or B get zero votes in the first round of the election.

Duverger's Law equilibria are most prevalent in $FC6$ (58.02%) and least prevalent in $MC6$ (13.6%) and $MU5$ (13.6%). This suggest that moving from the Majority threshold to the flexible threshold is increasing the occurrence of Duverger's Law equilibria especially when there is certainty about the second round voting population. This finding along with the payoff differences[31] between reaching Duverger's Law equilibria and the sincere equilibria (the other Symmetric Perfect Bayesian Equilibria) informs conjecture 1.1.

A similar finding supports conjecture 1.2. Uncertainty seems to increase the occurrence of the Duverger's Law equilibria when there is a majority threshold (comparing $MC6$ and $MU6$). Here uncertainty reduces the payoffs expected from sincere voting as seen in appendix F. The

[31] see appendix F for details of the different payoffs from each treatment at each voting level

opposite effect is found when there is just the majority threshold. This would seem to be due to the extreme difference between $MC6$ and $FC6$. The importance of the threshold here seems to outweigh the effect of the uncertainty. This result can be seen in table 1.7 and the interaction effect of the two variables is discussed in the conjectures. Finally conjecture 1.3 is supported by the fact that lowering the γ value reduces the occurrence of Duverger's Law equilibria by 7.4% and 17.3% respectively for the flexible threshold and the majority threshold.

While the theory in this chapter is silent about equilibrium selection based on the payoffs from Duverger's Law equilibria and the sincere equilibrium in each treatment, an informed conjecture regarding comparative statics can be made. The rationale for these conjectures is given below the conjecture and uses the summary of payoffs for each outcome in appendix F. Three conjectures regarding the dimensions of the experiment are presented. Table 1.7 presents the results to all three conjectures which are stated after the table.

TABLE 1.7: The Effect of threshold, uncertainty, and γ type group size on the probability that Duverger's Law Equilibria exist

DV: Duverger's Law Equilibrium probability of existence in round one of an election	Threshold effect & Uncertainty effect	Threshold effect & Uncertainty effect with interaction effect	Threshold effect & number of γ type voter
Lowering the Threshold level (from 50% to 40%)	0.278*** (0.101)	0.444*** (0.111)	0.209** (0.083)
Moving from Certainty to Uncertainty	0.006 (0.04)	0.173*** (0.066)	- -
Threshold and Uncertainty Interaction effect (moving from $MC6toFU6$)	- -	-0.333*** (0.093)	- -
Increasing the number of γ type voters	- -	- -	0.173** (0.049)
Threshold and γ type voters Interaction effect (moving from $MC6toFU6$)	- -	- -	-0.098 (0.098)
Constant	0.219*** (0.074)	0.136* (0.078)	0.136** (0.058)
Number of Observations	324	324	324

Notes: The Regression is a Random-effects GLS regression with Group Variable being the 18 independent voting groups. The results are the variables coefficient and standard error. An Observation is a single election result. The number of observations equals total observations 3888 divided by 8 for total number of elections 486 times $\frac{2}{3}$ as in each specification 2 of the 6 treatments are not relevant

* : P value < 0.1 , ** : P value < 0.05 , *** : P value < 0.01

Conjecture 1.1. The proportion of Duverger's Law equilibria is increased by lowering the threshold for first round victory when there are 6 γ type voters.

With and without uncertainty lowering the threshold leads to the minority candidate winning the election unless at least 6 of the 8 voters vote for the same candidate. This means that the sincere equilibrium gives a very low payoff. In comparing the payoffs, lowering the threshold under certainty reduces the payoffs from 1.35 down to 0.2 for the sincere equilibrium. Under uncertainty a similar reduction in payoff can be seen from 0.96 down to 0.2. This fall is smaller but still significant.This incentivises voters to co-ordinate to Duverger's Law equilibria. While a single vote does not change this the conjecture here argues that at such a low payoff voters are more likely to deviate and try to reach a Duverger Law equilibrium.

The first column in table 1.7 shows that lowering the threshold increases the probability of Duverger's Law equilibrium occurring by 27.8%, an effect that is significant at the 99% level. This supports the idea that lowering the threshold incentivises voters to try to coordinate and reach a Duverger's Law equilibrium. More interestingly move to column two when the interaction effect between lowering the threshold and introducing uncertainty increases the effect of the lower threshold to 44.4% showing that the effect of a lower threshold is greater when there is certainty. Both this effect and the interaction effect are significant at the 99% level. Looking at the change in payoff this is to be expected as the drop in payoff for the sincere equilibrium with certainty is larger.

Conjecture 1.2. The proportion of Duverger's Law equilibria is increased by introducing uncertainty over the round two voters types.

With and without a lower threshold the introduction of uncertainty decreases the expected return in round two (as candidate C is expected to win a positive proportion of second round elections) and incentivises deviation to a Duverger's Law equilibrium to win the election in round one. The results only partially support this conjecture. Uncertainty has no effect without the interaction effect. This suggests that the threshold effect is greater than the uncertainty effect so without the interaction of these two the effect of uncertainty is absorbed. This can be explained mostly by the very large effect of lowering the threshold with certainty. This can be seen when the interaction effect is included in column two. Uncertainty increases the probability of a Duverger's Law equilibrium by 17.3% which is 99% significant. However this effect is smaller than the interaction effect of 33.3% so while uncertainty has an effect it is not as large.

Conjecture 1.3. The proportion of Duverger's Law equilibria is increased by increasing the number of γ type voters.

Increasing the number of γ voters to 6 means candidate C wins in round one unless at least 6 of the 8 voters vote for the same candidate. The expected payoff with a lower γ is 1.06 while the payoff with a higher γ is 0.2 and 0.96 for the flexible threshold and majority threshold respectively. This incentivises voters to co-ordinate to Duverger's Law equilibria. The effect of increasing the number of γ voters to 6 increases the probability of a Duverger's Law equilibrium by 12.3% supporting the idea that with a larger minority support the likelihood of the minority candidate winning increases and this incentivises the majority to coordinate and see a rise in the occurrence of a Duverger's Law equilibrium.

The number of γ types does seem to have an effect on the occurrence of the Duverger's Law equilibria. Increasing the number of γ types to 6 increases the occurrence of the Duverger's Law equilibria by 17.3%. This rise has an effect for the majority threshold and the flexible threshold but the effect is stronger for the majority threshold. If the interaction effect is included this effect of increasing γ to six goes down to 7%. One issue with this finding is that the interaction effect while large is not significant and this makes interpreting the different effects with or without the change in threshold harder.

1.6.3 The Sincere Equilibrium

Theorem 1.8. *The Sincere equilibrium exist for all 6 treatments in the experiment.*

The theory in this chapter proves that there exists a set of values for the threshold, uncertainty and γ population parameters such that for all voters voting sincerely is not an equilibrium. None of the treatments in the experiment have the necessary parameter values such that the sincere equilibrium will not exist[32]. The prediction is therefore that: The sincere equilibrium should exist in each treatment. The same test of falsifying the null in favour of sincere voting will be conducted here. The probability will measure the probability that a population of 8 will

[32] see appendix H for the experimental example that shows sincere equilibrium existence

all vote sincerely. The probability that 8 random voters vote sincerely given the two choices is 0.39% while the probability that 7 voters vote sincerely is 3.5%. Using these numbers I reject the idea that a vote is random in favour of a sincere voting type effect if and only if all 8 voters are sincere. In the cases when this happens we can be 99% confident that a sincere voting type effect did happen as opposed to a random effect.

Table 1.8 shows the percentage of elections where all 8 voters are sincere. Therefore it shows the percentage of elections where random voting is rejected in favour of a sincere voting type effect. This is broken down by treatment to allow for comparison for the prevalence of the sincere equilibrium when the threshold or uncertainty is changed. They are also broken down by period to see if the sincere equilibrium exists more at the start or end of the treatment.

TABLE 1.8: The percentage of elections with a sincere equilibrium in the first round of voting for each of the treatments

Treatment	Percentage of elections that ended in the sincere equilibrium		
	All 9 Periods	Period 1-3	Period 7-9
MC6 (Majority Threshold, Certainty and 6 γ types)	25.93%	25.93%	29.63%
MU6 (Majority Threshold, Uncertainty and 6 γ types	6.17%	3.7%	7.41%
MU5 (Majority Threshold, Uncertainty and 5 γ types	19.75%	25.93%	14.81%
FC6 (Flexible Threshold, Certainty and 6 γ types)	2.47%	0%	3.70%
FU6 (Flexible Threshold, Uncertainty and 6 γ types	0%	0%	0%
FU5 (Flexible Threshold, Uncertainty and 5 γ types	1.23%	0%	3.70%
All treatments	9.26%	9.26%	9.88%

Each treatment has 81 elections and 9 of the 18 independent groups participated in each treatment. The value is the percentage of times that the sincere equilibrium exists, where candidate A or B get zero votes in the first round of the election.

For the whole experiment the sincere equilibrium does exist in 9.26% of elections however the distribution among treatments is not equal. The sincere equilibrium does not exist in treatment $FU6$ and is very low in $FC6$ and $FU5$. This suggests that moving from the majority threshold to the flexible threshold reduces the probability of the sincere equilibrium. This seems to suggest that the fact that the payoff for the sincere equilibrium with a flexible threshold is much lower[33] than the payoff for the Duverger's Law equilibria is having an effect. This supports conjecture 1.4 and will be expanded on there.

Uncertainty reduces the occurrence of the sincere equilibrium with certainty (moving from $MC6$ to $MU6$) from 25.93% to 6.17%. Introducing uncertainty reduces the payoff received from the sincere equilibrium and this seems to incentivise coordination and supports conjecture 1.5. The effect of γ on the sincere equilibrium is only found with the majority threshold. Increasing the value of γ increases the occurrence of the sincere equilibrium. This supports conjecture 1.6 but does not support this conjecture when there is a flexible threshold as the expected fall is not seen.

Finally the effect of learning is smaller than for the Duverger's Law equilibria. The change between the first three rounds and the last three rounds is much smaller. The occurrence only changes by more than 5% in one treatment as it falls from 25.93% to 14.81% for $MU5$. This is in part due to the lower occurrence in general but also suggests that if reached (or not reached) voters do not learn more about the sincere equilibrium.

As previously stated the theory section is silent on equilibrium selection but again using payoff from the different sincere equilibria in each treatment (see appendix F) a set of informed conjectures regarding the comparative statics can be made. Again three conjectures are presented. Table 1.9 give the result of all three conjectures.

Conjecture 1.4. The proportion of sincere equilibria is decreased by lowering the threshold for first round victory when there are 6 γ type voters.

For both certainty and uncertainty, lowering the threshold when there are 6 γ type voters leads to candidate C winning the election in the sincere equilibrium. This removes the incentive to remain sincere and voters have an incentive to try to get at least 6 votes for one of the two candidates. The payoffs from the sincere equilibrium with a majority threshold are 1.35 and 0.96 for certainty and uncertainty respectively and both fall to 0.2 with a lower threshold (see appendix F for values).

[33] see appendix F

TABLE 1.9: The Effect of threshold, uncertainty, and γ type group size on the probability that the sincere equilibria exists

DV: Sincere Equilibrium probability of existence in round one of an election	Threshold effect & Uncertainty effect	Threshold effect & Uncertainty effect with interaction effect	Threshold effect & number of γ type voter
Lowering the Threshold level (from 50% to 40%)	-0.148*** (0.054)	-0.235*** (0.061)	-0.123*** (0.057)
Moving from Certainty to Uncertainty	-0.111*** (0.028)	-0.120*** (0.039)	- -
Threshold and Uncertainty Interaction effect (moving from $MC6toFU6$)	- -	0.173*** (0.055)	
Increasing the number of γ type voters	- -	- -	-0.074*** (0.024)
Constant	0.216*** (0.041)	0.259*** (0.043)	0.166*** (0.042)
Number of Observations	324	324	324

Notes: The Regression is a Random-effects GLS regression with Group Variable being the 18 independent voting groups. The results are the variables coefficient and standard error. An Observation is a single election result. The number of observations equals total observations 3888 divided by 8 for total number of elections 486 times $\frac{2}{3}$ as in each specification 2 of the 6 treatments are not relevant

* : P value < 0.1 , ** : P value < 0.05 , *** : P value < 0.01

The results in column one of table 1.9 support this. The effect of moving from the majority threshold to the flexible threshold is to reduce the occurrence of the sincere equilibrium by 14.8%. This effect is more pronounced once the interaction between threshold and uncertainty is introduced (the difference between MC6 and FU6). This shows that the effect of threshold on when there is certainty shown in column two is 23.5%. This makes sense as the effect on payoff (a fall from 1.35 to 0.2) is largest here. The effect with uncertainty is much smaller once the interaction effect is included (0.235 - 0.173 = 0.062, there a fall of 6.2%). This again makes sense as the fall in payoff is smaller here (from 0.96 down to 0.2).

Conjecture 1.5. The proportion of sincere equilibria is decreased by introducing uncertainty over the round two voters types.

For both thresholds introducing uncertainty increases the probability that candidate C wins when all voters are sincere introducing an incentive to try to reach a point where candidate A or B wins in round one.

This conjecture is partially supported by the data. In column one of table 1.9 introducing uncertainty does have an effect when there is a majority threshold. The occurrence falls by 11.1% for the whole study; however this effect disappears when the interaction effect is included. Considering column 2 of table 1.9 the fall from MC6 to MU6 is 12% but the change from FC6 to FU6 is negligible and is absorbed by the interaction effect. This suggests that as with the results on Duverger's Law equilibria the threshold effect outweighs the uncertainty effect.

Conjecture 1.6. The proportion of sincere equilibria is decreased by increasing the number of γ type voters.

Increasing the number of γ type voters when there is a 40% threshold means that candidate C wins in round one when there is a sincere equilibrium. This creates an incentive for the voters to get one candidate at least 6 votes.

Column three in table 1.9 shows that increasing γ decreases the occurrence of the sincere equilibrium. If the interaction effect is included (not shown in the table) the effect of γ is absorbed by the interaction effect. This can be explained by the fact that with a flexible threshold the occurrence of the sincere equilibrium is extremely low in both cases so the floor for sincere equilibrium occurrence has been reached.

1.6.4 Non-symmetric Equilibria

Under the Symmetric Perfect Bayesian Equilibrium the sincere equilibrium and Duverger's Law equilibrium are the only equilibria. However if the Symmetric condition is dropped there is another set of possible non-symmetric equilibria; the 'non-pivotal' equilibria. At least one such equilibrium exists in each treatment[34]. The existence of such equilibria can in part be explained by error in the experiment but if they are significant it suggests that symmetry may be an unreasonable assumption to make about the set of equilibria that exists.

Theorem 1.9. *The non-symmetric equilibrium exists in all 6 treatments.*

For this theorem there is no limit to be tested as there was for Duverger's Law equilibria (a candidate getting zero votes compared to random voting) or the sincere equilibrium (all 8 voters voting sincerely). Therefore the table below simply reports the percentage of the time

[34] see appendix I for the experimental example that shows non-symmetric equilibrium existence in all treatments

that a non-symmetric equilibrium exists without being able to make a statistical claim about this not being random. It will be the case that some of the occurrence of these equilibria will be due to randomness making the results less concrete. Therefore the comparison here just considers how the occurrences changes between treatments instead of making a claim about the existence of an effect compared to randomness. Table 1.10 presents the occurrence of the non-symmetric equilibria for each treatment.

TABLE 1.10: The percentage of elections with non-symmetric equilibrium in the first round of voting for each of the treatments

	Percentage of elections that ended in the non-symmetric equilibrium		
Treatment	All 9 Periods	Period 1-3	Period 7-9
MC6 (Majority Threshold, Certainty and 6 γ types)	19.8%	11.1%	18.5%
MU6 (Majority Threshold, Uncertainty and 6 γ types	14.8%	11.1%	11.1%
MU5 (Majority Threshold, Uncertainty and 5 γ types	23.5%	22.2%	33.3%
FC6 (Flexible Threshold, Certainty and 6 γ types)	14.8%	22.2%	11.1%
FU6 (Flexible Threshold, Uncertainty and 6 γ types	6.1%	7.4%	0%
FU5 (Flexible Threshold, Uncertainty and 5 γ types	33.3%	44.4%	25.9%
All treatments	18.7%	19.8%	16.7%

Notes: Each treatment has 81 elections and 9 of the 18 independent groups participated in each treatment. The value is the percentage of times that either Duverger's Law equilibrium exists, where candidate A or B get zero votes in the first round of the election.

The first finding to observe is that the non-symmetric equilibria occurs reasonably often, in 18.7% of elections. This is twice as often as the sincere equilibrium and about half as often as the Duverger's Law equilibria which is the most prevalent type of equilibrium. While it is not possible to prove that this is not random this does suggest that voters are reaching these equilibria though they may not persist over time. Looking at the effect of learning there is no clear theme. In three treatments ($FC6$, $FU6$ and $FU5$) the occurrence of the non-symmetric equilibria goes down but in two treatments ($MC6$ and $MU5$) it goes up and in one treatment ($MU6$) it is unchanged. It is worth noting that in the three treatments where the non-symmetric equilibrium occurs less over time there is a lower threshold. This coincides with a rise in Duverger's

Law equilibria which supports the findings in the rest of the chapter that lowering the threshold seems to have a larger effect on the occurrence of Duverger's Law equilibria and this seems in part to come from a reduction in the occurrence of the non-symmetric equilibrium.

The possible conjecture regarding the effect of threshold level on non-symmetric equilibrium occurrence is not as clear as it was for a Duverger's Law equilibrium or sincere equilibrium existence but one can still be made. As thresholds seem to have a large positive effect on deviation to the Duverger's Law equilibrium it is reasonable to assume that the occurrence of the non-symmetric equilibrium will go down under a lower threshold (as the occurrence of the Duverger's Law equilibrium goes up). One caveat is worth including for this. In treatments $MU5$ and $FU5$ a vote total of 7 votes for one majority candidate and one vote for the other majority candidate is a non-symmetric equilibrium. These are the only treatments where this is the case and as a lower threshold seems to increase the occurrence of Duverger's Law equilibrium it would be reasonable to assume that it also increases the probability of this type of non-symmetric equilibrium. Therefore the effect of a lower threshold will be predicted to have the opposite effect. These predictions are outlined in conjecture 1.7. Table 1.10 seems to support these assumptions. Table 1.11 presents the results of conjecture 1.7 which is discussed after the table.

The final conjecture, conjecture 1.8 regarding the non-symmetric equilibrium, aims to predict the effect of uncertainty. If we consider appendix I we see an outline of the vote shares where the non-symmetric equilibrium exists[35]. Three of the treatments have 4 vote totals that are non-symmetric equilibrium ($MC6$, $FC6$ and $FU5$) while three treatments have 2 vote totals that are non-symmetric equilibrium ($MU6$, $MU5$ and $FU6$). Of the treatments where there are 4 possible non-symmetric equilibrium two are under certainty. Their corresponding treatments with uncertainty (where the only difference between the treatments is uncertainty) both only have 2 vote totals that are non-symmetric equilibrium. So while the paper does not claim to prove a specific equilibrium selection method there is a case to be made that the non-symmetric equilibrium will occur less often under uncertainty. This is the final conjecture regarding the non-symmetric equilibrium.

[35]The first set of non-symmetric equilibria are where candidates A or B have seven votes. This is a non-symmetric equilibrium in $MC6$, $MU5$, $FC6$ and $FU5$. The second set of non-symmetric equilibria are when candidate A or B have six votes. This is a non-symmetric equilibrium in $MC6$, $MU6$, $FC6$, $FU6$ and $FU5$.

TABLE 1.11: The Effect of threshold, uncertainty, and γ type group size on the probability that Non-symmetric Equilibria exist

DV: Non-symmetric Equilibrium existence probability in round one of an election	Threshold effect & Uncertainty effect	Threshold effect & Uncertainty effect with interaction effect	Threshold effect when the number of γ types equals five
Lowering the Threshold level (from 50% to 40%)	-0.068 (0.042)	-0.049 (0.057)	- -
Moving from Certainty to Uncertainty	-0.068* (0.038)	-0.049 (0.053)	- -
Threshold and Uncertainty Interaction effect (moving from $MC6toFU6$)	- -	-0.037 (0.076)	- -
Increasing the number of γ type voters	- -	- -	0.099 (0.088)
Constant	0.206*** (0.035)	0.198*** (0.076)	0.235*** (0.062)
Number of Observations	324	324	162

Notes: The Regression is a Random-effects GLS regression with Group Variable being the 18 independent voting groups. The results are the variables coefficient and standard error. An Observation is a single election result. The number of observations equals total observations 3888 divided by 8 for total number of elections 486 times $\frac{2}{3}$ as in specification one and two 2 of the 6 treatments are not relevant. In specification three only 2 of the 6 treatments are relevant so there are only 162 observations.
* : P value < 0.1 , ** : P value < 0.05 , *** : P value < 0.01

Conjecture 1.7. The non-symmetric equilibrium occurs less often with a flexible threshold if and only if there are six γ type voters. When there are five γ type voters the non-symmetric equilibrium will occur more often with a flexible threshold.

As stated above this conjecture has less motivation than those regarding the sincere equilibrium and the Duverger's Law equilibrium. This conjecture is based on the fact that a flexible thresholds increase the occurrence of the Duverger's Law equilibrium. With six γ type voters there is no non-symmetric equilibrium that is similar to the Duverger's Law equilibrium (no non-symmetric equilibrium where one majority candidate has 7 votes and the other has 1). For this reason the threshold is predicted to decrease the occurrence of the non-symmetric equilibrium. With five γ type voters there is a non-symmetric equilibrium where one majority candidate has 7 votes and the other has 1 and therefore the flexible threshold should increase the occurrence of the non-symmetric equilibrium. Table 1.11 shows that the effect of the flexible threshold does have the sign that is expected. It is negative with six γ types and positive with five γ types. Beyond this though the table does not support the conjecture. None of these effects are significant due in large part to large standard errors. As this conjecture has the weakest justification it is perhaps not surprising that this conjecture has the weakest support from the data. So while these non-symmetric equilibrium do seem to exists (see table 1.10) there seems to be limited evidence that changing the threshold has a significant effect.

Conjecture 1.8. The non-symmetric equilibrium occurs less often when there is uncertainty about who votes in the second round of the election.

The reasoning for this conjecture relates to the number of vote totals that represent non-symmetric equilibria. There are more vote totals that represent non-symmetric equilibria when there is certainty than when there is uncertainty. As with conjecture 1.7 this justification is weaker than the justification for the conjectures that relate to the sincere equilibrium and the Duverger's Law equilibrium. Table 1.11 shows that uncertainty does have the negative effect that is predicted. Introducing uncertainty decreases the occurrence of the non-symmetric equilibrium by 6.8%. Unlike the effect of the flexible threshold this effect is significant but only at the 90% level. So there is some support for this conjecture but the support from the data is again weaker than the support for the earlier conjectures.

1.6.5 Non-equilibrium Outcomes

A non-equilibrium outcome is any set of vote totals such that at least one voter can increase their payoff by changing their voting decision. In the theory section the non-equilibrium outcomes is not expected to occur as the election is expected to be in equilibrium.

In experiments there is an element of learning involved and the equilibrium is not expected to be reached in all cases. Table 1.12 shows the existence of non-equilibrium events where the vote totals in the election are such that an α or β type voter can increase their payoff by changing their vote in round one[36].

Conjecture 1.9. In the experiment non-equilibrium outcomes will occur as voters make mistakes.

[36]Such an outcome can occur in the second round but in any such case a voter will be taking a dominated vote and this is addressed in subsection 1.6.6.

The first thing to note about this table is that non-equilibria outcomes occur often. The occurrence of non-equilibria outcomes ranges from 24.7% of elections to 51.8% of elections among the treatments. This suggests as expected that voters do not act perfectly in the experiment and do not reach equilibria outcome in all elections.

TABLE 1.12: The percentage of elections with non-equilibrium outcomes in the first round of voting for each of the treatments

	Percentage of elections that ended in the non-equilibrium outcome		
Treatment	All 9 Periods	Period 1-3	Period 7-9
MC6 (Majority Threshold, Certainty and 6 γ types)	40.7%	51.8%	37.0%
MU6 (Majority Threshold, Uncertainty and 6 γ types	48.1%	48.1%	55.5%
MU5 (Majority Threshold, Uncertainty and 5 γ types	43.2%	44.4%	29.6%
FC6 (Flexible Threshold, Certainty and 6 γ types)	24.7%	25.9%	29.6%
FU6 (Flexible Threshold, Uncertainty and 6 γ types	51.8%	59.3%	40.7%
FU5 (Flexible Threshold, Uncertainty and 5 γ types	30.8%	37.0%	40.7%
All treatments	39.9%	44.4%	38.9%

Notes: Each treatment has 81 elections and 9 of the 18 independent groups participated in each treatment
The value is the percentage of times that the non-equilibrium exists,

The occurrence of non-equilibria outcomes is lowest in treatment *FC*6 which is also the treatment that has the largest percentage of Duverger's Law equilibrium outcomes. This suggests that when Duverger's Law equilibrium is most appealing (in terms of payoffs achieved at the Duverger's Law equilibrium compared to the sincere equilibrium) the non-equilibrium events occur less often. In *FC*6 the payoff from the sincere equilibria and the non-equilibria adjacent to the sincere equilibria (where one candidate gets 5 votes and the other gets 3) is 0.2 while the payoff from the non-symmetric equilibria and the Duverger's Law equilibria is either 1.5 or 1.2. With this in mind it is then surprising that the treatment with the largest non-equilibria outcomes is *FU*6 as this has a similar cost to being sincere; however in this case a vote tally of one candidate getting seven votes and the other getting one is not an equilibrium. This is the most prevalent non-equilibrium outcome for this treatment suggesting voters see the benefit of reaching Duverger's Law equilibria but fail to fully coordinate on it.

Conjecture 1.10. The proportion of non-equilibria outcomes will decrease during a treatment as players better learn the game and select an equilibrium strategy.

It is expected that over time voters learn the game better so the non-equilibrium outcomes should occur less frequently. All treatments have outcomes that are not equilibria[37]. Table 1.12 outlines the prevalence of non-equilibrium outcomes in each treatment over time.

The occurrence of non-equilibrium outcomes can be seen in table 1.12 falling by 5.5% from the first three periods to the last three periods. This fall is not as large as expected from learning. Yet another form of learning does seem to happen and be more important: learning between treatments. Each participant took part in three treatments and seems to learn more between treatments.

To test this the order of the treatment is included as a variable as is the order of periods in the treatment. The first of these variables takes the value one, two or three. The participants participated in three treatments and the order they participated was randomly assigned to them. When this variable takes the value one it means the treatment was the first treatment a participant participated in. When it takes the value two it is the second treatment a participant participated in. Then when it takes the value three it is the last treatments a participant participated in. The second of the new variables takes the value one to nine. For each treatment the participants repeated the election nine times. This variable equals where in that repetition a given election occurred. So if period equals one it was the first time they participated in that treatment and so on.

Table 1.13 shows that between studies the occurrence of non-equilibria outcomes goes down by 10.5% and this is significant at the 99% level. This effect does go down slightly if the effect of thresholds and uncertainty are taken into account but the effect is still 99% significant and has an effect of 9.4%. Equally interesting from this table is that the effect of threshold and uncertainty are not significant though both have quite large magnitudes. Non-equilibrium outcomes seem to be effected most by learning between treatments and not as much by the threshold or uncertainty. This supports conjecture 1.10 that non-equilibrium outcomes goes down as voters learn the game better and improve their strategy.

[37] see appendix J for details about the set of non-equilibrium outcomes in the experimental treatments

TABLE 1.13: Effect of time, threshold and uncertainty on the non-equilibrium existence

DV: Non-equilibrium outcome occurrence	Specification 1	Specification 2
Variables		
Threshold level	-	-0.160
(50% down to 40%)	-	(0.101)
Uncertainty	-	0.074
	-	(0.072)
Interaction	-	0.196*
	-	(0.102)
Treatment order in session	-0.105***	-0.094***
(1-3)	(0.026)	(0.034)
Period order in the treatment	-0.007	-0.006
(1-9)	(0.008)	(0.010)
Constant	0.646***	0.626***
	(0.072)	(0.111)
N=	486	324

Notes: Study order refers to order that participants did the treatments in if study order is 1 this treatments was the first the participants did etc Period order refers to order that participants did the treatments in. If study order is 1 this treatments was the first the participants did etc. There are less observations in specification two as treatment *MU*5 and *FU*5 have to be dropped

*** significant at 99%, ** significant at 95%, * significant at 90%

1.6.6 Key Assumptions to Test

Underpinning the results above is the assumption that voters do not take weakly dominated strategies. For the α type voters and the β type voters there are two strategies that are weakly dominated. Providing results to support these two propositions is not a contribution of this chapter but is necessary for the other results. The results of both these propositions are in table 1.14. Both results use the same tests that Forsythe, Myerson, Rietz, and Weber (1993) use to test dominated voting against random voting.

Proposition 1.10. All α type voters and β type voters always vote for candidate A or candidate B in all second rounds that either candidate contests against candidate C.

This has been proven in Besley and Coate (1997). The results should show that whenever a voter votes in round two they do not vote for candidate C. If we consider the percentage of votes for candidate C in round two in all treatments the percentage of dominated votes is below 5% and in 2 of the 6 treatments it is 0%. This supports the idea that voters do not vote for candidate C in round two.

Proposition 1.11. All α type voters and β type voters never vote for candidate C in the first round of elections. This is a weakly dominated vote and should never occur.

As in all treatments the percentage of voters that are γ type voters is greater than $\frac{1}{3}$ candidate C always enters the second round if it occurs. Therefore there are no cases where it is better for α and β type voters to vote C in round one. This is a dominated vote and should never occur. Table 1.14 shows that if we consider the percentage of votes for candidate C in round one in all treatments the percentage of dominated votes is below 5% and in 5 of the 6 treatments near 2%. This supports the idea that voters do not vote for candidate C in round one.

The two propositions that dominated voting does not occur in round one or round two seem to be supported by the results of the experiment. This is important as these are two theoretical assumptions that underpin the theorems in this chapter. These results are also similar to those found in similar experimental papers where there is a small percentage of dominated votes found in Morton and Rietz (2007). In their paper dominated voting is slightly larger but this can be explained in part by the fact that they make the γ type voters active and this group have more dominated votes so the likelihood of error is increased.

TABLE 1.14: The percentage of the votes in round one and round two where a voter took a dominated strategy by voting for candidate C

Treatment	Round One			Round Two		
	number of votes	Dominated Votes	% of Dominated votes	number of votes	Dominated Votes	% of Dominated votes
MC6	648	17	2.62%	560	19	3.39%
MU6	648	13	2.01%	161	3	1.86%
MU5	648	16	2.47%	142	4	2.82%
FC6	648	27	4.17%	112	2	1.79%
FU6	648	16	2.47%	100	0	0%
FU5	648	16	2.47%	46	0	0%

A third assumption of the experiment is that polling is a successful coordination mechanism. Forsythe, Myerson, Rietz, and Weber (1993, 1996) both give evidence that polling can work as a successful coordination mechanism. Finding the effect of polling is not a primary aim of this chapter and no treatments were run that did not include polling. All that needs to be found here is that polling improves coordination as the experiment needs a coordination device. Coordination here means that a voter votes for the candidate, between candidate A and candidate B, that wins the poll. This is the dependant variable in table 1.15. Table 1.15 is split into two groups. To measure how voters change their likelihood of voting for candidate A and candidate B. The table then reports the effects of candidate A winning the poll. This is in comparison to a poll tie. The same effect is tested when candidate B wins the poll. Specification one and two are tested under these conditions allowing for individual random effects and for individual clustering. The final specification then includes the margin of victory for each candidate. This measures if coordination is more likely when a candidate wins the poll by a large amount. Here the variable measure the difference between the winning candidate and the losing candidate and the effect is an increase of one in the difference. Conjecture 1.11 is that polling will have an effect and the winner of the poll will receive more votes in the main election. IF this is found to be the case then polling has worked as a coordination device. Election histories was also considered as a possible coordination mechanism as it was in Forsythe, Myerson, Rietz, and Weber (1993, 1996) but when the two were analysed for correlation[38] to first round votes it was found that polling better explained the action of the voters in the experiment and so polling is the only coordination mechanism considered in the results.

[38] See appendix E for the method used to check correlation

TABLE 1.15: The effect of polls result and poll lead on probability to coordinate votes

DV: Coordinated voting level in round one	Candidate A			Candidate B		
	Spec 1	Spec 2	Spec 3	Spec 1	Spec 2	Spec 3
Variables						
A wins the poll	0.372***	0.354***	0.285***	-0.359***	-0.336***	-0.278***
	(0.020)	(0.015)	(0.022)	(0.020)	(0.015)	(0.022)
B wins the poll	-0.398**	-0.387***	-0.307***	0.398***	0.396***	0.295***
	(0.023)	(0.017)	(0.025)	(0.025)	(0.016)	(0.025)
Margin of Victory A	-	-	0.028***	-	-	-0.024***
	-	-	(0.006)	-	-	(0.006)
Margin of Victory B	-	-	-0.040***	-	-	0.051***
	-	-	(0.009)	-	-	(0.009)
Constant	0.522***	0.527***	0.525***	0.446***	0.435***	0.437***
	(0.018)	(0.012)	(0.012)	(0.018)	(0.012)	(0.012)
Individual random effects	No	Yes	Yes	No	Yes	Yes
Cluster by individual	Yes	No	No	Yes	No	No
Notes	N is always 3888 and number of groups is always 144					
	All regressions are OLS with a form of individual fixed effect/clustering					
	*** significant at 99%, ** significant at 95%, * significant at 90%					

Conjecture 1.11. Polling is successfully used as a coordination mechanism.

There is a small in-built ballot effect: when considering the constant we see that with no poll winner, candidate A gets 52.2% of the vote, while candidate B gets 44.6%. This ballot effect is found in other papers that tests its effect in the real world in a number of election scenarios. Meredith and Salant (2013) finds it in a study of local elections in California and gets an estimate of 4% to 5% for ballot effect. Lutz (2010) finds similar effects in Swiss open ballot PR elections. Then finally Brockington (2003) suggests the effect is anything from 0% to 5% in his study of city council elections in Illinois; this paper argues it is primarily low information voters that cause a ballot effect which seems reasonable though in this experiment I would argue ballot effect has a useful role in assisting coordination and is not just due to low information voters. This could explain why the ballot effect is slightly higher here than in the real world studies where it is not a coordination tool.

Specification one and two respectively test the effect of candidate A or B winning the poll on the % of votes candidates A and B receive. This is run with both individual fixed effects and clustering by participant. These are two methods to allow for the predicted participant level correlation. The effect of candidate A winning the poll is an increase of 35.4% and 37.2% points in candidate A's votes respectively and a similar fall of 33.6% and 35.9% points in candidate B's votes. When candidate B wins the poll, it has an almost identically effect in reverse.

Specification three adds a second explanatory variable, the margin of the poll result. This refers to the difference between the number of poll votes that candidate A wins and the number that candidate B wins. Specification three is only run under an OLS with individual fixed effect. Both the GLS with random effects and clustering were also tested and gave results with similar magnitude and significance. As such, including all three seemed superfluous.

When margin of victory is included it reduces the magnitude of the simple victory variable under all three controls and all four coefficients. All remain significant and positive. The effect of candidate A winning the poll is now a rise of 28.5% points for candidate A's vote number and a 27.8% points fall in candidate B's vote %. A similar reversed effect happens when candidate B wins the poll (a fall of 30.7% for candidate A and 29.5% rise for candidate B).

1.7 Conclusion

The main aims of this chapter are: 1) To find experimental evidence of the existence of Duverger's Law equilibrium and the sincere equilibrium in two round divided majority election games with uncertainty and flexible thresholds, 2) Assuming these equilibria exist test the effect of flexible thresholds and uncertainty on the occurrence of these two equilibrium. 3) to measure the effect of changing the level of uncertainty on two round elections. The secondary aims of this chapter are: 4) to redesign the theoretical model in Bouton (2013), which initially proposed these predictions, to be testable by a lab experiment. 5) to investigate the assumption of symmetry in the equilibrium concept and search for the existence of non-symmetric equilibrium in the experiment 6) to show that voters do not take dominated strategies in the experiment under these new parameters of uncertainty and a lower threshold, therefore supporting the assumptions necessary for the other results.

To achieve these aims I ran a lab based experiment with six treatments. Three treatments with a majority threshold and three with a flexible threshold. Participants then participated in either the majority threshold treatments or the flexible threshold treatments. Within these three treatments one was run with certainty about the second round participants and six supporters of the minority candidate ($MC6$ and $FC6$ respectively) on was was run with uncertainty about the second round participants and six supporters of the minority candidate ($MU6$ and $FU6$ respectively) and one was run with uncertainty about the second round participants and five supporters of the minority candidate ($MU5$ and $FU5$ respectively). This gave the framework for the experiment.

The theoretical section of the chapter finds that both the Duverger Law equilibrium and the sincere equilibrium should occur in all 6 treatments in this experiment. This finding is supported by the results in the chapter. Using the same method as Forsythe, Myerson, Rietz, and Weber (1993) to compare a Duverger's Law type effect against random voting the chapter finds that a Duvergers Law type effect occurs in 32.10% of all elections in the experiment. This effect is found to occur most often in treatments $FC6$, $FU6$ and $FU5$. Therefore the flexible threshold seems to increase the occurrence of the Duverger's Law equilibrium. This supports the conjecture in this chapter that a lower threshold creates a greater cost to no reaching the Duverger's Law equilibrium and therefore incentivises participants to try to reach it.

When we test the effect of the flexible threshold on the occurrence of Duverger's Law equilibrium we find the effect to be a 27.8% increase in occurrence. This effect goes up when the interaction effect with uncertainty is included. Uncertainty also has a significant effect of the occurrence of Duverger's Law equilibrium but only when the interaction effect is taken into account. This suggest that of the two variables changing the threshold has a dominant effect. The effect of uncertainty, 17.3%, is smaller than the interaction effect, 33.3%, which is negative. So the flexible threshold and uncertainty effect the Duverger's Law equilibrium in the same direction. Both variables have a positive effect on the occurrence of the Duverger's Law equilibrium.

The second Symmetric Perfect Bayesian equilibrium that is predicted by the theory to exist is the sincere equilibrium. The findings of the experiment support this prediction. Using the same method as Forsythe, Myerson, Rietz, and Weber (1993) but now to compare random voting with sincere voting it is found that the sincere equilibrium occurs in 9.26% of elections. This is less than the Duverger's Law equilibrium but still a significant percentage of the time. The effect of threshold and uncertainty on the existence of the sincere equilibrium is also significant but smaller. The effect of the flexible threshold on the occurrence of the sincere equilibrium is 14.8% which rises to 23.5% when the interaction effect is considered. Unlike the Duverger's Law equilibrium uncertainty does has a significant effect, 11.1% without the interaction effect. The effect of uncertainty rises to 12% with the interaction effect, 17.3%, which again is negative. So the flexible threshold and uncertainty effect the sincere equilibrium in the same direction. Both variables have a negative effect on the sincere equilibrium

The first half of this paper present an altered version of Bouton (2013); the main theoretical differences are the use of a known and exogenous voting population. This is compared to a Poisson distribution in Bouton (2013). The other differences include a slightly altered definition of a variable threshold that retains the majority threshold as a victory condition. These two changes do not change the results of the Symmetric Perfect Bayesian equilibrium. As such the predictions of Bouton (2013) remain valid and will be tested.

The assumption of symmetry simplifies the analysis in Bouton (2013) but if removed the results still hold. However if a fixed number of voters is assumed this is not the case. This chapter finds a set of non-symmetric equilibrium that can exist in each of the treatments of the experiment. The non-symmetric equilibria occurs reasonably often, in 18.7% of elections in the experiment. This can be partially explained by error and some degree of randomness in real world voters. The fact that such equilibrium exist more often than the sincere equilibrium does raise questions about the validity of the assumption of symmetry. This chapter does not aim to address these questions but that does not mean they are not valid and worth investigation.

The set of dominated strategies in this experiment come from the results of Besley and Coate (1997). The occurrence of dominated voting has been investigated in other experimental literature and this chapter does not claim any innovation in this area. It is merely worth noting that this chapter finds a similar level of dominated voting, among its subjects, as similar experiments such as Morton and Rietz (2007) among others.

To finish this chapter finds that uncertainty and flexible thresholds matter to the results of elections. Yet neither is investigated extensively in the literature. Uncertainty in particular is to some degree inevitable in two round elections yet the risk of upset in these election is not well documented. This chapter supports the idea that a lower flexible threshold can be used by a election designer to incentivise coordination among voters and reduce the cost of elections by having less second round elections. The contribution of this chapter is to add experimental evidence to existing theoretical literature on the effect of threshold and uncertainty in two round elections with divided majorities. in doing so it expands the understanding of these models, highlights some questions regarding the assumption of symmetry among voting types, finds supporting evidence for dominated voting being rare in experiments. without making any theoretical claims about equilibrium selection it lays out a number of conjectures regarding the effect of threshold and uncertainty and the likely effect on equilibrium selection and supports these with data.

Chapter 2

The Weakness of Plurality in Electoral Systems

2.1 Introduction

Under a winner takes all plurality election rule the candidate with the most votes wins the election. Duverger (1959) proves that in such an election only two candidates should win a positive share of the votes cast. This proof known now as Duverger's Law can be seen to hold to a varying degree in many Plurality elections. In the United States for example the Republican Party candidate for president and Democratic Party candidate for president win roughly 95% of the votes cast and the whole electoral college. In 1992 when Ross Perot, the most successful third party candidate in the last 100 years, ran the two dominant parties still gained 80% of the votes cast and the whole electoral college[1]. A less stark example is the United Kingdom where the two dominant parties, the Labour party and the Conservative party, have both received at least 30% of the vote cast in all post war elections with two exceptions[2]. In no post World War Two election has another party won or come in second the United Kingdom or the United States. With such a clear history of two party dominance a vote for any other party would appear to be

[1]For more information about United States elections see The Federal Election Commission, https://transition.fec.gov/pubrec/electionresults.shtml

[2]The election of 1983 where leader Michael Foot got only 27.3% of the national vote and the election of 2010 where leader Gordon Brown got only 29% of the national vote. For more information about UK elections see The Electoral Commission, https://www.electoralcommission.org.uk/our-work/our-research/electoral-data

a wasted vote. Yet such third party candidate voters still exist.

As increased polarisation has crept into modern politics the actions of these third party candidate voters has become more important, especially as previous elections have suggested that the votes the third party candidate gets can alter the outcome of elections if re-allocated. For example, Ralph Nader in 2000 gained only 2.74% of the popular vote but with such a tight election[3] his supporters, had they voted for Vice President Al Gore or Governor George Bush, could have swung the election[4].

A number of explanations have been suggested for why such 'wasted' votes exist. Castanheira (2003) suggests that people vote for losing candidates to try to set the agenda. A number of papers including Chamberlain and Rothschild (1981) and Ferejohn and Fiorina (1974) focused on the pivotal probability and the fact that voters may choose a wasted vote as they do not think their vote matters. Fey (1997) among others suggests that there is a lack of information and that voters are unable to work out which of the two candidates they should coordinate to. Finally, and linked more closely to the work in this paper, Tavits and Annus (2006) suggest that it takes time for voters to learn especially in new democracies and so over time 'wasted votes' go down. These ideas among others will be considered in the literature review.

This chapter fits into the literature on third candidate voting. There is a trade-off between strategic voting and sincere revelation of preferences that is at the heart of the third candidate voting literature. Voters wish to influence the outcome of elections if they are pivotal to those outcomes but when not pivotal they prefer to vote sincerely (for any one of the reasons mentioned in other literature). This chapter adds to this literature by considering if voters in fact would prefer to be strategic in elections but due to their inability to know when they are pivotal or inability to fully rationalise the actions of all the other voters they fail to do this. Voters in this case can be described as having bounded rationality and in this chapter I suggest that this is one rationale for 'wasted' votes.

I will then propose one solution to this trade-off between strategic decision making and sincere voting to be the instant run-off election rule. Making a similar argument as that of Piketty

[3] Florida was within 537 votes and Nader receiving 97,421 in the state, for data see https://transition.fec.gov/pubrec/2000presgeresults.htm

[4] Herron and JeffreyLewis (2007) argues that while Ralph Naders did not exclusively take votes from Al Gore his supporters were 60 % to 40 % in favour of the defeated candidate Al Gore, a difference that due to the unusual closeness of Florida's election would have swung the election for Al Gore

(2000). This chapter will show that this rule simplifies the strategic decisions allowing voters to sincerely reveal their preferences while making fully rational strategic decisions under bounded rationality.

To investigate the effect of bounded rationality and the benefits of the instant run-off I present in section 2.3 a three candidate winner takes all election model. This model allows for all types of voter preference order but makes a set of assumptions about the number of voters with each type of preference. Two candidates will have a 'large' number of supporters while one candidate will have a 'small' number of supporters. The model will implement a social choice function where each voter inputs a set of preferences that link to the different voting rules. Two voting rules will be analysed in this chapter, the plurality election rule and the instant run-off election rule.

Under the Plurality rule each voter casts a single ballot for a single candidate and the candidate with the most votes wins the election. Under the instant run-off election rule each voter casts a single list ranking all candidates on the ballot in order of preference. A candidate then wins if they have a strict majority of the first ranked votes. If no candidate has a strict majority the candidate with the least votes is eliminated. Their votes transfer to their second ranked candidate. This continues until one candidate has a strict majority. Section 2.4 gives more detail regarding these voting rules. The instant run-off rule follows on from the two round election rule discussed in chapter one. Chapter three will show that under the conditions assumed in chapter one and two these two rules are equivalent. Therefore Chapter two continues to add to the understanding of these run-of election rules.

The first aim of this paper is to fully characterise the sincere equilibrium under the conditions of two dominant parties and one small party. This builds on work by Besley and Coate (1997), Osborne and Slivinski (1996) by specifically looking at the restricted case of three candidates where only two have a large support[5]. The sincere equilibrium occurs if no voter wishes to reveal insincere preferences to the social choice function when all other voters reveal sincere preferences to the social choice function. If no voter wishes to reveal insincere preferences in such a case that voting distribution has a sincere equilibrium[6]. In characterising the sincere equilibrium this chapter will define the necessary conditions for such an equilibrium to not exist

[5]Defined as their support being at least 2 greater than the small candidate; while this is a broad definition it is the necessary condition, in reality the difference is almost certainly larger

[6]The distribution of voters is the only parameter of importance other than the voting rule itself as this defines the number of votes each candidate gets when everyone is sincere

for the two voting rules.

The main contribution of this chapter is the application of bounded rationality to voter decision making. This builds most directly on Nagel (1995). Nagel (1995) introduces the idea of level-k thinking. Applied to voting games this suggests that there exists a level-0 thinker who does not react to any other voter in the election. Instead they apply the most minimal thought process in making a decision. The chapter interprets such a voter in two ways as either naive or random. The random interpretation comes directly from Nagel (1995) and assumes that all level-0 thinkers randomly pick a candidate (or in the case of instant run-off a random ordering of candidates) assigning each candidate equal probability. The naive interpretation comes from Bassi (2015) who argues that the level-0 voter will naively assume that there is no possible strategy they could take and so will be sincere. This chapter will model both interpretations for both voting rules. After the level-0 thinker comes a level-1 thinker. They assume that all other voters act as level-0 thinkers and then best respond to this population. The level-2 thinker then assumes that all other voters act as level-1 thinkers and then best respond and so on.

This chapter will analyse the strategies at all levels of thinking until an equilibrium is reached where higher levels do not change their strategy compared to lower level thinking. The chapter will then compare when the two voting rules lead to different strategies at the same level and analyse which voting rule gives an outcome that is preferred by the majority of voters. It will also compare when a voting rule reaches a strategy that does not change at higher levels of thinking. The motivation for this being that if a voting rule reaches the same outcome with a lower level of thinking, this is preferable as it requires voters to think less while achieving the same utility.

The first result of the chapter regards the sincere equilibrium. Subsection 2.2 proves that the sincere equilibrium exists for all vote distributions discussed in this chapter for the instant run-off rule. Subsection 2.1 proves that for the plurality rule the sincere equilibrium exists only if a candidate has a strict lead and all voters that most prefer the third candidate either prefer the leader or are not pivotal. These results support the idea that it is easier to be sincere and not be casting a vote for a candidate that is not optimal. Additionally to this result, the chapter finds that when the two election rules give different outcomes the outcome reached by the instant run-off election rule is strictly preferred by a strict majority of voters. This is implicit in the results and the explanation can be found in the discussion after the proof of the sincere equilibrium existence theorem.

The second result of the chapter is a simple one. It is found that applied to this voting game the results from the two assumptions about 'naive' (Ns) and 'random' (Nr) level-0 voters are equivalent with one adjustment. The best response to a set of 'random' level-0 voters is to vote sincerely which is the strategy of the 'naive' level-0 voter. The strategy of the level-0 'naive' voter to sincerely reveal their preferences is equivalent to the strategy of the level-1 voter that responds to a level-0 'random' voter. As a voter at level-k best responds to a population of level-(k-1) voters this relationship between the 'naive' assumption and the 'random' assumption holds. The strategy at level-k under the assumption of 'random' level-0 voters is the same as the strategy at level-(k-1) under the assumption of 'naive' level-0 voters. Therefore the main results for the level-k thinking here will be defined in terms of the 'random' level-0 assumption (as this includes all the findings for the 'naive' level-0 assumption but the 'naive' assumption does not include the finds for the level-0 'random' voter).

The third and most important set of results in this chapter relates to the findings on the level-k strategies for each election rule and how they compare to each other. The key areas of interest are the strategies taken at each level, the outcome of elections at each levels, and how long it takes to reach a strategy that does not change at higher levels of thinking. Each of these three areas support the idea that 1) under the plurality rule voters often make 'wasted' votes for candidates that cannot win when they are at lower levels of thinking, 2) that the instant run-off has higher levels of sincere voting while also eliminating these 'wasted' votes solving the issue of 'wasted' votes being due to bounded rationality.

Subsection 2.6.1 proves that all higher levels above the random level-0 thinker vote sincerely and the sincere equilibrium is the only one that exists. Subsection 2.6.2 proves that there are 3 possible equilibria for the plurality rule. The three possible equilibria are the sincere equilibrium, Duverger Law equilibrium and the non-pivotal equilibrium. In comparing the two voting rules the results tell us that when the sincere equilibrium exists for the plurality election rule that the strategy that does not change at higher levels is reached at the same point for both rules. As with the results in section 2.5.1 the instant run-off election rule will sometimes give an outcome that is strictly preferred to the outcome under the plurality election rule. When Duverger's law equilibrium occurs for the plurality election rule the outcome of the plurality election rule is the same as that of the instant run-off election rule however it takes an additional level of thinking to reach this outcome. Finally when the non-pivotal equilibrium is reached under the plurality election rule the outcome under the instant run-off rule is either the same as this outcome or strictly preferred for a strict majority of voters. It also takes a higher level of thought to reach the outcome.

See the discussion at the end of section 2.6 for more of this discussion but the key take away points are that under the plurality election rule the 'small' candidate does get votes due to bounded rationality initially but also, due to voters not being pivotal, the 'small' candidate still gets votes at all levels of rationality. Secondly, that the instant run-off election rule solves this problem by leading to sincere voting at all levels of rationality as well as eliminating wasted votes. The instant run-off election rule also reaches the outcome that is strictly preferred by a strict majority with a lower level of thinking than the plurality election rule takes to reach the same outcome. The next section will move onto an analysis of the literature mentioned in this section regarding 'wasted' votes and bounded rationality as well as other papers in the related literature.

2.2 Literature

Downs (1957) and Hotelling (1929), establishes the baseline median voter theory[7]. This considers the party position and voting problem for the simplest elections with only 2 candidates. Plurality, instant run-off, and all other single winner systems without a dictator, give the same results. Black (1948), gave the formal analysis and the explicit instructions to Hotelling linear city. All systems are equally simple. Applying the Nash strategy to such a game shows that when strategic, voters' weakly dominant strategy is to vote sincerely, and when pivotal this strategy is strictly dominant in 2 candidate elections for both systems we focus on. This result, while critical to the understanding of political economics, is limited due to its two candidate nature. Expanding on it requires consideration of three or more candidates.

Duverger (1959) argues that actually a larger number of voters does not really matter. Under plurality a multi-candidate system with strategic voters should in equilibrium tend towards only two parties gaining a positive vote share therefore the outcome is identical, these two candidates

[7]On a linear policy space, given sincere voters (there is never an incentive to be insincere in two candidate elections) parties all take positions most preferred by the middle voter.

being perceived as the viable winners of that election. See Palfrey (1988) for the mathematical proof of Duverger's Law. In theory, this should solve the problem of 3 candidate elections. Even if many candidates enter the electorate work out the viable winners and vote for them exclusively. Yet cases where a third party gets votes do still exist. Duverger's Law does not always hold and the cases where it fails to hold are significant and of interest. The idea of 'wasted votes' (votes that do not support one of the two viable winning candidates) frame this part of the literature. The next section looks at why voters 'waste' their vote and present some papers that argue that at times these votes are not 'wasted'.

2.2.1 Wasted Votes

Castanheira (2003) argues that agenda setting is a reason such 'wasted votes' might exist. In this model voters stick with candidates that are not expected to win so that in the future other more viable candidates will realise that such voters exist and might move to a position to win some of these 'wasted votes'.

One reason voters 'waste' votes is voters do not expect to be pivotal. If a voter does not expect to be pivotal they decide they do not need to worry about who wins, instead they opt to be sincere preferring honesty when all their voting options give the same outcome. Attempts were first made to calculate the probability of a decisive vote by Beck (1975). He noted that a vote "'will only be significant if an individual assumes all other voters are totally indifferent between the two [alternatives]."'. Chamberlain and Rothschild (1981) then showed that to the limit, a voter must assume two voting groups are of equal size to believe they might be pivotal. This case will be considered in this chapter and will be shown to be a key scenario where such 'wasted votes' are eliminated and the sincere equilibrium does not hold. As such they are of key importance if the aim is to incentivise both sincere revelation of preferences and voter optimal decision making. Since then the pivotal voting model has been more widely used, that introduces a degree of uncertainty over group types to allow for more than just this knife edge result. From this Ferejohn and Fiorina (1974), have shown that due to low pivotal probability for certain groups voting is not always rational. The pivotal voting models generally introduce a small cost of voting which is not recovered due to the low odds of changing the vote.

Fey (1997) suggests that such wasted votes appear due to a lack of information and that with pre-election polls voters will have enough information to move to the Duverger equilibrium. This lack of information is somewhat different to the idea posed in this chapter regarding bounded rationality. It suggests that voters are fully rational but do not know enough to work out who is a viable winner. This lack of information leading to wasted votes is seen in a lot of the divided majority literature of Myerson and Weber (1993) later tested by Forsythe, Myerson, Rietz, and Weber (1993, 1996) and for two round elections[8] by Morton and Rietz (2007). This literature supports the idea that the key issue is communication among a group that would like to coordinate but fail to. This failure to coordinate then leads to wasted votes.

There are a number of other rationales given for 'wasted votes' that I will not go into detail on such as split tickets for ethnic parties suggested in Chandra (2009). The strength of the candidates is also a generation of uncertainty as suggested in Myatt, Fisher, and others (2002) which tests this uncertainty on data from UK general elections.

In an argument that is closer to the one made in this chapter Tavits and Annus (2006) argue that voters take a period of time to learn what to do especially in new democracies. They show that over time, referring to Eastern European countries, as countries have democracy for longer the number of wasted votes goes down. This could be seen as a process of going up levels of thinking in the model in this chapter. As voters spend longer voting they learn more about themselves and more importantly the other voters and this leads to them applying higher levels of thinking and reduces the number of 'wasted votes'. This leads on to the literature around level-k thinking. This is the method by which bounded rationality is introduced into this chapter. The next section highlights the key papers on this topic and the alternative interpretations of bounded rationality including those chosen by this chapter and some that are not.

2.2.2 Bounded Rationality and Level-k

This chapter will analyse the level-k bounded rationality in Nagel (1995) in an election voting game. In her paper she proposes that people do not act with complete rationality they are in fact bounded by their rationality. She starts with the most bound person who with a level-0 strategy just randomly picks a candidate. The next level of person, level-1, assumes everyone that votes other than them are of level-0. They then best respond to such a group of voters.

[8]A voting rule that with 3 candidates is equivalent to the instant run-off

Each higher level best responds to the group one level below them until a point is reached where higher levels of voters do not change their voting strategy from those taken at a lower level. The idea of randomness being the lowest rationality is contentious in the literature. Bassi (2015) for example counters that even the least rational voter should know their own preferences. Therefore she proposes that the lowest level of rationality is to naively assume that there is no strategy better than sincerity. Bassi separates behaviour into strategic and sincere. With the aim of limiting manipulation which is defined as a voter behaving strategically (insincerely) to change the outcome of the election, naive voters in her model do not try to change the outcome of the election with strategy. They are therefore equivalent to voters in the sincere equilibrium from the last section. Where Bassi calls voters Naive we will simply call them sincere. Her paper uses this assumption about the level-0 to compare Approval voting[9] and Borda count[10] voting rules with Plurality. This chapter will add to this literature by looking at the instant run-off rule under similar conditions.

Other work in bounded rationality has been done by Camerer, Ho, and Chong (2004) where they defined the cognitive hierarchy. This model will not be considered in this chapter but it takes the assumptions laid out by Nagel (1995) and then expands her conditions. Where Nagel assumes that a voter at a higher level best responds to a group one level below them Camerer, Ho, and Chong (2004) assume that a voter best responds to a set of voters that are distributed among the levels below that voter. This gives a more general rule for such cognitive hierarchy models and an extension of this chapter could be to include such analysis alongside that of Nagel's level-k model.

2.3 Model

2.3.1 Voters and Candidates

The election consists of a set of players N with the size n. Each player is a voter in the election and an individual player, hereafter called a voter, is represented by i. The model will assume

[9]Approval voting is a winner takes all electoral system where each voter selects all candidates that she approves of. The winner is then the candidate that gets the most approval votes

[10]Borda count is a winner takes all electoral system where each voter ranks the candidates. Each candidate gets a fixed number of points from each voter based on where they appear in the ranking. The winner is then the candidate with the most points.

that n is odd[11].

$$N := \{1, , , n\} \text{ (N set of all voters)}$$
$$i \in N \text{ (i individual voter)}$$

There are three candidates in the set Ω and an individual candidate is represented by ω. These candidates will be called, candidate A, candidate B, and candidate C.

$$\Omega := \{A, B, C\}$$
$$\omega \in \Omega \text{ (} \omega \text{ individual candidate)}$$

2.3.2 Individual Preferences

Voter i's preference, $\succsim_i$, is defined over a set of feasible winning lotteries of the three candidates.

$$\succsim_i \; on \; \Delta\{A, B, C\}$$

There are 7 possible lotteries. These 7 lotteries are one of three types.

1. One candidate wins with certainty; in which case the lottery generates a single winner and the singleton lottery is written as A, B, or C depending on the candidate that wins.
2. Two candidates have a positive winning probability; in which case the odds of victory are uniform and each candidate has a one half probability of victory. These will be written as AB, AC or BC, depending on the two candidates that could win.
3. All three candidates have a positive winning probability; again the odds of victory are uniform and each candidate has a one third probability of victory. This is written as TIE[12]. For example:

$$\text{Lottery } \{1, 0, 0\} \equiv (A)$$
$$\text{Lottery } \{0.5, 0.5, 0\} \equiv (AB)$$
$$\text{Lottery } \{\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\} \equiv (TIE)$$

[11] This condition simplifies analysis but is not necessary for any of the results

[12] the reason for using TIE instead of (ABC) is to avoid confusion with the utility order (ABC) that is also defined in this section

The utility function $U_i(\omega)$ represents the voters' preferences; if a voter prefers a candidate, ω, over an alternative, ω', then the utility from a singleton lottery for candidate ω is greater than a singleton lottery for candidate ω'. The voter has strict preferences over the singleton lotteries therefore the utility relationship is strict.

$$U_i(\omega) > U_i(\omega') \iff (\omega) \succ_i (\omega')$$

The relationship is also transitive therefore the utility from singleton lotteries can be ordered/ranked.

$$\begin{gathered} U_i(\omega) > U_i(\omega') > U_i(\omega'') \\ \text{s.t. } \omega \neq \omega' \\ \omega \neq \omega'' \\ \omega' \neq \omega'' \\ \text{for some } \omega, \omega', \omega'' \in \Omega \end{gathered}$$

The strict utility order/rank will then be summarised with a three component list. For example:

$$U_i(A) > U_i(B) > U_i(C) \equiv (ABC)$$

This strict utility order explains the utility relationship over the singleton lotteries. This order also explains the utility relationship for two of the other lotteries. For the same example if the voter has utility order (ABC) then the utility from lottery (A) is strictly greater than the utility from lottery (AB) (as this lottery has a one half chance that B wins instead of A which is strictly worse), similarly the utility from lottery (AB) is strictly greater than the utility from lottery (B) etc. The full utility relationship that is now explained is;

$$\text{for } (ABC): \; U_i(A) > U_i(AB) > U_i(B) > U_i(BC) > U_i(C)$$

The utility relationship for the other two candidate tie, (AC), and the three way tie lottery, (TIE), are ambiguous under the definition usedhere to summarise voter preferences as a strict utility order over the singleton lotteries. Three cases must be considered that depend on the relationship between the singleton lottery (B) and the two candidate tie (AC) for the voter with utility order (ABC)[13]. More generally the case to consider is the relationship between the middle ranked candidate and a two way tie between the first and third ranked candidate. The three possible relationships are;

$$\text{For } (ABC): \ (B) \succ_i (AC), \ (B) \sim_i (AC) \ or \ (AC) \succ_i (B)$$

This ignores the three way tie but it can be shown that if the relationship between lottery B and AC is known then the position of the lottery TIE is also known. This is summarised below but for the full proof to this see appendix K.

$$\text{if } U_i(B) > U_i(AC) \text{ then } U_i(B) > U_i(TIE) > U_i(AC)$$
$$\text{if } U_i(B) = U_i(AC) \text{ then } U_i(B) = U_i(TIE) = U_i(AC)$$
$$\text{if } U_i(B) < U_i(AC) \text{ then } U_i(B) < U_i(TIE) < U_i(AC)$$

For a given voter with a given strict utility order it is not possible to assume that one of these three possible relationships always occurs. It will be shown in this paper that these differences never matter and all results hold for all three possible orders (AC), (B) and (TIE).

2.3.3 Preference profile and Support Vote

The $\succsim$ is a preference profile of the $\succsim_i$ preferences for all the i voters in N.

$$\succsim = \{\succsim_1, \succsim_2, \dots \succsim_n\}$$

The support vote, SV, for a candidate is then the number of voters that order that candidate first. These voters most preferred outcome is the singleton lottery for this candidate. For example the support vote for candidate A:

$$SV_A^1(\succsim) = \left\{ \#i \mid \succsim_i = \begin{Bmatrix} ABC \\ ACB \end{Bmatrix} \right\}$$

It is assumed that the support votes for candidate A and candidate B are 'large' and the support vote for candidate C is 'small' to represent the difference in 'large' and 'small' candidates in the model. Formally, a 'large' candidates' support vote will be defined as strictly greater than a

[13]The utility from (AC) compared to the other four lotteries is unambiguous: lottery (AC) is strictly worse than (A) and (AB) as half the time candidate C wins instead of candidate A and candidate B respectively and this is worse. By the same logic (AC) is strictly better than lottery (BC) and (C) making (B) the lottery where the utility relationship is ambiguous

'small' candidates' support vote plus two. With two 'large' candidates and one 'small' candidate, a 'large' candidates' support vote will be strictly greater than n divided by three. Finally, it will be assumed that no candidate has a zero support vote.

The difference here between a 'large' and a 'small' candidate is very minor and they can be very similar. The reason for this is that the distinction as defined here is sufficient for the cases discussed in this paper. Generally in the real world the difference between the 'large' and the 'small' candidates is much greater but this paper aims to show that this is not necessary and a much smaller difference is all that is required for the results. A larger difference between the 'large' and 'small' candidate does not change the results in this paper.

$$\mathfrak{R} \text{ is the set of all} \succsim \text{s.t. } V_A^1(\succsim),\ V_B^1(\succsim) > \tfrac{n}{3} > V_C^1(\succsim) + 2$$
$$V_\omega^1(\succsim) > 0\ \forall\, \omega \in \Omega$$

2.3.4 Strategies

This paper will analyse two strategies: firstly the sincere voting strategy and then the set of thinking levels for level-k thinking. These two strategies are introduced and defined below. The sincere voting strategy will then be analysed to see if all players playing this strategy is a Nash equilibrium which is outlined in in part 2.3.5. The strategy take by a voter at a given level of thinking is by definition a best response given their bounded rationality. So the analysis will focus on different levels of thinking and answer the question: if all voters take the strategy that is the best response at a given level is this a Nash equilibrium when their rationality is not bounded?

Sincere Voting strategy

A voter is sincere if they truthfully reveal their preferences to the social choice function. The profile $\succsim$ is sincere if all voters truthfully reveal their preferences. The model restricts the vote total such that when all voters truthfully reveal their preferences the round one vote total for A and for B are strictly more than two above the vote total C.

Level-K voting strategy

The second strategy considers the set of actions voters take when they have a finite depth of reasoning. It is assumed that voters are unable or unwilling to employ the necessary level of thinking to enact the completely rational strategy taken in equilibrium. Instead voters will be defined based on the level of thinking they implement in selecting their strategy. At a given level a voter has a set of beliefs about how rational the other voters are in the population.

The beliefs are calculated using the same belief orders as the players in Nagel (1995). Firstly the level-0 voter; they have $zero-order\ beliefs$. These voters select a strategy at random. They form no beliefs about other voters and do not pick a salient number that is important to them. They randomise over the action set; each candidate has a one third probability of being ordered first, ordered second or ordered third. An alternative definition will also be considered; level-0 voters will be considered naively sincere as suggested in Bassi (2015). In this case they will reveal a sincere set of preferences to the social choice function.

Using these two definitions of level-0 gives two specifications to consider. These will be defined as:

1. (Ns): level-0 voters reveal sincere preferences
2. (Nr): level-0 voters reveal random preferences

Next level-1 voters; they have $first$-$order\ beliefs$ on the behavior of the other players. These voters believe that all other voters make a vote at random. These voters then make a best response to this belief. The level-1 voter assumes all other voters are level-0. They reveal a set of preferences that is a best response to a population that vote randomly.

The level-2 voter has $second$-$order\ beliefs$. These beliefs are on the $first$-$order\ beliefs$ of the other voters. These voters then best respond to the belief that all other voters have $first$-$order\ beliefs$. The level-2 voter assumes all other voters are level-1. The belief of a voter at level-k can be generalized as: all other voters are thinking at level-(k-1) and hold one level of belief below themselves. The strategy at any given level-k is the best response to a population of voters at a level-(k-1).

A voter will take a strategy at a given level of thinking that is a best response to the population of voters one level below them. At a given level it can be ambigious what strategy is the best response. In such cases the voters will only deviate from a lower level strategy if it gives a strictly better outcome for them. When an agent thinking at level-k is indifferent between actions, one of which is their level-(k-1) action, they will remain with the decision at level-(k-1). This avoids the issue of non-pivotal equilibria in levels higher than k when an equilibrium has been reached at k.

2.3.5 Equilibrium Concept

The equilibrium concept used will be the Nash equilibrium[14]. The paper will not aim to fully define the set of Nash equilibria but analyse if the sincere voting strategy and each level of the level-k voting strategy when played by all voters reach a Nash equilibrium. The outcome of the SCF is a Nash equilibrium when no voter can give an alternative preference ranking that changes the outcome to one they prefer.

$$\succsim' \text{ is a Nash equilibrium if at } \succsim\text{: } scf\ (\succsim')\ \succsim_i\ scf\ (\succsim_i'', \succsim_{-i}')\ \forall\, i,\ \forall \succsim_i''$$

The SCF will represent either the SCF under plurality, $PL(\succsim)$, or the SCF under instant run-off, $IRO(\succsim)$.

2.4 Voting Rules

2.4.1 Plurality

$\Re$ represents the set of all $\succsim$. Then the social choice function f for the plurality voting rule PL maps $\Re$ to the lottery of candidate that wins the election.

$$\text{Social choice function: } PL\ f:\ \Re\ \rightarrow\ \Delta\{A,\ B,\ C\}$$

The function calculates the number of votes for each candidate: $V_A^1(\succsim)$, $V_B^1(\succsim)$ and $V_C^1(\succsim)$. For a given voter their vote will be cast for the candidate that they reveal as their most preferred

[14]see Nash (1950) for more details

candidate when they reveal their preferences. When all voters submit sincere preferences these values will equal the support vote.

Under plurality the candidate with the most votes wins the election. In this case the outcome of the SCF is a singleton lottery and a deterministic outcome. When more than one candidate tie on the most votes, the outcome of the SCF is a mixed lottery with a non-deterministic outcome. Each candidate that is tied for first has an equal chance of winning.

The set M represents the set of candidates that are tied for first. The value $|M|$ equals the number of candidates that are tied for first.

$$M(\succsim) := \{\omega | V^1_\omega(\succsim) \geq V^1_{\omega'}(\succsim)\ \forall \omega' \ \in \ \Omega\}$$

Each candidate that is in the set M has an equal probability of winning. This probability is equal to one divided by $|M|$.

$$\begin{aligned} PL(\succsim)(\omega) \ &= \ \tfrac{1}{|M|}\ if\ \omega \in M \\ PL(\succsim)(\omega) \ &= \ 0\ if\ V^1_\omega(\succsim) < V^1_{\omega'}(\succsim)\ for\ some\ \omega'\ in\ \Omega \end{aligned}$$

When $|M|$ equals one, the SCF has a deterministic outcome. If $|M|$ is greater than one, the SCF has a non-deterministic outcome. $|M|$ cannot be less than or equal to zero.

2.4.2 The Instant run-off

For the instant run-off, there are at most two calculation steps that can have different vote totals. Under the Instant run-off rule, a candidate wins by having more than half of the vote at a given step of the SCF. Once a candidate has over half the vote at a given step, later steps are not needed. The function f maps $\Re$ to the lottery of winners as it did under plurality. There is an odd number of voters, therefore there are no ties in an IRO election. The SCF only generates singleton lotteries. The outcome of an IRO election is always deterministic.

$$\text{Social choice function: } IRO\ f : \ \Re \ \rightarrow \ \{A,\ B,\ C\}$$

To select a winner, the function calculates the votes at two steps. In step one, the values $V^1_A(\succsim)$, $V^1_B(\succsim)$ and $V^1_C(\succsim)$ are calculated as they were for $PL(\succsim)$. These vote totals will equal the support vote if all voters reveal sincere preferences.

If any of these values exceed half the vote, that candidate wins. The only outcome that is generated from step one is a singleton for one candidate giving a deterministic victory for that candidate. When the election ends in step one the outcome of the SCF is:

$$IRO(\succsim)(\omega) \ = \ 1 \, if \ V^1_\omega(\succsim) \ > \ \tfrac{n}{2}$$

If no candidates exceeds half the vote then step two is needed. The social choice function generates a random linear order over the set Ω. This randomly assigns an order over the candidates A, B and C such that no candidate is repeated and all candidates are included.

Two candidates are selected, ω and ω' such that neither is the unique candidate with the lowest vote share in the first round. When two candidates have the equal lowest vote share ω is the candidate that is not the candidate with the lowest vote share and ω' is the higher ranked candidate from the random order among the candidates tied in last place. When three candidates are tied with one third of the vote each then neither ω or ω' is the unique candidate that comes last in the random order. The candidate ω'' that is not selected is eliminated from the election.

At this stage all votes must go to ω or ω'. This process generates the step two values for the two remaining candidates $V^2_\omega(\succsim)$, $V^2_{\omega'}(\succsim)$. Candidate ω'' vote total is zero.

$$\begin{aligned} V^2_\omega(\succsim)(\omega,\omega') \ &= \#\{\ i \mid \omega \succsim_i \omega'\} \\ V^2_{\omega'}(\succsim)(\omega,\omega') \ &= \#\{\ i \mid \omega' \succsim_i \omega\} \end{aligned}$$

There is no chance to change preference order once a candidate is eliminated. As such the only votes that can change between steps are those of voters who gave ω'' as their first preference. Voters had to give a full preference list over the three candidates and therefore the vote total at this stage still equals n.

$$V^2_\omega(\succsim) \ + \ V^2_{\omega'}(\succsim) \ = \ n$$

When the election ends in step two the outcome of the SCF is:

$$\begin{aligned} SCF(\succsim)(\omega) \ &= \ 1 \text{ if } V^2_\omega(\succsim) \ > \ \tfrac{n}{2} \\ SCF(\succsim)(\omega) \ &= \ 0 \text{ if } V^2_\omega(\succsim) \ < \ \tfrac{n}{2} \end{aligned}$$

As n is odd: $V_\omega^2(\succsim) \neq \frac{n}{2}$

2.5 Results

2.5.1 Sincere Voting results

Theorem 2.1. *Under the Plurality rule sincere revelation of preferences is an equilibrium if a candidate has a strict lead and all voters that most prefer candidate C are either not pivotal or all prefer the candidate with the strict lead.*

Proof: When voters reveal sincere preferences the social choice function, PL, outcome can only be a lottery over A and B. Without loss of generality, we will assume candidate A has a greater than or equal number of votes than candidate B. All results hold when candidate B has a greater than or equal number of votes than candidate A. Therefore the outcome of PL is A or AB.

$$PL(\succsim) = (A)$$
$$PL(\succsim) = (AB)$$

First consider$PL(\succsim) = (A)$: This is a Nash equilibrium under one of two conditions:

(1) No voter is pivotal

$$PL(\succsim_i', \succsim_{-i}) = (A) \; \forall \; i \text{ and } \forall \; \succsim_i' \neq \succsim$$

(2) all pivotal voters prefer A to B.

$$PL \; (\succsim_i', \succsim_{-i}) \prec_i (A) \; \forall \; i \text{ and } \forall \; \succsim_i' \neq \succsim \; s.t. \; PL(\succsim_i', \succsim_{-i}) \neq (A)$$

Table 2.1 outlines the preference profiles for which sincere voting is an equilibrium. Column one gives the necessary vote total conditions. Column two defines which voters are pivotal. Column three then gives the conditions needed for this to be a sincere equilibrium.

Vote Total Condition	Pivotal voters Preferences	Individual Preference Conditions
$V_A^1(\succsim) \geq V_B^1(\succsim) + 3$	No Pivotal Voters	No conditions
$V_A^1(\succsim) = V_B^1(\succsim) + 2$	ABC and ACB	$A \succ B$
$V_A^1(\succsim) = V_B^1(\succsim) + 1$	ABC and ACB CAB and CBA	$A \succ B$

TABLE 2.1: Pivotal voters and their necessary preferences for a set of vote totals to be a sincere equilibrium

$$V_A^1(\succsim) \;\geq\; V_B^1(\succsim)$$

The first-row states that if candidate A leads candidate B by more than two votes then no voter is pivotal. The second-row states that if candidate A leads candidate B by more than one but at most two votes than anyone that voted A must prefer candidate A to candidate B. All such voters do prefer candidate A to candidate B by definition. The final row states that if candidate A leads candidate B by one or less votes then all those that vote A and C must prefer candidate A to candidate B. This is true by the definition of their type. For candidate C's voters, it means that none can be type (CBA). As such this type must not exist in the election. There is no sincere equilibrium if candidate A leads candidate B by at most one vote and any of the voters that voted C prefer B, type (CBA).

Now consider $PL(\succsim) = (AB)$: This is never an equilibrium. When candidate A and candidate B have the same vote total and therefore the outcome is the lottery (0.5,0.5,0) all voters are pivotal. Therefore for this to be an equilibrium the candidates that voted for C must both prefer A to B (CAB) and B to A (CBA) or they will deviate to one of these candidates. This is not possible therefore this is never an equilibrium. ■

Theorem 2.2. *Under instant run-off sincere revelation of preferences is always an equilibrium.*

Proof: The social choice function, IRO, outcome can come from the first step of calculations or the second step of calculations.

Outcome in first step:

Occurs when a candidate has a strict majority. Only candidate A or B can have over half the vote when voters honestly reveal their preferences. Given the assumption that candidate A has weakly more votes than candidate B, then only candidate A can win in step one.

A candidate wins at step one:

$$V_A^1(\succsim) \geq \tfrac{n+1}{2}$$
$$IRO(\succsim) = (A)$$

This holds as an equilibrium under one of two conditions:

(1) No voter is pivotal

$$V_A^1(\succsim) > \tfrac{n+1}{2}$$
$$IRO(\succsim) = (A)$$

(2) all pivotal voters prefer A to B.

$$V_A^1(\succsim) = \tfrac{n+1}{2} \cap IRO(\succsim_i', \succsim_{-i}) \prec_i (A)\ \forall i \text{ and } \succsim_i' \ s.t\ IRO(\succsim_i', \succsim_{-i}) \neq (A)$$
$$IRO(\succsim) = (A)$$

The only types where $IRO(\succsim_i', \succsim_{-i}) \neq (A)$ are (ABC) and (ACB). All voters reveal their preferences honestly, therefore all those that are pivotal prefer candidate A to all other candidates. None of these voters can benefit from deviating away from the preferences they revealed. Sincere voting by all voters in such a case is therefore an equilibrium.

A candidate wins at step two:

This leaves any case where candidate A does not win in the first step and no candidate has a strict majority of the votes. This requires both steps of the social choice function to find the solution. The analysis starts with step two:

All voters reveal their preferences honestly therefore candidate C will have the least votes. Candidate A and candidate B progress to step two. In step two there can be no tie as there are an odd number of voters. It is assumed that candidate A has weakly more votes than candidate B in step one. This does not mean candidate A must have weakly more votes than candidate B in step two. Here it will be assumed that the candidate that wins in the second step is ω and the other candidate is ω'.

$$V_\omega^2(\succsim) \geq \tfrac{n+1}{2}$$

$$IRO(\succsim) = (\omega)$$

No one will deviate at step two (by changing the position they order candidate A and candidate B in):

(1) No voter is pivotal

$$V_\omega^2(\succsim) > \tfrac{n+1}{2}$$
$$IRO(\succsim) = (\omega)$$

(2) all pivotal voters prefer ω to ω'.

$$V_\omega^2(\succsim) = \tfrac{n+1}{2} \cap IRO(\succsim_i', \succsim_{-i}) \prec_i (\omega) \forall i \text{ and } \succsim_i' \; s.t \; IRO(\succsim_i', \succsim_{i-1}) \neq (\omega)$$
$$IRO(\succsim) = (\omega)$$

ω wins in the second step of the social choice function IRO with sincere voting. It must now be shown that knowing that ω wins in step two no voter will reveal an insincere first choice which will be used in the first step of analysis.

In step one when voters reveal preferences honestly, candidate C will be behind candidate A and candidate B by more than two votes. Therefore there is no deviation by a single voter such that candidate C is not the candidate that comes last.

The only way to change the outcome of function IRO is to get a candidate a strict majority so they win in the first step. For this to be possible this candidate must be within one vote of a strict majority. As candidate A has weakly more votes than candidate B it will first be assumed that just candidate A is within one vote of a strict majority. After this it is assumed that both candidate A and candidate B are within one vote of a strict majority.

$$V_A^1(\succsim) = \tfrac{n-1}{2}$$
$$V_B^1(\succsim) < \tfrac{n-1}{2}$$
$$IRO(\succsim) = (\omega)$$

There are then two cases, when $(\omega) = (A)$ and when $(\omega) = (B)$.

Case 1: The candidate that wins in the second step is candidate A: $(\omega) = (A)$.

This is always an equilibrium, no voter can change the outcome of the election. A deviation to A gives candidate A a strict majority in step one and candidate A wins, $IRO(\succsim_i', \succsim_{-i}) = (A)$. There is no benefit to this deviation as candidate A will win in step two without deviation therefore the result is not changed. Therefore in this case sincere voting by all voters is an equilibrium.

$$IRO(\succsim_i', \succsim_{-i}) = IRO(\succsim) \ \forall i \text{ and } \forall \ \succsim_i'$$
$$IRO(\succsim) = (A)$$

Case 2: The candidate that wins in the second step is candidate B: $(\omega) = (B)$.

This is always an equilibrium, no voter that can change the outcome wants to. If candidate A is within one vote of having a strict majority and candidate B wins in step two there must be no voters whose vote moves to candidate A in step two. There are no (CAB) types.

$$\text{if } V_A^1 = \tfrac{n-1}{2} \text{ and } V_B^2 \geq \tfrac{n+1}{2} \text{ then } V_A^1 = V_A^2$$

All the voters that most prefer C can deviate to A but all prefer candidate B to candidate A and will therefore not benefit from deviation. Therefore in this case sincere voting by all voters is an equilibrium.

$$IRO(\succsim_i', \succsim_{-i}) \prec_i IRO(\succsim) \ \forall i \text{ and } \succsim_i' \ s.t \ IRO(\succsim_i', \succsim_{-i}) \neq IRO(\succsim)$$
$$IRO(\succsim) = (B)$$

Finally, assume both candidate A and candidate B are within one vote of having a strict majority at step one and candidate ω still wins the election.

$$V_A^1(\succsim) = \tfrac{n-1}{2}$$
$$V_B^1(\succsim) = \tfrac{n-1}{2}$$
$$IRO(\succsim) = (\omega)$$

In all such cases $V_C^1 = 1$ as the sum of all three candidates vote share must equal n. This single voter must prefer ω to ω' as $V_\omega^2 = \frac{n+1}{2}$. They can deviate to ω' and ω' will win in step one however they prefer ω so such a deviation gives a lower utility. There is also no benefit from changing their most preferred choice to ω as this means ω wins in step one but they will win in step two anyway.

$$IRO(\succsim_i', \succsim_{-i}) \prec_i (\omega) \ \forall i \text{ and } \succsim_i' \ s.t \ IRO(\succsim_i', \succsim_{-i}) \neq (\omega)$$
$$IRO(\succsim) = (\omega)$$

Therefore in all such cases sincere voting by all voters is an equilibrium. ■

Discussion: The above analysis shows that a profile where all voters honestly reveal their preferences is always an equilibrium for the IRO but there is a set of vote distributions where the plurality rule does not lead to a sincere equilibrium. In moving to an IRO voting rule, a designer can guarantee that sincere voting will always be an equilibrium. This is beneficial if the aim of designing the election rules is to induce sincere revelation of preferences.

The next step is to compare the results of the election under sincere revelation of preferences. In this analysis, we can exclude any voting distribution where the two voting rules give the same outcome as sincere voting as in such cases all voters are indifferent between voting rules. When a sincere equilibrium exists for plurality, the two rules give different equilibria outcomes when the candidate (or candidates) that leads in step one does not equal the candidate that wins in step two. Otherwise they will give the same outcome.

Keeping the assumption that candidate A has weakly more support than candidate B the plurality election can have one of two outcomes: $O = (A), (AB)$. Of these only (A) exists in equilibrium. The instant run-off election can have one of two outcomes: either candidate A wins or candidate B wins: $O = (A), (B)$. When the outcome of the plurality election is (A) then the outcome of the instant run-off election must be (B) as the alternative outcome, (A), would mean the two rules give the same outcome.

If candidate B wins in step two and all voters honestly reveal their preferences then it must be the case that a strict majority prefer candidate B to candidate A in the population. Therefore when the outcome of plurality is candidate A winning and the outcome of the instant run-off election is that candidate B wins then a strict majority of voters prefer the outcome of the election under the instant run-off election rule compared to the plurality election rule when they are different. When the rules do not give different results all voters are indifferent.

Finally, consider the voter distributions where plurality does not lead to a sincere equilibrium. In such cases the lead candidate A has over candidate B is less than 1 (as before it is assumed without loss of generality that candidate A has weakly more sincere support than candidate B).

No voter that most prefers candidate A or candidate B will deviate but a voter that most prefers candidate C might. Doing so changes the result from (A) to (B) or vice versa. Importantly this means that the outcome is still only $O = (A), (B)$. So after deviation either candidate A or candidate B wins. When this new result (which can be the same as the old result but does not have to be) gives different results to the result under the instant run-off election rule then the instant run-off election rule leads to the candidate that a strict majority prefer between candidate A and candidate B. Therefore when the results are different the instant run-off election rule is preferred by a strict majority compared with the result of the plurality election rule. When the rules do not give different results all voters are indifferent.

2.6 Level - K voting Results

Here the results consider the optimal strategy at different levels of thinking where voters are bounded in their rationality. In these results the strategies at different levels of thinking are considered and compared for different voting rules. One way to compare the equilibrium at different levels of thinking will be to analyse if such outcomes are equilibrium when voters are not bounded by their rationality.

2.6.1 Instant Run-off

For instant run-off only one equilibrium is reached as k goes to infinity and that is the sincere equilibrium where all voters reveal their preferences sincerely.

Theorem 2.3. *With 'naive' sincere, (Ns), level-0 voters the instant run-off rule reaches an unbounded equilibrium at level-0. Sincere voting is the optimal strategy at all levels. The Nash equilibrium when k goes to infinity is the sincere equilibrium.*

Proof: For the (Ns) specification the level-0 voter is sincere, honestly revealing their preferences. Therefore when all voters are level-0 the IRO generates a profile with honest revelation of preferences.

$$IRO(\succsim) = (\omega) \; iff \; V^1_\omega(\succsim) \geq \frac{n+1}{2} \; and/or \; V^2_\omega(\succsim) \geq \frac{n+1}{2}$$

The level-1 voter assumes all voters are level-0. Their expectations are then a profile with honest revelation of preferences. From section 2.2 it is known that for instant run-off all voters voting sincerely leads to an equilibrium. Therefore level-1 voter will not deviate away from sincere voting as the election is in equilibrium. The level-2 voter assumes all voters are level-1. As all level-1 voters remain sincere the level-2 voter's expectations are also the sincere profile.

The Nash equilibrium when k goes to infinity is the sincere equilibrium and this is reached at level-0. ■

Theorem 2.4. *With randomizing, (Nr), level-0 voters the instant run-off rule reaches an unbounded equilibrium at level-1. Sincere voting is the optimal strategy at all levels from level-1 and above. The Nash equilibrium when k goes to infinity is the sincere equilibrium.*

Proof: For the Nr specification the level-0 voter is random: they present a preference at random. These preference only need to be complete, transitive and strict in singleton lotteries. The SCF generates a random profile where each candidate has an equal chance of victory.

$$IRO(\succsim) \implies \{\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\}$$

The level-1 voter assumes all voters are level-0 and their expectations are then the uniformly random profile. The best response to a random profile where each candidate has an equal chance of victory is to vote sincerely. A deviation to ω increases the probability ω wins by a fixed amount δ and decreases the probability of all other candidates winning by an equal amount, For example if for $\succsim_i'$ the voter deviates to A:

$$IRO(\succsim_i', \succsim_{-i}) = (\tfrac{1}{3} + \delta, \tfrac{1}{3} - \tfrac{\delta}{2}, \tfrac{1}{3} - \tfrac{\delta}{2})$$

The probability candidate A wins is $\frac{1}{3} + \delta$ etc. The value for δ is not affected by which candidate a voter deviates to. So, the odds of the candidate that is a voter's lowest preference winning is not affected by whether the voter selects their first or second most preferred candidate.

Therefore, it is best to present a profile where the candidate that gets the vote is the candidate that maximizes utility. Therefore a level-1 voter will vote sincerely.

Finally the level-2 voter assumes all voters are level-1. Their expectation is then the sincere profile. Therefore, the level-2 voter under the Nr specification is equivalent to the level-1 voter under Ns. The remaining results follow the same process as Ns.

The Nash equilibrium when k goes to infinity, is the sincere equilibrium and it is reached at level-1. ■

2.6.2 Plurality

For Plurality there will be one of three equilibria reached as k goes to infinity (voters rationality is unbounded): the sincere equilibrium, the Duverger's Law equilibrium and the non-pivotal equilibrium (the sincere equilibrium and the Duverger's Law equilibrium can be cases of the non-pivotal equilibrium. Here the non-pivotal equilibrium will be all equilibria excluding these two types of equilibrium.)

Definition 2.1. The Duverger's Law Equilibrium: This is an equilibrium where only two candidates get a positive share of the vote.

Definition 2.2. The Non-pivotal Equilibrium: This is an equilibrium where no voter is able to change the outcome of the election so there is no incentive to deviate.

Theorem 2.5. *With 'naive' sincere, (Ns), level-0 voters the plurality election has a sincere equilibrium under the same conditions as theorem 2.1. The strategy played in equilibrium is reached at level-0.*

Proof: All level-0 voters are sincere. It will again be assumed that candidate A has weakly more sincere votes than candidate B.

$$V_A^1(\succsim) \geq V_B^1(\succsim)$$
$$PL(\succsim) = (A)$$

The level-1 voters have to best respond to a population of sincere voters. If all level-1 voters best response is to vote sincerely, then there is a sincere equilibrium at level-1. The level-2 voters are then identical to the level-1 voters and so on as k goes to infinity. The sincere equilibrium occurs under one of two conditions.

1) No voter is pivotal:

$$PL(\succsim_i', \succsim_{-i}) = (A) \; \forall \; i \text{ and } \forall \; \succsim_i' \neq \succsim$$
$$PL(\succsim) = (A)$$

(2) All pivotal voters prefer A to B.

$$PL\ (\succsim_i', \succsim_{-i}) \prec_i (A)\ \forall\ i \text{ and } \forall\ \succsim_i' \neq \succsim\ s.t.\ PL(\succsim_i', \succsim_{-i}) \neq A$$
$$PL(\succsim) = (A)$$

These are the same results as those found in theorem 2.1. When a sincere equilibrium exists it will be reached at level-0 and all voters at higher levels of thinking will vote sincerely. ■

Theorem 2.5 means that the remaining results will only consider the cases where there is no sincere equilibrium. This occurs in one of two types of vote total: 1) $V_A^1(\succsim) = V_B^1(\succsim)$ regardless of the preferences of the voters that most prefer candidate C and 2) $V_A^1(\succsim) = V_B^1(\succsim) + 1$ if there exists at least one voter with preferences (CBA). The next two proofs calculate the necessary conditions for the other two.

Theorem 2.6. *With 'naive' sincere, (Ns), level-0 voters(preference profile $(\succsim)$) the plurality election has a Duverger's Law equilibrium under one of three conditions:*

1. $V_A^1(\succsim) = V_B^1(\succsim)$
2. $V_A^1(\succsim) = V_B^1(\succsim) + 1 \cap (\#i|\succsim_i = (CAB)) = 0$
3. $V_A^1(\succsim) = V_B^1(\succsim) + 1 \cap (\#i|\succsim_i = (CBA)) = 1, 2$

In condition one and two the Duverger's Law equilibrium strategy is reached at level-1 and in condition three it is reached at level-2.

Proof: The level-0 voters are sincere. There are two possible outcomes of the election with level-0 voters (that do not end in the sincere equilibrium):

1. $PL(\succsim) = (A)$ if $V_A^1(\succsim) = V_B^1(\succsim) + 1$
2. $PL(\succsim) = (AB)$ if $V_A^1(\succsim) = V_B^1(\succsim)$

No voter that most prefers candidate A, $((ABC), (ACB))$, will prefer to deviate. The outcomes of a deviation by a voter that most prefers candidate A are:

1. When $V_A^1(\succsim) = V_B^1(\succsim) + 1$; $PL(\succsim_i', \succsim_{-i}) = (AB)$ or (B) depending on if the deviation is to making C or B the first choice respectively. In either case deviating is worse than not deviating; $PL(\succsim_i', \succsim_{-i}) \prec_i PL(\succsim)$.

2. When $V_A^1(\succsim) = V_B^1(\succsim)$; the same logic holds as the outcome of any deviation is now $PL(\succsim_i', \succsim_{-i}) = (B)$. Deviating is again worse than not deviating; $PL(\succsim_i', \succsim_{-i}) \prec_i PL(\succsim)$.

No voters that most prefer candidate B, $((BAC), (BCA))$, will prefer to deviate when they can. The outcomes of a deviation by a voter that most prefers candidate B are:

1. When $V_A^1(\succsim) = V_B^1(\succsim) + 1$; there is no deviation that changes the result, $PL(\succsim_i', \succsim_{-i}) = PL(\succsim)$ so there will be no deviation.

2. When $V_A^1(\succsim) = V_B^1(\succsim)$; the same logic holds for voters that most prefer B that held for voters that most prefer A. The outcome of any deviation is $PL(\succsim_i', \succsim_{-i}) = (A)$. Deviating is again worse than not deviating; $PL(\succsim_i', \succsim_{-i}) \prec_i PL(\succsim)$.

This leaves the preference revelation of the voters that most prefer candidate C, $((CAB), (CBA))$. One or both of these types will deviate and will do so when pivotal. Both deviations are such that their second most preferred option is presented as their most preferred option:

$$(CAB) \text{ types deviate to } (ACB)$$
$$(CBA) \text{ types deviate to } (BCA)$$

If both opt to be insincere in their preference revelation in equilibrium this is a Duverger's Law equilibrium. If only one opts to be insincere in their preference revelation in equilibrium this is a non-pivotal equilibrium, which is covered in the next theorem.

Case 1: When candidate A and candidate B are tied all voters are pivotal and will deviate.

$$PL(\succsim) = (AB) \text{ if } V_A^1(\succsim) = V_B^1(\succsim)$$

If a (CAB) type voter reveals preferences (ACB) then the outcome of the social choice function is candidate A wins; $IRO(\succsim_i', \succsim_{-i}) = (A)$. They prefer candidate A to candidate B therefore this is better than the outcome without deviation; $IRO(\succsim) = (AB)$. Therefore the deviation happens. The same logic holds for the (CBA) type voters. Therefore all voters that prefer candidate C have deviated to candidate A or candidate B.

$$
\begin{aligned}
V_A^1(\succsim') &= \#(i|A \succsim_i B) \\
V_B^1(\succsim') &= \#(i|A \succsim_i B) \\
PL(\succsim') = \omega \text{ s.t. } \omega \in \{A, B\} \cap V_\omega^1 &> V_{\omega'}^1 \text{ s.t. } \omega' \in \{A, B\} \backslash \omega
\end{aligned}
$$

This is the same strategy as the second step of the IRO social choice function with sincere revelation of preferences and this has been shown to always be an equilibrium in theorem 2.2. This does not need to be repeated here. The level-1 voters take the strategy where they vote for the candidate they prefer between candidate A and candidate B. All players doing this is an equilibrium therefore the level-2 voters who best respond to the population of level-1 voters do so by taking the same strategy as the level-1 voter.

Case 2: When candidate A has one more vote than candidate B.

$$
\begin{aligned}
V_A^1(\succsim) &= V_B^2(\succsim) + 1 \\
PL(\succsim) &= (A)
\end{aligned}
$$

As with the last case, and for the same reason, all voters that most prefer candidate A and candidate B will not deviate. Additionally, no voter that most prefers candidate C and ranks candidate A second, (CAB), will deviate. If they deviate to $\succsim'$ so that their preference revelation is B as their first choice then the outcome is $IRO(\succsim_i', \succsim_{-i}) = (AB)$ this gives a lower utility than the outcome without deviation. If they deviate so that their preference reveals A as their first choice then the outcome is unchanged.

$$PL(\succsim_i, \succsim_{-i}) \prec (A) \; \forall \, i \text{ s.t. } (\succsim_i = (CAB) \text{ when } PL(\succsim_i, \succsim_{-i}) \neq A$$

The voters that most prefer candidate C and rank candidate B second, (CBA), will deviate. If they deviate the change to the outcome is the same as when a (CAB) type deviates. The difference is that they prefer the outcome (AB) to (A).

$$PL(\succsim_i, \succsim_{-i}) \succ (A) \; \forall \, i \text{ s.t. } (\succsim_i = (CBA) \text{ when } PL(\succsim_i, \succsim_{-i}) \neq A$$

Therefore with level-1 thinking all those that sincerely prefer candidate A and candidate B

remain sincere as does the type (CAB) but type (CBA) gives insincere preferences and reveals (BCA) instead. This is a Duverger's Law equilibrium if there are no (CAB) type voters. Then all voters vote for either candidate A or candidate B and as shown in 2.2 this is an equilibrium. Then all levels above level-1 play the same strategy as level-1.

Case 3: This continues on from **case 2**. Candidate A has one more vote than candidate B at level-0 with preference revelation $\succsim$.

$$V_A^1(\succsim) + V_B^1(\succsim) + 1$$
$$PL(\succsim) = A$$

Then in level-1 the preferences that are revealed, $\succsim'$, are still sincere for all types of voter except (CBA) who reveal (BCA). Now assume that there are a strictly positive number of (CAB) and (CBA)type voters. The level-2 thinking will lead to the Duverger's Law equilibrium if and only if the (CAB) type voters deviate. This occurs if candidate A is tied with candidate B or one behind candidate B. This occurs if there are exactly one or two voters that at level-0 revealed preferences of type (CBA).

$$V_A^1(\succsim') = V_B^1(\succsim') \text{ if } (\#i| \succsim_i = (CBA)) = 1$$
$$PL(\succsim') = (AB)$$
$$\text{or}$$
$$V_A^1(\succsim') = V_B^1(\succsim') - 1 \text{ if } (\#i| \succsim_i = (CBA)) = 2$$
$$PL(\succsim') = (B)$$

In either outcome all (CAB) types will want to reveal alternative preferences $\succsim''$ of (ACB). The outcome when $(\#i| \succsim_i = (CBA)) = 1$ is; $PL(\succsim_i'', \succsim_{-i}') = (A)$. The outcome when $(\#i| \succsim_i = (CBA)) = 2$ is; $PL(\succsim_i'', \succsim_{-i}') = (AB)$. These outcomes are better than (AB) and (B) respectively. Therefore all level-2 voters that reveal sincere preferences (CAB) at level-0 and level-1 now deviate to insincere revelation of preferences and now all voters reveal preferences that lead to a vote for candidate A or B and the Duverger's Law equilibrium has been reached and as with the other two cases this is an equilibrium for all higher levels of thinking. ■

Theorem 2.7. *With 'naive' sincere, (Ns), level-0 voters the plurality election has a non-pivotal equilibrium if three conditions hold:*

1. $V_A^1(\succsim) = V_B^1(\succsim) + 1$

2. $(\#i| \succsim_i = (CAB)) \geq 1$

3. $(\#i| \succsim_i = (CBA)) \geq 3$.

The strategy played in equilibrium is reached at level-1.

Proof: For there to be a non-pivotal equilibrium that is not the sincere equilibrium it must be the case that one of the groups that most prefer candidate C, $((CAB), (CBA))$, deviates to a preference revelation that ranks their second choice first (ACB) and (BCA) respectively. It cannot be the case that both change the preference they reveal as this is then the Duverger's Law equilibrium. Therefore this can only happen when candidate A has one more vote than candidate B from the sincere revelation of preferences $\succsim$. The outcome of sincere revelation of preferences is therefore candidate A winning.

$$V_A^1(\succsim) = V_B^1(\succsim) + 1$$
$$PL(\succsim) = (A)$$

The voters with sincere preferences (CBA) will then deviate at level-1 to $\succsim_i' = (BCA)$. The voters with preference (CAB) does not deviate as in all cases the outcome is either the same or worse. These two results were shown in 2.6.2 and are not repeated here. The outcome of the level-1 election is then either a tie between candidate A and candidate B or victory for candidate B.

$$PL(\succsim') = (AB), (B)$$

If there are no voters with sincere preferences (CAB) then this leads to the Duverger's Law equilibrium therefore there must be at least one such voter.

$$(\#i| \succsim_i = (CAB)) \geq 1$$

At level-2; if the result of a level-1 election is a tie between candidate A and candidate B then all the (CAB) type voters will want to deviate their revealed preferences to (ACB). Therefore the election can not be tied and there cannot only be one voter with sincere preferences (CBA). For the same reason candidate B cannot win by a single vote or the voters with sincere preferences equal to (CAB) will want to deviate. Therefore the number of (CBA) type voters cannot be one or two (it can also not be zero as this will always lead to a sincere equilbrium see 2.2).

$(\#i|\succsim_i = (CBA)) \geq 3$

When these three conditions hold, candidate B wins the election, no voter is pivotal and all voters vote sincerely except from those voters with sincere preferences (CAB). This non-pivotal equilibrium is reached at level-1 thinking, $\succsim'$. If all voters are level-0, $\succsim$, candidate A wins but this is not an unbounded equilibrium (it only exists due to voter's inability to reason a better response).

$$PL(\succsim) = (A)$$
$$PL(\succsim') = (B)$$

Theorem 2.8. *With randomized, (Nr), level-0 voters the result is identical to that of the (Ns) specification except each strategy occurs at a level of thinking one above that of the (Ns) specification and the level-0 thinker is random.*

Proof: At level-0 all voters vote randomly. As shown in proof 2.6.1 the best response to a population of random voters is to reveal sincere preferences. Therefore all level-1 voters vote sincerely. Therefore level-1 voters under the assumptions of (Nr) act in the same way as level - 0 voters under the (Ns) set of assumptions. All results found in proof 2.5, proof 2.6.2 and proof 2.7 hold for randomized level-0 voters except each strategy is reached one level higher as are the equilibria.

Discussion: Subsection 2.6.2 proves that there are 3 possible equilibria for the plurality rule: the sincere equilibrium, which exists under the same conditions as those laid out in subsection 2.1, the Duverger's Law equilibrium, where only two candidates get a positive share of the votes in step one and then finally a non-pivotal equilibrium where some percentage of supporters of the 'small' candidate do shift their vote to one of the 'large' candidates but not all do.

The sincere equilibrium is reached at level-0 and level-1 respectively for Ns and Nr. There is always a sincere equilibrium under the Instant Run-off election rule, see subsection 2.2 for the proof. This is not the case for the plurality election rule which occurs under the same conditions as those in subsection 2.1. The sincere equilibrium outcome is always candidate A or candidate

B. The equilibrium outcome under the instant run-off election rule will always be the candidate, between *A* and *B*, that is preferred by a strict majority of voters. Then, in any case where the two mechanisms give different outcomes, the outcome under the instant run-off election rule is preferred by a strict majority of voters compared to the plurality election rule.

The Duverger's Law equilibrium is reached when all supporters of the 'small' candidate have an incentive to deviate to one of the 'large' candidates. Duverger's Law equilibrium is reached at level-(1 or 2) and level-(2 or 3) respectively for Ns and Nr. The outcome of this election is the same as the outcome of the instant run-off election under the assumption of sincere voting. To reach this equilibrium requires an additional level of rationality regardless of specification. If we restrict to lower levels of rationality, then the two mechanisms can give different outcomes. When they do, the outcome under instant run-off election rule is preferred by a strict majority. When they give the same outcome the instant run-off election rule requires a lower level of rationality.

Finally the non-pivotal equilibrium exists when only some of the supporters of the 'small' candidate have an incentive to shift their vote, specifically those that prefer one 'large' candidate do deviate to that candidate while those that prefer the other 'large' candidate do not deviate. This will always require a higher level of thinking than the instant run-off election rule to reach equilibrium. The instant run-off election rule reaches equilibrium at level-0 or level-1 in Ns and Nr respectively while the non-pivotal equilibrium, will require at least level-1 and level-2 for Ns and Nr respectively. The outcome of the non-pivotal equilibrium will always be victory for the candidate that came second in the profile with honest revelations of preferences. Unlike the Duverger's Law equilibrium, the outcome of the non-pivotal equilibrium can be different from the outcome under instant run-off election rules. When the rules give different outcomes the outcome under the instant run-off election rule leads to an outcome that is strictly better for a strict majority of the population compared to the plurality election rule.

So at best, the outcome of the plurality election rule is the same as the instant run-off election rule while taking the same number of levels to reach equilibrium and is preferred by a strict majority to the other 'large' candidate. There exists a set of voter type distributions, such that the outcome is strictly worse for a strict majority and/or takes longer to be reached.

In summary, this supports the argument that the plurality election rule leads to 'wasted' votes due to bounded rationality at lower levels of thinking. It also suggests that even at higher levels of thinking with unbounded rationality 'wasted' votes occur due to voters not being pivotal. This is one rationale for the phenomenon of wasted votes that is suggested as a problem/ contradiction to Duverger's Law as well as being seen in the real world. One solution to this problem is the instant run-off election rule; this eliminates the 'wasted' vote problem as interpreted in this paper. It also requires a lower level of thinking to implement the strategy played with compete rationality. Finally it gives a better payoff to a strict majority of voters compared to the plurality election rule when these rules have different outcomes.

2.7 Conclusion

The chapter presents a social choice function voting model with three candidates with three objectives; (1) defining the sincere equilibrium for cases with two 'large' candidates and one 'small' candidate to compare plurality and instant run-off (2) to apply level-k thinking to election models to shed light on bounded rationality in voting games (3) to argue that the instant run-off election rule is superior with voters of bounded rationality in election systems with two dominant parities/candidates.

The model investigated the effect a 'small' candidate that is not expected to win can have in an election. The first aim of the chapter was to characterises the full equilibrium under two alternative voting rules. The Instant Run-off rule was found to give a sincere equilibrium for all vote shares under the condition of 'large' and 'small' candidates. Plurality by comparison only sometimes has a sincere equilibrium. The chapter then defined the cases where plurality fails to have a sincere equilibrium.

The second aim of the paper was to give the outcomes for the level-k bounded rationality thinking in Nagel (1995) in a three candidate voting game. This adds to the literature of the idea of 'wasted votes'. The argument in this paper is that some voters make 'wasted votes' because they are bounded in their rationality and are unable to consider the higher levels of thinking necessary to reach a Duverger's Law equilibrium. It also finds that under certain conditions voters still 'waste' their vote due to not being pivotal.

It was shown that the Nash equilibrium when k tends to infinity for the instant run-off election rule was always the sincere equilibrium. All three candidates got a positive vote share as predicted by Duverger (1959) in the corollary he gave regarding run-off elections. This requires a low level of thinking. When Nagel's random level-0 voter is assumed the sincere equilibrium strategy is reached by level-1 thinkers. Alternatively if the 'naive' voter from Bassi (2015) is assumed the sincere equilibrium strategy is reached by level-0 thinkers.

The results for plurality support Duverger's Law in part. The Duverger's Law equilibrium does exist for certain distributions of voters but the sincere equilibrium also still exists as does a set of non-pivotal equilibria.

When the Duverger equilibrium occurred under the plurality election rule, it gave the same results as the instant run-off election rule did but required a higher level of thinking to reach the equilibrium. When the plurality election rule ended in the sincere equilibrium, this was reached at the same level of thinking as the instant run-off election but not always with the same outcome.

A third set of equilibria were reached for plurality. In this set of non-pivotal equilibria all three candidates get a positive vote share but not all voters are sincere. This supports the idea of the 'wasted' vote not due to lack of information but rather due to voters not being pivotal. Reaching this equilibrium requires a higher level of rationality than the equilibrium reached under the instant run-off rule.

The final aim of the chapter was to argue that the instant run-off election rule is preferable when voters are bounded in their rationality and there are two large parties. This chapter finds that there are a number of reasons to prefer the instant run-off election rule. Firstly, when the equilibrium outcome of the plurality rule and the instant run-off rule are different the outcome of the instant run-off election rule was always strictly preferred by a strict majority of voters. So the instant run-off election rule would be considered better by anyone that wants a social choice function to represent the majority of votes.

The second reason to prefer the instant run-off rule is that it takes a weakly higher level of thinking to reach the equilibrium under the plurality voting rule. What this means is that the strategy that would be played by a perfectly rational voter is reached with a lower level of thinking using the instant run-off election rule (or the same level of thinking). Therefore even when both rules give the same outcome and can be argued to be equal it requires voters with a higher level of rationality to reach this equilibrium. So the instant run-off election rule would be considered better by anyone that wants an election rule that requires the voters to be able to act optimally without thinking as much. This makes voting more accessible.

The final reason to prefer the instant run-off rule in this case is that it allows sincere voting at the same time as letting voters optimise their strategy. Voters do not have to make a trade off between honest revelation of preferences and optimal decision making. This is preferable if the aim of elections is to receive the honest opinion of the voter and not just the outcome of tactical voting that obscures the real will of the people.

Chapter 3

Candidate Divergence, Entry, and the Run-off Rule

3.1 Introduction

The first two chapters analysed voter strategy in three candidate elections. In chapter one the Divided majority problem was tested experimentally under two related run-off rules. The first is the two round run-off with a majority threshold rule $(2RM)$; this is used in France's presidential election along with the majority of other presidential systems. The other is a flexible threshold $(2RF)$; this is loosely similar to that of Argentina and a number of other central and south American countries. In chapter two Sincere voting and level-k bounded rational thinking were analysed. The first rule analysed was the Plurality (P); this is used in 19 presidential systems and is loosely similar to the United States electoral system[1]. The second rule was the Instant run-off system (IRO); this is used to elect the Irish president and the parliament of Australia. These four systems represent the vast majority of non-proportional representation elections in the world.

The first two chapters focused on the voter's strategy under these rules and other conditions, in both cases a degree of policy divergence between candidates was assumed. Prior to this chapter no discussion on what can motivate such a divergence was had. A large number of papers have discussed what causes policy divergence and I present a non-exhaustive list in the literature review in section 3.2. One prominent paper by Palfrey (1984) proposes that candidate

[1] theirs is a plurality of the electoral college not a population plurality

divergence can be used as a crowding out method by incumbents to defeat entrant parties. In Palfrey's paper two candidates position themselves simultaneously preempting entry by a third candidate. The paper proves that for Plurality the two candidates take symmetrically opposite positions around the median voter and the entrant takes a position between the two candidates. When the population is more concentrated around the median the entrant will take the median position. Palfrey (1984) assumes sincere voting and shows that the entrant gets a smaller share of the vote than the incumbents and never wins the election.

The model in Palfrey (1984) puts additional restrictions on the utilities of the voters, compared to my first two chapters, by using the Downs (1957) and Hotelling (1929) linear city structure and making voters' utility single peaked. In spite of this, the model still gives a good structure to understand the policy positions that the candidates take in my first two chapters and explain the divergence. This Chapter, using the same structure as Palfrey (1984), will analysis the candidate strategies and equilibrium outcome for the four election rules from chapter one and two; 3.4.1 Plurality, 3.4.2 Instant Run-off, 3.4.3 two Round with majority Threshold, and 3.4.4 two round with flexible threshold.

Sincere voting has been shown to be a viable strategy under all four rules and will be assumed in each case for this chapter. The strength of this assumption and the possible effect of dropping it will be discussed in the results but will not be modeled. This chapter will compare the candidate strategies under the four rules. It will present the effect that changing the election rules has on the policy positions of different candidates as well as how it changes who win the election and the policy that is implemented. As the first two chapters set the policy of the candidates and the utility that a voter gets from each candidate exogenously this chapter will aim to see if a simple Downs-Hotelling model with entry leads to candidates taking policy positions that replicate those seen in the first two chapters. Neither of the first two chapters used single peaked preferences, as such there will be gaps or omission where the results of the analysis in this chapter does not match perfectly with that of the other chapters. Where this occurs it will be highlighted and discussed.

The simplest observation of the chapter is that if candidates cannot change their position between rounds. With three candidates the strategy of the candidates is identical under the (IRO) rule and the $(2RM)$ rule. Under the (IRO) rule voters rank their candidates sincerely, if a candidate has a strict majority that candidate wins, if no candidate has a strict majority the candidate with the least votes is eliminated. The votes for the candidate that got eliminated get instantly moved to the second ranking of the voters that had supported the eliminated candidate. With three candidates this only occurs once. Alternatively under $(2RM)$ voters vote for their most preferred candidate. If a candidate has a strict majority that candidate wins. If no candidate has a strict majority the two candidates with the most votes go through to a second election. In this election everyone votes again and votes for their most preferred candidate. For those whose most preferred candidate from round one is still in the election their vote has not changed. For those whose most preferred candidate was eliminated in the first round their vote will be for their second choice. These two definitions are equivalent to the candidate under the conditions of three candidates, no policy shift between voting rounds and sincere voting.

The first result of this chapter, section 3.6.1, is that under the Plurality rule the outcome is that the two incumbents have larger vote shares than the entrant. This fits well with the structure of the model used in chapter two. Here the 'large' candidates have strictly larger vote shares than the 'small' candidate. In equilibrium the two incumbents have an equal vote share on average. This relates well to the key area of interest in chapter two; when the two 'large' candidates have similar support levels. This is the only case in chapter two where voters are pivotal and may be strategic. In chapter two the supporters of the small candidate when pivotal will deviate to the larger candidates. This strategic thinking is not analysed in this chapter but intuition suggests that the policy divergence is not contradictory with strategic voting as both incumbents still fear the entrant could make itself one of the 'large' candidates and will position themselves accordingly. Therefore entry supports policy divergence under plurality and is a possible explanation for the policy positions of the candidates found in chapter two.

The second finding in the results, section 3.6.2, is less supporting of the structure of chapter 2. Under Instant Run-off the entrant takes a position that is identical to one of the incumbents. This creates two candidates with a 'small' vote share and one candidate that is 'large'. It suggests that the 'small' candidate in chapter two can not be explained by entry as easily with the (IRO) rule. The key factor here is that under (IRO) the entrant has a 25% chance of victory and victory for the 'small' candidate is not possible in chapter two. This issue is solved if it is assumed that candidates can not enter at the same position as an incumbent but there is no clear rationale for this and it was not modeled.

Moving to the two round elections an interesting finding arises. The fact that the entrant takes the same position as one of the incumbents supports in part the divided majority design in chapter one under the two round strict majority threshold rule. There are three components with the equilibrium found for this rule that do not fit the structure of chapter one. Firstly here the divided group is not a strict majority as they represent exactly 50% of the electorate and secondly the minority candidate does not represent a strict minority as they also have 50% of the electorate. Finally the two majority candidates have exactly the same policy position. All these issues suggest that the policy positions found in the divided majority problem in chapter one is not explained by the equilibrium found in this chapter.

The structure does however fit very well with a non-equilibrium outcome in this chapter. Specifically, when one of the incumbents (for simplicity candidate B) takes a position further from the centre than the other incumbent (A), candidate C then enters near to A but not taking the identical policy position. This creates the divided majority that are split between two options. Additionally the voters that prefer candidate A and candidate C over candidate B do now represent a strict majority as candidate B is further from the centre than A and C. The two majority candidates now do not have the same policy positions but are close. Finally as in chapter one the minority candidate now has a zero probability of victory. This link is not perfect. The voter utilities in chapter one are not single peaked on a single dimension however there is some evidence to suggest that the divided majority problem can be explained by the entry problem where one candidate takes a non equilibrium position.

The final result of the chapter involves the flexible threshold. The equilibrium of the game under this rule is identical to that of Plurality. This is due to any reduction in the threshold leading to it being impossible for the entrant to win the election and they return to being vote maximisers instead of trying to win. This result therefore fits better with the design of chapter one. It shows policy divergence among the majority candidates (in this case they are the entrant and the incumbent they move closer to). It also creates an actual minority candidate (they have strictly less than half the vote when the entrant moves near to the other candidate). It also supports the claims in chapter one that lowering the threshold increases the odds that the minority candidate wins. However it still has shortcomings, most obviously that the minority candidate can still win in round two when one is reached. Like the result regarding the majority threshold the divided majority can be partially explained by a entry problem with one candidate in a non-equilibrium position. In doing so it creates the case where the minority candidate exceeds the flexible threshold and always wins as it does in the first chapter. The entrant and the incumbent that is closer to the centre act as the majority and the minority is the other candidate. As with the last case this is not perfect as the voters do not have single peaked preferences but there is some evidence that an entry game partially explains the divided majority problem. This is also a stronger case for the divided majority with a flexible threshold being an entry game in equilibrium.

When considering two round elections; if candidates can shift their policy position in the second round, both candidates will shift to the median voter position and the first round will have no effect. This case will not be considered in this chapter as it is concerned with political promises and cheap talk and not primarily candidate positioning. A possible extension would be to allow for complete policy shifting and/or for a small shift between elections (which could be fixed). Another extension not considered here is including participation uncertainty between rounds. This was tested in chapter one for the divided majority but is not detailed here. With sincere voting the intuition would be that it will have a limited effect as in this model candidates rarely win in round one.

3.2 Literature

This chapter uses the spatial market proposed Hotelling (1929) that presents a linear city for firms to compete with and uses the political interpretation from Downs (1957). The model presented in these papers assumes a single policy space from zero to one hundred (in this chapter that will be scaled to zero to one) and introduces single peaked preferences. Downs and Hotelling have been the basis of a huge number of papers in the literature on political parties and is a well established framework. The finding of Hotelling (1929) is that parties in a two candidate system inevitably converge to the centre, now known as the median voter theory. Downs (1957) supports this main finding but suggests that if abstention is introduced for voters that have no candidate near them then with polarized voters the candidates will become polarized. Downs (1957) also allowed for stable spatial equilibria with 3 parties if voters were grouped as it forced parties to retain a degree of policy consistency (they were unable to jump over other candidates taking a more right wing position than a candidate that had previously been right of them etc). This abstention risk is one possible motivator for policy divergence but was not included in either of the first two chapters so will not be expanded on here.

If the conditions imposed by Downs are dropped and Hotelling's original model is considered, a three candidate election has parties converge to the middle but does not have a stable equilibrium. The median voter theory has been shown to explain a tenancy towards the centre for political parties but no political system has complete convergence to the centre. Downs proposes that this is due to the risk of abstention and voters not being approximately normally distributed. Since the introduction of the Downs-Hotelling model there have been numerous papers aimed at explaining why the median voter theorem does not appear perfectly in reality.

This chapter aims to consider one of these explanations to justify the policy divergence used in the other two chapters. The model this chapter analyses is the extension proposed by Palfrey (1984). This introduces candidate entry. He reverts to the conditions used by Hotelling and therefore with three candidates simultaneous movement is not stable. Instead of simultaneous policy positions his paper instead shows that if two incumbent parties move first and a third candidate moves subsequently the first two act to minimize the entrant's vote share and take divergent policy positions. The citetpalfrey1984entry paper shows this for plurality and the extension of this chapter is to implement the same policy for alternative winner takes all elections to compare the results and relate them to the models in my other chapters. Brusco, Dziubiński, and Roy (2012) analyses the run-off voting system under the Downs-Hotelling model without entry and three or more candidates. They find multiple equilibria including the divergent equilibria found in Palfrey (1984) for plurality and find cases with convergent outcomes like those in Hotelling (1929). This supports the set of equilibria found in this chapter. Specifically this is that median voter convergence will exist in such a model with simultaneous moves and that the median voter position is an equilibrium as getting the most votes in round one is meaningless unless the candidate also exceeds the threshold.

In contrast to Palfrey's findings, Feddersen, Sened, and Wright (1990) present a similar model of voter entry but with strategic voters. Additionally, they do not have two established candidates. Instead candidates enter one at a time and are not fixed. With single entry, they get complete convergence to the median voter. This model is in direct opposition to that of Palfrey

(1984) but supports the original findings for Hotelling (1929).

The rest of this section presents some of the alternative explanations for candidate divergence. These alternatives will not be modeled in this chapter but in cases where the findings in this chapter do not transfer to the analysis in the other chapters some of these alternatives will be discussed.

Castanheira, 2003 argues that candidates take divergent positions to give voters a chance to agenda set. Here candidates take divergent positions so that voters can vote for them and signal that there is a large percentage of the voting population that prefers this divergent position. Voting for a candidate that will not win can increase the chance that a similar candidate will win in the future.

Another reason given for divergent positions is the incumbent effect. Bernhardt and Ingerman, 1985 present a model where candidates get a benefit from maintaining their current position. Here divergence can occur as candidates take a position for a given set of voters and are committed to it once the voters' preference might change. There is then a premium on reputation; where reputation is bolstered by candidates showing consistency and being perceived as being less risky. In this case candidates diverge as the entrant will not want the same policy as the incumbent. Victory then goes to the candidate that is currently closer to the voters' tastes.

Aragones and Palfrey (2000) present a similar concept to incumbency effect by considering the effect of one candidate being a favoured candidate. The favoured candidate will get all the votes from any voters that are indifferent over the two candidates. They find divergent equilibria where the favoured candidate takes a more moderate position.

The final paper considered in this short summary of other possible motivations for policy divergence concerns the primary process; Gerber and Morton (1998) propose that policy divergence is caused by primaries prior to the election and the level of ideological extremism in the primary dictates the level of divergence.

The next section goes on to outline the model that will be analysed.

3.3 Model

3.3.1 Candidates, Policy Space and Voters

This first section outlines the elements of the model that are common to all four voting rules that will be analysed. This section has the same structure as Palfrey (1984) which is an extensive form game with two stages of policy positioning.

There will be three players in the model, A, B and C. A player, ω, is a candidate in the election. The set Ω represents all candidates. Players will henceforth be referred to as candidates.

$$\omega \in \Omega = \{A, B, C\}$$

There is a single dimension continuous policy space that will be normalized to the closed range $[0, 1]$. λ will be a single policy position on this policy space.

$$\lambda \in [0, 1]$$

Each voter, i, has a ideal 'bliss' point. This bliss point is a single policy position in the policy space that will be called λ_i. Voters are entirely policy motivated. The only utility they get comes from the policy that is implemented and not the candidate that implements it, $U_i(\lambda)$. Their utility decreases linearly the further a policy, λ, is from their bliss point. The utility curve is symmetric around the bliss point and complete. The size of the set of voters will equal n.

$$\lambda_i \in [0, 1]$$
$$U_i(\lambda) = -|\lambda_i - \lambda|$$

The voters' bliss points are distributed according to the CDF function F on the issue space, $[0, 1]$. For the analysis of this chapter the distribution of bliss points will be assumed to be uniform.

$$F(\lambda) = \lambda$$

Voters are not assumed to be active players in this game. They will vote sincerely and act automatically. Sincere voting is defined by the voting rule and will be expanded on in the section 3.4 on voting rules. The voting rule and voters' sincere voting are common knowledge to the candidates.

3.3.2 Histories

H is a set of finite histories that are terminal when a candidate has won the election. Z represents the set of terminal histories. The set of actions available at a non-terminal history, h, are denoted $A(h)$. The action set $A(h)$, at all non terminal histories, are all policy positions within the policy space. The action of a candidate at history h is then a single policy position within the policy space. A candidate's policy position will be denoted λ_ω

$$A(h) = [0,1] \; \forall \, h \in H \backslash Z$$
$$a_\omega(h) = \lambda_\omega \in [0,1]$$

P is the function that assigns to each non-terminal history a subset of players from $\Omega \cup \{E\}$. This is the player function and $P(h)$ is the player that takes an action at history h. When $P(h) = E$ the election rule is implemented. No player acts at this stage but using the information at history h the exogenous rules of the election select a winner.

The $(\emptyset)$ history occurs prior to any candidate setting a policy. At this history P assigns the candidates A and B to set their policy position. These are the incumbent candidates. They set their policy position knowing that after doing so a third candidate will enter and set a policy position.

$$P(\emptyset) = A, B$$

The next history follows on from candidates A and B setting their respective policy positions, λ_A and λ_B. At this history, $(\lambda_A \lambda_B)$ the function P selects candidate C to set their policy position. Candidate C knows the policy position that both candidates A and B have taken.

$$P(\lambda_A \lambda_B) = C$$

After all three candidates have set their policy position at history $(\lambda_A \lambda_B \lambda_C)$ the function P selects E to play. The election rule is then implemented using the three policy positions at that history and the distribution of voters.

$$P(\lambda_A \lambda_B \lambda_C) = E$$

The election rule selects the outcome, O. O is the candidate that wins the election and is an element of Ω. The election is over and the history $(\lambda_A \lambda_B \lambda_C O)$ is terminal.

The utility a candidate gets from the election is denoted $U(O|\omega)$. A candidate, ω, is purely office seeking. They get no utility from the policy, λ_ω that is implemented. They get a positive utility when they win the election, $O = \omega$. They get no utility when another candidate, ω', wins the election, $O = \omega'$.

$$U(\omega|\omega) \ > U\ (\omega'|\omega)\ \forall\ \omega \in \Omega \cap \forall\, \omega' \in \Omega \backslash \omega$$

The equilibrium concept that will be tested is the Nash equilibrium. An equilibrium is a set of actions such that no candidate can increase their utility by deviating while all other players' strategies are fixed.

3.3.3 Strategies

Candidate A's and Candidate B's strategy, σ_ω is pure strategy selecting one element of $[0,1]$ for the non-terminal history where $h = (\emptyset)$.

$$\sigma_\omega : (\emptyset) \to [0,1]\ \forall\ \omega \in \{A, B\} \tag{3.1}$$

Candidate C's mixed strategy, σ_C is a probability measure over the set $[0,1]$ for the non-terminal history where $h = (\lambda_A \lambda_B)$, such that all elements of the mixed strategy are selected with equal probability.

$$\sigma_C : (\lambda_A \lambda_B) \to \Delta[0,1] \tag{3.2}$$

3.3.4 Summary

The set up presented in section 3.3 defines the game from Palfrey (1984). The game has a Stackelberg leader relationship with candidate A and candidate B as the leaders and candidate C as the follower. There is also a Cournot-Nash relationship between candidate A and candidate B. These relationships are presented in figure 3.1.

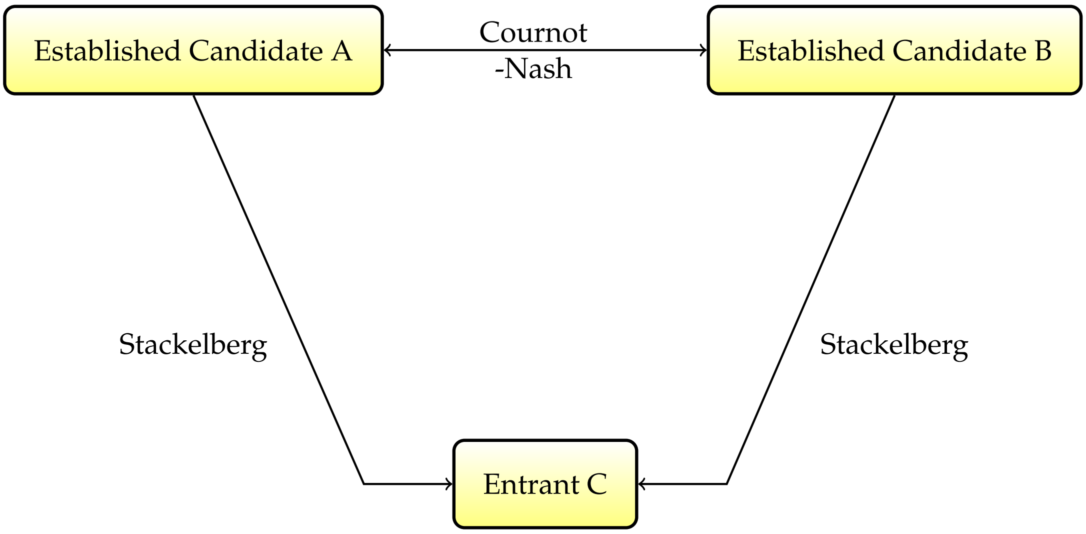


FIGURE 3.1: Strategic Interactions in the Palfrey Model

3.4 Voting Rules

This chapter compares the 4 voting rules that have been analysed in the first two chapters of the thesis: single round plurality, two round majority threshold, two round flexible threshold and Instant run-off. In this section each rule will be defined as will how they are implemented by the voters.

3.4.1 Plurality (P):

Each voter has a single vote, v_i, that they cast for a single candidate. The total number of votes for a candidate are summed and divided by the number of voters n, to give a candidate's vote percentage, V_ω.

$$v_i = \omega \in \Omega$$
$$V_\omega = \frac{(\#i|v_i=\omega)}{n}$$

The candidate with the largest V_ω wins the election. When two candidates tie with the largest V_ω one is randomly picked as the winner. Each has a one half probability of victory. If all three candidates are tied with one third of the vote the winner is randomly picked from all three candidates and each candidate has a one third probability of victory.

$$V_\omega > V_{\omega'} \, \forall \omega' \in \Omega \backslash \omega \;\; \Rightarrow \;\; O = \omega$$
$$V_\omega = V_{\omega'} > V''_\omega \, s.t. \, \omega \neq \omega' \;\; \Rightarrow \;\; Prob(O = \omega) = Prob(O = \omega') = \tfrac{1}{2}$$
$$V_\omega = V_{\omega'} = V''_\omega \, s.t. \, \omega \neq \omega' \neq \omega'' \;\; \Rightarrow \;\; Prob(O = \omega) = Prob(O = \omega') = Prob(O = \omega'') = \tfrac{1}{3}$$

Sincere Voting under Plurality

A voter gets utility from the outcome of the election. Their utility is dependent on their bliss point, λ_i, the candidate that wins and the policy they implement; $U_i(\lambda_\omega | O = \omega)$. To simplify notation this will be written $U_i(\omega)$.

$$U_i(\lambda_\omega | O = \omega) = -|\lambda_i - \lambda_\omega|$$
$$U_i(\lambda_\omega | O = \omega) \equiv U_i(\omega)$$

Voters are sincere; they will therefore vote for the candidate that is closest to their bliss point. If multiple candidates are tied as closest to a voter's bliss point the voter will randomly select one of the candidates and each candidate has an equal probability of winning.

$$U_i(\omega) > U_i(\omega') \forall \omega' \in \Omega \backslash \omega \;\; \Rightarrow \;\; v_i = \omega$$
$$U_i(\omega) = U_i(\omega') > U(\omega'') \omega \neq \omega' \;\; \Rightarrow \;\; Prob(v_i = \omega) = Prob(v_i = \omega') = \tfrac{1}{2}$$
$$U_i(\omega) = U_i(\omega') = U(\omega'') \omega \neq \omega' \neq \omega'' \;\; \Rightarrow \;\; Prob(v_i = \omega) = Prob(v_i = \omega') = Prob(v_i = \omega') = \tfrac{1}{3}$$

Voters cannot abstain from voting therefore the sum of the three candidates' vote shares will equal one.

$$\sum_{\omega = A,B,C} V_\omega = 1$$

3.4.2 Instant Run-off (IRO):

Each voter gives a single vote v_i that ranks the three candidates (v_i^1, v_i^2, v_i^3). A vote v_i^r is the candidate that voter i ranks in position $r \in \{1,2,3\}$. The total number of first votes, v_i^1 are summed for each candidate's first round vote total and divided by n to give V_ω^1. Each rank must be unique and no candidate can be repeated in the ranking and therefore this is a complete ranking of all candidates.

$$
\begin{gathered}
v_i = (v_i^1, v_i^2, v_i^3) \\
v_i^1, v_i^2, v_i^3 \in \Omega \;\; s.t. \;\; v_i^1 \neq v_i^2 \neq v_i^3 \\
V_\omega^1 = \tfrac{(\#i|v_i^1=\omega)}{n}
\end{gathered}
$$

If $V_\omega^1 > 0.5$ then candidate ω has a strict majority and wins the election. Only one candidate can get a strict majority so there are no tie breaker rules for victory.

$$
\begin{gathered}
V_\omega^1 > \tfrac{1}{2} \;\; \Rightarrow \;\; O = \omega \\
V_\omega^1 \leq \tfrac{1}{2} \; \forall \; \omega \in \Omega \;\; \Rightarrow \;\; round\ two
\end{gathered}
$$

The candidate with the lowest share of the vote is eliminated from the set of options. The eliminated candidate is denoted ϵ. When more than one candidate are tied with the lowest vote share one of these candidates will be eliminated at random. The two remaining candidates have their vote share re-calculated for round two, V_ω^2. Their vote share now includes the second votes, v_i^2, of any voter that had voted for candidate ϵ. If a candidate gets a strict majority that candidate win the election. If the two candidates get 50% of the vote each such that neither has a strict majority then a candidate will be selected at random to win the election.

$$
\begin{gathered}
V_\omega^2 = 0 \; iff \; \omega \; = \; \epsilon \\
V_\omega^2 = V_\omega^1 + \tfrac{(\#i|v_i^2=\omega \, \cap \, v_i^1=\omega'=\epsilon)}{n} \; \forall \omega \neq \epsilon \\
V_\omega^2 > \tfrac{1}{2} \;\; \Rightarrow \;\; O = \omega \\
V_\omega^2 = V_{\omega'} = 0.5 \;\; \Rightarrow \;\; Prob(O = \omega) = 0.5, \; Prob(O = \omega') = 0.5
\end{gathered}
$$

Sincere Voting under Instant Run-off

The utility calculations are identical to (P); so will not be defined again. The only factor that is different is how thee voters implement their vote. Now they have to rank all three candidates. The candidate that they chose for v_i^1 is the same candidate that they would vote for under (P). The candidate chosen for v_i^2 is then the candidate that they would select in (P) if the candidate they chose for v_i^1 was not possible. The final selection v_i^3 is whatever candidate remains. The third selection is trivial and is included here for completeness. If there were more than 3 candidate this selection would not be trivial.

As before abstention is not allowed therefore the sum of the three vote shares equals one.

$$\sum_{\omega=A,B,C} V_{\omega}^{r} = 1 \, \forall r \in \{1,2\}$$

3.4.3 Two Round Strict Majority Threshold (2RM):

Each voter has two votes, one per round of voting. Their first vote, v_i^1 is conducted in the first round and is identical to (P) and will not be expanded on here. If a candidate gets a strict majority of first round votes, $V_{\omega}^{1} > \frac{1}{2}$, that candidate wins the election. If no candidate gets a strict majority the election moves to round two.

$$V_{\omega}^{1} > \tfrac{1}{2} \quad \Rightarrow \quad O = \omega$$
$$V_{\omega}^{1} \leq \tfrac{1}{2} \, \forall \, \omega \in \Omega \quad \Rightarrow \quad \textit{round two}$$

In round two the two candidates with the most votes will be on the ballot. Each voter's second vote is then a binary choice between the two candidates that got the most votes in round one. A candidate will win a strict majority of the votes in this round[2]. Therefore with three candidates and no policy shifting between rounds the $(2RM)$ rule is equivalent to the (IRO) rule.

Sincere Voting under Two Round Strict Majority threshold

The sincere voting decision in round one for $(2RM)$ is identical to (IRO); they select their most preferred candidate. Additionally with three candidates and no policy shifting between rounds the sincere voting decision for the second round of voting is identical to the second ranked candidate for (IRO). The only difference between (IRO) and $(2RM)$ is that for (IRO) the third ranked candidate is also identified. This was only done for completeness and therefore the actions of the voter are identical in both models.

[2]For the same reason as (IRO) there must be a candidate with a strict majority in round two.

3.4.4 Two round elections with flexible threshold (2RF):

This has the same structure as $(2RM)$. There are two rounds of voting but a candidate now wins if they are the only candidate to surpass a percentage of the total votes cast. This percentage is exogenously decided and denoted, $\tau \in [0,1]$. A candidate wins if they exceed τ with their round one vote total. If no candidate exceeds the threshold or two candidates exceed the threshold[3] the election moves to round two and from that point is identical to $(2RM)$.

$$(V^1_\omega > \tau) \;\cap\; (V^1_{\omega'} \leq \tau \;\forall\; \omega' \neq \omega) \quad \Rightarrow \quad O = \omega$$
$$V^1_\omega \leq \tau \;\; \forall \omega \in \Omega \quad \Rightarrow \quad \textit{round two}$$
$$(V^1_\omega > \tau) \;\cap\; (V^1_{\omega'} > \tau \;\; s.t. \;\; \omega \neq \omega') \quad \Rightarrow \quad \textit{round two}$$

Sincere Voting under Two Round flexible threshold

This is identical to sincere voting under the two round strict majority threshold.

3.4.5 Comments

There are a number of comments to be made about these rules and how they will be analysed.

1. The majority threshold is a special case of the flexible threshold. These are separated out for two reasons. Firstly this chapter is in part aiming to support findings in the first two chapters and chapter one tests the majority threshold as a special case therefore it makes sense to analyse the differences from the candidate perspective having done so from the voter perspective. Secondly the majority threshold is by far the most prevalent two round system in the world and while the work here and in chapter one acknowledge it is just a special case of a flexible threshold its prevalence motivates any analysis to single it out.

2. The set of rules with three candidates and no policy shifting between voting rounds as well as the set of sincere voters mean that (IRO) and $(2RM)$ are equivalent from the perspective of the players and for a given strategy the outcomes will be identical. For this reason the two rules will be treated as the same in the analysis.

[3] τ is assumed to be $> \frac{1}{3}$ as a threshold lower than a third can lead to three candidates passing the threshold for entry into round two but the aim of two round elections is to limit the second vote to two candidates

3.5 Vote Shares

To calculate how voters vote when they have a candidate to their left and right it is generally necessary to calculate the mid point between two candidates. However, as the voters are uniformly distributed it is not necessary to calculate the mid point and instead just halve the voters with bliss points between two candidates. This is the number of voters that will vote for each of the candidates. Calculating the vote share of any population range just requires taking the upper limit minus the lower limit.

3.5.1 With Candidate Divergence

Histories: $h = (\lambda_A, \lambda_B) = (0.5, 0.5)$

Candidate C will enter in one of five positions in relation to the position of candidate A (λ_A) and candidate B (λ_A). The five possible positions are: $\lambda_C = (\lambda_A - \delta)$, (λ_A), (λ_B), $(\lambda_B + \delta)$ or $\lambda_C \in (\lambda_A, \lambda_B)$, where δ is strictly positive.

When $\lambda_C = (\lambda_A - \delta)$:

Candidate C gets all voters where $\lambda_i \in [0, \lambda_C]$ and half of the votes where $\lambda_i \in [\lambda_C, \lambda_A]$. The vote share is then:

$$V_C = (\lambda_C - 0) + (\tfrac{\lambda_A - \lambda_C}{2}) = (\tfrac{\lambda_A + \lambda_C}{2})$$
$$\lambda_C = (\lambda_A - \delta)$$
$$\therefore\ V_C = (\tfrac{\lambda_A - \delta + \lambda_A}{2}) = (\lambda_A - \tfrac{\delta}{2})$$

Candidate A then gets the other half of the votes where $\lambda_i \in [\lambda_C, \lambda_A]$ and half of the votes where $\lambda_i \in [\lambda_A, \lambda_B]$. Finally candidate B gets the other half of voters where $\lambda_i \in [\lambda_A, \lambda_B]$ and all the votes where $\lambda_i \in [\lambda_B, 1]$. So in terms of λ_A and λ_B their vote shares are;

$$V_A = (\tfrac{\lambda_B - \lambda_A}{2}) + \tfrac{\lambda_A - (\lambda_A - \delta)}{2}) = (\tfrac{\lambda_B - \lambda_A + \delta}{2})$$
$$V_B = (1 - \lambda_B) + (\tfrac{\lambda_B - \lambda_A}{2}) = (1 - \tfrac{\lambda_B + \lambda_A}{2})$$

To maximise vote share candidate C makes δ as small as possible.

When $\lambda_C = (\lambda_B + \delta)$:

The same logic as $\lambda_C = (\lambda_A - \delta)$ holds when $\lambda_C = (\lambda_B + \delta)$. Candidate C gets all of the votes where $\lambda_i \in [\lambda_C, 1]$ and half of the votes where $\lambda_i \in [\lambda_B, \lambda_C]$, while candidate A's and B's vote shares are flipped. The vote share is:

$$V_C = (1 - \lambda_B - \delta) + (\tfrac{\lambda_B + \delta - \lambda_B}{2}) = (1 - \lambda_B - \tfrac{\delta}{2})$$
$$V_A = (\tfrac{\lambda_A + \lambda_B}{2})$$
$$V_B = (\tfrac{\lambda_B - \lambda_A + \delta}{2})$$

To maximise vote share candidate C makes δ as small as possible again.

When $\lambda_C \in (\lambda_A, \lambda_B)$:

Candidate C gets half the vote where $\lambda_i \in [\lambda_A, \lambda_C]$ and half the voters where $\lambda_i \in [\lambda_C, \lambda_B]$. Therefore their vote share is:

$$V_C = (\tfrac{\lambda_C - \lambda_A}{2}) + (\tfrac{\lambda_B - \lambda_C}{2}) = (\tfrac{\lambda_B - \lambda_A}{2})$$

Candidate C gets the same vote share in all cases and will therefore randomize within this range. The vote share for candidate A and B is dependent on λ_C. Each gets the votes outside of their position, respectively $\lambda_i \in [0, \lambda_A]$ and $\lambda_i \in [\lambda_B, 1]$. They then also get the other half of the votes between themselves and candidate C, respectively $\lambda_i \in [\lambda_A, \lambda_C]$ and $\lambda_i \in [\lambda_C, \lambda_B]$.

$$V_A = (\lambda_A - 0) + (\tfrac{\lambda_C - \lambda_A}{2}) = (\tfrac{\lambda_A + \lambda_C}{2})$$
$$V_B = (1 - \lambda_B) + (\tfrac{\lambda_B - \lambda_C}{2}) = (1 - \tfrac{\lambda_B + \lambda_C}{2})$$

When $\lambda_C = \lambda_A$:

Candidate C gets half the votes where $\lambda_i \in [0, \lambda_A]$ as all voters are indifferent between candidate A and candidate C. They also get a quarter of the votes where $\lambda_i \in [\lambda_A, \lambda_B]$[4]. Their vote share is therefore:

$$V_C = (\tfrac{\lambda_A - 0}{2}) + (\tfrac{\lambda_B - \lambda_A}{4}) = (\tfrac{2 * \lambda_A + \lambda_B - \lambda_A}{4}) = (\tfrac{\lambda_A + \lambda_B}{4})$$

[4]In this range all the voters that are close to λ_B will vote for candidate B and this is half of the voters. Of the other half that are closer to λ_A half will vote for candidate A and half for candidate C therefore a quarter of the voters

Candidate A gets an identical vote share to candidate C. Then candidate B gets half the votes where $\lambda_i \in [\lambda_A, \lambda_B]$ and all the votes where $\lambda_i \in [\lambda_B, 1]$

$$V_A = (\tfrac{\lambda_A+\lambda_B}{4})$$
$$V_B = (1 - \tfrac{\lambda_A+\lambda_B}{2})$$

When $\lambda_C = \lambda_B$:

Candidate C gets half of the votes where $\lambda_i \in [\lambda_B, 1]$ and a quarter of the votes where $\lambda_i \in [\lambda_A, \lambda_B]$. Candidate B now has the same vote share as candidate C and candidate A gets the rest. This gives vote share:

$$V_C = (\tfrac{1-\lambda_B}{2}) + (\tfrac{\lambda_B-\lambda_A}{4}) = (0.5 - \tfrac{\lambda_A+\lambda_B}{4})$$
$$V_A = (\tfrac{\lambda_A+\lambda_B}{2})$$
$$V_B = (0.5 - \tfrac{\lambda_A+\lambda_B}{4})$$

The values are summarised in the table below.

Position Candidate C λ_C	Candidate A Vote Share V_A	Candidate B Vote Share V_B	Candidate C Vote Share V_C
$\lambda_A - \delta$	$\frac{\lambda_B-\lambda_A+\delta}{2}$	$1 - \frac{\lambda_B+\lambda_A}{2}$	$\lambda_A - \frac{\delta}{2}$
λ_A	$\frac{\lambda_A+\lambda_B}{4}$	$1 - \frac{\lambda_A+\lambda_B}{2}$	$\frac{\lambda_A+\lambda_B}{4}$
$\in (\lambda_A, \lambda_B)$	$\frac{\lambda_A+\lambda_C}{2}$	$1 - \frac{\lambda_B+\lambda_C}{2}$	$\frac{\lambda_B-\lambda_A}{2}$
λ_B	$\frac{\lambda_A+\lambda_B}{2}$	$0.5 - \frac{\lambda_A+\lambda_B}{4}$	$0.5 - \frac{\lambda_A+\lambda_B}{4}$
$\lambda_B + \delta$	$\frac{\lambda_A+\lambda_B}{2}$	$\frac{\lambda_B-\lambda_A+\delta}{2}$	$1 - \lambda_B - \frac{\delta}{2}$

TABLE 3.1: Summary of Vote Shares for set of feasibly optimal strategies for candidate C when the incumbent candidates take divergent policy positions

3.5.2 Without Candidate Divergence

Histories: $h = (\lambda_A, \lambda_B) = (0.5, 0.5)$

In this case there are only 3 possible points for candidate C to maximise vote share (for plurality or round one in a Run-off); $\lambda_C = (0.5 - \delta),\ (0.5)\ or\ (0.5 + \delta)$. As this is for a single policy position of candidate A and B the real numbers can be used and the calculations are very simple. When candidate C takes position $\lambda_C = (0.5 - \delta)\ or\ (0.5 + \delta)$; candidate C gets all votes to the outside of their policy position (below or above respectively) plus half the distance between themselves and the middle. They therefore set δ to be as low as possible. This equals;

$V_C = (0.5 - \frac{\delta}{2})$. The rest of the population is split equally between the other two candidates; $V_A = V_B = (0.25 + \frac{\delta}{4})$.

Position Candidate C λ_C	Candidate A Vote Share V_A	Candidate B Vote Share V_B	Candidate C Vote Share V_C
$0.5 - \delta$	$0.25 + \frac{\delta}{4}$	$0.25 + \frac{\delta}{4}$	$0.5 - \frac{\delta}{2}$
0.5	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$0.5 + \delta$	$0.25 + \frac{\delta}{4}$	$0.25 + \frac{\delta}{4}$	$0.5 - \frac{\delta}{2}$

TABLE 3.2: Summary of Vote Shares for set of feasibly optimal strategies for candidate C when the incumbent candidates take the median voter position

3.6 Results

The results are divided by voting rule. Each is broken into a theorem and its proofs. Running in parallel to the proof is a numerical example in Appendix L. When a step has a connected example the relevant appendix section will be referenced. The proof is then followed by a discussion of that result and how it relates to the other chapters.

For simplicity of notation it will be assumed throughout that when candidate A and candidate B do not take the same position on the spectrum that candidate A takes the lower position. This is for simplification and is not the primary cause of any of the results.

3.6.1 Plurality (P)

The first result is a repetition of the Palfrey (1984) paper. The reason it is repeated here is to frame the discussion on plurality and how this result fits with chapter two.

Theorem 3.1. *Under the Plurality rule* $(\lambda_A, \lambda_B, \lambda_C)$ *is an equilibrium if and only if:* $\begin{cases} \lambda_A = 0.25 \\ \lambda_B = 0.75 \\ \lambda_C \in [0.25, 0.75] \end{cases}$

Proof: When $h = (0.5, 0.5)$; $\lambda_A = \lambda_B$ and vote shares can be found in table 3.2. Candidate C's best response is to play: $\lambda_C = (\lambda_A - \delta)$ *or* $(\lambda_B + \delta)$. In either case they get $V_C = 0.5 - \frac{\delta}{2}$ and win with certainty.

This is not an equilibrium for candidate A or candidate B. They will deviate away from the centre. The vote shares for candidates C's strategies can be found in table 3.1 For small deviations candidate C will still take the outside position $\lambda_C = (\lambda_A - \delta)$ *or* $(\lambda_B + \delta)$. They select the option that gives the largest vote share. This will be outside the candidate (assume A) that is closer to the centre. Candidate C no longer wins as they have more votes than candidate A but less votes than candidate B. Candidate B wins with certainty.

Candidate A now loses the election with certainty due to being the closer candidate to the center. If they deviate to a position further from the center so that candidate B is now closer; candidate C will enter with $\lambda_C = (\lambda_B + \delta)$ for the same reason as the previous paragraph. Candidate A now wins with certainty. See appendix L.1.

The incentive to deviate exists with and without symmetric policy positions. This divergence will continue until there is no incentive to move further from the centre. This will only occur when the vote share that candidate C gets from $\lambda_C \in [\lambda_A, \lambda_B]$ is greater than the vote share they get from $\lambda_C = (\lambda_A - \delta)$ *or* $(\lambda_B + \delta)$. Taking vote shares:

$$\frac{\lambda_B - \lambda_A}{2} > \lambda_A - \delta$$
$$\frac{\lambda_B - \lambda_A}{2} > 1 - \lambda_B - \delta$$

δ is the smallest amount possible. It is also necessary that both equations hold. If only one holds then candidate C will still take $\lambda_C = (\lambda_A - \delta)$ *or* $(\lambda_B + \delta)$. Additionally we want the first point either candidate stops diverging away from the center therefore for both equations the vote share from $\lambda_C \in [\lambda_A, \lambda_B]$ should be greater than that from $\lambda_C = (\lambda_A - \delta) or (\lambda_B + \delta)$ by the smallest possible amount. Putting these three points together gives the two equalities:

$$\frac{\lambda_B - \lambda_A}{2} = \lambda_A$$
$$\frac{\lambda_B - \lambda_A}{2} = 1 - \lambda_B$$

Solving for these simultaneous equations gives $\lambda_A = 0.25$ and $\lambda_B = 0.75$. Candidate C's best response at $h = (0.25, 0.75)$ is; $\lambda_C \in [\lambda_A, \lambda_B]$. They randomise in this range so half the time they are closer to candidate A and half the time they are closer to candidate B. If an incumbent deviates away from the centre; candidate C's strategy is unchanged and the other candidate will win as their vote share will be higher as they are closer to the centre[5]. Deviation further

[5]This result comes from the condition that candidate C randomises when indifferent so will randomise between λ_A and λ_B so candidate A and B will get a quarter of the inside region on average but if A goes lower there are less voters where $\lambda_i \in [0, \lambda_A]$ than there are where $\lambda_i \in [\lambda_B, 1]$

from the centre leads to a worse outcome (certain defeat instead of a 50% chance at victory). If an incumbent deviates to the centre they will lose with certainty as candidate C will take the outside position on their side. See appendix L.2 for an example. Therefore $\lambda_A = 0.25$, $\lambda_B = 0.75$ and $\lambda_C \in [\lambda_A, \lambda_B]$ is the only Nash equilibrium. ■

Discussion: The key findings of this result are that there is a stable equilibrium where three candidates take different policy positions and that one of these three candidates gets strictly less votes than the other two candidates. This mirrors the set of assumptions used in chapter two. In chapter two there were two 'large' candidates but no motivation was given for why they were 'large'. One possible motivator for these 'large' candidates is that they are the incumbents and have an advantage from moving first. This chapter suggest that the entrant will take a position between the two incumbents. This finding does not transfer to chapter two as there is no equivalence for 'between' when there is not a linear policy space. Where is does relate is the fact that the 'small' candidate supporters can be split over which 'large' candidate they prefer. This is analogous to being between the two candidates. Chapter two aims to be more general than this but the specific cases presented in this chapter are included in the analysis in chapter two.

The second finding of this result that supports the work done in chapter two is that the equilibrium in this chapter has the two incumbent candidates hold equal vote shares on average. This fits with the central cases of importance in chapter two. Most of the results in chapter two where sincere voting as a strategy is not an equilibrium or performs less well than level-k thinking are when the two 'large' candidates have a similar vote share. What this says about the equilibrium found here is that if they acted strategically the voters that support candidate C will probably stop voting for candidate C and move to the incumbent candidates. As there is no cost to entry this will not stop the candidate from entering but makes their vote share smaller (with full rationality zero). Introducing strategic voting will not be done here but the intuition is that some form of candidate divergence will still exist as both incumbents would still need to avoid being seen as the small candidate.

3.6.2 Majority run-off Rules (IRO and 2RM)

This result concerns both majority run-off rules; the instant run-off and the two round rule with majority threshold.

Theorem 3.2. *Under the instant run-off rule (and the two round elections with majority threshold) there are multiple symmetric equilibria where $\lambda_A = 1 - \lambda_B$ and $\lambda_A \in [0.25, 0.5]$. Then candidate C's best response is; $\lambda_C = \lambda_A$ or λ_B.*

A summary of vote share in round one for each of candidate C's strategies can be seen in table 3.1

Proof: While the vote shares that candidate C gets from different positions has not changed their motivation has; now they want to not come last in round one and then be closer to the centre than the candidate that they face in round two. Their best response must be examined for two types of history; 1) symmetric $h = (\lambda_A, \lambda_B)$ $s.t.$ $\lambda_A = 1 - \lambda_B$, 2) non-symmetric $h = (\lambda_A, \lambda_B)$ $s.t.$ $\lambda_A \neq 1 - \lambda_B$.

Starting with the non-symmetric $h = (\lambda_A, \lambda_B)$ $s.t.$ $\lambda_A \neq 1 - \lambda_B$. Without loss of generality it will be assumed that $\lambda_A > 1 - \lambda_B$, so that candidate A is closer to the centre than candidate B. Candidate C can guarantee victory by eliminating candidate A and being closer to the centre than candidate B. This is achieved in one of two ways:

1) $\lambda_C = \lambda_A - \delta$ when $\lambda_A + \delta > \frac{\lambda_B - \lambda_A}{2}$.

When candidate A is close enough to the centre; candidate C takes $\lambda_C = \lambda_A - \delta$. Their vote total $V_C^1 = \lambda_A - \frac{\delta}{2}$ is greater than candidate A's vote share of $V_A^1 = \frac{\lambda_B - \lambda_A + \delta}{2}$. Therefore candidate A is eliminated and candidate C is closer to the centre than candidate B. See appendix L.3 for an example.

2) $\lambda_C = \lambda_A + \delta$ when $\lambda_A + \frac{\delta}{2} < \frac{\lambda_B - \lambda_A}{2}$.

They now take an inside option, they get the same vote share $(\frac{\lambda_B - \lambda_A}{2})$ at any position where $\lambda_C \in (\lambda_A, \lambda_B)$ but the closer they move to λ_A the lower candidate A's vote share $(\lambda_A + \frac{\delta}{2})$ is so they will go just inside candidate A to eliminate them. Candidate A is closer than candidate B and candidate C is closer than candidate A therefore candidate C must be closer to the center than candidate B so candidate C gets over half the round two votes and wins with certainty. See appendix L.4 for an example.

There is an overlap between the two ways that C can win with certainty but there is always at least one way.

Therefore candidate A and candidate B know that they will lose with certainty if they take non-symmetric policy positions. So their strategies will be symmetric; $h = (\lambda_A, \lambda_B)\ s.t\ \lambda_A = 1 - \lambda_B$.

In all such histories candidate C cannot enter outside the two candidates. While they might be able to eliminate one of the candidates they then enter round two further from the centre than their opponent and lose. See appendix L.5 for an example.

Candidate C can still win with certainty when they enter on the inside if the other candidates are sufficiently far from the centre. They would take the position $\lambda_C = (\lambda_A + \delta)\ or\ (\lambda_B - \delta)$ and as long as they do not come last they will be closer to the centre than their opponent. See appendix L.6 for an example.

Therefore there is a limit at which symmetric policy positions by candidate A and B will stop candidate C taking an inside position and winning. This occurs when the vote share candidate A gets, $(\lambda_A + \frac{\delta}{2})$, when candidate C takes $\lambda_C = \lambda_A + \delta$, exceeds the vote share that candidate C gets, $(\frac{\lambda_B - \lambda_A}{2})$, at that position.

$$\lambda_A + \frac{\delta}{2} > \frac{\lambda_B - \lambda_A}{2}$$

due to the symmetric policy position: $\lambda_A = 1 - \lambda_B$.

$$\lambda_A + \frac{\delta}{2} > \frac{1 - \lambda_A - \lambda_A}{2}$$
$$\lambda_A + \frac{\delta}{2} > \frac{1}{2} - \lambda_A$$
$$2 * \lambda_A + \frac{\delta}{2} > \frac{1}{2}$$

δ tends towards zero and is strictly positive therefore for the equation above to hold it must be the case that:

$$2 * \lambda_A = \frac{1}{2}$$

So $\lambda_A = 0.25$ and using the symmetric policy position $\lambda_B = 0.75$. Therefore at any symmetric policy pair closer to the centre if candidate C enters inside or outside one of the other candidates they will lose with certainty in either round one or round two.

This leaves two remaining strategies $\lambda_C = (\lambda_A)\ or\ (\lambda_B)$. The two cases are equivalent so just $\lambda_C = \lambda_A$ is considered. In such a case candidate C enters in the same place as candidate A and they split the vote 50% each where $\lambda_i \in [0, 0.5]$. The other half of the vote goes to candidate B. Candidate A and C have a 50% chance of being eliminated and the same chance of reaching round two. In round two the two candidates left are the same distance from the centre so split the second round vote 50% each and have a 50% chance of victory.

Candidate C's overall odds of victory are 25%. As candidate C is indifferent over who they copy they will randomise. Therefore the odds of candidate A or B winning is $(\frac{50+25}{2})\% = 37.5\%$. If either candidate A or B moved closer or further from the centre their odds of victory would be zero.

Therefore under the instant run-off rule (and the two round elections with majority threshold) there are multiple symmetric equilibria where $\lambda_A = 1 - \lambda_B$ and $\lambda_A \in [0.25, 0.5]$ then candidate C's best response is; $\lambda_C = \lambda_A\ or\ \lambda_B$. ■

Discussion: These results are relevant to two of the election rules that are being analysed in this chapter. Firstly the instant run-off. The interesting point of discussion for this rule is how it relates to plurality. In chapter two it was argued that (IRO) leads to a policy as close or closer to majority support compared with plurality. This was due to the way sincere voting was implemented. The results here suggest an additional way that (IRO) leads to a position nearer the median voter (The policy that gets majority support over all other policies in a head to head). (IRO) leads to a set of equilibria where candidates take positions closer to the middle. In essence candidates are less worried about the extremist candidates winning so equilibria exist where they can play closer to the centre. It is important to note that this relates to equilibrium selection and this chapter does not aim to make any claims regarding which equilibria will be selected. all that is said here is that the unique equilibrium under (P) is the most extreme[6] equilibrium $(0.25, 0.75)$ that is possible under (IRO) and all other equilibria are closer to the median. One result that contradicts the structure in chapter two is that for (IRO) there is no 'small' candidate. The fact that with entry the entrant takes an identical position to one of the incumbents suggests

[6] furthest from the centre

that the policy divergence in chapter two must also be due to some alternative motivation such as those proposed in Castanheira (2003) for example.

The second rule to discuss is $(2RM)$ and how the findings of this chapter link to the work in chapter one. The first thing to note is that the divided majority is at least in part found in this chapter. The entrant takes an identical position to that of one of the other candidates. In doing so they divide up the vote of half the electorate. This does not fit perfectly with the analysis in chapter one where the two candidates divided up a strict majority. This does introduce a interesting link. It can be argued that the divided majority problem does not resemble the equilibrium outcome found in this chapter. It does however link closely to a disequilibrium case where one of the entrants takes a position that is too extreme. In such a case the entrant takes a position close to the candidate that is nearer the middle and this entrant along with that candidate represent the divided majority while the candidate that took the extreme position is the minority candidate. This fits clearly with the structure of the first chapter and therefore one justification for the structure in that chapter that causes the policy divergence is entry with one incumbent not taking a equilibrium position. This is not a perfect comparison due to the single dimension policy space in this model that is not used in chapter one but elements of entry game support the divided majority. It asks an additional question of the divided majority game of why the minority candidate does not take a slightly more central policy position to win some of the time as they currently lose with certainty with a majority threshold. This suggests that as with the 'small' candidate in chapter two the minority candidate in chapter one is motivated by more than just victory odds.

3.6.3 Flexible Threshold rule (2RF)

This result relates to the flexible threshold τ for any value where τ is below $\frac{1}{2}$. It is also assumed but not necessary that τ is above $\frac{1}{3}$.

Theorem 3.3. *Under the Flexible threshold rule $(\lambda_A, \lambda_B, \lambda_C)$ is an equilibrium if and only if:* $\begin{cases} \lambda_A = 0.25 \\ \lambda_B = 0.75 \\ \lambda_C \in [0.25, 0.75] \end{cases}$

A summary of the vote share in round one can be found in table 3.1.

Proof: First consider when $\lambda_A = (1 - \lambda_B)$ *and* $\lambda_A \leq \frac{\tau}{2}$. Candidate C can take a position that stops both candidates passing the threshold as long as it is in the range $[1 - \frac{3\tau}{2}, \frac{3\tau}{2}]$. Assume $\lambda_A = \frac{\tau}{2}$ and candidate C playing $\lambda_C = \frac{3\tau}{2}$ first. Then candidate A gets all votes of voters in the range $[0, \lambda_A]$ and half the votes of voters in the range $[\lambda_A, \lambda_C]$. This means a vote share:

$$V_A = (\tfrac{\tau}{2} - 0) + (\frac{\frac{3\tau}{2} - \frac{\tau}{2}}{2}) = \tfrac{\tau}{2} + \tfrac{\tau}{2} = \tau$$

So candidate A has exactly the threshold but is not above it so will not win in round one. Candidate B gets all votes of voters in the range $[\lambda_B, 1]$ and half the votes in $[\lambda_C, \lambda_B]$[7]. Finally candidate C gets half the votes of voters in $[\lambda_A, \lambda_B]$.

$$V_B = (1 - (1 - \tfrac{\tau}{2})) + (\frac{1 - \frac{\tau}{2} - \frac{3\tau}{2}}{2}) = \tfrac{\tau}{2} + 0.5 - \tau = 0.5 - \tfrac{\tau}{2}$$
$$V_C = \frac{(1 - \frac{\tau}{2}) - \frac{\tau}{2}}{2} = 0.5 - \tfrac{\tau}{4} - \tfrac{\tau}{4} = 0.5 - \tfrac{\tau}{2}$$

So candidate B and C have an equal vote share. So one of them will be eliminated. When candidate B is eliminated candidate C is closer to the centre than candidate A and will win with certainty. When candidate C is eliminated candidate A and B are equally close to the centre and will have a 50% chance of victory. In total, candidate A and B win with 25% likelihood and candidate C wins with a 50% likelihood. Either candidate A or B wins with certainty if they move to the centre slightly (assume candidate A); candidate C is not able to both stop candidate A from passing the threshold and come second. Candidate A is guaranteed victory as either they win passing the threshold or candidate C is eliminated and they are closer to the centre than candidate B. So both candidates would move slightly closer to the centre. See appendix L.7 for an example. The same logic holds for non-symmetric policy positions such that neither incumbent candidate will take a position where $\lambda_A \in [0, \frac{\tau}{2}]$ and $\lambda_B \in [1 - \frac{\tau}{2}, 1]$.

Now analysis moves to symmetric policy positions where candidates are closer to the centre; $\lambda_A = (1 - \lambda_B)$ *and* $\lambda_A \in (\frac{\tau}{2}, 0.5)$. Candidate C can not win; they can not stop one candidate from passing the threshold and come second. So they will just aim to maximise vote share.

When $\lambda_A \leq 0.25$ this is achieved by taking an inside position and randomizing within that range. While candidate C takes the inside position there is an incentive for candidate A (assume

[7] as $\tau < 0.5 \;\Rightarrow\; (\frac{3\tau}{2} < 0.75) \cap (1 - \tau) > 0.75 \;\Rightarrow\; \lambda_B > \lambda_C$

here A but the logic also holds for B) to deviate towards the centre as by being closer than candidate B they beat candidate B in round two[8]. There is no equilibrium until the inside option and the outside option for both candidates are equal. This has been shown in 3.6.1 to be at $h = (0.25, 0.75)$. For the same reason[9] all points where $\lambda_A \neq 1 - \lambda_B$ when $\lambda_A < 0.25$ are not equilibria as candidate A will move towards the centre. See appendix L.8 for an example.

When $\lambda_A = (1 - \lambda_B)$ *and* $\lambda_A \in (0.25, 0.5]$ the best response of candidate C is to take the position $\lambda_C = (\lambda_A - \delta)$ *or* $(\lambda_B + \delta)$. If candidate A (can be candidate B as well) moves further from the centre to $\lambda_A = 0.25 - \delta$ candidate C will now take the position $\lambda_C = (\lambda_B + \delta)$ and candidate A gets $V_A = (0.5 - \frac{\delta}{2})$ this is then above the threshold and candidate A wins with certainty. Both candidates use this logic until they reach the point where candidate C will not take the outside position any more; $h = (0.5, 0.5)$. This is then the only equilibrium.

Therefore the equilibrium is a symmetric policy position where $\lambda_A = 0.25$ *and* $\lambda_B = 0.75$. Then candidate C's best response is $\lambda_C \in [0.25, 0.75]$. ■

Discussion: The obvious result of interest is that any threshold lower than a half leads to the same policy positions as Plurality. Policy divergence is guaranteed with anything but a forced majority requirement. The reason for this is that with the exception of a majority requirement the incumbents can eliminate any probability that candidate C wins the election. As such they do not try to win but to vote maximise. In doing so they force the incumbents to stop the entrant from taking their outside position and this forces divergence equal to that of plurality. In chapter one a lower threshold increased the odds that the minority candidate (in this chapter the candidate that does not have the entrant move to its side) would win if their sincere support exceeds the threshold. In this chapter voting is assumed to be sincere so a candidate only wins in the first round if their sincere support exceeds the threshold. This event does occur in this chapter when the entrant moves too far from the centre. It was not calculated in the result but could easily be worked out how often this occurs. It was also the case in chapter one that the odds that the minority candidate wins in round one increases as the threshold goes down and this is true here. As the threshold goes down the entrant must be closer and closer to the centre to stop a candidate winning in round one and as they are random the odds that they are close

[8] the only reason they stop moving to the centre is candidate C taking the position to their outside
[9] that candidates keep moving to the centre and being further from the centre is the weaker position

enough go down.

As with the last result there are short comings in this result. First and most obvious is that as with the last result the minority candidate can still win in round two if it is reached. Unlike the last result though the policy difference of the majority is explained in this chapter.

3.7 Conclusion

This chapter analysed the candidate strategies in three candidate elections with entry. It assumed that two candidates start in the election and play against each other. A third candidate then enters the election. When candidates enter is common knowledge as is the position they take once they have taken it. This was modeled as a extensive form game.

I proposed using this model based on the entry game in Palfrey (1984) as a support to the assumptions laid out in the first two chapters of this thesis. The first two chapters present respectively a divided majority problem with three candidates under varying thresholds and a three candidate election with two 'large' candidates who are the only candidates that can viably win. The aim was to analyse how candidates positioned themselves under the 4 voting rules proposed in the two other chapters; Plurality (P), Instant run-off (IRO), two round strict majority threshold (2RM) and two round elections with flexible threshold (2RF). In doing so I hoped to find some insight to justify the assumptions made about the candidates in those chapters.

The first observation, though this is no innovation, is to note that the instant run-off and the two round election with majority threshold are, with three candidates, identical as long as candidates cannot change their position between voting rounds. Therefore only three voting rules need to be analysed.

The first result of this chapter is that under the plurality rule the two incumbents who move first end up with a larger share of the vote than the entrant. This links well with the assumptions in chapter two that assumes two 'large' candidates. One rationale for these two 'large' candidates in chapter two could be that they are incumbents with a first mover advantage. Following on from this these two incumbents are the only two candidates that can win the election. This is the key assumption of the second chapter. Here the two 'large' candidates are assumed to be the only candidates with enough underlying support to win. The results for the second rule in chapter two are less supportive. Under the instant run-off rule the entrant copies the strategy of one of the incumbents. This creates two 'small' candidates and one 'large' candidate in this chapter which does not fit with the assumptions of chapter two. Therefore incumbent first mover advantage does not seem to explain the set up of the instant run-off rule in chapter two. Similarly in this chapter the entrant has a chance of victory which the 'small' candidate in chapter two does not.

Moving to the results regarding chapter one the evidence is again supportive. Under the two round election rule with strict majority the entrant takes the same position as one of the incumbents. This mirrors the assumption in chapter one of a divided majority ,where the entrant and the incumbent they copy represent the divided majority while the incumbent they do not copy represents the minority candidate. A few issues do arise in this comparison. Firstly the divided majority in this chapter is not a strict majority as it is in chapter one. Directly linked to this the minority candidate is not a strict minority in this chapter as it is in chapter one. Then finally the minority candidate in this chapter has a positive probability of victory. This is not to say that chapter one cannot be in part explained by this entry model. A better comparison to chapter one would be an entry model that is not in equilibrium. In such a case the entrant moves to a position very close to a incumbent and selects the incumbent that is closer to the centre. Now they do represent a strict majority and the incumbent that was further from the centre represents a strict minority. Additionally the candidate further from the centre that represents a minority never wins.

The final result is that lowering the threshold by any amount reverts the strategy in this game to that strategy played under the plurality rule. Any lower threshold makes it impossible for an entrant to enter and win. This then supports the assumptions of chapter one. We have a divided majority with slightly different policies as we see in chapter one (the entrant and the incumbent they randomly position themselves closer to) and a unified minority (the other incumbent). Other than the divided majority minor policy difference this theory again has the same issues as the last argument. This issue is solved in the same way as the last one. You view the divided majority game as a entry game being played marginally out of equilibrium.

This raises an interesting question about the divided majority game. Namely, why does the minority candidate in the divided majority game not change their policy to try to get enough votes to represent a majority? This is what they would do in the entry model proposed in this chapter. The suggestion is that in the divided majority game due to ideological factors the minority candidate is making a decision that is not fully rational. This is especially relevant with a lower threshold as in such a case the minority candidate can move slightly closer to the centre and win with certainty while holding a policy position close to its theoretical ideological position.

The entry game with incumbent first mover advantage can clearly explain parts of the assumptions made in the first two chapters. In each case there are also weaknesses which do not fit as nicely. As with any such proposed solution that uses a different set of rules these findings and any support they give can only go so far. This chapter therefore does not claim to fully explain why candidates in the first two chapters take the positions they take but it does give a suggestion that seems to partially explain where the policy divergence in the first two chapters could come from.

Appendix A

Summary of world voting rules

Of the 217 countries, dependencies and other territories on which iDEA (institute for democracy and electoral assistance) have data 104 do not directly elect a president. Sudan and Libya are currently undergoing a transition without a clear election system known for the future. This leaves 111 countries that elect a president directly. Of these, 87 use a form of two round run-off. Of the rest 21 use the first past the post voting mechanism, one uses a party list, one uses single transferable vote and one a simplified version of the single transferable vote.

Of the 87 countries that use a run-off, 78 use the basic absolute majority run-off. If there are three or more candidates and no candidate wins an absolute majority of first round votes the top two candidates based on first round vote share move to round two and the country votes again. The other 9 have a variety of other mechanisms with a variable threshold. Kenya, Nigeria, and Sierra Leone are the only countries that have a higher threshold than absolute majority in round one. In Sierra Leone, a candidate requires 55% of the vote to win in round one. The second round reverts to plurality (or 50% as they are equivalent in round two) as the other run-offs do. In Kenya, they have the absolute majority requirement but also require that a candidate have at least 25% of the vote in at least a half of the state's counties (there are 57 counties). Nigeria have a majority requirement and the same 25% rule but require at least two thirds of the state's regions (there are 36 regions). Argentina, Costa Rica, and Nicaragua have all lowered their threshold level: Argentina have a threshold of 45% while Costa Rica and Nicaragua have a threshold of 40%. On top of a lower threshold, they have a conditional threshold in Argentina and Nicaragua. In Argentina and in Nicaragua a candidate wins with 40% or 35% if no other candidate is above 30%. Finally, in Bolivia, Haiti and Ecuador they retain the 50% threshold but introduce a conditional threshold as well. In Bolivia and Ecuador, a candidate wins with above 40% of the vote if no other candidate has more than 30% of the vote. Then in Haiti, a candidate can win with a plurality if every other candidate is behind them by 25% of the vote.

Of the 21 countries with a form of first past the post (Plurality), Cameroon, Equatorial Guinea, Gabon, Iceland, Kiribati, South Korea, Malawi, Panama, Philippines, Rwanda, Mexico, Singapore, Taiwan, Togo and Venezuela have the standard 'most votes win' system of plurality. The only countries with alternative plurality elections are Bosnia and Herzegovina and the United States. Bosnia and Herzegovina elect a three member body using plurality based on specific groups, one Bosniak, one Croat and one Serb. They rotate the role every 8 months. Finally, within Plurality the United States elect their president using an indirect plurality using the Electoral College.

The only countries that do not use either Plurality or two round runoff elections are Ireland, Sri Lanka and Angola. In Ireland they use the single transferable vote where voters give a list of preferred candidates and a candidate is eliminated one at a time (the lowest vote total leading to elimination) until a candidate has an absolute majority (Also known as the instant run-off). Sri Lanka use a similar system but only list 3 candidates. Finally, Angola elect a president as the leader of the largest party in the national assembly akin to a Prime Minister but with executive power.

Appendix B

Round two Victory probability for sincere voting in the experiment

The probability that a voter is selected first in treatment 1, 2, 3 and 4 is $\frac{1}{14}$. There are four α type voters so the probability that a α voter is selected first is $\frac{4}{14}$. Similarly, there are four β type voters and six γ type voters. So their probabilities are $\frac{4}{14}$ and $\frac{6}{14}$ respectively. Having been selected a voter cannot be selected again.

Therefore if a voter is not selected first their probability of being selected second is $\frac{1}{13}$. As such if a ? voter is selected first the probability than an α, β and γ voter are selected second are $\frac{3}{13}$, $\frac{4}{13}$ and $\frac{6}{13}$ respectively and so on. From these we can calculate the probability that a specific population is drawn.

In the experiment the second-round population is either everyone or 5. When it is everyone then the odds that A (or B) beat C is certain. The calculations only need to happen when $k = 5$. To do so it is necessary to consider the number of ways that a specific population can be drawn. For example, there are no ways that five α types can be drawn, as there are only four α type voters. There are then 5 ways four α type voters and one β voter can be selected.

$$\alpha\ \alpha\ \alpha\ \alpha\ \beta$$

$$\alpha\ \alpha\ \alpha\ \beta\ \alpha$$

$$\alpha\ \alpha\ \beta\ \alpha\ \alpha$$

$$\alpha\ \beta\ \alpha\ \alpha\ \alpha$$

$$\beta\ \alpha\ \alpha\ \alpha\ \alpha$$

Formally the number of ways that a population with $\#\alpha$ types, $\#\beta$ types and $\#\gamma$ types can be generated is:

$$(\#\alpha, \#\,\beta, \#\gamma) = \frac{5!}{((\#\alpha)! * (\#\beta)! * (\#\gamma)!)}$$

For example, the number of ways there can be 2α, 2β and 1γ type voter in the second round is: $\frac{5!}{2!*2!*1!} = \frac{120}{2*2*1} = 30$. This is then multiplied by the probability a specific population of this type is generated to calculate the probability that such a population can be calculated in general. Using the same example if the order of the draw was $\alpha\ \alpha\ \beta\ \beta\ \gamma$:

$$30 * \frac{4}{14} * \frac{3}{13} * \frac{4}{12} * \frac{3}{11} * \frac{6}{10} = 0.107892$$

The table below then summarises the probability of each possible population to 3 decimal places.

Probability a population is drawn		Number of votes for β.					
		0	1	2	3	4	5
Number of votes for α	0	0.003	0.030	0.060	0.030	0.030	0
	1	0.030	0.160	0.180	0.048	0.002	
	2	0.060	0.180	0.108	0.012		
	3	0.030	0.048	0.012			
	4	0.003	0.002				
	5	0					

TABLE B.1: Summary of probabilities for all possible second round voting populations

From this we can calculate the probability that each candidate wins. If A (or B) faces C all α and β types will vote A (or B) and all γ type voters will vote for C. So C wins when there are 3 or more type voters. This occurs with a probability of 0.342. Therefore A (or B) wins with a 0.658 probability. When A and B are in round two the probability each win is 0.5 as they are symmetrical in the size of their population and in the probability that a γ type voter votes for them.

These calculations can be repeated when there are 13 voters. The table below summarises this.

Additionally, voters can update their probabilities with additional information. The only additional information they get entering the final round is that if they are voting then at least one of their type is voting (as they have been selected). So of the 5 voters one is them. This changes the odds that a candidate will win. The calculations can be repeated under these new conditions and these are summarised in the table as well.

	Round two candidates	Probability that A wins	Probability that B wins	Probability that C wins
14 voters No updating	AB	0.5	0.5	0
	AC	0.658	0	0.342
	BC	0	0.658	0.342
14 voters α type updating	AB	0.642	0.358	0
	AC	0.783	0	0.217
	BC	0	0.783	0.217
14 voters β type updating	AB	0.358	0.642	0
	AC	0.783	0	0.217
	BC	0	0.783	0.217
14 voters γ type updating	AB	0.5	0.5	0
	AC	0.490	0	0.510
	BC	0	0.490	0.510
13 voters No updating	AB	0.5	0.5	0
	AC	0.751	0	0.249
	BC	0	0.751	0.249
13 voters α type updating	AB	0.640	0.360	0
	AC	0.848	0	0.152
	BC	0	0.848	0.152
13 voters β type updating	AB	0.360	0.640	0
	AC	0.848	0	0.152
	BC	0	0.848	0.152
13 voters γ type updating	AB	0.5	0.5	0
	AC	0.594	0	0.407
	BC	0	0.594	0.407

TABLE B.2: Second round probability a candidate wins based on sincere voting

The important information to see here (in the highlighted cells) is that with less γ type voters (when there are 13 voters there are 5 γ types) the probability that C wins goes from 0.342 down to 0.249. This shows firstly that the probability of an upset victory is smaller and by how much. This can help in understand how voters respond to level of uncertainty.

Appendix C

Full set of non-symmetric equilibrium

When symmetry is dropped there are a number of other possible equilibrium cases. These are called here the non-symmetric equilibrium. The five sets of necessary conditions for a non-symmetric equilibrium too exist are outlined in this appendix.

No player can effect the result at all:

This refers not just to who wins but when they win(1st or 2nd round). Any case where no voter can change the result at all requires that:

1. No candidate is 1 or less votes below passing the threshold.
2. No candidate is 1 or less votes above passing the threshold.
3. No candidate is within 2 or less votes of another candidate. With the exception of 2nd and 3rd if and only if the 1st place candidate is above the threshold.

No player can change the winner of the election:

This only occurs when there is certainty over the voters in the 2nd round, and is derived from the fact that C can't win the 2nd round, this requires:

1. γ voters vote C such that C has at least 1/3 of the vote.
2. Candidate C is not 1 or less votes below passing the threshold.

3. Candidate C is not 1 or less votes above passing the threshold.

4. No candidate is within 2 or less votes of another candidate. With the exception of 2nd and 3rd if and only if the 1st place candidate is above the threshold.

5. If A or B are within one of or above passing the threshold(or γ is not 1 or less above or below the majority threshold.)

No player that can wants to stop a candidate win in the 1st round:

This refers to a case where a candidate is 1 or less above the threshold such that if one voter deviates away from this candidate they go below the threshold:

1. A candidate is above the threshold by 1 or less votes

2. No one that voted for this candidate would prefer for the other viable option to win more than this candidate. This can be either another candidate that is above the threshold (only 2 can be) or the candidate that would come 2nd after the deviation.

No player that can wants a candidate above the threshold, this requires:

This refers to a case where a candidate is 1 or less below the threshold, such that is one voter deviated to it it would go above the threshold, this requires:

1. A candidate is below the threshold by 1 or less votes.

2. All voters that do not vote for this candidate prefer the other viable option. This is either the candidate that is above the threshold (this can be two candidates but one will fall below the threshold if one of its supporters deviate) or the other candidate that is in the top 2 (it could be tied with this candidate or coming 2nd).

No voter than can wants to change the two candidates that reach the 2nd round:

This refers to the case no voter wishes to deviate to to change the two candidates that reach the 2nd round. This requires:

1. No candidate is above the threshold
2. 2nd and 3rd place are 2 or less votes apart.
3. No one that votes for 2nd would prefer a head to head between 1st and 3rd instead of a head to head between 1st and 2nd.
4. No one that votes for 1st would prefer the head to head in that deviation[1] than the current one, in this case 2nd and 3rd must be only 1 vote or less apart.

At least one of these cases exists for all 6 treatments in the experiment as is shown in appendix I.

[1] either between 1st and 3rd or between 2nd and 3rd.

Appendix D

Experimental Form

D.1 Experimental Instructions Form:

Experimental Instructions:

Introduction

This experiment is part of a study of voting rules. Along with the participation payment the remainder of your payment comes from the decisions you and your fellow participants make. The Instructions are simple, if they are followed carefully and participants make good decisions you should make a decent sum of money. If you have any questions during the experiment please put up your hand. The experimenter will come and answer them for you. Other than these questions participants must keep silent while the experiment is in progress.

There are going to be 27 separate voting periods. The votes will be conducted under 3 different rules. These will change every 9 periods. The new rules will be explained before the 1st period they are implemented. At the end of 9 periods the computer will select 2 period at random. These periods will be used to calculate your payment, you will not get payments from the other periods.

Total Payments = Attendance Fee + 6 selected payment periods

Summary

There are 8 participants. These 8 will vote in an election along with a number of pre-programmed voters (ranging from 5-6). Therefore each period has 13-14 voters.

The objective is to elect a single candidate. There will be 3 candidates (voting options) A, B and C. At the start of each new rule you will be assigned a type α or β. All pre-programmed voters are assigned a third type γ. The payment you get is dependent on your type and the candidate that wins. All of this information along with the size of each type is shown in the payment table (see example below *not real payoffs):

Type	Candidate A	Candidate B	candidate C	# of voters
α	1.50	1.20	0.05	4
β	0.50	0.85	0.00	4

You have a single vote that you must cast for 1 of the 3 candidates. A candidate wins the period if they get a high enough share of the votes cast. The share required will change during the experiment. In any case where none of the candidates reaches the required share you will be asked to vote again in a 2nd election (this is still the same period). In this 2nd election you will be asked to choose between just 2 of the candidates. The candidate that got the least votes in the 1st round is no longer an option. In this 2nd round the candidate with the largest vote share wins.

At any given round the pre-programmed players will select the party that offers them the largest payment. They will always vote for candidate C. If C is not an option it will randomly select A or B.

Any case where a tiebreaker is needed one participant will be asked to flip a coin.

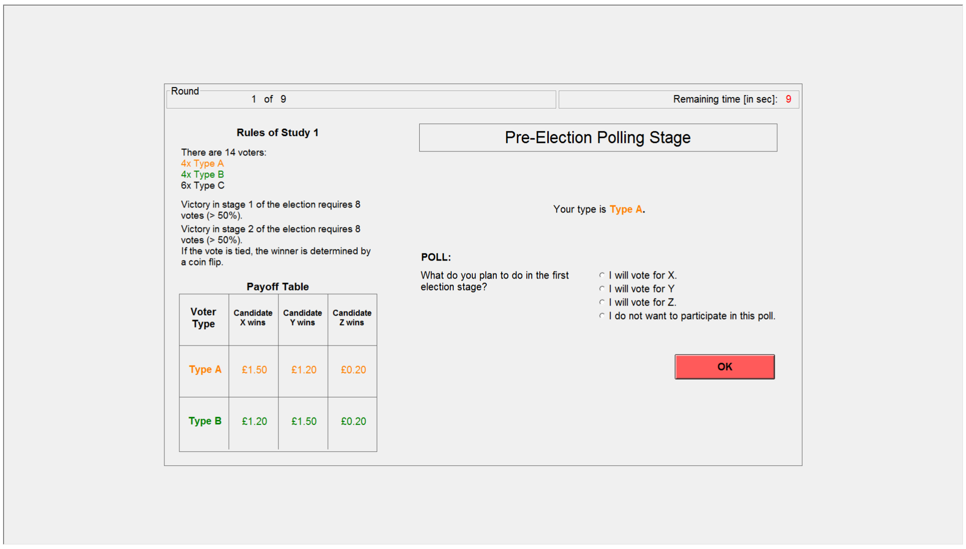
<u>Procedure</u>

In each period there can be up to 3 rounds, a pre-election round, the first election, and the second election. At the start of the period you will be informed which of these rounds you are required to vote in. These rules will also be at the top of each voting page.

The procedure is explained below:

Prior to any voting you will be given the payment table for that election. This informs you of the payments and the number of each type in that election. At the top of this page you will be given your ID as well as the group you have been selected into (α, β) and your type. To the right hand side there will be a box to vote in the first round, this vote is a pre-election poll. Everyone can vote in this poll. The poll is optional and has no effect on the election result for that round.

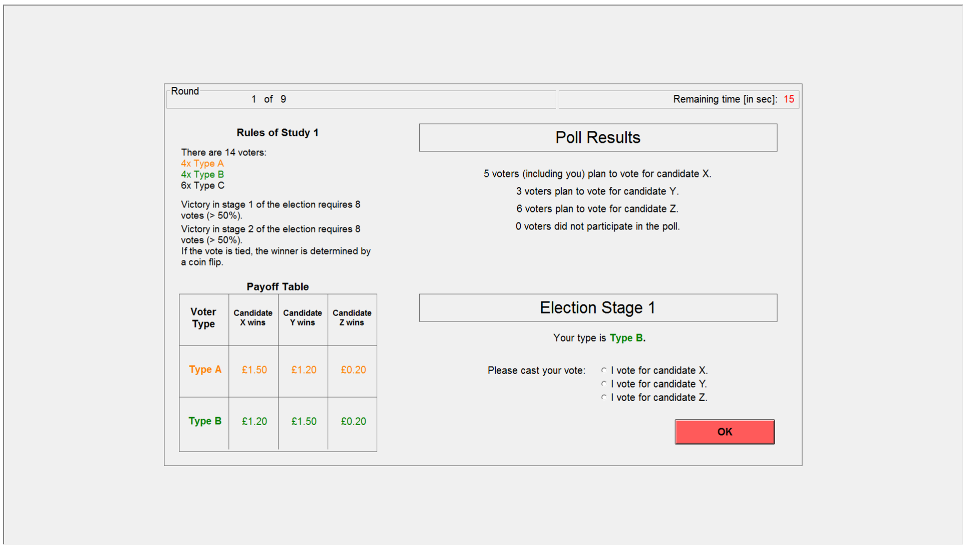
Image: payment table and Pre-election poll



note that the voter types here are A and B and the candidates X, Y and Z. This is to reduce the bias of α types assuming they should vote for A.

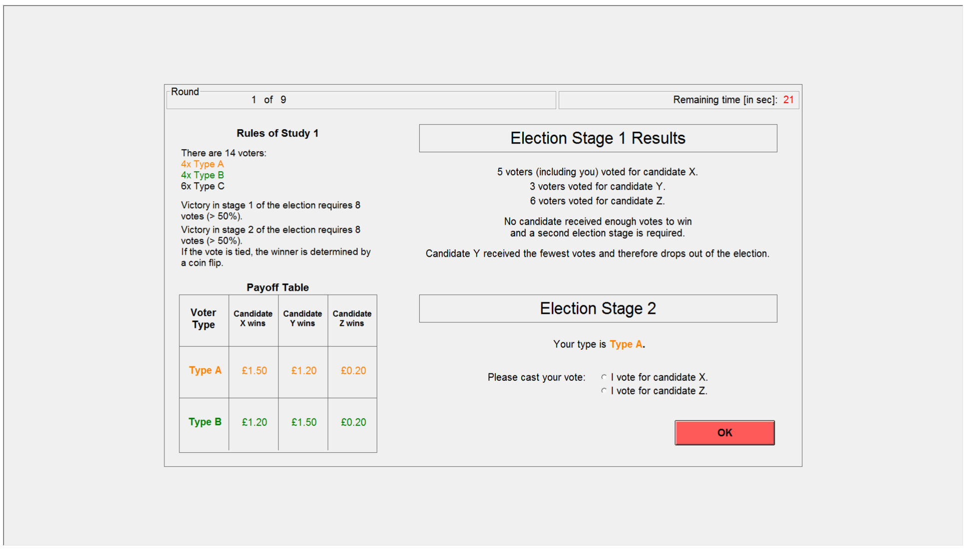
The next page will show the results of the pre-election poll as well as giving you the 2nd chance to vote. This vote is the first that can decide the election. In this round you must vote for 1 of the 3 candidates. There will be a box to type your vote into. This period is compulsory: abstention is not an option. On the screen you will be reminded of the number of each voter type and your type for that period. Once you have voted you will have to press the submit button at the bottom of the page.

Image: Pre election poll results page and Primary election voting page



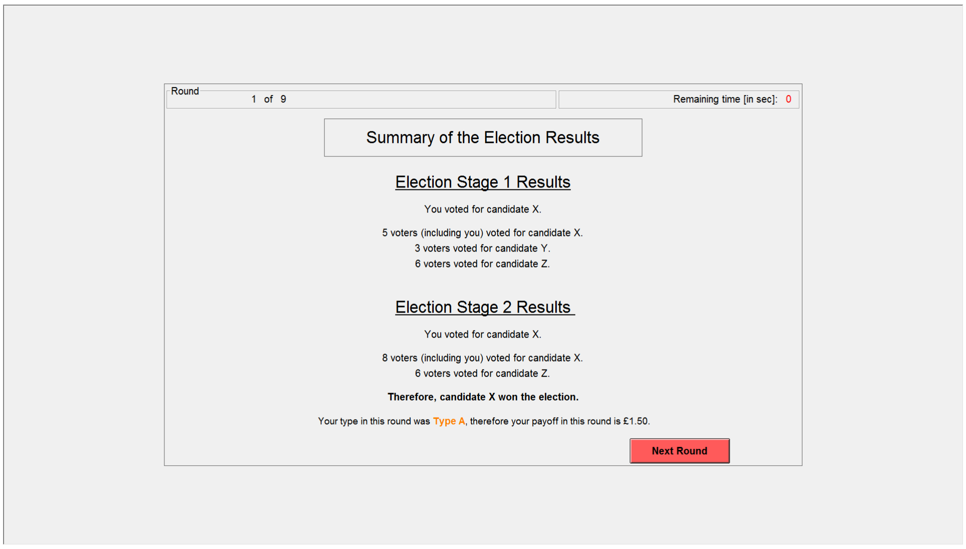
After the 2nd round there is a results page (1st Election History Box). This will tell you how many votes each candidate achieved in the 1st election. It will also tell you if someone passed the required number of votes to win. In the 3rd and final round you may or may not be asked to vote. This is dependent on the voting rules. If you are not selected you must wait for those who are voting. If you are selected you are taken to a 2nd voting page. On this page you will be reminded of the 2 remaining candidates in the election and your type. You are not told who else has been selected to vote in the 2nd period. There will again be a voting box for you to enter your vote in. If selected, voting is compulsory in the second period. Once you have selected your choice you have to press the submit button at the bottom of the page.

Image: Primary election results and second election voting page



After the 2nd vote all participants are told the results (Final Election History Box). The final page informs voters of the winning candidate and the payment they get if this period is randomly selected. If there is no need for a 2nd period (a candidate won in the 1st period) this page comes after the 1st period results page. All payments are based on the latest period that was required. The results page will also tell you the number of votes each candidate got in the 2nd period but not who was selected to vote in the second period.

Image: Final Election Results page



The history boxes will only give details about the current period. It presents the results from earlier election rounds in that period. Paper will be given for you to make notes of previous periods if you wish to.

Payment

When the experiment session is completed the computer will sum the earnings from the 6 selected periods (one for each voting rule). This total will be displayed on your screen. It will inform you of the 6 periods that were randomly selected. This plus the attendance fee will be the amount you are paid.

Any Questions.

D.2 Experiment Control Questions form

Control Questions

Please Circle your answer:

For the following set of control questions assume that the voting group preferences is such:

Type	R	Y	G	# of voters
α	0.90	0.50	0.05	5
β	0.40	0.95	0.20	4
γ	0.05	0.10	1.00	5

1) With a 50% threshold, if all voters are sincere who is eliminated in the first round?[1]

A Candidate G

B Candidate R

C Candidate Y

D Candidate G or R

2) With a 50% threshold, if all voters are sincere who will an α voter vote for in the second round?[2]

A Candidate G

B Candidate R

C Candidate Y

D Candidate G or R

3) With a 50% threshold, if all voters are sincere who will a β voter vote for in the second round?[3]

A Candidate G

B Candidate R

C Candidate Y

D Candidate G or R

[1]Checking people understand: The basics of a two round election elimination process.
[2]That preferences do not change in the second round and therefore sincere preferences do not change
[3]The preference ordering, how elimination works and decision making in the second round.

4) With a 40% threshold, if all voters are sincere who will win? When will they win?[4]

A Candidate G In the second round

B Candidate R In the first round

C Candidate Y In the first round

D Candidate R In the second round

5) If R and Y reach the 2nd round and everyone votes sincerely who will win? What pay-off does a γ voter get?[5]

A Candidate Y Wins, a γ voter gets 0.10.

B Candidate R Wins, a γ voter gets 0.10.

C Candidate R Wins, a γ voter gets 0.05.

D Candidate Y Wins, a γ voter gets 0.00.

6) Will an α voter ever have a reason for voting insincerely in the second round?[6]

A No

B Yes, only if R is eliminated in the first round

C Yes, only if the threshold is 40%

D Yes, only if there is a pre poll that says her first choice will be knocked out.

7) Now assume in the second round vote only 5 randomly selected people from the 14 vote. If R and Y reach the second round who is more likely to win?[7]

A They are equally as likely as it is random.

B R, they have more primary support, more alpha than beta voters.

C Y, They have more overall support as gamma voters as well as beta voters prefer them.

D We do not know as it is random

8) Now assume there is a pre-poll for the first round, a 40% threshold and the second round is the same as that described in question 7. You are one of the 5 α types. The poll says that Y will get 5 votes, R will get 3 and G 5. If you believe these predictions what is your best response?[8]

[4] The affect of the 40% threshold and how it effects the election.

[5] Preference relationships over two options in the second round and can read the pay-off table.

[6] The strategy in the second round and therefore understand how an election should go depending on who reaches the second round.

[7] How random selection works and therefore the effect of uncertainty in electorates in the second round.

[8] The different elements when all thrown together. So the fact that 40% is enough to win, that in the second round there is a risk that G can win and finally that sincere voting is optimal in the second round but not in the first.

A It does not matter, Y and G are going to get into the second round regardless.

B Vote R, sincere voting is always optimal

C Vote Y, The threshold is only 40% so Y can win if you give them a 6th vote.

D Vote G, the first round is not important as in the second everyone will vote sincerely and Y wins anyway.

Appendix E

Correlation between Voting decision and Election history/poll results

When considering the correlation between the voting decision and the election history/polling results it is simple to start by considering the whole data set. However 422 observations must be eliminated before any results are possible as the first election someone participates in has no Victor from the last election. This gives a starting point for the correlation table as shown in E.1. As we can see from this very broad picture correlation is higher for the polling.

TABLE E.1: Correlation table for all data points excluding the first round of the session

Corr	Round one vote	Poll Leader	Previous election winner
Round one vote	1		
Poll Leader	0.3016	1	
Previous election winner	0.1784	0.0828	1
number of observations = 3456			

This analysis is incomplete however as there are multiple data points that should be removed. Firstly Consider the data points that should be removed to accurately represent the correlation between poll vote leader and voting decision. When there is no poll leader there can be no correlation. So data points where this is the case can be dropped. Correlation between poll leader and the first round vote has risen to 0.7071. It is also possible to add back the first round votes to give a complete picture and the correlation is very similar at 0.7071. The total number of observations if the first round is included is 2968. It can also be seen that correlation with previous election winner and voting decision has gone up but not by much.

TABLE E.2: Correlation table for all data points excluding the first round of the session and all votes where there was no poll winner

Corr	Round one vote	Poll Leader	Previous election winner
Round one vote	1		
Poll Leader	0.7071	1	
Previous election winner	0.2216	0.2007	1
number of observations = 2648			

In the previous table there are still issues. When considering the correlation between last election's winner and current vote it makes no sense to correlate with the last election winner when that winner was C. So these observations can be dropped. This is done in table E.3. As the table E.3 shows the results are very similar to the results shown in the other tables. The level of correlation between poll result and voting decision is much higher at 0.7164 compared to the correlation between last election result and voting decision at 0.2450.

TABLE E.3: Correlation table for all data points excluding: the first round of the session, all votes where there was no poll winner and whenever candidate C won the last election

Corr	Round one vote	Poll Leader	Previous election winner
Round one vote	1		
Poll Leader	0.7164	1	
Previous election winner	0.2450	0.2779	1
number of observations = 2064			

Appendix F

Expected payoffs from each outcome in the experiment

The experiment has the payoff table shown in table F.1.

Type	A	B	C	# of voters
α	1.50	1.20	0.20	4
β	1.20	1.50	0.20	4

TABLE F.1: Subject's payoff table

Each payoff is known with certainty as is the probability that each outcome occurs for a given vote share from the first round. The table F.2 summarise the payoff for each possible first round vote total by calculating the probability each candidate wins for a given vote total times the payoff at that vote total.

The table is simplified by taking the two assumptions that all γ type voters vote for candidate C and no α or β type voter ever votes for candidate C. Therefore the table can be simplified to 8 values for each treatment.

The 8 values here represent the number of votes that candidate A gets (the number of votes that candidate B gets will equal 8 minus this number). The table shows expected payoffs for an α type voter but the β type voter's payoffs can be calculated by reversing the table (the payoff for α type voter when candidate A has 8 votes and candidate B has 0 votes is the same as the payoff for β type voter when candidate B has 8 votes and candidate A has 0 votes and so on).

The table then summarise the payoffs for all 6 treatments.

TABLE F.2: Table to summarise the expected payoffs for the 6 tratments from the 8 possible vote distributions

A's vote	payoff for treatment					
total	MC6	MU6	MU5	FC6	FU6	FU5
8	1.5	1.5	1.5	1.5	1.5	1.5
7	1.5	1.05	1.5	1.5	1.05	1.5
6	1.5	1.05	1.17	1.5	1.05	1.5
5	1.5	1.05	1.17	0.2	0.2	1.17
4	1.35	0.96	1.06	0.2	0.2	1.06
3	1.2	0.86	0.95	0.2	0.2	0.95
2	1.2	0.86	0.95	1.2	0.86	1.2
1	1.2	0.86	1.2	1.2	0.86	1.2
0	1.2	1.2	1.2	1.2	1.2	1.2

note: all values are in £.

This table assumes that all elections in round two with certainty will be won by whichever candidate A or B enters round two. Then with a γ equal to 6 the probability that in round two enough γ types are drawn to win the election for C is 34%. The probability that in round two enough γ tpes are drawn when γ equals 5 is 25%.

Appendix G

The Duverger Law Equilibrium Existence in the Experiment

When $X_A^1 = (0, 8)$ there is a Duverger's Law equilibrium in all treatments.

When the total number of voters equals thirteen ($MU5$, $FU5$) the majority threshold is six and a half. If candidate A (or B) gets eight, a single deviation will lead to candidate A (or B) getting seven votes. This is still above the threshold. No voter is pivotal and the Duverger's Law equilibrium exist.

When the total number of voters equals fourteen ($MC6$, $MU6$, $FC6$ and $FU6$) the majority threshold is seven. If candidate A (or B) gets eight, a single deviation will lead to candidate A (or B) getting seven votes. This is now equal to the majority threshold so they do not pass it.

In $MC6$ and $MU6$: There is no 40% threshold so round two will occur. In $FC6$ and $FU6$: The 40% threshold is 5.6 and candidate C will pass it so round two will occur.

Then in round two:

In $MC6$ and $FC6$: Candidate A (or B) will win with 8 so the deviation has no effect. In $MU6$ and $FU6$: Candidate C now has 34.2% chance of victory. This round two is strictly worse than candidate A (or B) winning in round one.

In all six treatments, there is no incentive to deviate and the Duverger Law is an equilibrium.

Appendix H

The Sincere Equilibrium

When all voters are sincere this is an equilibrium in all treatments.

In $MC6$, $MU6$ and $MU5$ there is no 40% threshold. The only way to win in round one is to pass the majority threshold. In $MC6$ and $MU6$ this is seven, in $MU5$ it is six and a half. Both candidate A and candidate B have four votes. No single deviation can take a candidate above the majority threshold.

The only other effect deviation can have is to change who progresses to round two. No single deviation can lead to candidate A and candidate B reaching round two against each other. A voter can only change who faces candidate C. All voters currently vote for their most preferred candidate. If they deviate, this candidate will never progress to round two as their total will go down to three, which is strictly worse than the other candidate who must have at least four. Therefore, no voter has an incentive to deviate and the sincere equilibrium exists.

In $FC6$, $FU6$ and $FU5$ there is a variable threshold. For $FC6$ and $FU6$ this threshold is five point six and for $FU5$ it is five point two. In all cases, a candidate needs six to pass the variable threshold. Candidates A and B both have four votes. Therefore, no single deviation can lead to candidate A or B passing the variable threshold.

In $FC6$ and $FU6$ candidate C does pass the variable threshold. The outcome is that candidate C wins the election. This still holds as an equilibrium as no voter can deviate to force round two.

In $FU5$ candidate C does not pass the variable threshold. The election goes to round two where candidate C has a 24.9% chance of victory. This is a sincere equilibrium as voters for candidate A (or B) will not deviate to B (or A) as currently candidate A and candidate B both have a 37.5% of victory as they will be randomly selected to progress to round two. If an α (or β) type voter deviated then candidate A (or B) would have three votes. They never reach round two and have no chance of victory. Candidate B (or A) now has a 75.1% chance of victory. This is strictly worse for α (or β) type voters.

So while the outcomes may not be favourable in some treatments, there is always a sincere equilibrium.

Appendix I

The non-symmetric Equilibrium Outcomes in the Experiment

For an equilibrium to be non-symmetric all voters must either be unable to change the outcome of the election or when they can change it not want to.

The first set of non-symmetric equilibria are where candidates A or B have seven votes. This is a non-symmetric equilibrium in $MC6$, $MU5$, $FC6$ and $FU5$.

In $MC6$ and $FC6$ if candidate A (or B) has seven votes, they equal the majority threshold but do not pass it. They then progress to round two along with candidate C. In round two, candidate A (or B) wins with certainty eight votes to six. A voter can deviate such that candidate A (or B) wins in round one. The outcome of the election is the same in either case. They have no incentive to deviate. In $MU5$ and $FU5$ if candidate A (or B) has seven votes they pass the majority threshold. Candidate A (or B) wins the election in round one. In $MU5$ if they deviate such that candidate A (or B) has six votes, the election goes to round two and here candidate A (or B) wins, so they do not change the result. In $FU5$ if they deviate candidate A (or B) has six votes while candidate C has five, so candidate A still wins in round one. These four cases are all non-symmetric equilibria.

The second set of non-symmetric equilibria are when candidate A or B have six votes. This is a non-symmetric equilibrium in $MC6$, $MU6$, $FC6$, $FU6$ and $FU5$.

In $MC6$ and $MU6$ there is no variable threshold and the majority threshold is seven. So, for a candidate to win in round one they need eight votes. Therefore, no voter can deviate to lead to a candidate winning in round one. Additionally, if candidate A (or B) has six votes, then B (or A) has two votes. Therefore, no voter can deviate and change the candidate that progresses to round two. No voter is pivotal so the election is in a non-symmetric equilibrium.

In $FC6$, $FU6$ and $FU5$ there is a variable threshold. In $FC6$ and $FU6$ it is five point six and in $FU5$ it is five point two. So, in all three cases candidate A (or B) passes the variable threshold. In $FC6$ and $FU6$ candidate C also passes the variable threshold. In $FC6$ candidate A beats candidate C with certainty in round two. In $FU6$ candidate C's odds of winning are 34.2%. A deviation that leads to candidate A (or B) only getting five will lead to candidate C being the only candidate above the variable threshold and candidate C will win with certainty. This is strictly worse. In $FU5$ candidate A (or B) is the only candidate above the variable threshold so candidate A (or B) wins with certainty. A deviation that reduces candidate A's (or B's) vote share to five, leads to a round two between candidate A (or B) and candidate C. Candidate C's odds of victory are 24.9%. This is strictly worse. All pivotal voters prefer to not deviate so the election is in a non-symmetric equilibrium.

Appendix J

The non-equilibrium events

A non-equilibrium event is any where there is a deviation that a voter can make that leads to a preferred outcome.

The first set of events are when candidate A or B has five votes. This is not an equilibrium in any treatment.

In $MC6$, $MU6$ and $MU5$ if candidate A (or B) has five votes then at least one β (or α) voter must be insincere. The election goes to round two between candidate A (or B) and candidate C. In $MC6$ candidate A (or B) wins in round two. In $MU6$ and $MU5$ candidate C's chances of victory are 34.3% and 24.9%. If the β (or α) voter deviates this does not change candidate C's odds. They still reach round two. However, both candidate A and candidate B have four votes. One will be randomly selected to progress to round two. Therefore in $MC6$ candidate B's (or A's) chance of victory rises to 50%. For $MU6$ and $MU5$ candidate B's (or A's) chance of victory rises to 32.9% and 37.5% respectively. This outcome is preferred for the β (or α) type voter. This is therefore not an equilibrium.

In $FC6$, $FU6$ and $FU5$ it takes six votes to pass the variable threshold. Currently neither candidate A nor B passes the threshold. In $FC6$ and $FU6$ candidate C will pass the variable threshold. Candidate C wins with certainty. In $FU5$ no one passes the variable threshold. Candidates A (or B) and C progress to round two. Candidate C has 24.9% chance of victory. If candidate A (or B) has five votes then any voter that currently votes B (or A) has an incentive to deviate. If they deviate to A (or B) then candidate A (or B) now has six votes. In $FC6$ and $FU6$ the election goes to round two. Here candidate C now has a 34.2% chance of victory. In $FU5$

candidate A (or B) now wins with certainty. In both cases there is an incentive to deviate and this is not an equilibrium.

The second set of events are when candidate A or B is less than one vote below the majority threshold in the uncertainty treatments, $MU6$, $MU5$ and $FU6$. This is not an equilibrium in $MU5$. In $MU5$ there is no variable threshold. The majority threshold is six point five. When candidate A (or B) has six votes no candidate is above the majority threshold. Candidates A (or B) and C progress to round two and candidate C has a 24.9% chance of victory. A single deviation by either of the voters that voted B (or A) to A (or B) takes candidate A (or B) above the majority threshold with seven votes. They now win with certainty. Both types prefer this.

This is not an equilibrium in $MU6$ and $FU6$. When candidate A (or B) has seven votes no candidate is above the majority threshold. Candidates A (or B) and C progress to round two and candidate C has a 34.2% chance of victory. A single deviation by the one voter that voted B (or A) to A (or B) takes candidate A (or B) above the majority threshold with eight votes. They now win with certainty. Both types prefer this.

The final set of events are when candidate A and candidate B have four votes but not all voters are sincere. This is not an equilibrium in $MC6$, $MU6$, $MU5$ and $FU5$. It is necessary that candidate C is not above the variable threshold. This is true in these four treatments. Therefore no candidate passes the variable threshold. Candidate A or B is randomly selected to progress to round two. In $MC6$ candidate A and candidate B have a 50% chance of victory. Then in $MU5$ and $FU5$ it is 37.5% and for $MU6$ it is 32.9%. At least one α type voter is insincere. If they deviate from B to A then candidate A has five votes. They then reach round two in all cases. Now their chance of victory is double. This is therefore preferable for the α voter. The same rationale means that any β voter will deviate to being sincere. Therefore any case where candidate A and candidate B have four votes but not all voters are sincere, this is not an equilibrium.

Appendix K

Preference order of the lottery TIE only depends on the relationship between lotteries B and AC

It will be assumed that the voter i's strict utility relationship over the singleton lotteries is:

$$U_i(A) > U_i(B) > U_i(C)$$

Therefore their strict utility order is (ABC). It is known that for a voter with utility order (ABC) the lotteries that have a known position are:

$$U_i(A) > U_i(AB) > U_i(B) > U_i(BC) > U_i(C)$$

The position of AC is then ambiguous but takes one of three positions

$$B \succ_i AC\ ,\ \ B \sim_i AC\ \ or\ \ AC \succ_i B$$

Theorem K.1. *The utility relationship for the lottery TIE is known if the relationship between AC and B is known.*

Proof: The utility from TIE is:

$$U_i(TIE) = \frac{U_i(A)}{3} + \frac{U_i(B)}{3} + \frac{U_i(C)}{3}$$

The utility from a two way tie for example AB is:

$$U_i(AB) = \frac{U_i(A)}{2} + \frac{U_i(B)}{2}$$

The utility from lottery A is always higher than lottery TIE regardless of the relationship between AC and B:

$$U_i(A) > U_i(B) \text{ and } U_i(A) > U_i(C)$$
$$\therefore\ 2 * U_i(A) > U_i(B) + U_i(C)$$

Divide both sides by 3

$$\frac{2*U_i(A)}{3} > \frac{U_i(B)}{3} + \frac{U_i(C)}{3}$$

Add $\frac{U_i(A)}{3}$ to both sides

$$U_i(A) > \frac{U_i(A)}{3} + \frac{U_i(B)}{3} + \frac{U_i(C)}{3}$$
$$U_i(A) > U_i(TIE)$$

The utility from lottery C is always lower than lottery TIE regardless of the relationship between AC and B:

$$U_i(A) > U_i(C) \text{ and } U_i(B) > U_i(C)$$
$$\therefore\ U_i(A) + U_i(B) > 2 * U_i(C)$$

Divide both sides by 3

$$\frac{U_i(A)}{3} + \frac{U_i(B)}{3} > \frac{2*U_i(C)}{3}$$

Add $\frac{U_i(C)}{3}$ to both sides

$$\frac{U_i(A)}{3} + \frac{U_i(B)}{3} + \frac{U_i(C)}{3} > U_i(C)$$
$$U_i(ABC) > U_i(C)$$

The utility from lottery AB is always higher than the utility from lottery TIE

$$U_i(A) > U_i(C) \text{ and } U_i(B) > U_i(C)$$
$$\therefore U_i(A) + U_i(B) > 2(U_i(C))$$

Add $2 * (U_i(A) + U_i(B))$ to both sides

$$3 * (U_i(A) + U_i(B)) > 2(U_i(A) + U_i(B) + U_i(C))$$

Divide both sides by 6.

$$\frac{U_i(A)}{2} + \frac{U_i(B)}{2} > \frac{U_i(A)}{3} + \frac{U_i(B)}{3} + \frac{U_i(C)}{3}$$
$$U_i(AB) > U_i(TIE)$$

The utility from lottery BC is always lower than the utility from lottery TIE

$$U_i(A) > U_i(B) \text{ and } U_i(A) > U_i(C)$$
$$\therefore 2 * (U_i(A)) > U_i(B) + (U_i(C)$$

Add $2 * (U_i(B) + U_i(C))$ to both sides

$$2 * (U_i(A) + U_i(B) + U_i(C)) > 3 * (U_i(B) + U_i(C))$$

Divide both sides by 6.

$$\frac{U_i(A)}{3} + \frac{U_i(B)}{3} + \frac{U_i(C)}{3} > \frac{U_i(B)}{2} + \frac{U_i(C)}{2}$$
$$U_i(TIE) > U_i(BC)$$

As such there is no ambiguity in regards to TIE and any of the 4 lotteries, A, C, AB and BC. Additionally these all hold regardless of the relationship between AC and B. This just leaves the relationship between TIE, B and AC. All give lower utility than A and AB and all give greater utility than BC and C. Now the specific cases must be considered.

$$\text{assume: } U_i(B) > U_i(AC)$$
$$U_i(B) > \frac{U_i(A)}{2} + \frac{U_i(C)}{2}$$

Add $\frac{U_i(B)}{2}$ to both sides

$$1.5 * U_i(B) > \frac{U_i(A)}{2} + \frac{U_i(B)}{2} + \frac{U_i(C)}{2}$$

Divide both sides by 1.5.

$$U_i(B) > \frac{U_i(A)}{3} + \frac{U_i(B)}{3} + \frac{U_i(C)}{3}$$
$$U_i(B) > U_i(TIE)$$

Doing the same thing to compare TIE and AC

$$U_i(B) > \frac{U_i(A)}{2} + \frac{U_i(C)}{2}$$

Add $U_i(A) + U_i(C)$ to both sides.

$$U_i(A) + U_i(B) + U_i(C) > \frac{3{*}U_i(A)}{2} + \frac{3{*}U_i(C)}{2}$$

Divide both sides by 3.

$$\frac{U_i(A)}{3} + \frac{U_i(B)}{3} + \frac{U_i(C)}{3} > \frac{U_i(A)}{2} + \frac{U_i(C)}{2}$$
$$U_i(TIE) > U_i(AC)$$

Therefore When $U_i(B) > U_i(AC)$ then $U_i(B) > U_i(TIE) > U_i(AC)$.

The same logic holds such that if $U_i(B) = U_i(AC)$ then $U_i(B) = U_i(TIE) = U_i(AC)$. Similarly when $U_i(B) < U_i(AC)$ then $U_i(B) < U_i(TIE) < U_i(AC)$.

Therefore the relative position of TIE is solely dependent on the utility relationship between B and AC. ■

Appendix L

Numerical examples of results for Chapter Three

Assume $\delta = 0.01$ (this is for simplicity of the example. It can be smaller if the candidate needed it to be)

L.1 Plurality: Small Incumbent divergence with candidate C's optimal strategy being an outside position.

Take $h = (0.49, 0.52)$ as an example.

$$\text{When } \lambda_C = (\lambda_A - \delta);\, V_C = 0.49 - \tfrac{\delta}{2} = 0.485$$
$$\text{When } \lambda_C = (\lambda_B + \delta);\, V_C = 1 - 0.52 - \tfrac{\delta}{2} = 0.475$$

Therefore the vote share is greater when candidate C takes the position just outside candidate A which is the candidate that is closer to the centre.

$$\text{When } \lambda_C \in (\lambda_A, \lambda_B);\, V_C = \tfrac{0.52-0.49}{2} = 0.015$$
$$\text{When } \lambda_C = \lambda_A;\, V_C = \tfrac{0.49+0.52}{4} = 0.2525.$$
$$\text{When } \lambda_C = \lambda_B;\, V_C = 0.5 - \tfrac{0.49+0.52}{4} = 0.2475.$$

Therefore the vote share is greater on the outside than the inside/copying so candidate C will take the outside position. The vote shares are:

$$V_A = 0.02,\ V_B = 0.495\ \textit{and}\ V_C = 0.485$$

Candidate B wins with certainty.

Candidate A now has an incentive to deviate so the new history is for example $h = (0.47, 0.52)$. The new vote shares are:

$$V_A = 0.495,\ V_B = 0.03\ \textit{and}\ V_C = 0.475$$

Candidate A now wins with certainty. This process of increased divergence will continue until candidate C no longer takes the outside position.

L.2 Plurality: Incumbent divergence with candidate C's optimal strategy being an inside position.

$h = (0.25, 0.75)$ is the equilibrium policy position. At this history:

$$\text{When } \lambda_C = (\lambda_A - \delta);\ V_C = 0.25 - \tfrac{\delta}{2} = 0.245$$
$$\text{When } \lambda_C = (\lambda_B + \delta);\ V_C = 1 - 0.75 - \tfrac{\delta}{2} = 0.245$$

Candidate C gets a vote share, $V_C = 0.245$ taking the outside positions.

$$\text{When } \lambda_C \in (\lambda_A, \lambda_B);\ V_C = \tfrac{0.75-0.25}{2} = 0.25.$$
$$\text{When } \lambda_C = \lambda_A;\ V_C = \tfrac{0.75+0.25}{4} = 0.25.$$
$$\text{When } \lambda_C = \lambda_B;\ V_C = \tfrac{0.75+0.25}{4} = 0.25.$$

Candidate C gets a vote share of $V_C = 0.25$ taking the inside position or copying one of the two incumbents. So their optimal strategy is $\lambda_C \in [\lambda_A, \lambda_b]$

Candidate A and candidate B both get $V_A = V_B = 0.375$ on average and each has a 50% chance of victory. Candidate C will take a random position within the range $[0.25, 0.75]$

Assume Candidate A tries to deviate away from this point.

If $\lambda_A = 0.24$; candidate C randomises in the range $[0.24, 0.75]$, the vote shares are: $V_C = 0.255$ and the average for $V_A = 0.37$ and $V_B = 0.375$. Candidate A now has a less than 50% chance of victory and is worse off.

If $\lambda_A = 0.26$; candidate C moves to $\lambda_C = 0.25$, the vote shares are: $V_A = 0.25,\ V_B = 0.495\ and\ V_C = 0.255$. Candidate B wins with certainty and candidate A is worse off.

This is then the only equilibrium.

L.3 Majority Run-off: Candidate C winning against non-symmetric incumbents with the inside option

Take $h = (0.4, 0.65)$ as an example.

To win with certainty candidate C can play $\lambda_C = 0.39$ and the vote shares in round one are;

$$V_A = 0.13,\ V_B = 0.475\ and\ V_C = 0.395$$

Candidate A is eliminated and then the vote shares in round two are;

$$V_B = 0.48\ and\ V_C = 0.52.$$

Therefore candidate C wins with certainty.

L.4 Majority Run-off: Candidate C winning against non-symmetric incumbents with the outside option

Take $h = (0.25, 0.8)$ for example:

To win with certainty candidate C can play $\lambda_C = 0.26$ the vote shares in round one are;

$$V_A = 0.255, \; V_B = 0.47 \; and \; V_C = 0.275$$

Candidate A is eliminated and then the vote shares in round two are;

$$V_B = 0.47 \; and \; V_C = 0.53.$$

Therefore candidate C wins with certainty.

L.5 Majority Run-off: Candidate C always loses against symmetric incumbents with the outside option.

Take $h = (0.45, 0.55)$ for example.

If candidate C plays $\lambda_C = 0.44$ the vote shares in round one are:

$$V_A = 0.055, \; V_B = 0.5 \; and \; V_C = 0.455$$

Candidate A is eliminated and then the vote shares in round two are;

$$V_B = 0.505 \; and \; V_C = 0.495.$$

Candidate B therefore wins.

L.6 Majority Run-off: Candidate C always wins against symmetric incumbents with the inside option.

Take $h = (0.2, 0.8)$ for example

To win with certainty candidate C can play $\lambda_C = 0.21$. The vote shares in round one are:

$$V_A = 0.205,\ V_B = 0.495\ and\ V_C = 0.4$$

Candidate A is eliminated and then the vote shares in round two are;

$$V_B = 0.495\ and\ V_C = 0.505$$

Candidate C wins with certainty.

L.7 Flexible Run-off: Candidate C wins against incumbents that are too far from the centre.

Take for example $\tau = 0.4$ and $h = (0.2, 0.8)$

Candidate C will take position; $\lambda_C \in \{0.4, 0.6\}$. Take $\lambda_C = 0.6$ [1], the vote shares are;

$$V_A = 0.4,\ ,V_B = 0.3\ and\ V_C = 0.3.$$

In half the elections candidate C is eliminated, candidate A and candidate B are equally close to the centre and have a 50% chance of victory each. In the other half candidate B is eliminated. Candidate C is closer to the centre than candidate A and wins with certainty. The odds of victory for candidate A and B are 25% each and for candidate C the odds are 50%.

This is not an equilibrium. If either candidate A or B moves towards the centre (assume for the example A) then $h = (0.21, 0.8)$; if $\lambda_C = 0.6$ then:

$$V_A = 0.405$$

[1] The result is mirrored with $\lambda_C = 0.4$

Candidate A passes the threshold and wins with certainty. If candidate C moves towards A to stop them winning in round one e.g. $\lambda_C = 0.59$ the vote shares are:

$$V_A = 0.4,\ V_B = 0.305 \ and \ V_C = 0.295$$

No candidate wins in round one but candidate C is eliminated and the vote shares are:

$$V_A = 0.505 \ and \ V_B = 0.495$$

Candidate A wins in round two.

L.8 Flexible Run-off: Candidate A moves to the centre until candidate C takes an outside position.

Take $\tau = 0.4$ and $h = (0.21, 0.75)$; candidate C cannot win;

When, $\lambda_C = 0.71 \quad \Rightarrow \quad V_C = V_B \ and \ V_A = 0.46$

Candidate A passes the threshold and then wins in round one.

Alternatively $\lambda_C = 0.59 \quad \Rightarrow \quad V_A = 0.4,\ V_B = 0.33 \ and \ V_C = 0.27$

No candidate passes the threshold but candidate C comes last and is eliminated and candidate B wins in round two.

Candidate C can not stop both other candidates passing the threshold and come second. So they can not win and instead vote maximise. They take $\lambda_C \in (0.21, 0.75)$.

When $0.59 \geq \lambda_C > 0.21$ candidate B wins the election in either round one by passing the threshold (When $0.45 > \lambda_C > 0.21$) or winning in round two (when $0.59 > \lambda_C \geq 0.45$).

When $0.75 > \lambda_C > 0.59$ candidate A wins in round one. So candidate A's probability of winning is roughly 30%. This is not optimal for them as if they move to for example $\lambda_A = 0.25$ then their odds of victory are 50%[2]. The same logic holds with symmetry.

[2]Due to symmetry and the fact that C can not win

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