Online Appendix for Random Serial Dictatorship: The One and Only

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Abstract

The set of all Pareto optimal, strategy proof and non-bossy mechanisms is characterized as the set of trading and braiding mechanisms. Fix a matching problem with more than three houses and a profile of preferences. At the start of a trading and braiding mechanism at most one house is brokered; all other houses are owned. In the first trading round, owners point to their most preferred houses, the broker - if there is one - points to his most preferred owned house, and houses point to the agents who control them. At least one cycle forms. The agents in such a cycle are matched to the houses they point to. The process is repeated with the remainder. Once there are only three houses left the mechanism might turn into a braid, a device that avoids a particular matching.

1 Introduction

This online appendix contains the proofs of Theorem 2, Lemma 3 and Lemma 7. I prove Theorem 2 and Lemma 3 by induction over the number of agents n.

Start: If $n = 1$ there is exactly one good mechanism: match the agent with his highest ranked house. This mechanism has a unique representation as a trading and braiding mechanism with one control rights function that assigns ownership of every house to the single agent. All statements on lax trading and braiding mechanisms (including Lemma 3) are trivially satisfied with only one agent.

Hypothesis: Theorem 2 and Lemma 3 hold with fewer than n agents.

Step: Theorem 2 and Lemma 3 hold with n agents given that the hypothesis holds.

The proof of the Step for Theorem 2 and Lemma 3 has many chapters. Braids are good (Section 3). Lax mechanisms are well-defined and good (Sections 4 and 5). Representations as trading and braiding mechanisms are unique (Section 6). The proof of the converse direction of Theorem 2, that any good mechanism M can be represented as a trading and braiding mechanism, starts with a list of arguments that are used repeatedly in the sequel (Section 7). For M to be representable as a trading and braiding mechanism a function $c_\emptyset : H \to N \times \{o, b\}$ needs to be defined (Section 8). This function $c_\emptyset$ is shown to have a set of properties in line with the properties of control rights functions in trading and braiding mechanisms (Section 9). These properties are used to show that $c_\emptyset$ satisfies (C1), (C2'), and (C3) implying in particular that braids are the only alternative to trading rounds (Section 10). If M is not a braid, then its outcome at R is consistent with any submatching achieved in a first trading round given $c_\emptyset$ and R. Any such submatching is followed by a well-defined trading and braiding (sub-)mechanism, represented by a tight structure $c[\nu]$ (Section 11). These tight structures $c[\nu]$ together with $c_\emptyset$ define a tight structure c that satisfies all conditions (C1)-(C6) (Section 12). If c is a tight structure derived from a control rights structure $\bar{c}$ via the rules outlined in Lemma 3, then $M^c(R) = M^{\bar{c}}(R)$ holds for each profile R (Section 13).

Pycia and Unver [5] also sets out to characterize the set of all good mechanisms. While their result turned out to be wrong (their trading cycles mechanisms do not admit braids) their approach and in particular the insight that houses might be brokered are indispensable for the present characterization. Many of the following chapters lean on the Pycia and Unver [5] proof. To disentangle the present contributions from Pycia and Unver's [5], let me give an overview of the differences between the two proofs.

First, my proof is by induction over the number of all agents n. This makes the present proof that all trading and braiding mechanisms are good more concise than the Pycia and Unver [5] proof of the equivalent claim. Next, Pycia and Unver [5] does not consider braids. Neither the statement that braids are good nor its proof appear in Pycia and Unver [5]. The first steps of the current proof that any good mechanism has a representation as a trading and braiding mechanism exactly follow Pycia and Unver [5]. I state the first result in this context, Lemma 12 [Pycia and Unver [5]: Lemma 9], without proof. This result formalizes the pivotal idea in Pycia and Unver [5]: if all agents' rankings agree on the two top ranked houses then the recipient of the top ranked house does not depend on the agents' rankings of the other houses. For all remaining lemmas I do state proofs even though some of these lemmas appear in Pycia and Unver [5]. In some cases the new proofs are shorter. Moreover, the complete proof can be understood without going back and forth between two different approaches and sets of notation. I will revisit the comparison between the two proofs in Sections 8 through 12.

Some ancillary statements in the current paper do not correspond to results in Pycia and Unver [5]. Given that the definition of a lax mechanism only requires the matching of at least one cycle at every trading round, it needs to be shown that lax mechanisms are well-defined in the sense that outcomes do not depend on the order in which trading cycles are removed (Section 4). While Section 6 shows that any good mechanism has a unique representation as a trading and braiding mechanism, Pycia and Unver [5] makes no such uniqueness claim. The fact that braids cannot be represented as Pycia and Unver [5] trading cycles mechanisms follows from the uniqueness result in Section 6.

Lemma 7 has already been established by Bogomolnaia and Moulin [2]. For convenience I also state a proof here. Differently from the Bogomolnaia and Moulin [2] proof, the present proof does not use the concept of ordinal efficiency. Otherwise the present proof is identical to the Bogomolnaia and Moulin [2] proof.

2 Further Notation and Concepts

In addition to the notation $R_i : e, g$ for a preference that ranks e and g in first and second place, I use superscripts when all agents agree on the top ranked houses: Any profile of preferences in which all agents agree that h is the best house is denoted R^h, so $hR_i^h H$ for all $i \in N$. Similarly the profiles R^{gh} and R^{egh} are such that all agents agree on the ranking of the top two and respectively top three houses: $gR_i^{gh}hR_i^{gh}H\setminus\{g\}$ and $eR_i^{egh}gR_i^{egh}hR_i^{egh}H\setminus\{e,g\}$ for all $i \in N$. Different profiles where all agents rank the same house at the top, say e, are denoted $\hat{R}^e$ and $\tilde{R}^e$. Two preferences R_i and R_i' **coincide** on $H' \subset H$ if $eR_ig \Leftrightarrow eR_i'g$ holds for all $e, g \in H'$. Two profiles of preferences R and R' coincide on $H' \subset H$ if R_i coincides with R_i' on H' for all $i \in N$. If R_i and R_i' coincide on H' then their restrictions to H', $\overline{R}_i$ and $\overline{R}_i'$, are identical. Let $\mathcal{N}(c_\emptyset)$ be the the set of direct c-successors to $\emptyset$ and let $\mathcal{N}(c_\emptyset)(R)$ be the subset of direct c-successors to $\emptyset$ that are reachable under c at R.[1]

For the proof, let me be precise about requirements (C1), through (C6) on control right functions c_ν.

(C1) If $c_\nu(e) = (\cdot, b)$ and $c_\nu(g) = (\cdot, b)$ hold for two different $e, g \in H$, then $H = \{e, f, g\}$ and $c_\nu(e) = (i, b)$, $c_\nu(f) = (j, b)$, and $c_\nu(g) = (j', b)$ holds for three different agents $i, j, j' \in N$.

(C2) If $c_\nu(h_b) = (i_b, b)$ holds for exactly one $h_b \in H$ and $i_b \in N$ then $c_\nu(h) = (i, o)$ holds for some $i \in N \setminus \{i_b\}$ and $h \in H \setminus \{h_b\}$.

(C2') If $c_\nu(h_b) = (i_b, b)$ holds for exactly one $h_b \in H$ and $i_b \in N$ then $c_\nu(e) = (i, o)$, $c_\nu(f) = (j, o)$ holds for two different agents $i, j \in N \setminus \{i_b\}$ and two houses $e, f \in H \setminus \{h_b\}$.

(C3) There are no houses $h_b, h \in H$ and agent $i \in N$ with $c_\nu(h_b) = (i, b)$ and $c_\nu(h) = (i, o)$.

Fix a c-relevant ν° and a direct c-successor ν to ν°. Then the following implications holds for all agents i, j, i_b and all houses h_b, h, e

(C4) If $c_{\nu^\circ}(h) = (i, o)$ and $i \notin N_\nu$ then $c_\nu(h) = (i, o)$.

[1] Since these sets coincide for two lax trading and braiding mechanisms c and c' with $c_\emptyset = c'_\emptyset$ only the control rights function $c_\emptyset$ is used to define $\mathcal{N}(c_\emptyset)$ and $\mathcal{N}(c_\emptyset)(R)$.

(C5) If $c_{\nu^\circ}(h_b) = (i_b, b)$ $c_{\nu^\circ}(h) = (i, o) \neq c_{\nu^\circ}(e) = (j, o)$, and $i_b, i, j \notin N_\nu$ then $c_\nu(h_b) = (i_b, b)$.
(C6) If $c_{\nu^\circ}(h_b) = (i_b, b)$, $c_{\nu^\circ}(h) = (i, o)$, $i_b, i \notin N_\nu$ and $c_\nu(h_b) \neq (i_b, b)$, then $c_\nu(h_b) = (i, o)$ and $c_{\nu\cup\{(i,h_b)\}}(h) = (i_b, o)$.

3 Braids are good

The following claim was first made in Bade [1]. Neither the claim nor its proof appear in Pycia and Unver [5]. The published revision, Pycia and Unver [6], proves that braids (formalized as mechanisms with 3 brokers) are good.

Lemma 11 *The braid B^ω is good.*

Proof Since $B^\omega(R)(i) = \emptyset$ and $B^\omega(R'_i, R_{-i}) = B^\omega(R)$ holds for all $i \notin N_\omega$, R'_i and R it is w.l.o.g to assume $N = \{1, 2, 3\}$. To fix ideas, let $\omega = (e, f, g)$.[2] B^ω is Pareto optimal as $B^\omega(R) \in Mini(R) \subset PO(R)$ holds for any $R \in \mathcal{R}$. The alternative representation B' is useful in the upcoming arguments.

I. If $\{\omega\} = PO(R)$ then $B'(R) := \omega$.

for II and III let $\{i, j, j'\} = \{1, 2, 3\}$ and say ω' and ω'' are the two matchings that maximally avoid ω.

II. If $\omega'(i) R_i \omega''(i)$ and $R_j : \omega(i) = \omega'(j)$, then $B'(R) := \omega'$.

III. If $\omega'(i) R_i \omega''(i)$, $R_j : \omega(j)$ and $R_{j'} : \omega(i) = \omega'(j)$, then $B'(R) := (\omega'(i), \omega(j), \omega(i))$.

Cases I, II, and III partition the set of all preference profiles, as $PO(R) \cap \{\omega', \omega''\} \neq \emptyset$ holds if and only if R is covered by case II, whereas case III applies if and only if $\{\omega\} \neq PO(R) \subset \mathcal{R} \setminus \{\omega', \omega''\}$.

To see that $B^\omega(R) = B'(R)$ holds for all $R \in \mathcal{R}$, first assume $PO(R) = \{\mu\}$, implying $B^\omega(R) = \mu$. If $\mu = \omega$, then $B'(R) = \omega$ holds by I. If $\mu = \omega' = (g, e, f)$, then $B'(R) = \omega'$ holds by II as agent 2 ranks $\omega'(2) = e = \omega(1)$

[2] Recall that matchings can be denoted as vectors, where the i-th component stands for agent i's match.

highest and as $R_1 : g$ implies $g = \omega'(1) R_1 \omega''(1) = f$. If $\mu \notin \{\omega, \omega', \omega''\}$, then some agent, say 2, must rank his avoidance match highest: $R_2 : f = \omega(2) = \mu(2)$. For $\mu \neq \omega$ to be the unique Pareto optimum, we must have $R_1 : g = \omega(3) = \mu(1)$ and $R_3 : e = \omega(1) = \mu(3)$. By III we obtain $B'(R) = \mu$.

Now assume that $PO(R)$ is not a singleton, say two agents rank $\omega(1) = e$ highest. Assume w.l.o.g that $g = \omega'(1) R_1 \omega''(1) = f$. First let R be covered by II, so $R_2 : e = \omega'(2)$, $B'(R) = \omega'$, and $\omega' \in Mini(R) \subset \{\omega', \omega''\}$. If in addition $R_3 : e$, then $B^\omega(R) = \omega'$ holds since 1 prefers ω' to ω''. If $R_3 : h \neq e$ then $B^\omega(R) = \omega'$ holds since ω' matches 2 (the only agent other than 1 who ranks e at the top) with e. Finally consider the case that R is covered by III, where the preceding assumption that $g = \omega'(1) R_1 \omega''(1) = f$ is now combined with $R_2 : f$ and $R_3 : e = \omega'(2) = \omega(1)$, so that $B'(R) = (g, f, e)$. Since $PO(R)$ is not a singleton $R_1 : e,\ g$ must hold and $B^\omega(R)$ equals (g, f, e), the only Pareto optimum that matches 3 to e. In sum, B^ω and B' define the same mechanism called B in the remainder of the proof of Lemma 11.

Only in two cases is an agent j matched with his avoidance match $\omega(j)$: Either ω is the unique Pareto optimum at R (Case I) or $\omega'(i) R_i \omega''(i)$, $R_j : \omega(j)$, and $R_{j'} : \omega(i) = \omega'(i)$ hold (Case III). So we have

$(O):$ $B(R)(j) = \omega(j)$ only holds if j but no other agent ranks $\omega(j)$ highest.

To see that B is strategy proof and non-bossy fix a profile R. If R is covered by I all agents obtain their best house. Moreover $B(R) \neq B(R'_i, R_{-i})$ holds only if $R'_i : h \neq \omega(i)$. But in that case (O) implies $B(R)(i) = \omega(i) \neq B(R'_i, R_{-i})(i)$.

As an example of a profile that is covered by II, consider R with $g = \omega'(1) R_1 \omega''(1) = f$ and $R_2 : e = \omega(1) = \omega'(2)$, so $B(R) = \omega' = (g, e, f)$. Given $R_2 : e = \omega(1)$ and (O) 1 cannot obtain e, the only house he might possibly prefer to $g = B(R)(1)$, 2 obtains his best house, and 3 has no impact on the outcome. By the last statement $B(R)(3) = B(R'_3, R_{-3})(3) \Rightarrow B(R) = B(R'_3, R_{-3})$ trivially holds. For $B(R)(2) = e = B(R'_2, R_{-2})(2)$ and $B(R) \neq B(R'_2, R_{-2})$ to hold $B(R'_2, R_{-2})$ must equal (f, e, g). So (R'_2, R_{-2}) is covered by III with $f R_1 g$ a contradiction to the assumption that $g R_1 f$. $B(R) = (g, e, f) \neq B(R'_1, R_{-1})$ and $B(R)(1) = B(R'_1, R_{-1})(1)$ imply $B(R'_1, R_{-1})(2) \neq$

e. The latter only holds if $R_3 : e$ and $f R'_1 g$. But then $B(R'_1, R_{-1}) = \omega''$ holds by II and we obtain the contradiction $B(R'_1, R_{-1})(1) = \omega''(1) \neq \omega'(1) = B(R)(1)$.

As an example of a profile that is covered by III, consider R with $g = \omega'(1) R_1 \omega''(1) = f$, $R_2 : f = \omega(2)$, and $R_3 : e = \omega'(2)$, so $B(R) = (g, f, e)$. By $R_3 : e$ and (O), 1 cannot obtain $e = \omega(1)$, the only house he might possibly rank above $B(R)(1) = g$; 2 and 3 obtain their best houses. Fixing the match between 1 and g, (g, f, e) is the only Pareto optimal matching, and fixing the match between 3 and e, (g, f, e) is the only Pareto optimal matching. So $B(R)(i) = B(R'_i, R_i)(i) \Rightarrow B(R) = B(R'_i, R_{-i})$ holds for $i = 1, 3$ and any R'_i. Finally (O) implies $B(R'_2, R_{-2})(1) \neq e$ for any R'_2. Since (g, f, e) is the only such Pareto optimum at (R'_2, R_{-2}), the implication also holds for $i = 2$.

Mutatis mutandum the above arguments apply to generic profiles R and B is strategy proof and non-bossy. □

4 Lax mechanisms are well-defined

Fix a control rights structure c for n agents and a profile of preferences R. To see that the lax mechanism M^c is well-defined we need to first check that the algorithm always specifies some step to be taken and terminates with a matching. Secondly, if the algorithm allows for multiple choices at some step, we need to check that all these choices lead to the same matching.

To see the first note that c_ν is specified for any possible round of the algorithm.[3] Now consider a c-relevant ν with at most one broker. By the definition of c_ν, every unmatched house is controlled by an unmatched agent. So every $h \in \overline{H}_\nu$ points to an $i \in \overline{N}_\nu$. Every owner in $\overline{N}_\nu$ points to his most preferred house in $\overline{H}_\nu$. Every broker in $\overline{N}_\nu$ points to his most preferred owned house in $\overline{H}_\nu$. Assumption (C2) ensures that there is at least one

[3] A control rights structure c specifies c_ν for any c-relevant ν. A submatching ν is in turn c-relevant if there exists a profile R, such that ν it is reachable under the algorithm given by R and c. Finally ν is reachable given R and c if one can match trading cycles in such an order that some round of the trading algorithm used to calculate $M^c(R)$ starts with ν.

owned house at ν. Since there are finitely many houses and agents, at least one cycle forms. Since at least one cycle must be matched in every round the algorithm terminates with a matching.

We, secondly, need to see that the outcome of the algorithm does not depend on the order in which trading cycles are removed. Assume that c does not define a braid and let ν and ν' arise out of matching different (mutually exclusive, possibly empty) sets of cycles that form at $\emptyset$ given c and R. By the hypothesis of the induction the outcomes $M^{c[\nu\cup\nu']}$ (at any preferences) do not depend on the order in which trading cycles are matched. It therefore suffices to show ν' is reachable given $c[\nu]$ and $\overline{R}$, the restriction of R to $\overline{N}_\nu$ and $\overline{H}_\nu$. If ν or ν' equals $\emptyset$ the result trivially holds. So suppose $\nu \neq \emptyset \neq \nu'$. Since no broker may point to the house he brokers at least one agent in $N_{\nu'}$ is an owner at $\emptyset$. Given (C4) this agent must also be an owner at ν and $c[\nu]$ cannot define a braid.

Case 1: $c_\emptyset(h) = c_\nu(h)$ holds for all $h \in H_{\nu'}$. So any $h \in H_{\nu'}$ points to the same agent at $\emptyset$ and at ν. Any agent $i \in N_{\nu'}$ points to the same house at $\emptyset$ and at ν (since for each $i \in N$, $h, h_b \in H$: hR_iH implies $hR_i\overline{H}_\nu$ and since $hR_iH \setminus \{h_b\}$ implies $hR_i\overline{H}_\nu \setminus \{h_b\}$, which matters if i brokers h_b). So any cycle at $\emptyset$ that is not immediately removed remains a cycle at ν. As the inductive hypothesis applies to $c[\nu]$ we may match the cycles that result in ν' before any other cycles that may form at ν given c and ν' reachable under c at R.

Case 2: $c_\emptyset(h_b) \neq c_\nu(h_b)$ holds for some $h_b \in H_{\nu'}$. By (C4) $c_\emptyset(h_b) = (i_b, b)$ holds for some $i_b \in N$. As a broker i_b must point to some owned house h at $\emptyset$. Let $c_\emptyset(h) = (i, o)$. By (C4) agent i continues to own h at ν. By (C5) i is the only agent who owns houses at $\emptyset$ and ν. So $i_b \to h \to i \to h_b$ is the only cycle that forms at $\emptyset$ but remains unmatched at ν and ν' must equal $\{(i, h_b), (i_b, h)\}$. By (C6) i owns h_b at ν. Since i points to h_b at $\emptyset$ we have h_bR_iH and consequently $h_bR_i\overline{H}_\nu$. So at ν, h_b points to i and i to h_b. By the inductive hypothesis, we may match this one cycle and $\{(i, h_b)\}$ is reachable under $c[\nu]$ and $\overline{R}$. (C6) and $c_\emptyset(h) = (i, o)$ imply that i_b owns h at $\nu \cup \{(i, h_b)\}$. Since $hR_{i_b}H \setminus \{h_b\}$ implies $hR_{i_b}\overline{H}_\nu \setminus \{h_b\}$ a cycle just involving i_b and h forms. By the inductive hypothesis we may match this one cycle at $\nu \cup \{(i, h_b)\}$ and $\nu \cup \{(i, h_b), (i_b, h)\} = \nu \cup \nu'$ is reachable give c and R.

5 Lax mechanisms are good

Fix a lax mechanism M^c for n agents, a profile of preferences R, an agent i, and a preference R'_i. If $\nu \in \mathcal{N}(c_\emptyset)(R)$ and $i \notin N_\nu$ then $M^c(R) = \nu \cup M^{c[\nu]}(\overline{R})$ and $M^c(R'_i, R_{-i}) = \nu \cup M^{c[\nu]}(\overline{R}'_i, \overline{R}_{-i})$ hold for $\overline{R}$ and $\overline{R}'_i$ the restrictions of R and R'_i to $\overline{N}_\nu, \overline{H}_\nu$. Since there are fewer than n agents in $M^{c[\nu]}$, it is by the inductive hypothesis good. The proof that c is strategyproof and non-bossy is split into two cases.

Case 1: There exists a $\nu \in \mathcal{N}(c_\emptyset)(R)$ that does not match i. Fix such a ν and recall that $M^{c[\nu]}(\overline{R})(i) = M^c(R)(i)$ and $M^{c[\nu]}(\overline{R}'_i, \overline{R}_{-i})(i) = M^c(R'_i, R_{-i})(i)$. Since $M^{c[\nu]}$ is strategyproof, we have $M^{c[\nu]}(\overline{R})(i)\overline{R}_i M^{c[\nu]}(\overline{R}'_i, \overline{R}_{-i})(i)$ and consequently $M^c(R)(i)R_iM^c(R'_i, R_{-i})(i)$. Moreover $M^c(R)(i) = M^c(R'_i, R_{-i})(i)$ holds if and only if $M^{c[\nu]}(\overline{R})(i) = M^{c[\nu]}(\overline{R}'_i, \overline{R}_{-i})(i)$. Since $M^{c[\nu]}$ is non-bossy the latter implies $M^{c[\nu]}(\overline{R}) = M^{c[\nu]}(\overline{R}'_i, \overline{R}_{-i})$ and we have $M^c(R) = \nu \cup M^{c[\nu]}(\overline{R}) = \nu \cup M^{c[\nu]}(\overline{R}'_i, \overline{R}_{-i}) = M^c(R'_i, R_{-i})$.

Case 2: The only $\nu \in \mathcal{N}(c_\emptyset)(R)$ is such that $i \in N_\nu$. Since any cycle not involving i at $\emptyset$ given (R'_i, R_{-i}) would also form at $\emptyset$ given R, agent i must be part of the unique cycle that forms at $\emptyset$ given (R'_i, R_{-i}). So $M^c(R'_i, R_{-i})(i)$ is the house that i points to at $\emptyset$ given R'_i. Since $M^c(R)(i)$ is the R_i-best house among all houses that i may point to at $\emptyset$ under c, $M^c(R)(i)R_iM^c(R'_i, R_{-i})(i)$ holds and M^c is strategyproof. If $M^c(R'_i, R_{-i})(i) = M^c(R)(i)$ then the cycle that yields ν also forms at $\emptyset$ under (R'_i, R_{-i}) and $M^c(R'_i, R_{-i})$ equals $\nu \cup M^{c[\nu]}(\overline{R}) = M^c(R)$. So c is non-bossy.

To see that $M^c(R)$ is Pareto optimal at R, suppose there existed a matching $\mu \neq M^c(R)$ with $\mu(i)R_iM^c(R)(i)$ for all $i \in N$ and $\mu(i_b) \neq M^c(R)(i_b)$ for some $i_b \in N$. If $i_b \in N_\nu$ with $\nu \in \mathcal{N}(c_\emptyset)(R)$, then i_b is a broker as $\nu(i)R_iH$ holds for any owner $i \in N_\nu$. Say h_b is such that $c_\emptyset(h_b) = (i_b, b)$. Under c, h_b is the only house that i_b may not point to at $\emptyset$ and $\mu(i_b)$ must equal h_b. Since $i_b \in N_\nu$, $\nu(i^*) = h_b$ must hold for some i^*. By (C1) i^* is an owner and we have $h_bR_{i^*}H$, so that i^* strictly prefers $M^c(R)(i^*) = \nu(i^*) = h_b$ to $\mu(i^*) \neq h_b$ - a contradiction to $\mu(i)R_iM^c(R)(i)$ for all $i \in N$. So $\mu(i)$ must equal $\nu(i) = M^c(R)(i)$ for all $i \in N_\nu$. But $M^c(R) \setminus \nu$ equals $M^{c[\nu]}(\overline{R})$ which is by the inductive hypothesis Pareto optimal. So no matching μ can Pareto domiante $M^c(R)$.

6 Uniqueness

Fix two tight structures c and c^* with $M^c(R) = M^{c^*}(R)$ for all $R \in \mathcal{R}$. First suppose that $c_\emptyset(e) = (i, b)$ holds for some $e \in H$ and $i \in b$. So there exist two houses f and g such that $M^c(R^{ef})$ and $M^c(R^{eg})$ match e with two different agents. Since $M^c(R) = M^{c^*}(R)$ holds for all $R \in \mathcal{R}$, e is also brokered according to $c^*_\emptyset$. Since an agent brokers e if and only if $M^{c^*}(R^{ef})$ matches him with house f and since $M^c(R^{ef}) = M^{c^*}(R^{ef})$, we have that $c_\emptyset(e) = (i, b) = c^*_\emptyset(e)$. Since e was fixed arbitrarily c^* defines a braid if and only if c does. When c and c^* both define braids, they share the same avoidance matching ω where $(i, h) \in \omega$ if $c_\emptyset(h) = (i, b)$.

So suppose that at most one house is brokered according to $c_\emptyset$. If such a house exists then this house has, by the above arguments, the same broker under $c^*_\emptyset$. If $c_\emptyset(h) = (i, o)$ holds for some house h and agent i then $(i, h) \in M^c(R^h)$ holds for all profiles R^h. Since $M^c(R) = M^{c^*}(R)$ holds for all $R \in \mathcal{R}$, $(i, h) \in M^{c^*}(R^h)$ holds for all profiles R^h, and h must also under c^* be owned by i at $\emptyset$: $c^*_\emptyset(h) = (i, o)$. In sum we obtain $c_\emptyset = c^*_\emptyset$, so that the sets of direct c-successors and c^*-successors to $\emptyset$ coincide. For any such $\nu \in \mathcal{N}(c_\emptyset)$, $c[\nu]$ is by the hypothesis of the induction identical to $c^*[\nu]$, so that c and c^* are identical.

7 A collection of arguments

Fix an arbitrary good mechanism M, a profile of preferences R, an agent i, a preference R'_i and a house e such that $M(R)(i) = e$. The following arguments are used throughout the next sections. Strategy proofness implies that nothing changes for agent i when he ranks $e = M(R)(i)$ at least as high under R'_i as under R_i:

SP-I If $eR_ih \Rightarrow eR'_ih$ for all $h \in H$, then $M(R'_i, R_{-i})(i) = e$.

Since M is non-bossy we additionally obtain:

SP-NB If $eR_ih \Rightarrow eR'_ih$ for all $h \in H$, then $M(R'_i, R_{-i}) = M(R)$.

Let $g \neq e$ rank directly below e according to R_i, let e rank directly below g according to R'_i and let this be the only difference between R_i and R'_i (so

$eR_i g$, $gR'_i e$, and $hR_i h' \Leftrightarrow hR'_i h'$ if $\{h, h'\} \neq \{e, g\}$), then

SP-II $M(R'_i, R_{-i})(i) \in \{e, g\}$.

In combination with Pareto optimality the preceding observation yields

SP-PO If $M(R)$ is not Pareto optimal at (R'_i, R_{-i}), then $M(R'_i, R_{-i})(i) = g$.

8 The Definition of $c_\emptyset$

Lemma 12 below shows that if $M(\hat{R}^{eg})(i) = e$ holds for a particular $\hat{R}^{eg}$ then $M(R^{eg})(i) = e$ holds for any R^{eg}. This lemma is identical to Pycia and Unver [5] Lemma 9 and I do not provide a proof. The lemma crucially simplifies the problem of characterizing all good mechanisms. Thanks to Lemma 12 only a few top ranked houses matter in the upcoming arguments.

Lemma 12 *[Pycia and Unver [5], Lemma 9] Fix any $\hat{R}^{eg}$, $\tilde{R}^{eg}$, $\hat{R}^{feg}$, and $\tilde{R}^{feg}$. Then $M(\hat{R}^{eg})(i^*) = e$ implies $M(\tilde{R}^{eg})(i^*) = e$, and $M(\hat{R}^{feg})(i^*) = e$ implies $M(\tilde{R}^{feg})(i^*) = e$.*

Following Pycia and Unver [5], define a function $c_\emptyset : H \to N \times \{o, b\}$. If $M(R^{eg})(i) = e$ holds for all R^{eg} with $g \neq e$ let $c_\emptyset(e) = (i, o)$ if not let $c_\emptyset(e) = (i_b, b)$ where i_b is such that $M(R^{eg})(i_b) = g$ for some R^{eg}. To see that $c_\emptyset$ is well-defined we need to check that there exists a unique agent i_b who obtains the second best house in any profile R^{eg} when $M(R^{eg})(i) = e = M(R^{eh})(j)$ holds for some g, h and $i \neq j$. Lemma 13, which is equivalent to Pycia and Unver [5] Lemma 10, does this.

Lemma 13 *[Pycia and Unver [5], Lemma 10] Let $M(\hat{R}^{eg})(1) = e = M(\hat{R}^{ef})(2)$ for some $\hat{R}^{eg}, \hat{R}^{ef}$. Then there exists an agent i_b such that $M(R^{eh})(i_b) = h$ for any R^{eh} with $h \neq e$.*

Proof Fix an arbitrary $\tilde{R}^{efg}$ and R^{eg}. Say i_b is such that $M(\tilde{R}^{efg})(i_b) = f$ and say R^{eg} coincides with R^{efg} and R^{egf} on $H \setminus \{f\}$. Lemma 12 yields $M(R^{efg})(i_b) = f$. Switching R_i^{efg} to R_i^{eg} for all $i \neq i_b$ SP-NB yields $M(R_{i_b}^{efg}, R_{-i_b}^{eg}) = M(R^{efg})$. If $M(R_{i_b}^{egf}, R_{-i_b}^{eg})(i_b) = f = M(R_{i_b}^{efg}, R_{-i_b}^{eg})(i_b)$ then $M(R_{i_b}^{egf}, R_{-i_b}^{eg}) = M(R^{efg})$ as M is non-bossy. Lemma 12 and $M(\hat{R}^{eg})(1) = e = M(\hat{R}^{ef})(2)$

imply the contradiction $M(R^{egf}_{i_b}, R^{eg}_{-i_b})(1) = e = M(R^{efg})(2)$. By SP-II $M(R^{egf}_{i_b}, R^{eg}_{-i_b})(i_b)$ is either g or f. SP-I then implies $g = M(R^{egf}_{i_b}, R^{eg}_{-i_b})(i_b) = M(R^{eg})(i_b)$. So M matches agent i_b and g for any profile R^{eg}.

Switching the roles of g and f in the above arguments and using $M(R^{egf})(i_b) = g$ we obtain $M(R^{ef})(i_b) = f$ for any R^{ef}. To prove $M(R^{eh})(i_b) = h$ for all R^{eh} with $h \notin \{e, g, f\}$ apply the above arguments to R^{eh} and R^{egh} if $M(R^{eh})(e) = 2$ and to R^{eh} and R^{efh} otherwise. □

9 Properties of $c_\emptyset$

Lemmas 14, 15, and 16 show that $c_\emptyset$ satisfies a range of properties. Say i owns g and i_b brokers e. If i ranks e above all other houses and if i_b ranks g above all other houses (except possibly e), then i and i_b respectively obtain e and g. If i ranks g at the top he gets it. As a broker i_b does not control any house other than e. Lemmas 14, 15, and 16 condense the Lemmas 12 and 13 which are not (directly) used after the current section. The following proof of Lemma 15 significantly simplifies the Pycia and Unver [5] proof by induction over the set of unmatched agents. Lemmas 14 and part c) of Lemma 16 are small preliminary results whose content also appears interspersed in the arguments in Pycia and Unver [5]. Part a) of Lemma 16 corresponds to Pycia and Unver [5] Lemma 15. Part b) of Lemma 16) is not shown in Pycia Unver [5].

Lemma 14 *Let $c_\emptyset(e) = (1, b)$ and $M(\hat{R}^{eg})(2) = e$ for some $\hat{R}^{eg}$. If R such that $R_2 : e$ and either $R_1 : g$ or $R_1 : e,\ g$ then $M(R)(2) = e$ and $M(R)(1) = g$.*

Proof Let R^{eg} coincide with R on $H\backslash\{e, g\}$. Lemma 12 and $M(\hat{R}^{eg})(2) = e$ imply $M(R^{eg})(2) = e$. Lemma 13 and $c_\emptyset(e) = (1, b)$ imply $M(R^{eg})(1) = g$. Dropping e and g in all rankings SP-NB yields $M(R^{eg}) = M(R)$, in particular $M(R)(1) = g$ and $M(R)(2) = e$. □

Lemma 15 *[Pycia and Unver [5], Lemma 11] If $c_\emptyset(e) = (1, o)$ and $R_1 : e$, then $M(R)(1) = e$.*

Proof Fix a profile R with $R_1 : e$ and a house g. All profiles in the proof coincide with R on all statements not explicitly mentioned. Define i and f so that $M(R^{eg})(i) = g$ and $M(R^e)(i) = f$. **Case 1:** gR_if: By SP-NB $\{(1,e),(i,g)\} \subset M(R_i^{eg}, R_{-i}^e)$. Since $M(R_i^{eg}, R_{-i}^e)(i) = gR_if = M(R^e)(i)$ and since M is strategyproof, g must equal f. Since M is non-bossy we get $M(R_i^{eg}, R_{-i}^e) = M(R^e)$ and hence $M(R^e)(1) = e$. **Case 2:** fR_ig and $f \neq g$. SP-PO, SP-NB and $e = M(R_i^{efg}, R_{-i}^{ef})(1)$ yield $f = M(R_1^{fe}, R_i^{efg}, R_{-\{1,i\}}^{ef})(1) = M(R_1^{fe}, R_i^{efg}, R_{-\{1,i\}}^{e})(1)$. If $f = M(R_1^{ef}, R_i^{efg}, R_{-\{1,i\}}^{e})(1)$ then $f = M(R_1^{ef}, R_i^{eg}, R_{-\{1,i\}}^{e})(1)$ holds by SP-NB. A contradiction arises since $\{(1,e),(i,g)\} \subset M(R^{eg})$ and SP-NB imply $M(R_1^{ef}, R_i^{eg}, R_{-\{1,i\}}^{e})(1) = e$. So $M(R_1^{ef}, R_i^{efg}, R_{-\{1,i\}}^{e})(1) = e$ must hold by SP-II. This, $M(R^e)(i) = f$ and SP-NB imply $M(R_1^{ef}, R_i^{efg}, R_{-\{1,i\}}^{e}) = M(R_i^{efg}, R_{-i}^e) = M(R^e)$. So $M(R^e)(1) = e$ holds in either case. SP-NB and $R_1 : e$ then imply $M(R)(1) = e$. □

Lemma 16 *Let $c_\emptyset(e) = (1,b)$ for some $e \in H$. Then*

a) $c_\emptyset(h) \neq (1,o)$ holds for all $h \in H$.

b) $c_\emptyset(h) \neq (1,b)$ holds for all $h \in H \setminus \{e\}$.

c) If $c_\emptyset(g) = (i,o)$ then $M(R^{eg})(i) = e$.

Proof a) Suppose $c_\emptyset(g) = (1,o)$ for some $g \neq e$. Since $c_\emptyset(e) = (1,b)$ there exist $1 \neq i \neq j \neq 1$ and $f \notin \{e,g\}$ with $M(R^{eg})(i) = e = M(R^{ef})(j)$. SP-PO yields $M(R_i^{ge}, R_{-i}^{eg})(i) = g$. Since 1 owns g, we have $M(R_i^{ge}, R_{-i}^{eg})(1)R_1^{eg}g$. Since $M(R_i^{ge}, R_{-i}^{eg})(1) \neq g$, $M(R_i^{ge}, R_{-i}^{eg})(1)$ must equal e. SP-I then implies $M(R_i^{ge}, R_{-i}^{eg})(1) = e = M(R_1^{ef}, R_i^{ge}, R_{-\{1,i\}}^{eg})(1)$. Lemma 14 and $M(R^{ef})(j) = e$, yield the contradiction $M(R_1^{ef}, R_i^{ge}, R_{-\{1,i\}}^{eg})(j) = e$.

b) Suppose $c_\emptyset(g) = (1,b)$ for some $g \neq e$ and $M(R^{eg})(2) = e$. If there exist an agent $i \notin \{1,2\}$ and a house $f \notin \{e,g\}$ such that $M(R^{gf})(i) = g$, then Lemma 14 together with $c_\emptyset(e) = (1,b)$, $M(R^{eg})(2) = e$, $c_\emptyset(g) = (1,b)$ and $M(R^{gf})(i) = g$ yields the contradiction $M(R_{\{1,i\}}^{gf}, R_{-\{1,i\}}^{eg})(1) = g = M(R_{\{1,i\}}^{gf}, R_{-\{1,i\}}^{eg})(i)$. If no such i, f exist, then $M(R^{gh})(2) = g$ must hold for all $h \neq e$. Since g is brokered $M(R^{ge})(i) = g$ must hold for $i \notin \{1,2\}$. But then Lemma 14 together with $c_\emptyset(e) = (1,b)$, $M(R^{eg})(2) = e$, $c_\emptyset(g) = (1,b)$, and $M(R^{ge})(i) = g$ yields the contradiction $M(R_i^{ge}, R_{-i}^{eg})(1) = g = M(R_i^{ge}, R_{-i}^{eg})(i)$.

c) Lemma 15, $c_\emptyset(g) = (i, o)$, and SP-II imply $M(R^{eg})(i) \in \{e, g\}$. By part a) or the current Lemma $i \neq 1$. Since $c_\emptyset(e) = (1, b)$ we have $M(R^{eg})(1) = g$ so that $M(R^{eg})(i) = e$ must hold. □

10 Braids, (C1), (C2), and (C3)

If there is at most one brokered house according to $c_\emptyset$, then part a) of Lemma 16 implies that the broker of this house does not own any house as required by (C3). The definition of $c_\emptyset$ implies that there must be at least two owners under $c_\emptyset$ if there is exactly one broker under $c_\emptyset$ as required by (C2). Finally, Lemma 17 shows that M is a braid if at least two houses are brokered according to $c_\emptyset$ which therefore satisfies (C1).

The following Lemma 17 has no counterpart in Pycia and Unver [5] which claims in Lemmas 12 and 13, that a stronger version of (C1), according to which there is at most one broker at any ν, holds for any good mechanism. Pycia and Unver [6] corrects this error in their Appendix G.

Lemma 17 *Let $c_\emptyset(e) = (1, b)$ and $c_\emptyset(g) = (k, b)$ for $e \neq g$ and some $k \in N$. Then $\mid H \mid= 3$ and M is a braid.*

The upcoming proof makes extensive use of Lemma 14 to derive contradictions: throughout I define profiles of preferences R which must by Lemma 14 match some house to two different agents.

Proof W.l.o.g. assume that $M(R^{eg})(2) = e = M(R^{ef})(3)$ for some R^{eg}, R^{ef} and say $M(R^{ge})(j) = g$.

Claim 1: $\{1, 2\} \cap \{j, k\} \neq \emptyset$.

If $\{1, 2\} \cap \{j, k\} = \emptyset$, then $M(R^{eg}_{\{1,2\}}, R^{ge}_{-\{1,2\}})$ must by Lemma 14 match e with 2 as well as with k - a contradiction.

Claim 2: House g is brokered by $k \in \{2, 3\}$.

Since 1 brokers e, 1 cannot broker g by part b) of Lemma 16. Suppose $k > 3$. By Claim 1, $j \in \{1, 2\}$. If $j = 2$, then $M(R^{ge}_{\{2,k\}}, R^{ef}_{-\{2,k\}})$ must by Lemma 14 match e with 3 as well as with $k \neq 3$ - a contradiction. If $j = 1$, then $M(R^{ge}_{\{1,k\}}, R^{eg}_{-\{1,k\}})$ must by Lemma 14 match e with 2 as well as with $k \neq 2$ - a contradiction.

Claim 3: House g is brokered by $k = 3$.

Suppose not. So suppose $k = 2$ (given that $k \in \{2, 3\}$ by Claim 2). **Case 1:** $j = 1$. Lemma 14 yields $M(R^{ge}_{\{1,2\}}, M^{eg}_{-\{1,2\}})(1) = g$, which in light of SP-II implies $M(R^{ge}_2, M^{eg}_{-2})(1) \in \{e, g\}$. The assumption $M(R^{eg})(2) = e$ together with SP-II implies $M(R^{ge}_2, M^{eg}_{-2})(2) \in \{e, g\}$. Since M is Pareto optimal $M(R^{ge}_2, M^{eg}_{-2})(2) = g$ and $M(R^{ge}_2, M^{eg}_{-2})(1) = e$ must hold. By SP-NB $M(R^{ef}_1, R^{ge}_2, M^{eg}_{-\{1,2\}}) = M(R^{ge}_2, M^{eg}_{-2})$ and consequently $M(R^{ef}_1, R^{ge}_2, M^{eg}_{-\{1,2\}})(1) = e$. The assumption $M(R^{ef})(3) = e$ together with Lemma 14 yields the contradiction $M(R^{ef}_1, R^{ge}_2, M^{eg}_{-\{1,2\}})(3) = e$. **Case 2:** $j > 3$. $M(R^{ef}_{\{1,3\}}, R^{ge}_{-\{1,3\}})$ must then by Lemma 14 match e with 2 as well as with 3 - a contradiction. **Case 3:** $j = 3$. $M(R^{eg}_{\{1,2\}}, R^{ge}_{-\{1,2\}})$ must then by Lemma 14 match g with 1 as well as with 3 - a contradiction.

Claim 4: Agent $j = 2$ obtains house g at $M(R^{ge})$.

SP-NB implies $M(R^{eg})(2) = M(R^{eg}_2, R^{ge}_{-2})(2) = e$. By SP-PO $M(R^{ge})(2)$ must either be e or g. Since $k = 3$ brokers g, we have $M(R^{ge})(3) = e$ and consequently $M(R^{ge})(2) = g$.

Claim 5: There is no $h \in H$ and $i > 3$ such that $c_\emptyset(h) = (i, \cdot)$.

Case 1: $c_\emptyset(h) = (i, o)$ for some $i > 3$ and $h \in H$. By Part c) of Lemma 16 we than have $M(R^{eh})(i) = e$. This combined with Lemma 14, $c_\emptyset(e) = (1, b)$, $c_\emptyset(g) = (3, b)$ and $M(R^{ge})(2) = g$ (as established in Claims 3 and 4) implies that $M(R^{eh}_{\{1,i\}}, R^{ge}_{-\{1,i\}})$ matches e with i as well as with 3 - a contradiction. **Case 2:** $c_\emptyset(h) = (i, b)$ for some $i > 3$ and $h \in H$. Define j such that $M(R^{he})(j) = h$. Lemma 14 implies $M(R^{he}_{\{i,j\}}, R_{-\{i,j\}})(i) = e$ for any R. If $j \notin \{1, 2\}$, Lemma 14, $c_\emptyset(e) = (1, b)$, and $M(R^{eg})(2) = e$ imply the contradiction $M(R^{he}_{\{i,j\}}, R^{eg}_{-\{i,j\}})(2) = e$. Lemma 14 also leads to a contradiction in the remaining two cases: If $j = 2$, then $c_\emptyset(e) = (1, b)$ and $M(R^{ef})(3) = e$ imply $M(R^{he}_{\{i,j\}}, R^{ef}_{-\{i,j\}})(3) = e$. If $j = 1$, then $c_\emptyset(g) = (3, b)$ and $M(R^{ge})(2) = g$ (established in Claims 3 and 4) imply $M(R^{he}_{\{i,j\}}, R^{ge}_{-\{i,j\}})(3) = e$.

Claim 6: $c_\emptyset(f) = (2, b)$ and $H = \{e, f, g\}$.

Let $c_\emptyset(h) = (i, \cdot)$ for some $h \notin \{e, g\}$. Claim 5 implies $i \leq 3$. By part b) of Lemma 16 no broker controls more than one house. So $c_\emptyset(e) = (1, b)$ and $c_\emptyset(g) = (3, b)$ imply $i \neq 1, 3$, in particular $c_\emptyset(f) = (2, \cdot)$. Part c) of Lemma 16 and $M(R^{ef})(2) \neq e$ then imply $c_\emptyset(f) = (2, b)$. By Lemma 16, 2 does not control any other house and H equals $\{e, f, g\}$.

Claim 7: Fix $\omega = (e, f, g)$. If $R \in \{R^{eg}, R^{gf}, R^{fe}\}$ then $M(R) = B^\omega(R) = (g, e, f)$, if $R \in \{R^{ef}, R^{ge}, R^{fg}\}$ then $M(R) = B^\omega(R) = (f, g, e)$.

Claims 3 and 4 imply $c_\emptyset(g) = (3, b)$ and $M(R^{ge})(2) = g$, so that some $h \notin \{e, g\}$ and $j \notin \{2, 3\}$ with $M(R^{gh})(j) = g$ must exist. Since there are only three houses $h = f$. If $j \neq 1$ Lemma 14 implies that $M(R^{eg}_{\{1,2\}}, R^{gf}_{-\{1,2\}})$ matches g with 1 as well as with j - a contradiction. Claim 6 implies $M(R^{fg})(2) = g$ and $M(R^{fe})(2) = e$. By SP-NB $e = M(R^{ef})(3) = M(R^{ef}_3, R^{fe}_{-3})(3)$. SP-II then implies $M(R^{fe})(3) \in \{e, f\}$. Given $M(R^{fe})(2) = e$, $M(R^{fe})(3)$ must equal f. Applying the same arguments mutatis mutandis we obtain $M(R^{fg})(1) = f$. In sum we know which agents in $\{1, 2, 3\}$ are matched with the two top ranked houses for any $R^{hh'}$ with $\{h, h'\} \subset \{e, f, g\}$. Lemma 14, $c_\emptyset(g) = (3, b)$, and $M(R^{gf})(1) = g$ imply $M(R^{ge}_1, R^{eg}_2, R^{gf}_3)(3) = f$. SP-NB implies $M(R^{eg}) = M(R^{ge}_1, R^{eg}_2, R^{gf}_3)$ and in sum we obtain $M(R^{eg}) = (g, e, f)$. Mutatis mutandis the same arguments prove the claim for R^{ef}, R^{ge}, R^{gf}, R^{fe}, and R^{fg}.

Claim 8: $M(R) = B^\omega(R)$ holds for all $R \in \mathcal{R}$.

To show that M equals B^ω with $\omega = (e, f, g)$, $\omega' = (g, e, f)$, and $\omega'' = (f, g, e)$ I separately consider some profiles R that are covered by cases I, II, and III as defined in Lemma 11. The arguments for each one of these examples apply mutatis mutandis to all other profiles that are covered by the same case. As a profile that is covered by II consider R with $g = \omega'(1) R_1 \omega''(1) = f$ and $R_2 : \omega(1) = e$. Claim 7 and SP-NB imply $\omega' = M(R^{eg}) = M(R) = B^\omega(R)$.

As an example that is covered by III consider R with $g R_1 f$, $R_2 : f$, and $R_3 : e = \omega(1) = \omega'(2)$. The preceding paragraph yields $M(R^{eg}_1, R^{ef}_{-1}) = M(R^{eg}_1, R^{fe}_{-1}) = (g, e, f)$. SP-PO then implies $M(R^{eg}_1, R^{fe}_2, R^{ef}_3)(2) = f$ and $M(R^{eg}_1, R^{fe}_2, R^{ef}_3)(3) = e$. The preceding paragraph also yields $M(R^{fe}_2, R^{ef}_{-2})(1) = f$, and by SP-I $M(R^{eg}_1, R^{fe}_2, R^{ef}_3)(1) \neq \emptyset$. In sum we obtain $M(R^{eg}_1, R^{fe}_2, R^{ef}_3) =$

(g, f, e) and by SP-NB $M(R_1^{eg}, R_2^{fe}, R_3^{ef}) = M(R) = B^\omega(R)$.

Finally let $R_1 : e$, $R_2 : f$ and $R_3 : g$, so R is covered by I. Since (R_i', R_{-i}) for $R_i' : \omega(j)$ is covered by II or III (as analysed above), i is matched under $M(R_i', R_{-i})$. Since M is strategyproof $M(R)(i) \neq \emptyset$. Since ω is the only Pareto optimum matching $\{1, 2, 3\}$, $M(R)$ equals $\omega = B^\omega(R)$. □

11 Pointing

Assume that M is not a braid and fix some R°. Lemma 18 shows that the outcome $M(R^\circ)$ contains any submatching achieved at R° in the first round of any trading and braiding mechanism with $c_\emptyset$ the control rights function at $\emptyset$, so $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$ implies $\nu \subset M(R^\circ)$. Lemma 19 shows that the calculation of $M(R^\circ)$ can be split into the calculation of such a submatching and the outcome of a well-defined submechanism. This submechanism inherits the property of being good from M. By the inductive hypothesis this sub-mechanism is a trading and braiding mechanism which can, moreover, be represented via a unique tight structure $c[\nu]$. In sum we obtain that $M^c(R) = M(R)$ for any $R \in \mathcal{R}$. Lemmas 18 and 19 correspond to Pycia and Unver [5] Section F3 which shows that at any R the outcome of the trading cycles mechanism constructed in the preceding sections equals the outcome of the underlying good mechanism. For the next two Lemmas fix a submatching $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$. W.l.o.g. assume $N_\nu = \{1, \cdots, m\}$, $H_\nu = \{h_1, \cdots, h_m\}$ and $c_\emptyset(h_i) = (i, \cdot)$ for all $i \leq m$.

Lemma 18 *Any submatching that arises out of matching one cycle under $c_\emptyset$ at R° is part of the outcome $M(R^\circ)$; so $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$ implies $\nu \subset M(R^\circ)$.*

Proof **Case 1**: $m = 1$. So $\nu = \{(1, h_1)\}$. Since only an owner may point to a house he controls and since an owner may point to any house, we have $c_\emptyset(h_1) = (1, o)$ and $R_1^\circ : h_1$. Lemma 15 then implies $(1, h_1) \in M(R^\circ)$ and $\nu \subset M(R^\circ)$.

For the remaining cases 2, 3, and 4 let $m > 1$. W.l.o.g assume that $\nu(i) = h_{i+1}$ for all $i < m$ and $\nu(m) = h_1$. **Case 2**: $c_\emptyset(h_i) = (i, o)$ for all $i \leq m$. For all $i \leq m$ let $R_i^* : \nu(i), h_i$ and let R_i^* and R_i° coincide on

$H \setminus \{h_i\}$. Suppose we had $M(R^*)(i^*) \neq \nu(i^*)$ for at least one $i^* \leq m$, say $i^* = m$. Lemma 15 and $c_\emptyset(h_m) = (m, o)$ imply $M(R'_m, R^*_{-m})(m) = h_m$ for $R'_m : h_m$. SP-II implies that $M(R^*)(m) \in \{h_1, h_m\}$. The assumption $M(R^*)(m) \neq \nu(m) = h_1$ then implies $M(R^*)(m) = h_m$. So $M(R^*)(m-1)$ differs from $\nu(m-1) = h_m$. Inductively applying these arguments to all agents in the cycle we obtain that $M(R^*)(i) = h_i$ for all $i \leq m$. This contradicts the Pareto optimality of M since each $i \leq m$ strictly prefers $\nu(i)$ to h_i. So $\nu \subset M(R^*)$ must hold. Dropping h_i in the rankings of all agents $i \leq m$ SP-NB yields $M(R^*) = M(R^\circ)$.

For the remaining cases 3 and 4 assume that $c_\emptyset(h_m) = (m, b)$. By (C2) $c_\emptyset(h_i) = (i, o)$ holds for all $i < m$. **Case 3**: $m = 2$: $R^\circ_1 : h_2$ and either $R^\circ_2 : h_1$ or $R^\circ_2 : h_2, h_1$ must hold for ν to arise out of matching a single cycle. Lemma 14 and part c) of Lemma 16 then imply $\nu \subset M(R^\circ)$.

Case 4: $m > 2$: Define $R^*_{m-2} : h_{m-1}, h_1,\ R^*_m : h_m, h_1, h_{m-1}$ and $R^*_i : \nu(i), h_i$ for all other $i \leq m$. For all preference statements that have not been explicitly mentioned let R^* and R° coincide. Under $M(R'_{m-2}, R^*_{-(m-2)})$ with $R'_{m-2} : h_1$ the owners $\{1, \cdots, m-2\}$ form a pointing cycle and $(m-2, h_1) \in M(R'_{m-2}, R^*_{-(m-2)})$ holds by Case 2. By Case 3 $(m, h_{m-1}) \in M(R'_m, R^*_{-m})$ holds for $R'_m : h_m,\ h_{m-1}$. Strategyproofness implies $M(R^*)(m-2) R^*_{m-2} h_1$, $M(R^*)(m) R^*_m h_{m-1}$, and $M(R^*)(m) \neq h_m$. So $m-2$ and m must be matched to the houses $\{h_1, h_{m-1}\}$ at R^*. Pareto optimality requires that $M(R^*)(m-2) = h_{m-1}$ and $M(R^*)(m) = h_1$. Lemma 15, $c_\emptyset(h_1) = (1, o)$, and SP-I imply $M(R^*)(1) R^*_1 h_1$. Since $M(R^*)(m) = h_1$, $M(R^*)(1)$ must equal h_2. Inductively applying these arguments to all other agents in N_ν we obtain $\nu \subset M(R^*)$. Applying SP-NB to drop h_1 in R^*_{m-2}, h_{m-1} as well as h_m in R^*_m, and h_i in each R^*_i with $i \in \{1, \ldots, m-3, m-1\}$ we obtain $M(R^*) = M(R^\circ)$. □

For the fixed ν and any R let $\overline{R}$ be its restriction to $\overline{N}_\nu$ and $\overline{H}_\nu$. Let $\overline{\mathcal{R}}$ be the set of all such restrictions. Let $\overline{\mathcal{M}}$ be the set of submatchings ν' such that $\nu \cup \nu'$ is a matching. Define a trading and braiding mechanism $M^{c[\nu]} : \overline{\mathcal{R}} \to \overline{\mathcal{M}}$ by $M^{c[\nu]}(\overline{R}) := M(R) \setminus \nu$ whenever $\nu \in \mathcal{N}(c_\emptyset)(R)$.

Lemma 19 *The tight structure $c[\nu]$ is well-defined.*

Proof It suffices to show that $M^\circ : \overline{\mathcal{R}} \to \overline{\mathcal{M}}$ with $M^\circ(\overline{R}) := M^\circ(R) \setminus \nu$ for each $\overline{R} \in \overline{\mathcal{R}}$ where R is such that $R_i : \nu(i)$ for each $i \in N_\nu$ and $\overline{R}$

the restriction of R to $\overline{N}_\nu$ and $\overline{H}_\nu$ is a well-defined good mechanism: By the inductive hypothesis there is a unique tight structure $c[\nu]$ the represents such a mechanism M° as $M^{c[\nu]}$. To show that M° is well-defined and good, fix some profiles R^*, R, $\tilde{R}$, and R'', such that $\nu \in \mathcal{N}(c_\emptyset)(R^*)$, such that $\overline{H}_\nu R''_i H_\nu$ holds for all $i \in \overline{N}_\nu$, and such that R, $\tilde{R}$ and R'' have the same restriction $\overline{R}$ to $\overline{H}_\nu$ and $\overline{N}_\nu$.

By Lemma 18, $\nu \subset M(R^\circ_{N_\nu}, R_{-N_\nu})$ and $\nu \subset M(R^\circ_{N_\nu}, \tilde{R}_{-N_\nu})$. By SP-NB $M(R^\circ_{N_\nu}, R_{-N_\nu}) = M(R^\circ_{N_\nu}, R''_{-N_\nu}) = M(R^\circ_{N_\nu}, \tilde{R}_{-N_\nu})$ and consequently $M(R^\circ_{N_\nu}, R_{-N_\nu}) \setminus \nu = M(R^\circ_{N_\nu}, \tilde{R}_{-N_\nu}) \setminus \nu$. So $M^\circ : \overline{\mathcal{R}} \to \overline{\mathcal{M}}$ with $M^\circ(\overline{R}) := M(R^\circ_{N_\nu}, R_{-N_\nu}) \setminus \nu$ for each $\overline{R}$ is a well-defined mechanism. By the same token $M^* : \overline{\mathcal{R}} \to \overline{\mathcal{M}}$ with $M^*(\overline{R}) := M(R^*_{N_\nu}, R_{-N_\nu}) \setminus \nu$ each $\overline{R}$ is a well-defined mechanism.

To see that $M^* = M^\circ$, note that for ν to arise at $\emptyset$ under $(R^\circ_{N_\nu}, R_{N_\nu})$ as well as under $(R^*_{N_\nu}, R_{N_\nu})$ the statements $R^*_i : \nu(i)$ and $R^\circ_i : \nu(i)$ must hold for all owners $i \leq m$. Moreover, $\nu(i) R^*_i H \setminus \{h_b\}$ and $\nu(i) R^\circ_i H \setminus \{h_b\}$ must hold for a broker $i \leq m$ (if there is one) where h_b is the house with $c_\emptyset(h_b) = (i, b)$. So the cycle yielding ν also arises under $c_\emptyset$ at any of the profiles $(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}) \cdots (R^\circ_{\{1,\cdots,m-1\}}, R^*_m, R_{-N_\nu})$. Lemma 18 implies that $\nu \subset M(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}), \cdots, \nu \subset M(R^\circ_{\{1,\cdots,m-1\}}, R^*_m, R_{-N_\nu})$. Due to the non-bossiness of M the matchings of all agents in $\overline{N}_\nu$ stay constant as well and we have

$$\begin{aligned} \nu \cup M^*(\overline{R}) = M(R^*_{N_\nu}, R_{-N_\nu}) = \\ M(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}) = \cdots = M(R^\circ_{\{1,\cdots,m-1\}}, R^*_m, R_{-N_\nu}) = \\ M(R^\circ_{N_\nu}, R_{-N_\nu}) = \nu \cup M^\circ(\overline{R}). \end{aligned}$$

To see that M° is good fix an agent $i \in \overline{N}_\nu$ and a preference R'_i with $\overline{R}'_i$ as its restrictions to $\overline{H}_\nu$. The definition of M° and the strategyproofness of M then imply that M° is strategyproof as we have

$$M^\circ(\overline{R})(i) = M(R^\circ_{N_\nu}, R_{-N_\nu})(i) R_i M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})})(i) = M^\circ(\overline{R}'_i, \overline{R}_{-i})(i).$$

If $M^\circ(\overline{R})(i) = M^\circ(\overline{R}'_i, \overline{R}_{-i})(i)$, then $M(R^\circ_{N_\nu}, R_{-N_\nu})(i)$ equals $M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})})(i)$. The non-bossiness of M then implies $M(R^\circ_{N_\nu}, R_{-N_\nu}) = M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})})$

and therefore

$$M^\circ(\overline{R}) = M(R^\circ_{N_\nu}, R_{-N_\nu}) \setminus \nu = M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})}) \setminus \nu = M^\circ(\overline{R}'_i, \overline{R}_{-i}).$$

So M° is non-bossy. Since $M(R^\circ_{N_\nu}, R_{-N_\nu}) = \nu \cup M^\circ(\overline{R})$ is Pareto optimal at $(R^\circ_{N_\nu}, R_{-N_\nu})$, M° is Pareto optimal. So M° is good. □

12 Defining a tight structure c

Combine the definition of $c_\emptyset$ in Section 8 with Lemma 19 on submechanisms to define a tight structure c via $c_\nu :\, = c[\nu^*]_{\nu'}$ if $\nu = \nu^* \cup \nu'$, $\nu^* \in \mathcal{N}(c_\emptyset)$, and ν' $c[\nu^*]$-relevant. Since a submatching $\nu \neq \emptyset$ is c-relevant if and only if it can be split into a submatching $\nu^* \in \mathcal{N}(c_\emptyset)$ and a $c[\nu^*]$-relevant ν', c is defined on all c-relevant submatchings.

We know from Section 10 that $c_\emptyset$ satisfies (C1), (C2) and (C3). Since $c[\nu^*]$ is a tight structure for any $\nu^* \in \mathcal{N}(c_\emptyset)$, (C1), (C2), and (C3) are satisfied for any c-relevant $\nu \neq \emptyset$. Moreover (C4), (C5), and (C6) are satisfied by any pair of a c-relevant $\nu^\circ \neq \emptyset$ with a direct c-successor ν of ν°. In the following Lemma 20 I show that (C4), (C5), and (C6) are also satisfied at any pair $\emptyset, \nu$ with $\nu \in \mathcal{N}(c_\emptyset)$. Lemma 20 corresponds to Pycia and Unver [5] Lemmas 16, 17, 18, and 19 on the requirements that link control rights functions.

Lemma 20 *(C4), (C5) and (C6) hold for $\emptyset$ and $\nu \in \mathcal{N}(c_\emptyset)$.*

Proof **(C4)** Fix $i \notin N_\nu$ with $c_\emptyset(h) = (i, o)$. Lemma 18 implies $(i, h) \in M(R^\circ_{N_\nu}, R^h_{-N_\nu})$ for any R^h. Since $M(R^\circ_{N_\nu}, R^h_{-N_\nu}) = \nu \cup c[\nu](\overline{R}^h)$, $(i, h) \in M^{c[\nu]}(\overline{R}^h)$ holds for any $\overline{R}^h$ and i owns h under $c[\nu]_\emptyset = c_\nu$.

(C5) Let $i_b, i, j \notin N_\nu$ be such that $c_\emptyset(h_b) = (i_b, b)$, $c_\emptyset(h) = (i, o)$, and $c_\emptyset(e) = (j, o)$. Lemma 18 implies $\{(i_b, h), (i, h_b)\} \in M(R^\circ_{N_\nu} R^{h_b h}_{-N_\nu})$ and $(j, h_b) \in M(R^\circ_{N_\nu} R^{h_b e}_{-N_\nu})$. Since $M(R^\circ_{N_\nu} R^{h_b h}_{-N_\nu}) = \nu \cup M^{c[\nu]}(\overline{R}^{h_b h})$ and $M(R^\circ_{N_\nu} R^{h_b e}_{-N_\nu}) = \nu \cup M^{c[\nu]}(\overline{R}^{h_b h})$, we have $(i, h_b) \in M^{c[\nu]}(\overline{R}^{h_b h g})$ and $(j, h_b) \in M^{c[\nu]}(\overline{R}^{h_b e})$ and h_b is not owned at $c[\nu]_\emptyset$. Since $(i_b, h) \in M^{c[\nu]}(\overline{R}^{h_b h})$, i_b brokers h_b under $c[\nu]_\emptyset = c_\nu$.

(C6) Let $c_\emptyset(h) = (i, o)$, $c_\emptyset(h_b) = (i_b, b) \neq c_\nu(h_b)$ and $i, i_b \notin N_\nu$. Lemma 18 implies $\{(i, h_b), (i_b, h)\} \subset M(R^\circ_{N_\nu}, R^{h_b h}_{-N_\nu}) = \nu \cup M^{c[\nu]}(\overline{R}^{h_b h})$ for any $R^{h_b h}$. So $(i, h_b) \in M^{c[\nu]}(\overline{R}^{h_b h})$ and $(i_b, h) \in M^{c[\nu]}(\overline{R}^{h_b h})$ holds for any $\overline{R}^{h_b h}$ and h_b can neither be brokered by i nor be owned by some $j \neq i$ at ν. If $c_\nu(h_b) = (j, b)$ with $j \in \overline{N}_\nu \setminus \{i_b, i\}$ the contradiction $(j, h) \in M^{c[\nu]}(\overline{R}^{h_b h})$ results. So $c_\nu(h_b) = (i, o)$ must hold and $M(R^\circ_{N_\nu}, R^{h_b h}_{-N_\nu})$ can be calculated as $\nu \cup \{(i, h_b)\} \cup M^{c[\nu \cup \{(i, h_b)\}]}(\overline{\overline{R}}^h)$ with $\overline{\overline{R}}^h$ the restriction of $R^{h_b h}$ to $\overline{H}_\nu \setminus \{h\}$ and $\overline{N}_\nu \setminus \{i\}$. Since i_b is matched with h under $M^{c[\nu \cup \{(i, h_b)\}]}$ whenever he ranks h above all remaining houses, he owns h under $c[\nu \cup \{(i, h_b)\}]_\emptyset = c_{\nu \cup \{(i, h_b)\}}$. □

13 Proof of Lemma 3

Fix a control rights structure $\overline{c}$ with n agents and use the rules given in Lemma 3 to define a tight structure c.

Case 1: $\overline{c}_\emptyset$ satisfies (C2'). The construction of c uniquely defines $c_\emptyset$ as $\overline{c}_\emptyset$. Applying the inductive hypothesis to any $\nu^* \in \mathcal{N}(c_\emptyset) = \mathcal{N}(\overline{c}_\emptyset)$ yields that $c[\nu^*]$ is a tight structure that represents the same mechanism as does $\overline{c}[\nu^*]$. To see that c is uniquely defined fix an arbitrary c-relevant $\nu \neq \emptyset$, so that there exists some $\nu^* \in \mathcal{N}(c_\emptyset)$ and $c[\nu^*]$-relevant ν' such that $\nu = \nu^* \cup \nu'$. If ν can alternatively be represented as $\overline{\nu}^* \cup \overline{\nu}'$ with $\overline{\nu}^* \in \mathcal{N}(c_\emptyset)$ and $\overline{\nu}'$ $c[\overline{\nu}^*]$-relevant, then $c[\nu^*]_{\nu'} = c[\overline{\nu}^*]_{\overline{\nu}'}$ holds since $c[\nu^*]$ and $c[\overline{\nu}^*]$ are both derived from $\overline{c}$ via Lemma 3.

Case 2: $\overline{c}_\emptyset$ does not satisfy (C2'), so $\overline{c}_\emptyset(h_b) = (i_b, b)$ and $\overline{c}_\emptyset(h) = (i^*, o)$ holds for two different $i_b, i^* \in N$, some $h_b \in H$ and all $h \in H \setminus \{h_b\}$. The construction of c requires $c_\emptyset(h) := (i^*, o)$ for all $h \in H$ and $c_{\{(i^*, h_b)\}}(h) := (i_b, o)$ for all $h \in H \setminus \{h_b\}$. To see that c_ν is defined for each c-relevant ν firstly note that exactly one submatching is c- but not $\overline{c}$-relevant: $\{(i^*, h_b)\}$. The control rights function $c_{\{(i^*, h_b)\}}$ is defined such that any direct c-successor of $\{(i^*, h_b)\}$ can be represented as $\{(i^*, h_b), (i_b, h)\}$ for some $h \neq h_b$. Since $\{(i^*, h)\}$ and $\{(i^*, h_b), (i_b, h)\}$ are $\overline{c}$-relevant for any $h \neq h_b$, $c[\{(i^*, h)\}]$ and $c[\{(i^*, h_b), (i_b, h)\}]$ are by the inductive hypothesis tight structures. For any c-relevant $\nu \notin \big\{\{(i^*, h)\}, \{(i^*, h_b), (i_b, h)\}$ for some $h \neq h_b\big\}$ there exists a unique $\nu^* \in \big\{\{(i^*,h)\},\{(i^*,h_b),(i_b,h)\}$ for some $h \neq h_b\big\}$ and a unique $c[\nu^*]$-relevant ν' such that $\nu = \nu^* \cup \nu'$ and c_ν is uniquely defined as $c_\nu = c[\nu^*]_{\nu'}$.

To see that c is a tight structure in either case, first note that $c[\nu^*]$ for any $\nu^* \in \mathcal{N}(\overline{c}_\emptyset)$ satisfies (C1), (C2'), (C3), (C4), (C5) and (C6) by the inductive hypothesis. I follow the above two cases to cover all other c-relevant submatchings. If Case 2 applies, any c-relevant ν that cannot be reached via some $\nu^* \in \mathcal{N}(\overline{c}_\emptyset)$ is an element of $\big\{\emptyset,\{(i^*,h)\},\{(i^*,h_b),(i_b,h)\}$ for $h \in H \setminus \{h_b\}\big\}$. Since any such ν is c-dictatorial it (together with its c-direct successors) trivially satisfies (C1)-(C6). If Case 1 applies then $c_\emptyset$ satisfies (C1), (C2'), and (C3), since $\overline{c}_\emptyset$ does. Now consider $\nu^\circ = \emptyset$ and a direct c-successor $\nu \in \mathcal{N}(c_\emptyset)$. If $c_\nu = \overline{c}_\nu$, then c satisfies (C4), (C5), and (C6) at $\emptyset,\nu$ since $\overline{c}$ does. If $c_\nu \neq \overline{c}_\nu$, $\overline{c}_\nu(h_b) = (i_b,b)$ and $\overline{c}_\nu(h) = (i^*,o)$ hold for two different $i_b,i^* \in \overline{N}_\nu$, some $h_b \in \overline{H}_\nu$ and all $h \in \overline{H}_\nu \setminus \{h_b\}$. By construction $c_\nu(h) = (i^*,o)$ holds for all $h \in \overline{H}_\nu$. Since $\overline{c}$ satisfies (C4) and since $\overline{c}_\nu(h) = (i,o) \Rightarrow c_\nu(h) = (i,o)$ holds for all $(h,i) \in \overline{H}_\nu \times \overline{N}_\nu$, (C4) is satisfied by c at $\emptyset,\nu$. (C5) trivially holds at $\emptyset,\nu$ since at most one agent owns houses under $c_\emptyset$ and c_ν. For (C6) to have any grip at $\emptyset,\nu$ $c_\emptyset(h_b) = (i_b,b)$ must hold for some h_b and i_b. In that case the construction of c implies $c_{\nu\cup\{(i^*,h_b)\}}(h) = (i_b,o)$ for all $h \in \overline{H}_\nu \setminus \{h_b\}$ exactly as required by (C6).

To see $M^c(R) = M^{\overline{c}}(R)$ for all $R \in \mathcal{R}$ fix an arbitrary R together with a $\nu \in \mathcal{N}(\overline{c}_\emptyset)(R)$. If $\nu \notin \mathcal{N}(c_\emptyset)(R)$ then ν equals $\{(i^*,h_b),(i_b,h)\}$ with $\overline{c}_\emptyset(h_b) = (i_b,b)$ and $\overline{c}_\emptyset(h) = (i^*,o)$. Moreover, agent i^* ranks h_b at the top and $\{(i^*,h_b)\}$ forms under c at $\emptyset$. By the construction of c we have $c_{\{(i^*,h_b)\}}(h) = (i_b,o)$. Since $\nu = \{(i^*,h_b),(i_b,h)\} \in \mathcal{N}(\overline{c}_\emptyset)(R)$ agent i_b prefers h to all other houses in $H\setminus\{h_b\}$. So $\nu = \{(i^*,h_b),(i_b,h)\}$ is reachable given c and R. Let $\overline{R}$ be the restriction of R to $\overline{N}_\nu$ and $\overline{H}_\nu$. So $\nu \subset M^c(R)$ holds whether $\nu \in \mathcal{N}(c_\emptyset)(R)$ or not. By the inductive hypothesis Lemma 3 holds for $\overline{c}[\nu]$ and $c[\nu]$ and $M^{\overline{c}}(R) = \nu \cup M^{\overline{c}[\nu]}(\overline{R})$ equals $\nu \cup M^{c[\nu]}(\overline{R}) = M^c(R)$.

14 Relations with other sets of good mechanisms

A trading and braiding mechanism qualifies as a Pycia and Unver [5] trading cycles mechanism if (C1) is replaced by the requirement that there is at most one broker at any given round of the mechanism. The set of trading and braiding mechanisms coincides with the revised set of Pycia and Unver [6] trading cycles mechanism. A trading and braiding mechanism without brokerage or braids belongs to the class of hierarchical exchange mechanisms, characterized by Papai [4]. In this subclass (C1), (C2), (C3), (C5), and (C6) are trivially satisfied. A trading and braiding mechanism is GTTC if $| N |=| H |$ and if there exists a matching μ such that $c_{\emptyset}(\mu(i)) = (i, o)$ for all $i \in N$.

Just like Shapley and Scarf's [7] original definition of Gale's top trading cycles, the definition of trading and braiding cycles requires that at least one cycle is matched at any trading round. Papai's [4] hierarchical exchange mechanisms and Pycia and Unver's [5] trading cycles mechanisms in contrast require that all cycles at any given round have to be matched at that round.[4] The freedom to remove trading cycles in any order considerably simplifies the proof that any lax mechanism is good and the proof of Theorem 1. Theorem 2 contains a *unique* representation of good mechanisms: any good mechanism can be represented as exactly one trading and braiding mechanism. In contrast Pycia and Unver's [5] trading cycles mechanisms and Papai's [4] hierarchical exchange mechanism have multiple representations.

Strategyproofness and Pareto optimality are better founded than nonbossiness as principles of mechanism design and one might wonder about a characterization of the set of all strategy proof and Pareto optimal mechanisms. Given that all three axioms are repeatedly used in the proof that any good mechanism can be represented as a trading and braiding mechanism a simple extension of the same proof to the grand set of all strategy proof and Pareto optimal mechanisms is out of the question. However, the representation of good mechanisms as trading and braiding mechanisms can be used to

[4] Carroll [3] shows that the order of the elimination of trading cycles does not matter in top trading cycles mechanisms.

construct a class of Pareto optimal and strategy proof mechanisms. Simply modify control rights structure such that the inheritance of houses not only depends on submatchings ν but also on the preferences of the matched agents, keeping (C1)-(C6) intact. So at any given round of the mechanism, where the submatching ν has been achieved, the control rights over unmatched house may not only depend on ν but also on the preferences R_{N_ν} of all agents that have already been matched. A serial dictatorship in which the second dictator depends on the first dictator's preferences over houses he did not choose is the simplest example of such a bossy mechanism. The inductive proof that any trading and braiding mechanism is good can be extended to show that any mechanism, defined through such a modified control rights structure, is strategyproof and Pareto optimal. Similarly, only a few extra steps are required to extend the proof of Theorem 1 to this wider class to strategy proof and Pareto optimal mechanisms. The question whether any Pareto optimal and strategyproof mechanisms can be represented by such a modified control rights structure awaits some new ideas and techniques of proof.

15 Proof of Lemma 7

Fix an arbitrary profile of preferences R. I show that the ex post Pareto optimality, ordinal strategy proofness and equal treatment of equals of $\mathfrak{M}$ uniquely determine the 9 values $\mathfrak{M}(R)[i, h]$. Since random serial dictatorship satisfies the named properties it equals $\mathfrak{M}$.

Case (I) there is a unique Pareto optimum at R. Ex post Pareto optimality requires that $\mathfrak{M}(R)$ assigns probability 1 to this matching. Case (II) $R_1 = R_2 = R_3$. Equal treatment of equals requires $\mathfrak{M}(R)[i, h] = \frac{1}{3}$ for all i, h. Case (III) (R_1^c, R_{-1}^{ab}). Ex post Pareto optimality implies $\mathfrak{M}(R_1^c, R_{-1}^{ab})[1, c] = 1$. Equal treatment of equals implies $\mathfrak{M}(R_1^c, R_{-1}^{ab})[2, \cdot] = \mathfrak{M}(R_1^c, R_{-1}^{ab})[3, \cdot]$ and therefore $\mathfrak{M}(R_1^c, R_{-1}^{ab})[i, h] = \frac{1}{2}$ for $i = 2, 3$ and $h = a, b$. Case (IV) (R_2^{ac}, R_{-2}^{ab}). Ex post Pareto optimality implies $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[2, b] = 0$; ordinal strategyproofness and (II) imply that $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[2, a] = \frac{1}{3}$. Equal treatment of equals implies $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[1, \cdot] = \mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[3, \cdot]$. The unique solution to this system of linear equations is $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[2,c] = \frac{2}{3}$, $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[i,a] = \frac{1}{3}$ for $i = 1,2,3$, $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[i,b] = \frac{1}{2}$ and $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[i,c] = \frac{1}{6}$ for $i = 1,3$.

Case (V) (R_1^b, R_{-1}^{ab}). Ex post Pareto optimality implies $\mathfrak{M}(R_1^b, R_{-1}^{ab})[1,a] = 0$. Equal treatment of equals implies $\mathfrak{M}(R_1^b, R_{-1}^{ab})[2,\cdot] = \mathfrak{M}(R_1^b, R_{-1}^{ab})[3,\cdot]$. Ordinal strategy-proofness and (II) imply $\mathfrak{M}(R_1^b, R_{-1}^{ab})[1,a] + \mathfrak{M}(R_1^b, R_{-1}^{ab})[1,b] = \mathfrak{M}(R^{ab})[1,a] + \mathfrak{M}(R^{ab})[1,b] = \frac{2}{3}$. The unique solution of this system of linear equations is $\mathfrak{M}(R_1^b, R_{-1}^{ab})[i,a] = \frac{1}{2}, \mathfrak{M}(R_1^b, R_{-1}^{ab})[i,b] = \frac{1}{6}, \mathfrak{M}(R_1^b, R_{-1}^{ab})[i,c] = \frac{1}{3}$ for $i = 2,3$, $\mathfrak{M}(R_1^b, R_{-1}^{ab})[1,b] = \frac{2}{3}$ and $\mathfrak{M}(R_1^b, R_{-1}^{ab})[1,c] = \frac{1}{3}$.

Case (VI) $(R_1^b, R_2^{ac}, R_3^{ab})$. By ex post Pareto optimality $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1,a]$ and $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[2,b]$ equal 0. By ordinal strategyproofness and (IV)

$$\frac{5}{6} = \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1,a] + \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1,b] = \mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[1,a] + \mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[1,b].$$

By ordinal strategyproofness and (V) $\mathfrak{M}(R_1^b, R_{-1}^{ab})[2,a] = \frac{1}{2} = \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[2,a]$. The unique solution of this system of linear equations is $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1,b] = \frac{5}{6}$, $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1,c] = \frac{1}{6}$, $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[2,c] = \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[2,a] = \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[3,a] = \frac{1}{2}, \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[3,b] = \frac{1}{6}$ and $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[3,c] = \frac{1}{3}$. Mutatis mutandis all profiles of preferences are covered by Cases (I) through (VI).

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