

ON THE h -INVARIANT OF CUBIC FORMS, AND SYSTEMS OF CUBIC FORMS

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ABSTRACT. We define a so-called ℓ -invariant for systems of homogeneous forms of the same degree, which coincides with the well known h -invariant for a single quadratic or cubic form, and bound the ℓ -invariant of a system of rational forms F_1, \dots, F_r in terms of the ℓ -invariant of a single form $\alpha_1 F_1 + \dots + \alpha_r F_r$ in their complex pencil in case of algebraic $\alpha_1, \dots, \alpha_r$. As an application, we show that a system of r rational cubic forms in more than $400000r^4$ variables has a non-trivial rational zero.

1. INTRODUCTION

Let K be a field of characteristic zero and $F(X_1, \dots, X_s) \in K[X_1, \dots, X_s]$ be a form of degree d at least 2. Then the h -invariant $h_K(F)$ of F (see [16], p. 245) is defined to be the smallest non-negative integer h such that there exist forms $G_i, H_i \in K[X_1, \dots, X_s]$ ($1 \leq i \leq h$) of degree strictly less than d such that

$$F = \sum_{i=1}^h G_i H_i.$$

Now let $\mathbf{F} = (F_1, \dots, F_r)$ be a system of forms $F_i(X_1, \dots, X_s) \in K[X_1, \dots, X_s]$ of the same degree $d \geq 2$. In the special case $d = 2$ Schmidt ([11], p. 285) introduced the joint rank of a system \mathbf{F} . One could generalize this to define the joint h -invariant of a system \mathbf{F} , but we want to strengthen the condition to the effect that all G_i are linear forms. This way we define the ℓ -invariant $\ell_K(\mathbf{F})$ to be the smallest non-negative integer h such that there exist linear forms $L_1, \dots, L_h \in K[X_1, \dots, X_s]$ and forms $H_i^{(j)} \in K[X_1, \dots, X_s]$ ($1 \leq i \leq h, 1 \leq j \leq r$) of degree $d - 1$ such that

$$(1) \quad F_j = \sum_{i=1}^h L_i H_i^{(j)} \quad (1 \leq j \leq r).$$

In the special case $r = 1$ of just one form $\mathbf{F} = F_1$, we write $\ell_K(F)$ for $\ell_K(\mathbf{F})$. Clearly, always $\ell_K(\mathbf{F}) \leq s$ and

$$h_K(F) \leq \ell_K(F),$$

where for $d = 2$ and $d = 3$ in the latter inequality always equality holds true. The following useful result is easy to prove and motivates our definition of the ℓ -invariant.

Lemma 1. *Let $F_i(X_1, \dots, X_s) \in K[X_1, \dots, X_s]$ ($1 \leq i \leq r$) be forms of degree d with $d \geq 2$, and let*

$$m = s - \ell_K(\mathbf{F}).$$

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Then m is the maximum dimension of any K -linear subspace $V \subset K^s$ on which F_1, \dots, F_r simultaneously vanish.

The proof follows immediately from the following two facts: On the one hand, if one has (1), then F_1, \dots, F_r simultaneously vanish on the K -linear space of dimension at least $s - h$ given by $\{\mathbf{x} \in K^s : L_i(\mathbf{x}) = 0 \ (1 \leq i \leq h)\}$. On the other hand, if F_1, \dots, F_r simultaneously vanish on a K -linear space V of dimension $s - h$, then after a suitable non-singular linear transformation of the variables X_1, \dots, X_s we may without loss of generality assume that V is given by $x_1 = \dots = x_h = 0$. Restricting the F_i to V we obtain homogeneous polynomials vanishing identically, which therefore must have all their coefficients equal to zero. This way we obtain a representation of the form (1) with $L_i = X_i \ (1 \leq i \leq h)$.

Our next observation addresses the behaviour of the ℓ -invariant under field extensions: suppose that $L|K$ is a field extension of K . Then of course if $F_1, \dots, F_r \in K[X_1, \dots, X_s]$, then also $F_1, \dots, F_r \in L[X_1, \dots, X_s]$. It is immediate from the definition that

$$\ell_L(\mathbf{F}) \leq \ell_K(\mathbf{F}),$$

but for our applications we are interested in inequalities in the other direction. The following result provides such a reverse inequality in the arithmetically relevant special case $K = \mathbb{Q}$ and $L = \mathbb{C}$.

Theorem 1. *Let $F_i(X_1, \dots, X_s) \in \mathbb{Q}[X_1, \dots, X_s]$ ($1 \leq i \leq r$) be forms of degree d where $d \geq 2$. Then*

$$\ell_{\mathbb{Q}}(\mathbf{F}) \leq 2d\ell_{\mathbb{C}}(\mathbf{F}) + 1.$$

The following result is a variant of Theorem 1, bounding $\ell_{\mathbb{Q}}(\mathbf{F})$ not in terms of $\ell_{\mathbb{C}}(\mathbf{F})$ but in terms of $\ell_{\mathbb{C}}(F)$ for a single form F in the $\overline{\mathbb{Q}}$ -pencil of F_1, \dots, F_r , where as usual we write $\overline{\mathbb{Q}}$ for the field of all algebraic numbers.

Theorem 2. *Let $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}$ be \mathbb{Q} -linearly independent, and let $F_1, \dots, F_r \in \mathbb{Q}[X_1, \dots, X_s]$ be forms of degree d where $d \geq 2$. Then*

$$\ell_{\mathbb{Q}}(\mathbf{F}) \leq 2dr\ell_{\mathbb{C}}(\alpha_1 F_1 + \dots + \alpha_r F_r) + 1.$$

Note that in the special case $r = 1$, Theorem 1 follows from Theorem 2. Though Theorem 1 and Theorem 2 are certainly of interest in their own right, one of our main motivations for Theorem 2 comes from considering systems of rational cubic forms. For a positive integer r , let $\gamma(r)$ be the smallest non-negative integer such that whenever $C_1, \dots, C_r \in \mathbb{Q}[X_1, \dots, X_s]$ are cubic forms where $s > \gamma(r)$, then there exists $\mathbf{x} \in \mathbb{Q}^s \setminus \{\mathbf{0}\}$ such that $C_1(\mathbf{x}) = \dots = C_r(\mathbf{x}) = 0$. The currently best known result for $r = 1$ is $\gamma(1) \leq 13$ by Heath-Brown (see Theorem 1 in [7]), improving on Davenport's long-standing bound $\gamma(1) \leq 15$ (see [2]). For $r = 2$, the author and Wooley established the bound $\gamma(2) \leq 827$ (see [5], Theorem 4 (a)) by injecting $\gamma(1) \leq 15 =: m$ into [5], Theorem 2 (a), improving on previous bounds by Schmidt and Wooley (see [15], formula (1.4) and [17], Corollary 1 (b)). Using Heath-Brown's new bound $\gamma(1) \leq 13$, one can apply Theorem 2 (a) in [5] with $m = 13$ and this way one immediately obtains $\gamma(2) \leq 654$. For general r , Schmidt (see formula (1.6) in [15]) has shown that $\gamma(r) < (10r)^5$. Using Theorem 2 in combination with Schmidt's results on local solubility of systems of cubic forms (see [12]–[14]), we are able to reduce the order of magnitude of r for large r as follows.

Theorem 3. *We have*

$$\gamma(r) \leq 400000r^4.$$

Let us remark that we were mainly interested in the exponent of r and did not try to optimize the constant 400000 too much, which could be lowered somewhat.

As it turns out, for systems of cubic forms the local problem is harder than establishing the Hasse principle: If $C_1, \dots, C_r \in \mathbb{Z}[X_1, \dots, X_s]$ are cubic forms and

$$N(P) = \#\{\mathbf{x} \in \mathbb{Z}^s : |\mathbf{x}| \leq P \text{ and } C_i(\mathbf{x}) = 0 \quad (1 \leq i \leq r)\},$$

then the expected asymptotic formula

$$(2) \quad N(P) = \mathfrak{J}\mathfrak{S}P^{s-3r} + O(P^{s-3r-\delta})$$

for some $\delta > 0$ holds true providing either some geometric condition is satisfied (see [1], and also [4] and [8] for some recent refinement), or, which is for our purposes more suitable, each cubic form in the \mathbb{Q} -rational pencil of C_1, \dots, C_r has h -invariant exceeding $8r^2 + 8r$ (see Theorem 1.3 in [4] and Theorem 2 in [15] for the previous weaker bound $10r^2 + 6r$ which would suffice for our purposes). This bound is quadratic in r . One can then show that the singular integral \mathfrak{J} is positive (see First Supplement in [15]), so the remaining problem is to prove that the singular series

$$\mathfrak{S} = \prod_p \chi_p$$

is positive. As shown by Schmidt (see the series of papers [12]–[14]), if $s \geq 5300r(3r+1)^2$ (see Theorem 1 in [3] for some improvement of the constant in front of r^3 for $p \neq 2$), then there are non-trivial simultaneous zeros of C_1, \dots, C_r over all local fields \mathbb{Q}_p . Existence of non-trivial local solutions for the system $C_1 = \dots = C_r = 0$ is only a necessary condition for $\mathfrak{S} > 0$, though. One possible approach to show that $\mathfrak{S} > 0$ is to combine the local result yielding non-trivial local solutions as soon as $s \geq 5300r(3r+1)^2$ with a slicing argument and bounds for cubic exponential sums. This approach was used by Schmidt in [15] and leads to a bound for $\gamma(r)$ of order of magnitude $O(r^5)$. Another sufficient condition for $\mathfrak{S} > 0$ is the existence of *non-singular* local solutions, but it seems difficult to construct non-singular p -adic zeros of the system $C_1 = \dots = C_r = 0$. We will show that Schmidt's method developed in [12]–[14] in fact can not only be used to find non-trivial p -adic zeros of $C_1 = \dots = C_r = 0$, but also to show that $\mathfrak{S} > 0$ under suitable conditions, without relying on slicing or constructing non-singular p -adic zeros. Schmidt's method shows that if the system C_1, \dots, C_r is 'bottomed', then for all rational primes p the density

$$(3) \quad \varrho(p^m) = \#\{\mathbf{x} \in (\mathbb{Z}/p^m\mathbb{Z})^s : C_i(\mathbf{x}) \equiv 0 \pmod{p^m} \quad (1 \leq i \leq r)\}$$

of simultaneous p -adic zeros of C_1, \dots, C_r is at least of the expected order of magnitude, i.e.

$$(4) \quad \varrho(p^m) \gg p^{(s-r)m}$$

for all positive integers m , with an implied constant independent of m (see Lemma 6 below). Since

$$\chi_p = \lim_{m \rightarrow \infty} p^{-(s-r)m} \varrho(p^m),$$

this shows that $\chi_p > 0$ for all rational primes p , whence $\mathfrak{S} > 0$, so in (2) the leading term dominates the error term and one finds (many) non-trivial simultaneous rational zeros of C_1, \dots, C_r . One can therefore concentrate on the case that the system

C_1, \dots, C_r is ‘bottomless’, i.e. not bottomed, and it is here that we introduce new ideas: It turns out that in this case there are algebraic numbers $\alpha_1, \dots, \alpha_r$ such that $\alpha_1 C_1 + \dots + \alpha_r C_r$ has ‘small’ ℓ -invariant over \mathbb{C} (see Lemma 4 and Lemma 7 below), so Theorem 2 comes into play and inductively allows one to reduce the number of cubic forms one has to consider. The proof of Theorem 1 and Theorem 2, in turn, possibly somewhat surprisingly, depends on techniques from Diophantine approximation, in particular approximations to systems of linear forms and Schmidt’s subspace theorem.

Notation: Our notation is fairly standard. We write $|\mathbf{x}|$ for the maximum norm of a vector \mathbf{x} , and we make use of the following equivalence relation \sim on the set of tuples (F_1, \dots, F_r) of forms F_i of degree d in s variables over a field K for fixed r, s, d and K : We define $(F_1, \dots, F_r) \sim (G_1, \dots, G_r)$ and say that F_1, \dots, F_r and G_1, \dots, G_r are K -equivalent systems if and only if there are non-singular linear maps $T : K^r \rightarrow K^r$ and $\tau : K^s \rightarrow K^s$ such that

$$\begin{aligned} & (G_1(X_1, \dots, X_s), \dots, G_r(X_1, \dots, X_s)) \\ &= T(F_1(\tau(X_1, \dots, X_s)), \dots, F_r(\tau(X_1, \dots, X_s))). \end{aligned}$$

It is easily seen that properties such as $\ell_K(F_1, \dots, F_r)$ and existence of a non-trivial (or non-singular) K -rational zero of $F_1 = \dots = F_r = 0$ are preserved under \sim . In case of a local field $K = \mathbb{Q}_p$ with ring of p -adic integers \mathbb{Z}_p , we also make use of Schmidt’s definition of ω -bottomed and ω -bottomless systems F_1, \dots, F_r , for which we refer to [10], §2 and [14], §2, and we write $|\cdot|_p$ for the usual p -adic absolute value.

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2. PROOF OF THEOREM 1

The following result is Lemma 1 in [6].

Lemma 2. *Let $L_1, \dots, L_h \in \mathbb{R}[X_1, \dots, X_s]$ be linear forms, and let $m \leq s$. Then for every $N \geq 1$ there exist linearly independent vectors $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{Z}^s$ such that $|\mathbf{x}_j| \leq N$ ($1 \leq j \leq m$) and*

$$|L_i(\mathbf{x}_j)| \ll N^{1 - \frac{s}{h} + \frac{m}{h}} \quad (1 \leq i \leq h; 1 \leq j \leq m),$$

where the implied O -constant only depends on s, m, h and L_1, \dots, L_h , but not on N .

Corollary 1. *Let $L_1, \dots, L_h \in \mathbb{C}[X_1, \dots, X_s]$ be linear forms, and let $m \leq s$. Then for every $N \geq 1$ there exist linearly independent vectors $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{Z}^s$ such that*

$$(5) \quad |\mathbf{x}_j| \leq N \quad (1 \leq j \leq m)$$

and

$$(6) \quad |L_k(\mathbf{x}_j)| \ll N^{1 - \frac{s}{2h} + \frac{m}{2h}} \quad (1 \leq k \leq h; 1 \leq j \leq m),$$

where the implied O -constant only depends on s, m, h and L_1, \dots, L_h , but not on N .

Proof. This follows immediately from writing $L_k = G_k + iH_k$ for suitable real linear forms G_k and H_k and applying Lemma 2 to $G_1, \dots, G_h, H_1, \dots, H_h$. \square

We are now in a position to prove Theorem 1. Since $\ell_{\mathbb{Q}}(\mathbf{F})$ and $\ell_{\mathbb{C}}(\mathbf{F})$ clearly do not change if one multiplies F_1, \dots, F_r with any positive integer, we can without loss of generality assume that $F_i \in \mathbb{Z}[X_1, \dots, X_s]$ ($1 \leq i \leq r$). Let $h = \ell_{\mathbb{C}}(\mathbf{F})$. We may assume that $h \geq 1$, because otherwise the statement is trivial since then the forms F_1, \dots, F_r must be identically zero, and we may suppose that $s \geq 2dh + 1$ as well as otherwise the statement again is trivial. Now

$$F_j = \sum_{i=1}^h L_i H_i^{(j)} \quad (1 \leq j \leq r)$$

for suitable linear forms $L_i \in \mathbb{C}[X_1, \dots, X_s]$ ($1 \leq i \leq h$), and forms $H_i^{(j)} \in \mathbb{C}[X_1, \dots, X_s]$ ($1 \leq i \leq h, 1 \leq j \leq r$) of degree $d-1$. Let C be a fixed sufficiently large constant. Then for any $N \geq 1$ and $m \leq s$, by Corollary 1, there exist linearly independent $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{Z}^s$ such that (5) and (6) hold true. Since the L_i are linear, this implies that

$$\left| L_i \left(\sum_{k=1}^m c_k \mathbf{x}_k \right) \right| \ll N^{1 - \frac{s}{2h} + \frac{m}{2h}} \quad (1 \leq i \leq h),$$

whenever c_1, \dots, c_m are integers with

$$(7) \quad |c_j| \leq C \quad (1 \leq j \leq m),$$

with an implied O -constant depending at most on $s, m, h, L_1, \dots, L_h, C$, but not on N . Moreover, under the same assumptions,

$$\left| H_i^{(j)} \left(\sum_{k=1}^m c_k \mathbf{x}_k \right) \right| \ll N^{d-1} \quad (1 \leq i \leq h, 1 \leq j \leq r),$$

by (5) and (7), and since the H_i are of degree $d-1$. We conclude that

$$\left| F_j \left(\sum_{k=1}^m c_k \mathbf{x}_k \right) \right| \ll N^{d - \frac{s-m}{2h}} \quad (1 \leq j \leq r),$$

whenever (7) is satisfied, again with an O -constant independent of N . On the other hand, as the $F_j \in \mathbb{Z}[X_1, \dots, X_s]$, $\mathbf{x}_k \in \mathbb{Z}^s$ and $c_1, \dots, c_m \in \mathbb{Z}$, the numbers

$$F_j \left(\sum_{k=1}^m c_k \mathbf{x}_k \right) \quad (1 \leq j \leq r)$$

are integers. We find that for

$$(8) \quad s - m > 2dh$$

and sufficiently large N we have

$$(9) \quad F_j \left(\sum_{k=1}^m c_k \mathbf{x}_k \right) = 0 \quad (1 \leq j \leq r)$$

whenever (7) holds true. As F_j is homogeneous of degree d , we can write it in the form

$$F_j(\mathbf{X}) = T_j(\mathbf{X}, \dots, \mathbf{X})$$

for a suitable symmetric d -linear form T_j . From (9) we then obtain that a certain homogeneous polynomial $P_j(c_1, \dots, c_m)$ of degree d attains the value zero for all

$\mathbf{c} \in \mathbb{Z}^m$ satisfying (7). The coefficients of this polynomial are all certain non-zero multiples of the expressions

$$T_j(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_d})$$

where $1 \leq i_1 \leq \dots \leq i_d \leq m$. By choosing C large enough in terms of s and d in the first place, we conclude that P_j must be the zero polynomial, whence

$$T_j(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_d}) = 0$$

whenever $1 \leq i_1 \leq \dots \leq i_d \leq m$, so the forms F_1, \dots, F_r simultaneously vanish on the \mathbb{Q} -linear space spanned by $\mathbf{x}_1, \dots, \mathbf{x}_m$. Lemma 1 therefore implies that

$$(10) \quad \ell_{\mathbb{Q}}(\mathbf{F}) \leq s - m.$$

The largest permissible m by (8) is $m = s - 2dh - 1$. The theorem therefore follows from (10).

3. PROOF OF THEOREM 2: AN APPLICATION OF THE SUBSPACE THEOREM

Our key ingredient is the following well known consequence of Schmidt's celebrated subspace theorem (see [9]).

Lemma 3. *Let $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}$ be \mathbb{Q} -linearly independent, and let $\delta > 0$. Then there are only finitely many $\mathbf{x} \in \mathbb{Z}^r \setminus \{\mathbf{0}\}$ such that*

$$|\alpha_1 x_1 + \dots + \alpha_r x_r| < |\mathbf{x}|^{1-r-\delta}.$$

We are now in a position to prove Theorem 2, using a similar idea as for the proof of Theorem 1. Let $h = \ell_{\mathbb{C}}(\alpha_1 F_1 + \dots + \alpha_r F_r)$. Again we may assume that $h \geq 1$ (if $h = 0$, then by \mathbb{Q} -linear independence of $\alpha_1, \dots, \alpha_r$ all the forms F_1, \dots, F_r must be identically zero), $s \geq 2drh + 1$ and $F_i \in \mathbb{Z}[X_1, \dots, X_s]$ ($1 \leq i \leq r$). Then there exist linear forms $L_i \in \mathbb{C}[X_1, \dots, X_s]$ ($1 \leq i \leq h$) and forms $H_i \in \mathbb{C}[X_1, \dots, X_s]$ ($1 \leq i \leq h$) of degree $d - 1$ such that

$$(11) \quad \sum_{i=1}^r \alpha_i F_i = \sum_{i=1}^h L_i H_i.$$

Let C be a fixed sufficiently large constant, and suppose that

$$(12) \quad s - m > 2dhr.$$

Then as in the proof of Theorem 1, for all $N \in \mathbb{N}$ we can find linearly independent $\mathbf{x}_1^{(N)}, \dots, \mathbf{x}_m^{(N)} \in \mathbb{Z}^s$ such that

$$(13) \quad |\mathbf{x}_j^{(N)}| \leq N \quad (1 \leq j \leq m, N \in \mathbb{N})$$

and

$$(14) \quad \left| L_i \left(\sum_{j=1}^m c_j \mathbf{x}_j^{(N)} \right) \right| \ll N^{1 - \frac{s}{2h} + \frac{m}{2h}} \quad (1 \leq i \leq h, N \in \mathbb{N}),$$

whenever (7) holds true, and under the same assumption,

$$(15) \quad \left| H_i \left(\sum_{j=1}^m c_j \mathbf{x}_j^{(N)} \right) \right| \ll N^{d-1} \quad (1 \leq i \leq h, N \in \mathbb{N}).$$

Let us introduce the notation

$$(16) \quad a_i^{(\mathbf{c}, N)} = F_i \left(\sum_{j=1}^m c_j \mathbf{x}_j^{(N)} \right) \quad (1 \leq i \leq r, \mathbf{c} \in \mathbb{Z}^m, N \in \mathbb{N}).$$

Note that $a_i^{(\mathbf{c}, N)} \in \mathbb{Z}$. Then (11), (14), and (15) imply that

$$(17) \quad \left| \sum_{i=1}^r \alpha_i a_i^{(\mathbf{c}, N)} \right| \ll N^{d(1 - \frac{s-m}{2dh})},$$

for all $N \in \mathbb{N}$ and whenever (7) is satisfied, with an implied O -constant independent of N . Now fix $\mathbf{c} \in \mathbb{Z}^m$ satisfying (7), and write

$$A_N = \max_{1 \leq i \leq r} |a_i^{(\mathbf{c}, N)}|.$$

We claim that the sequence A_1, A_2, \dots becomes 0 from some index onwards. For if not, then there exists a strictly increasing sequence N_1, N_2, \dots such that $A_{N_t} \neq 0$ for all $t \in \mathbb{N}$. Since $\alpha_1, \dots, \alpha_r$ are \mathbb{Q} -linearly independent, this implies that

$$(18) \quad \sum_{i=1}^r \alpha_i a_i^{(\mathbf{c}, N_t)} \neq 0$$

for all $t \in \mathbb{N}$. By (12), (17) and (18) we obtain

$$(19) \quad 0 < \left| \sum_{i=1}^r \alpha_i a_i^{(\mathbf{c}, N_t)} \right| \ll N_t^{d(1-r-\delta)}$$

for all $t \in \mathbb{N}$, where

$$\delta = \frac{1}{2dh} > 0.$$

Letting t (and thus N_t) tend to infinity, we conclude that the sequence A_{N_t} cannot be bounded. By going over to a subsequence if necessary, we may therefore without loss of generality assume that A_{N_t} is strictly increasing. Now (7), (13) and (16) give

$$|a_i^{(\mathbf{c}, N_t)}| \leq A_{N_t} \ll N_t^d \quad (1 \leq i \leq r; t \in \mathbb{N}),$$

as the F_i are of degree d , so from (19) we get

$$\left| \sum_{i=1}^r \alpha_i a_i^{(\mathbf{c}, N_t)} \right| \ll A_{N_t}^{1-r-\delta},$$

where the implied O -constant as well as $\delta > 0$ do not depend on t . Since A_{N_t} is strictly increasing with t , we obtain infinitely many $\mathbf{a} \in \mathbb{Z}^r \setminus \{\mathbf{0}\}$ such that

$$\left| \sum_{i=1}^r \alpha_i a_i \right| \ll \left(\max_{1 \leq i \leq r} |a_i| \right)^{1-r-\delta},$$

which contradicts Lemma 3. Therefore, the sequence A_1, A_2, \dots becomes 0 from some index onwards, say $A_N = 0$ for $N \geq u$. This translates into

$$(20) \quad F_i \left(\sum_{j=1}^m c_j \mathbf{x}_j^{(N)} \right) = 0 \quad (1 \leq i \leq r)$$

for all $N \geq u$, and fixed \mathbf{c} . Note that $u = u_{\mathbf{c}}$ may depend on \mathbf{c} . Let

$$M = \max_{\mathbf{c} \in \mathbb{Z}^r \text{ with } (7)} u_{\mathbf{c}}.$$

Then (20) holds true for all $N \geq M$, and for all $\mathbf{c} \in \mathbb{Z}^r$ satisfying (7). By choosing C large enough at the beginning of the argument, in the same way as in the proof of Theorem 1, we find that F_1, \dots, F_r simultaneously vanish on the \mathbb{Q} -linear space spanned by $\mathbf{x}_1^{(M)}, \dots, \mathbf{x}_m^{(M)}$. Lemma 1 and (12) on choosing $m = s - 2dhr - 1$ therefore imply that

$$\ell_{\mathbb{Q}}(F_1, \dots, F_r) \leq 2drl_{\mathbb{C}}(\alpha_1 F_1 + \dots + \alpha_r F_r) + 1,$$

which finishes the proof.

4. BOTTOMLESS SYSTEMS OF CUBIC FORMS

Let p be a rational prime, and let $C_1, \dots, C_r \in \mathbb{Z}_p[X_1, \dots, X_s]$ be cubic forms. To each C_i we associate a symmetric trilinear form T_i such that

$$C_i(\mathbf{X}) = T_i(\mathbf{X}, \mathbf{X}, \mathbf{X}) \quad (1 \leq i \leq r).$$

We refer to [10], §2 for the definition of an ω -bottomless system C_1, \dots, C_r ; note that being ω -bottomless is just the negation of being ω -bottomed.

Lemma 4. *Let p be a rational prime, and let $C_1, \dots, C_r \in \mathbb{Z}_p[X_1, \dots, X_s]$ be an ω -bottomless system of cubic forms where $\omega > 0$. Then there exists a cubic form C in the \mathbb{Q}_p -rational pencil of C_1, \dots, C_r such that*

$$\ell_{\mathbb{Q}_p}(C) < 3\omega r.$$

Proof. We adapt the proof of Theorem 3 in [10] to our setting. Without loss of generality we can assume that $s \geq 3\omega r$, because otherwise the statement is trivial. By Theorem 6 in [10], as (C_1, \dots, C_r) is ω -bottomless, we can find a system $(\tilde{C}_1, \dots, \tilde{C}_r)$ where $\tilde{C}_i \in \mathbb{Q}_p[X_1, \dots, X_s]$ ($1 \leq i \leq r$) that is \mathbb{Q}_p -equivalent to (C_1, \dots, C_r) and has the following property: Writing as above

$$\tilde{C}_i(\mathbf{X}) = \tilde{T}_i(\mathbf{X}, \mathbf{X}, \mathbf{X}) \quad (1 \leq i \leq r)$$

for suitable symmetric trilinear forms \tilde{T}_i , there exist non-negative integers a_1, \dots, a_s and b_1, \dots, b_r such that

$$a_1 + \dots + a_s < \omega(b_1 + \dots + b_r)$$

and

$$(21) \quad \tilde{T}_i(\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w) = 0$$

for every $i \in \{1, \dots, r\}$ and every triple $(u, v, w) \in \{1, \dots, s\}^3$ with

$$(22) \quad a_u + a_v + a_w < b_i.$$

(As usual, \mathbf{e}_i denotes the i -th unit vector.) By going over to an equivalent system, if necessary, we may without loss of generality assume that

$$(23) \quad a_1 \leq \dots \leq a_s, \quad b_1 \geq \dots \geq b_r,$$

whence

$$(24) \quad a_1 + \dots + a_s < \omega r b_1.$$

If $a_s < \frac{b_1}{3}$, then (22) is true for $i = 1$ and all $u, v, w \in \{1, \dots, s\}$. Put $n = s + 1$ in this case. Otherwise, let n be minimal with $a_n \geq \frac{b_1}{3}$. Then

$$a_1 + \dots + a_s \geq (s - n + 1) \frac{b_1}{3}.$$

Using (24), we obtain

$$(25) \quad n - 1 > s - 3\omega r.$$

This also holds true in case of $n = s + 1$. Now, by (21), (22) and (23), the form \tilde{C}_1 vanishes on the \mathbb{Q}_p -linear space spanned by $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$. Hence, by Lemma 1 and (25), we have

$$\ell_{\mathbb{Q}_p}(\tilde{C}_1) \leq s - (n - 1) < 3\omega r.$$

Since $(C_1, \dots, C_r) \sim (\tilde{C}_1, \dots, \tilde{C}_r)$, there exists a form C in the \mathbb{Q}_p -rational pencil of C_1, \dots, C_r such that $\ell_{\mathbb{Q}_p}(C) < 3\omega r$ as well, which completes the proof. \square

5. THE LOCAL PROBLEM FOR SYSTEMS OF CUBIC FORMS

Lemma 5. *Let $q, r \in \mathbb{N}$ such that $1 \leq q \leq r$. Then*

$$r^4 - qr^3 \geq (r - q)^4.$$

Proof. We have

$$\begin{aligned} r^4 - qr^3 - (r - q)^4 &= q(3r^3 - 6qr^2 + 4q^2r - q^3) \\ &= q(r - q) \left(3\left(r - \frac{q}{2}\right)^2 + \frac{1}{4}q^2 \right) \geq 0. \end{aligned}$$

\square

Lemma 6. *Let p be a rational prime, and suppose that $C_1, \dots, C_r \in \mathbb{Z}[X_1, \dots, X_s]$, regarded as cubic forms in $\mathbb{Z}_p[X_1, \dots, X_s]$, are ω_0 -bottomed, where*

$$(26) \quad \omega_0 = 1764(3r + 1)^2 + 1.$$

Then for the densities $\rho(p^m)$ as defined in (3) the lower bound (4) holds true, with an implied constant independent of m .

Proof. This is implicit in Schmidt's work [14], but as our setting is slightly different, let us indicate how to derive it: In [14], by a p -adic compactness argument one can without loss of generality assume that the system of cubic forms (C_1, \dots, C_r) is 'generic', which by Lemma 4 in [10] implies that it is ω_0 -bottomed where

$$\omega_0 = \frac{s}{3r}.$$

The assumption $s \geq 5300r(3r + 1)^2$ in formula (1.5) in [14] then makes sure that

$$\omega_0 > 1764(3r + 1)^2.$$

In a further preparatory step one can then *reduce* C_1, \dots, C_r . If (C_1, \dots, C_r) , where $C_i \in \mathbb{Z}_p[X_1, \dots, X_s]$ ($1 \leq i \leq r$), is a reduced ω_0 -bottomed system of cubic forms, then if $T : \mathbb{Q}_p^r \rightarrow \mathbb{Q}_p^r$ is a non-singular linear transformation acting on the system (C_1, \dots, C_r) , and $\tau : \mathbb{Z}_p^s \rightarrow \mathbb{Z}_p^s$ is a non-singular linear transformation of the variables X_1, \dots, X_s , such that $T^{-1}(C_1(\tau(\mathbf{X})), \dots, C_r(\tau(\mathbf{X})))$ is a system of \mathbb{Z}_p -integral forms, then

$$|\det T|_p^{\omega_0} \geq |\det \tau|_p.$$

An application of bounds for cubic exponential sums then provides the bound (4), at first only for reduced systems (see Theorem 3 in [14]), but then also shown to hold true for ω_0 -bottomed systems that are not necessarily reduced (see Theorem 2 in [14]). Finally, a ‘non-generic’ system (C_1, \dots, C_r) can then be approximated by a sequence of generic ones, each of them having a non-trivial p -adic zero by (4), whence by a compactness argument also the original one has a non-trivial p -adic zero. The latter argument, though, only provides one solution, not a lower bound as in (4), so is not useful for our purposes. We therefore cannot assume that (C_1, \dots, C_r) is generic, so cannot assume that it is ω_0 -bottomed where $\omega_0 = \frac{s}{3r}$, instead we explicitly assume (26), without any longer knowing that $s = 3\omega_0 r$. In fact, we will later apply the lemma in situations where s can be much bigger than $3\omega_0 r$. Fortunately, a careful analysis of the proofs in [14] reveals that the only condition that is really needed to establish (4) is that (C_1, \dots, C_r) is ω_0 -bottomed with ω_0 given by (26), regardless of whether $s = 3\omega_0 r$ or not. This is not surprising as using similar techniques an analogous result for systems of quadratic forms is explicitly stated in Theorem 2 in [10]. The only necessary minor adjustment is the proof of Lemma 4 in [14], in which on page 221, line 7, the relation $\omega_0 = \frac{s}{3r}$ is used. This can be avoided, though, so let us briefly explain what to do: In the proof of Lemma 4, familiarity of which is now assumed, a certain subspace Y of \mathbb{Z}_p^s is constructed, and a linear surjective map

$$\tau : \mathbb{Z}_p^s \rightarrow Y.$$

In fact, by construction $Y \subset p^{a-1}\mathbb{Z}_p^s$, so $C_i(\mathbf{x}) \equiv 0 \pmod{p^{3(a-1)}} \quad (1 \leq i \leq r)$ for all $\mathbf{x} \in Y$, and by construction $C_1(\mathbf{x}) \equiv 0 \pmod{p^\ell}$ for all $\mathbf{x} \in Y$ (note that we write C_i instead of \mathfrak{F}_i as Schmidt, but our other notation is the same). One can therefore take out a factor $p^{3(a-1)}$ from $C_i(\mathbf{x}) \quad (2 \leq i \leq r)$, and a factor p^ℓ from $C_1(\mathbf{x})$, leading to the definition of the linear map

$$\begin{aligned} T : \mathbb{Q}_p^r &\rightarrow \mathbb{Q}_p^r \\ \mathbf{e}_1 &\mapsto p^\ell \mathbf{e}_1, \quad \mathbf{e}_i \mapsto p^{3(a-1)} \mathbf{e}_i \quad (2 \leq i \leq r), \end{aligned}$$

where the \mathbf{e}_i are the unit vectors. Further, it is shown that

$$|\det \tau|_p = p^{(1-a)s} |(X(\ell)/K)_a|^{-1}$$

and

$$|\det T|_p = p^{-\ell - 3(a-1)(r-1)}.$$

Since (C_1, \dots, C_r) is ω_0 -bottomed and reduced, and $T^{-1}(C_1(\tau(\mathbf{x})), \dots, C_r(\tau(\mathbf{x})))$ is integral for all $\mathbf{x} \in \mathbb{Z}_p^s$, one deduces that

$$|\det T|_p^{\omega_0} \geq |\det \tau|_p,$$

so using $\omega_0 = \frac{s}{3r}$ one obtains

$$|(X(\ell)/K)_a| \geq p^{\omega_0(\ell - 3a + 3)},$$

which is Lemma 4 (i); parts (ii) and (iii) then follow from (i). However, the argument can be adjusted no longer to depend on $s = 3\omega_0 r$: Instead of $\tau : \mathbb{Z}_p^s \rightarrow Y$, consider $\tilde{\tau} : \mathbb{Z}_p^s \rightarrow \tilde{\tau}(Y)$, where

$$\begin{aligned} \tilde{\tau} : \mathbb{Q}_p^s &\rightarrow \mathbb{Q}_p^s \\ \mathbf{x} &\mapsto p^{(1-a)} \mathbf{x}. \end{aligned}$$

Since $Y \subset p^{a-1}\mathbb{Z}_p^s$, we have $\tilde{Y} := \tilde{\tau}(Y) \subset \mathbb{Z}_p^s$, and since the cubic form C_1 satisfies $C_1(\mathbf{x}) \equiv 0 \pmod{p^\ell}$ for all $\mathbf{x} \in Y$, we still have $C_1(\mathbf{x}) \equiv 0 \pmod{p^{\ell-3(a-1)}}$ for all $\mathbf{x} \in \tilde{Y}$, providing that $3(a-1) \leq \ell$; the latter condition is amply met in the later application of Lemma 4 (see formula (5.7) on page 222 in [14], where a is chosen such that $\frac{\ell}{6} \leq a < \frac{\ell}{6} + 1$). This means that for all $\mathbf{x} \in \tilde{Y}$ we can still take out a factor $p^{\ell-3(a-1)}$ from $C_1(\mathbf{x})$, so we can define

$$\begin{aligned} \tilde{T} : \mathbb{Q}_p^r &\rightarrow \mathbb{Q}_p^r \\ \mathbf{e}_1 &\mapsto p^{\ell-3(a-1)}\mathbf{e}_1, \quad \mathbf{e}_i \mapsto \mathbf{e}_i \quad (2 \leq i \leq r), \end{aligned}$$

and $\tilde{T}^{-1}(C_1(\tilde{\tau}\tau(\mathbf{x})), \dots, C_r(\tilde{\tau}\tau(\mathbf{x})))$ is integral for all $\mathbf{x} \in \mathbb{Z}_p^s$, so from the fact that (C_1, \dots, C_r) is ω_0 -bottomed and reduced, we obtain

$$(27) \quad |\det \tilde{T}|_p^{\omega_0} \geq |\det(\tilde{\tau}\tau)|_p.$$

Now

$$|\det \tilde{T}|_p = p^{-\ell+3(a-1)}$$

and

$$\begin{aligned} |(\det \tilde{\tau}\tau)|_p &= |(\det \tilde{\tau}) \cdot (\det \tau)|_p = |p^{s(1-a)} \det \tau|_p \\ &= p^{s(a-1)} |\det \tau|_p = |(X(\ell)/K)_a|^{-1}. \end{aligned}$$

From (27) we therefore obtain

$$|(X(\ell)/K)_a| \geq p^{\omega_0(\ell-3(a-1))}$$

as before. \square

Lemma 7. *Let $C_1, \dots, C_r \in \mathbb{Q}[X_1, \dots, X_s]$ be cubic forms, and let p be a rational prime. Suppose that there exist $\alpha_1, \dots, \alpha_r \in \mathbb{Q}_p$, not all zero, such that*

$$(28) \quad \ell_{\mathbb{Q}_p}(\alpha_1 C_1 + \dots + \alpha_r C_r) \leq m$$

for some $m \leq s$. Then there exist $\beta_1, \dots, \beta_r \in \overline{\mathbb{Q}}$, not all zero, such that

$$\ell_{\overline{\mathbb{Q}}}(\beta_1 C_1 + \dots + \beta_r C_r) \leq m.$$

Proof. Since not all α_i are zero, and

$$\ell_{\mathbb{Q}_p}(\alpha_1 C_1 + \dots + \alpha_r C_r) = \ell_{\mathbb{Q}_p}(\lambda(\alpha_1 C_1 + \dots + \alpha_r C_r))$$

for all $\lambda \in \mathbb{Q}_p \setminus \{0\}$, we can without loss of generality assume that $\alpha_1 = 1$. The condition (28) then is equivalent to the existence of $\alpha_2, \dots, \alpha_r \in \mathbb{Q}_p$ and linear forms $L_i \in \mathbb{Q}_p[X_1, \dots, X_s]$ and quadratic forms $Q_i \in \mathbb{Q}_p[X_1, \dots, X_s]$ ($1 \leq i \leq m$), such that

$$C_1 + \alpha_2 C_2 + \dots + \alpha_r C_r = \sum_{i=1}^m L_i Q_i.$$

The latter equation translates into a system of polynomial equations

$$(29) \quad F_i(\alpha_2, \dots, \alpha_r, \mathbf{a}, \mathbf{b}) = 0 \quad (1 \leq i \leq n),$$

for suitable $F_i(\alpha_2, \dots, \alpha_r, \mathbf{a}, \mathbf{b}) \in \mathbb{Q}[\alpha_2, \dots, \alpha_r, \mathbf{a}, \mathbf{b}]$ ($1 \leq i \leq n$), where the \mathbf{a} are the coefficients of L_1, \dots, L_m and the \mathbf{b} are the coefficients of Q_1, \dots, Q_m (note that this also works in the special case $m = 0$: we just get a system of equations for $\alpha_2, \dots, \alpha_r$). We conclude that the system (29) of polynomial equations with rational coefficients has a solution $\alpha_2, \dots, \alpha_r, \mathbf{a}, \mathbf{b}$ over \mathbb{Q}_p . We claim that it also has a solution $\alpha_2, \dots, \alpha_r, \mathbf{a}, \mathbf{b}$ over $\overline{\mathbb{Q}}$, which immediately implies Lemma 7. Now as

$\overline{\mathbb{Q}}$ is algebraically closed, if (29) has no solution over $\overline{\mathbb{Q}}$, then following the approach in [11], p. 291, by Hilbert's Nullstellensatz there are polynomials $G_1, \dots, G_n \in \overline{\mathbb{Q}}[\alpha_2, \dots, \alpha_r, \mathbf{a}, \mathbf{b}]$ such that

$$(30) \quad 1 = \sum_{i=1}^n F_i G_i.$$

The coefficients of G_1, \dots, G_n all lie in some finite algebraic extension of \mathbb{Q} , which can be assumed to be the splitting field of a finite set of polynomials with rational coefficients. We can also regard the latter polynomials as polynomials with coefficients in \mathbb{Q}_p and adjoin all their roots to \mathbb{Q}_p , so (30) can also be interpreted as an equation valid in some extension of \mathbb{Q}_p . However, then (29) would not have a solution over \mathbb{Q}_p , which is a contradiction. Therefore (29) also has a solution over $\overline{\mathbb{Q}}$. \square

Theorem 4. *Let p be a rational prime, and let $C_1, \dots, C_r \in \mathbb{Z}[X_1, \dots, X_s]$ be cubic forms. If (4) does not hold true, then there exists $q \in \{1, \dots, r\}$ and q linearly independent cubic forms F_1, \dots, F_q in the \mathbb{Q} -rational pencil of C_1, \dots, C_r with*

$$(31) \quad \ell_{\mathbb{Q}}(F_1, \dots, F_q) \leq 18q\omega_0 r,$$

where ω_0 is given by (26).

Proof. If (4) is false, then by Lemma 6 the system C_1, \dots, C_r is ω_0 -bottomless, with ω_0 given by (26). By Lemma 4, there exists a form F in the \mathbb{Q}_p -rational pencil of C_1, \dots, C_r with $\ell_{\mathbb{Q}_p}(F) < 3\omega_0 r$. Consequently, by Lemma 7, there also exists a form C in the $\overline{\mathbb{Q}}$ -rational pencil of C_1, \dots, C_r with

$$(32) \quad \ell_{\overline{\mathbb{Q}}}(C) < 3\omega_0 r,$$

say

$$C = \alpha_1 C_1 + \dots + \alpha_r C_r$$

for suitable $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}$, not all zero. Let $q \geq 1$ be the dimension of the \mathbb{Q} -vector space spanned by the numbers $\alpha_1, \dots, \alpha_r$. If $q < r$, then without loss of generality we can write

$$\alpha_r = a_1 \alpha_1 + \dots + a_{r-1} \alpha_{r-1}$$

for certain $a_1, \dots, a_{r-1} \in \mathbb{Q}$, so

$$C = \alpha_1 \tilde{C}_1 + \dots + \alpha_{r-1} \tilde{C}_{r-1}$$

where

$$\begin{pmatrix} \tilde{C}_1 \\ \vdots \\ \tilde{C}_{r-1} \end{pmatrix} = A \begin{pmatrix} C_1 \\ \vdots \\ C_r \end{pmatrix}$$

with the $(r-1) \times r$ matrix

$$A = \begin{pmatrix} 1 & & & a_1 \\ & 1 & & a_2 \\ & & \ddots & \vdots \\ & & & 1 & a_{r-1} \end{pmatrix}.$$

In particular, as A has rank $r-1$, the cubic forms $\tilde{C}_1, \dots, \tilde{C}_{r-1}$ are linearly independent forms in the rational pencil of C_1, \dots, C_r . Continuing this process if

necessary, we eventually obtain q linearly independent cubic forms F_1, \dots, F_q in the rational pencil of C_1, \dots, C_r , and \mathbb{Q} -linearly independent $\beta_1, \dots, \beta_q \in \overline{\mathbb{Q}}$ such that

$$C = \beta_1 F_1 + \dots + \beta_q F_q.$$

As C did not change, of course still (32) holds true. By Theorem 2, observing that $\ell_{\mathbb{C}}(C) \leq \ell_{\overline{\mathbb{Q}}}(C)$, we therefore obtain (31). \square

6. PROOF OF THEOREM 3

We prove Theorem 3 by induction on r . The base cases $r = 1$ and $r = 2$ follow from the bounds $\gamma(1) \leq 13$ and $\gamma(2) \leq 654$ mentioned in the introduction. Now suppose that Theorem 3 has already been established for systems of at most two cubic forms. Let $C_1, \dots, C_r \in \mathbb{Q}[X_1, \dots, X_s]$ be cubic forms with

$$(33) \quad s > 400000r^4$$

and $r \geq 3$. We want to find a non-trivial simultaneous rational zero of C_1, \dots, C_r and therefore without loss of generality can assume that $C_1, \dots, C_r \in \mathbb{Z}[X_1, \dots, X_s]$. If a form C in the rational pencil of C_1, \dots, C_r has $h_{\mathbb{Q}}(C) = \ell_{\mathbb{Q}}(C) \leq 8r^2 + 8r$, then C by Lemma 1 vanishes on a rational linear space V of dimension at least $s - (8r^2 + 8r)$, so we can substitute this linear space V into the $r - 1$ remaining cubic forms and find a non-trivial rational zero for them, as

$$s - 8r^2 - 8r > 400000(r - 1)^4$$

by (33). Otherwise, as explained in the introduction, the asymptotic formula (2) holds true, where $\mathfrak{J} > 0$. If $\mathfrak{S} > 0$, then we are done. On the other hand, if $\mathfrak{S} = 0$, then $\chi_p = 0$ for some p , so (4) must be false, and by Theorem 4 there exist $q \geq 1$ linearly independent forms F_1, \dots, F_q in the rational pencil of C_1, \dots, C_r such that

$$\ell_{\mathbb{Q}}(F_1, \dots, F_q) \leq 18q(1764(3r + 1)^2 + 1)r \leq 400000qr^3.$$

By Lemma 1, this implies that F_1, \dots, F_q simultaneously vanish on a rational linear space V of dimension at least $s - 400000qr^3$. If $q = r$, then we already found a non-trivial simultaneous rational zero of C_1, \dots, C_r because of $s - 400000qr^3 > 0$ by (33). Otherwise, as F_1, \dots, F_q are linearly independent forms in the rational pencil of C_1, \dots, C_r , there exist cubic forms $F_{q+1}, \dots, F_r \in \mathbb{Q}[X_1, \dots, X_s]$ such that $(F_1, \dots, F_r) \sim (C_1, \dots, C_r)$. By substituting V into F_{q+1}, \dots, F_r , we just need to find a non-trivial simultaneous rational zero of F_{q+1}, \dots, F_r on V , which is possible by our inductive assumption, since by (33) and Lemma 5 we have

$$\dim V \geq s - 400000qr^3 > 400000(r - q)^4.$$

This finishes the proof.

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