ON THE DIMENSION OF CLASSIFYING SPACES FOR FAMILIES OF ABELIAN SUBGROUPS

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Abstract. We show that a finitely generated abelian group G of torsion-free rank $n \geq 1$ admits a $n+r$ dimensional model for $E_{\mathfrak{F}_r}G$, where $\mathfrak{F}_r$ is the family of subgroups of torsion-free rank less than or equal to $r \geq 0$.

1. Introduction

In this note we consider classifying spaces $E_{\mathfrak{F}}G$ for a family of subgroups $\mathfrak{F}$ of G. We are particularly interested in the minimal dimension, denoted $\operatorname{gd}_{\mathfrak{F}} G$, such a space can have.
Let G be a group. We say a collection of subgroups $\mathfrak{F}$ is a family if it is closed under conjugation and taking subgroups. A G-CW-complex X is said to be a classifying space $E_{\mathfrak{F}}G$ for the family $\mathfrak{F}$ if, for each subgroup $H \leq G$, $X^H \simeq \{*\}$ if $H \in \mathfrak{F}$, and $X^H = \varnothing$ otherwise.
The spaces $\underline{E}G = E_{\mathfrak{F}}G$ for $\mathfrak{F} = \mathfrak{F}in$ the family of finite subgroups and $\underline{\underline{E}}G = E_{\mathfrak{F}}G$ for $\mathfrak{F} = \mathcal{V}cyc$ the family of virtually cyclic subgroups have been widely studied for their connection with the Baum-Connes and Farrell-Jones conjectures respectively. For a first introduction into the subject see, for example, the survey [3].
We consider finitely generated abelian groups G of finite torsion-free rank $r_0(G) = n$ and families $\mathfrak{F}_r$ of subgroups of torsion-free rank less or equal to $r < n$. Note that for $r = 0$, $\mathfrak{F}_0 = \mathfrak{F}in$ and that it is a well known fact, see for example [3], that $\mathbb{R}^n$ is a model for $\underline{E}G$ and that $\operatorname{gd}_{\mathfrak{F}_0} G = n$. For $r = 1$, $\mathfrak{F}_1 = \mathcal{V}cyc$ and it was shown in [5, Proposition 5.13(iii)] that $\operatorname{gd}_{\mathfrak{F}_1} G = n + 1$.
The main idea is to use the method developed by Lück and Weiermann [5] to build models of $E_{\mathfrak{F}_r}G$ from models for $E_{\mathfrak{F}_{r-1}}G$. We begin by recalling those results in [5] that we need for our construction. Let $\mathfrak{F}$ and $\mathfrak{G}$ families of subgroups of a given group G such that $\mathfrak{F} \subseteq \mathfrak{G}$.

Date: April 21, 2016.
2000 *Mathematics Subject Classification.* 55R35, 20J06, 18G99.
The fourth author gratefully acknowledges the support by the Italian *National Group for Algebraic and Geometric Structures and Their Applications* (GNSAGA – INDAM).

Definition 1.1. [5, (2.1)] Let $\mathfrak{F}$ and $\mathfrak{G}$ be families of subgroups of a given group G such that $\mathfrak{F} \subseteq \mathfrak{G}$. Let $\sim$ be an equivalence relation on $\mathfrak{G}\backslash\mathfrak{F}$ satisfying:

- For $H, K \in \mathfrak{G}\backslash\mathfrak{F}$ with $H \leq K$ we have $H \sim K$.
- Let $H, K \in \mathfrak{G}\backslash\mathfrak{F}$ and $g \in G$, then $H \sim K \iff gHg^{-1} \sim gKg^{-1}$.

Such a relation is called a strong equivalence relation. Denote by $[\mathfrak{G}\backslash\mathfrak{F}]$ the equivalence classes of $\sim$ and define for all $[H] \in [\mathfrak{G}\backslash\mathfrak{F}]$ the following subgroup of G:

$$N_G[H] = \{g \in G \mid [gHg^{-1}] = [H]\}.$$

Now define a family of subgroups of $N_G[H]$ by

$$\mathfrak{G}[H] = \{K \leq N_G[H] \mid K \in \mathfrak{G}\backslash\mathfrak{F}\,,\, [K] = [H]\} \cup (\mathfrak{F} \cap N_G[H]).$$

Here $\mathfrak{F} \cap N_G[H]$ is the family of subgroups of of $N_G[H]$ belonging to $\mathfrak{F}$.

Theorem 1.2. [5, Theorem 2.3] *Let $\mathfrak{F} \subseteq \mathfrak{G}$ and $\sim$ be as in Definition 1.1. Denote by I a complete set of representatives of the conjugacy classes in $[\mathfrak{G}\backslash\mathfrak{F}]$. Then the G-CW-complex given by the cellular G push-out*

$$\begin{array}{ccc}
\sqcup_{[H]\in I} G \times_{N_G[H]} E_{\mathfrak{F}\cap N_G[H]}(N_G[H]) & \xrightarrow{i} & E_{\mathfrak{F}}(G) \\
\Big\downarrow{\scriptstyle \sqcup_{[H]\in I} id_G \times_{N_G[H]} f_{[H]}} & & \Big\downarrow \\
\sqcup_{[H]\in I} G \times_{N_G[H]} E_{\mathfrak{G}[H]}(N_G[H]) & \longrightarrow & X
\end{array}$$

where either i or the $f_{[H]}$ are inclusions, is a model for $E_{\mathfrak{G}}(G)$.

The condition on the two maps being inclusions is not that strong a restriction, as one can replace the spaces by the mapping cylinders, see [5, Remark 2.5]. Hence one has:

Corollary 1.3. [5, Remark 2.5] *Suppose there exists an n-dimensional model for $E_{\mathfrak{F}}G$ and, for each $H \in I$, a $(n-1)$-dimensional model for $E_{\mathfrak{F}\cap N_G[H]}(N_G[H])$ and a n-dimensional model for $E_{\mathfrak{G}[H]}(N_G[H])$. Then there is an n-dimensional model for $E_{\mathfrak{G}}G$.*

Corollary 1.3 gives us a tool to find an upper bound for $\mathrm{gd}_{\mathfrak{G}}\, G$. A very useful tool to find a lower bound for $\mathrm{gd}_{\mathfrak{G}}\, G$ is the following Mayer-Vietoris sequence [4], which is an immediate consequence of Theorem 1.2, see also [1, Proposition 7.1] for the Bredon-cohomology version.

Corollary 1.4. *With the notation as in Theorem 1.2 we have following long exact cohomology sequence:*

$$\dots \to H^i(G\backslash E_{\mathfrak{G}}G) \to (\prod_{[H]\in I} H^i(N_G[H]\backslash E_{\mathfrak{G}[H]}N_G[H])) \oplus H^i(G\backslash E_{\mathfrak{F}}G) \to$$

$$\prod_{[H]\in I} H^i(N_G[H]\backslash E_{\mathfrak{F}\cap N_G[H]}N_G[H]) \to H^{i+1}(G\backslash E_{\mathfrak{G}}G) \to \dots$$

This note will be devoted to proving the following Theorem:

Main Theorem. *Let G be a finitely generated abelian group of finite torsion-free rank $n \geq 1$, and denote by $\mathfrak{F}_r$ the family of subgroups of torsion-free rank less than or equal to $r \geq 0$. Then*

$$\operatorname{gd}_{\mathfrak{F}_r} G \leq n + r.$$

The case for more general classes of groups G is going to be dealt with, using different methods, by the second author in his Ph.D thesis.

2. The Construction

Throughout, let G denote a finitely generated abelian group of torsion-free rank $r_0(G) = n$.
The idea is to construct models for $E_{\mathfrak{F}_r}G$ in terms of models for $E_{\mathfrak{F}_{r-1}}G$ using the push-out of Theorem 1.2 inductively. As a first step we shall define an equivalence relation in the sense of Definition 1.1.

Lemma 2.1. *Let $\sim$ denote the following relation on $\mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$:*

$$H \sim K \iff rk(H \cap K) = r.$$

Then $\sim$ is a strong equivalence relation.

Proof. We show that $\sim$ is transitive: If $H \sim K$ and $K \sim L$, this implies that both $H \cap K$ and $K \cap L$ are finite index subgroups of K. Hence also $H \cap K \cap L$ is a finite index subgroup of K, and in particular of $K \cap L$ and thus of L. Hence $H \cap L$ is finite index in both H and L. The rest is easily checked. $\square$

Lemma 2.2. *G satisfies $(M_{\mathfrak{F}_{r-1} \subseteq \mathfrak{F}_r})$, i.e. every subgroup $H \in \mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$ is contained in a unique $H_{max} \in \mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$, which is maximal.*

Proof. The existence follows from [6]. As regards uniqueness, suppose H is included in two different maximal elements $K, L \in \mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$: then $H \leq KL$. Note that, since $H \sim L$ and $H \leq L$, it follows that $|L \colon H| < \infty$. Hence

$$|KL \colon K| = |L \colon K \cap L| \leq |L \colon H| < \infty$$

implies $KL \in \mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$, contrary to the maximality of K and L. $\square$

Note that we always have maximal elements in $\mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$ as long as the ambient group is polycyclic [6], but uniqueness already fails for the Klein-bottle group K, which is non-abelian but contains a free abelian subgroup of rank 2 as an index 2 subgroup. Denote

$$K = \langle a, b \,|\, aba^{-1} = b^{-1} \rangle$$

and consider $\mathfrak{F}_1$ the family of cyclic subgroups. Since $a^2 = (ab^{-1})^2$, in follows that $\langle a^2 \rangle \leq \langle ab^{-1} \rangle$ as well as $\langle a^2 \rangle \leq \langle a \rangle$, both of which are maximal.
For $M \leq G$ a subgroup of G we denote by $\mathfrak{All}(M)$ the family of all subgroups of M.

Lemma 2.3. *Let M be a maximal subgroup of G of torsion-free rank r. Then $\mathbb{R}^{n-r}$ is a model for $E_{\mathfrak{All}(M)}G$, and $\mathrm{gd}_{\mathfrak{All}(M)} G = n - r$.*

Proof. Since M is maximal it follows that G/M is torsion-free of rank $n-r$ and hence $\mathbb{R}^{n-r}$ is a model for $E(G/M)$. The action of G given by the projection $G \twoheadrightarrow G/M$ now yields the claim. $\square$

Lemma 2.4. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be two families of subgroups of G. Then*

$$\mathrm{gd}_{\mathfrak{F}\cup\mathfrak{G}} G \leq \max\{\mathrm{gd}_{\mathfrak{F}} G, \mathrm{gd}_{\mathfrak{G}} G, \mathrm{gd}_{\mathfrak{F}\cap\mathfrak{G}} G + 1\}.$$

Proof. By the universal property of classifying spaces for families, there are maps, unique up to G-homotopy, $E_{\mathfrak{F}\cap\mathfrak{G}}G \to E_{\mathfrak{G}}G$ and $E_{\mathfrak{F}\cap\mathfrak{G}}G \to E_{\mathfrak{F}}G$. Now the double mapping cylinder yields a model for $E_{\mathfrak{F}\cup\mathfrak{G}}G$ of the desired dimension. $\square$

Lemma 2.5. *Given $r < n$, suppose there exists a $d \geq n$ such that $\mathrm{gd}_{\mathfrak{F}_{r-1}} G \leq d$ and that for all maximal subgroups N with $r_0(N) > r-1$ we also have $\mathrm{gd}_{\mathfrak{F}_{r-1}\cap\mathfrak{All}(N)} G \leq d$. Then*

$$\mathrm{gd}_{\mathfrak{F}_r} G \leq d+1 \quad \textit{and} \quad \mathrm{gd}_{\mathfrak{F}_r\cap\mathfrak{All}(M)} G \leq d+1$$

for all maximal subgroups M of $r_0(M) > r$.

Proof. We begin by applying Theorem 1.2 to the families $\mathfrak{G} = \mathfrak{F}_r$ and $\mathfrak{F} = \mathfrak{F}_{r-1}$. Lemma 2.2 implies that G satisfies $(M_{\mathfrak{F}_{r-1}\subseteq\mathfrak{F}_r})$. Denote by $\mathcal{N}$ the set of equivalence classes of maximal elements in $\mathfrak{F}_r\backslash\mathfrak{F}_{r-1}$. Then [5, Corollary 2.8] gives a push-out:

$$\begin{tikzcd}
\sqcup_{N\in\mathcal{N}} E_{\mathfrak{F}_{r-1}}(G) \arrow[r] \arrow[d] & E_{\mathfrak{F}_{r-1}}(G) \arrow[d] \\
\sqcup_{N\in\mathcal{N}} E_{\mathfrak{F}_{r-1}\cup\mathfrak{All}(N)}(G) \arrow[r] & Y,
\end{tikzcd}$$

and Y is a model for $E_{\mathfrak{F}_r}G$.
By assumption we have that $\mathrm{gd}_{\mathfrak{F}_{r-1}} G \leq d$ and $\mathrm{gd}_{\mathfrak{F}_{r-1}\cap\mathfrak{All}(N)} G \leq d$ for all $N \in \mathcal{N}$. Furthermore, by Lemma 2.3 we have that $\mathrm{gd}_{\mathfrak{All}(N)} G = n - r_0(N) < n$. Lemma 2.4 now implies that $\mathrm{gd}_{\mathfrak{F}_{r-1}\cup\mathfrak{All}(N)} G \leq d+1$. Applying Corollary 1.3 to the above push-out yields

$$\mathrm{gd}_{\mathfrak{F}_r} G \leq d+1.$$

The second claim is proved similarly applying Theorem 1.2 to the families $\mathfrak{G} = \mathfrak{F}_r \cap \mathfrak{All}(M)$ and $\mathfrak{F} = \mathfrak{F}_{r-1} \cap \mathfrak{All}(M)$. The argument of Lemma 2.2 applies here as well and hence G satisfies $(M_{(\mathfrak{F}_{r-1}\cap\mathfrak{All}(M))\subseteq(\mathfrak{F}_r\cap\mathfrak{All}(M))})$. We denote by $\mathcal{N}(M)$ the set of equivalence classes of maximal elements in $\mathfrak{F}_r \cap \mathfrak{All}(M)\backslash\mathfrak{F}_{r-1} \cap \mathfrak{All}(M)$. this now gives us a push-out:

$$\begin{tikzcd}
\sqcup_{N\in\mathcal{N}(M)} E_{\mathfrak{F}_{r-1}\cap\mathfrak{All}(M)}(G) \arrow[r] \arrow[d] & E_{\mathfrak{F}_{r-1}\cap\mathfrak{All}(M)}(G) \arrow[d] \\
\sqcup_{N\in\mathcal{N}(M)} E_{(\mathfrak{F}_{r-1}\cap\mathfrak{All}(M))\cup\mathfrak{All}(N)}(G) \arrow[r] & Z,
\end{tikzcd}$$

and Z is a model for $E_{\mathfrak{F}_r \cap \mathfrak{All}(M)}G$.
Since $N \leq M$, it follows that $(\mathfrak{F}_{r-1} \cap \mathfrak{All}(M)) \cap \mathfrak{All}(N) = \mathfrak{F}_{r-1} \cap \mathfrak{All}(N)$ and hence, by assumption $\mathrm{gd}_{(\mathfrak{F}_{r-1} \cap \mathfrak{All}(M)) \cap \mathfrak{All}(N)} G \leq d$ and Lemma 2.4 implies that $\mathrm{gd}_{(\mathfrak{F}_{r-1} \cap \mathfrak{All}(M)) \cup \mathfrak{All}(N)} G \leq d+1$. Now the same argument as above applies and

$$\mathrm{gd}_{\mathfrak{F}_r \cap \mathfrak{All}(M)} G \leq d+1.$$

□

Proof of Main Theorem: We begin by noting that for $r = 0$ we have that $\mathfrak{F}_r = \mathfrak{F}_0$ is the family of all finite subgroups of G. Then for all maximal subgroups M of rank 1, we have that $\mathfrak{F}_0 = \mathfrak{F}_0 \cap \mathfrak{All}(M)$. Furthermore, is is well known that $\mathrm{gd}_{\mathfrak{F}_0} G = n$, see for example [3].
Now an induction using Lemma 2.5 yields the claim.

□

Question 2.6. *Is the bound of our Main Theorem sharp, i.e. for $n > r$, is*

$$\mathrm{gd}_{\mathfrak{F}_r} G = n + r?$$

Since $\mathrm{gd}_{\mathfrak{F}_0} G = \mathrm{gd}_{\mathfrak{F}_0 \cap \mathfrak{All}(N)} G = n$ for all maximal subgroups N, we can assume equality in the inductive step (assumptions of Lemma 2.5). Then a successive application of the Mayer-Vietoris sequences to the push-outs in Lemmas 2.5 and 2.4, reduces the question to whether the map

$$H^d(G \backslash E_{\mathfrak{F}_{r-1}} G) \to H^d(G \backslash E_{\mathfrak{F}_{r-1} \cap \mathfrak{All}(N)} G)$$

is surjective or not.

We know by [5] that $\mathrm{gd}_{\mathfrak{F}_1} G = n+1$ and it was shown in [2] that the question has a positive answer for $G = \mathbb{Z}^3$, i.e. that $\mathrm{gd}_{\mathfrak{F}_2}(\mathbb{Z}^3) = 5$.

References

[1] Dieter Degrijse and Nansen Petrosyan, *Geometric dimension of groups for the family of virtually cyclic subgroups*, J. Topol. **7** (2014), no. 3, 697–726, DOI 10.1112/jtopol/jtt045.

[2] A. Lopes-Onorio, *Ph.D thesis, University of Southampton* (in preparation).

[3] W. Lück, *Survey on classifying spaces for families of subgroups*, Infinite groups: geometric, combinatorial and dynamical aspects, 2005, pp. 269–322. MR2195456 (2006m:55036)

[4] ———, *Algebraische Topologie. Homologie und Mannigfaltigkeiten*, Aufbaukurs Mathematik, Vieweg Studium, Wiesbaden, 2005.

[5] W. Lück and M. Weiermann, *On the classifying space of the family of virtually cyclic subgroups*, Pure App. Math. Q. **8** (2012), no. 2, 479–555.

[6] Daniel Segal, *Polycyclic groups*, Cambridge Tracts in Mathematics, vol. 82, Cambridge University Press, Cambridge, 1983.

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