# Finite generation of iterated wreath products in product action

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**Abstract.** Let $\mathcal{S}$ be a sequence of finite perfect transitive permutation groups with uniformly bounded number of generators. We prove that the infinitely iterated wreath product in product action of the groups in $\mathcal{S}$ is topologically finitely generated, provided that the actions of the groups in $\mathcal{S}$ are never regular. We also deduce that certain infinitely iterated wreath products obtained by a mixture of imprimitive and product actions of groups in $\mathcal{S}$ are finitely generated. Finally we apply our methods to find explicitly two generators of infinitely iterated wreath products in product action of certain sequences $\mathcal{S}$ of 2-generated perfect groups.

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**Keywords.** Wreath product, Product action, Profinite group, Generators.

**Notation 1.** All the actions will be right actions and $e$ will stand for the identity element of a group. We will write $\boldsymbol{n} = \{1, \dots, n\}$.

**Notation 2.** Let $d$ be an integer and let $\{m_k\}_{n\in\mathbb{N}}$ be a sequence of integers. Throughout this paper we will denote by $\mathcal{S}$ a sequence $\{S_k\}_{k\in\mathbb{N}}$ where $S_k$ is a transitive subgroup of $\mathrm{Sym}(m_k)$ that is perfect and at most $d$-generated as an abstract group.

## 1. Introduction

Infinitely iterated wreath products have been widely studied in the past (see [1, 2, 6]) and their generation properties proved to be of great interest. In particular, [2, Theorem 1] states that an infinitely iterated permutational wreath product of finite $d$-generated transitive permutation groups is finitely generated if and only if its abelianization is finitely generated. In this paper we prove two parallel results. We prove that an *infinitely iterated exponentiation* and an *infinitely iterated mixed wreath product of stride at most $m$* of finite $d$-generated perfect transitive permutation groups are topologically finitely generated under certain conditions.

**Definition.** The abstract wreath product $A \wr B$ of two permutation groups $A \leq \mathrm{Sym}(m)$ and $B \leq \mathrm{Sym}(n)$ can be considered as a permutation group in two (in general inequivalent) ways. One way is to consider $A \wr B$ acting on $\boldsymbol{m}^n$ with the product action: if $(a_1, \ldots, a_n)b \in A \wr B$ and $(x_1, \ldots, x_n) \in \boldsymbol{m}^n$ then the product action of $A \wr B$ is given by

$$(x_1, \ldots, x_n)^{(a_1, \ldots, a_n)} = (x_1^{a_1}, \ldots, x_n^{a_n}) \text{ and } (x_1, \ldots, x_n)^{b^{-1}} = (x_{1^b}, \ldots, x_{n^b}).$$

We will write $A \circledwr B \leq \mathrm{Sym}(m^n)$ for this permutation group and we will call it the **exponentiation** of $A$ by $B$. The second way is to consider $A \wr B$ acting on $\boldsymbol{m} \times \boldsymbol{n}$ with the permutational wreath action: if $(a_1, \ldots, a_n)b \in A \wr B$ and $(x, y) \in \boldsymbol{m} \times \boldsymbol{n}$ then the permutational wreath action of $A \wr B$ is given by

$$(x, y)^{(a_1, \ldots, a_n)b} = (x^{a_y}, y^b).$$

We will write $A \wr B \leq \mathrm{Sym}(mn)$ for this permutation group and we will call it the **permutational wreath product** of $A$ by $B$.

For other properties of the exponentiation and permutational wreath product we refer to [3] and references therein.

**Definition.** The **iterated exponentiation** $\widetilde{S}_n \leq \mathrm{Sym}(\widetilde{m}_n)$ of the groups in the sequence $\mathcal{S}$ is the permutation group inductively defined by: $\widetilde{m}_1 = m_1$, $\widetilde{S}_1 = S_1 \leq \mathrm{Sym}(\widetilde{m}_1)$ and $\widetilde{m}_k = m_k^{\widetilde{m}_{k-1}}$, $\widetilde{S}_k = S_k \circledwr \widetilde{S}_{k-1} \leq \mathrm{Sym}(\widetilde{m}_k)$ for $k \geq 2$. The groups $\widetilde{S}_k$, together with the projections $\widetilde{S}_k \to \widetilde{S}_{k-1}$, form an inverse system of finite groups. We will call the profinite group $\varprojlim \widetilde{S}_k$ the **infinitely iterated exponentiation** of the groups in $\mathcal{S}$.

In Section 2 we prove our main result.

**Theorem 1.** *Let $d$ be an integer. Let $\mathcal{S} = \{S_k\}_{k \in \mathbb{N}}$ be a sequence of transitive subgroups $S_k \leq \mathrm{Sym}(m_k)$ such that each $S_k$ is perfect and at most $d$-generated as an abstract group. Suppose that for every $k \in \mathbb{N}$ there exist elements $i, j \in \boldsymbol{m}_k$ such that $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Then the infinitely iterated exponentiation of the groups in $\mathcal{S}$ is topologically finitely generated.*

As customary, we denote by $\mathrm{d}(G)$ the minimal number of (topological) generators of a (topological) group $G$. The proof of Theorem 1 gives an explicit set of $d + \mathrm{d}(S_1)$ generators for $\varprojlim \widetilde{S}_k$ and this bound is asymptotically best possible (see Lemma 3). The groups under study here are very different from the ones in [2]. We cannot rely on the tree-like structure of iterated wreath products and the iterated exponentiation of permutation groups is not associative. This is the reason why we need to ask that the groups in the sequence $\mathcal{S}$ have non-regular actions. Using the same methods we can improve our bound for a sequence $\mathcal{S}$ of perfect 2-generated perfect groups (see Corollary 7).

In [6] it is proved that the infinitely iterated wreath product with arbitrary actions of a sequence $\mathcal{S}$ of finite non-abelian simple groups is two generated (in fact *positively* 2*-generated*). In particular, this implies that when $\mathcal{S}$

is a sequence of finite non-abelian simple groups the profinite groups considered in Theorem 1 are actually 2-generated. It is important to mention that [6] relies heavily on the classification of finite simple groups and does not provide an explicit set of generators, it is a probabilistic argument. On the other hand the proof of Theorem 1 is purely combinatorial and it produces an explicit set of generators for the groups considered.

In Section 3 we use [4, Theorem 3.1] to exchange the exponentiation and the permutational wreath product to obtain the finite generation of iterated wreath products with "mixed" action.

**Definition.** Let $\{k_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive integers. Define the sequence $\{G_n\}_{n\in\mathbb{N}}$ of perfect transitive subgroups of $\mathrm{Sym}(r_n)$ starting from the groups in $\mathcal{S}$ in the following way: $G_0 = \{e\}$ and for $k \geq 1$

$$G_k = \begin{cases} S_k \textcircled{\wr} G_{k-1} & \text{if } k \in \{k_1, k_2, \ldots\}, \\ S_k \wr G_{k-1} & \text{otherwise.} \end{cases}$$

The permutation groups $G_n$ are called **iterated mixed wreath product of type** $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$.

Let $m$ be an integer, if the sequence $\{k_n\}_{n\in\mathbb{N}}$ is such that $k_{n+1} - k_n \leq m$ for every $n \in \mathbb{N}$, we say that the iterated mixed wreath product $G_n$ of type $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$ has **stride at most** $m$.

The groups $G_n$, together with the projections $G_n \to G_{n-1}$, form an inverse system of finite groups. We say that the profinite group $\varprojlim G_n$ is an **infinitely iterated mixed wreath product of type** $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$. If the groups $G_n$ have stride at most $m$ we say that $\varprojlim G_n$ has stride at most $m$.

We remark that an infinitely iterated exponentiation is an infinitely iterated mixed wreath product of stride at most one.

Our second main result is the following.

**Theorem 2.** *Let $d$ be an integer. Let $\mathcal{S} = \{S_k\}_{k\in\mathbb{N}}$ be a sequence of transitive subgroups $S_k \leq \mathrm{Sym}(m_k)$ such that each $S_k$ is perfect and at most $d$-generated as an abstract group. Suppose that for every $k \in \mathbb{N}$ there exist elements $i, j \in \boldsymbol{m}_k$ such that $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Let $G = \varprojlim G_n$ be an infinitely iterated mixed wreath product of type $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$ of stride at most $m$. Then $G$ is topologically finitely generated.*

The hypotheses of Theorem 2 can be weakened in two ways (see Remark 1). We conclude with Section 4 where we use the techniques of this paper to find the minimal number of generators of infinitely iterated exponentiations of particular sequences $\mathcal{S}$ (see Corollary 12).

## 2. Proof of Theorem 1

First we find a lower bound for the minimal number of generators of a wreath product of perfect non-simple groups. This shows that the bound given by Theorem 1 can be improved only by a multiplicative and an additive constant. We will denote by ${}^{\mathrm{t}}x$ the transpose of a vector $x \in \mathbb{Z}^n$.

**Lemma 3.** *Let $N$ be a natural number. Let $A$ be a finite simple group and let $B \leq \mathrm{Sym}(n)$ be a finite permutation group. Then*

$$\mathrm{d}(A^N \wr B) \geq \max\left\{\frac{1}{n}\left(\mathrm{d}\left(A^N\right) - \mathrm{d}(A) - 1\right), \mathrm{d}(B)\right\}.$$

*Proof.* Set $G = A^N \wr B = (A^N)^n \rtimes B$ and $\mathrm{d}(G) = d$. It is clear that $d \geq \mathrm{d}(B)$, since $B$ is a quotient of $G$. Let

$$\gamma_j = ((x_{11}^{(j)}, \dots, x_{1N}^{(j)}), \cdots, (x_{n1}^{(j)}, \dots, x_{nN}^{(j)}))\sigma_j = (\gamma_1^{(j)}, \cdots, \gamma_n^{(j)})\sigma_j \in A^N \wr B,$$

for $j = 1, \dots, d$, be generators for $G$. Form the $N \times nd$ matrix $M$ with entries $M_{l,n(i-1)+j} = x_{jl}^{(i)}$ for $i = 1, \dots, d$, $j = 1, \dots, n$ and $l = 1, \dots, N$. For every number $m \in \{1, \dots, nd\}$ there exist unique $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, n\}$ such that $m = n(i-1)+j$ and the $(n(i-1)+j)$-th column of $M$ is the vector ${}^{\mathrm{t}}\gamma_j^{(i)}$:

$$M = \left({}^{\mathrm{t}}\gamma_1^{(1)}, \dots, {}^{\mathrm{t}}\gamma_n^{(1)}, \dots\dots, {}^{\mathrm{t}}\gamma_1^{(d)}, \dots, {}^{\mathrm{t}}\gamma_n^{(d)}\right).$$

Our goal is to show that $N \leq |A|^{nd}$. Suppose by contradiction that $N > |A|^{nd}$. Then, since the $x_{jl}^{(i)}$ are elements of $A$, we would have that two rows of $M$ are equal. Without loss of generality we can suppose that the first and the second rows are equal, in particular it follows that $x_{i1}^{(j)} = x_{i2}^{(j)}$ for every $i = 1, \dots, n$ and for every $j = 1, \dots, d$. Since the action of $B$ swaps rigidly the $n$ $N$-tuples of $(A^N)^n$, any element $((y_{11}, \dots, y_{1N}), \cdots, (y_{n1}, \dots, y_{nN}))\tau$ of the subgroup generated by the $\gamma_j$'s satisfies $y_{11} = y_{12}$. This is a contradiction with our assumption that the $\gamma_j$'s generate $G$.

Therefore $N \leq |A|^{nd}$ and applying logarithms on both sides of the inequality we have

$$d \geq \frac{1}{n \log|A|} \log N = \frac{1}{n} \log_{|A|} N > \frac{1}{n}\left(\mathrm{d}\left(A^N\right) - \mathrm{d}(A) - 1\right),$$

where the last inequality holds by [8, Lemma 2]. □

Before proving Theorem 1 we fix some notation.

**Notation 3.** We will denote an $\widetilde{m}_k$-tuple on elements in $\{1, \dots, m_{k+1}\}$ as $(i_1, \dots, i_{\widetilde{m}_k})_{\widetilde{m}_k}$. This notation will be convenient in particular when we will have to deal with $\widetilde{m}_k$-tuples where all the coordinates are equal, for example $(1, \dots, 1)_{\widetilde{m}_k}$. We will denote an element of the group $S_{k+1}^{\widetilde{m}_k}$ as $(\sigma_1, \dots, \sigma_{\widetilde{m}_k})_{\widetilde{m}_k}$.

**Definition.** For $(i_1, \dots, i_n), (j_1, \dots, j_n) \in \boldsymbol{m}^n$ we say that $(i_1, \dots, i_n)$ **precedes** $(j_1, \dots, j_n)$, if and only if there exists $1 \leq l \leq n$ such that $i_k = j_k$ for $1 \leq k \leq l-1$ and $i_l < j_l$. The relation "precedes" defines a total order on $\{1, \dots, m\}^n$ that is called the **lexicographic order**.

The following straightforward lemma is one of the key tricks to prove Theorem 1.

**Lemma 4.** *Let $G \leq \mathrm{Sym}(m)$ and $H \leq \mathrm{Sym}(n)$ be permutation groups. Then the subgroup $H$ of the exponentiation $G \circledcirc H$ acts trivially on the subset*

$$\{(i, \dots, i) \mid i \in \boldsymbol{m}\}.$$

Theorem 1 will now follow from an application of the next lemma.

**Lemma 5.** *Let $\mathcal{S} = \{S_k\}_{k \in \boldsymbol{n}}$ be a sequence of transitive subgroups $S_k \leq \mathrm{Sym}(m_k)$ and let $d$ be an integer. Suppose that each $S_k$ is perfect, at most $d$-generated and for every $k \in \boldsymbol{n}$ there exist elements $i, j \in \boldsymbol{m}_k$ such that $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Then the iterated exponentiation $\widetilde{S}_n$ of the groups $S_k$ satisfies $\mathrm{d}(\widetilde{S}_n) \leq d + \mathrm{d}(S_1)$.*

*Proof.* Let $S_1 = \langle \alpha_1(1), \dots, \alpha_{\mathrm{d}(S_1)}(1) \rangle$, $S_k = \langle \alpha_1(k), \dots, \alpha_d(k) \rangle$, for $k = 2, \dots, n$ and order the elements of $\{1, \dots, m_{k+1}\}^{\widetilde{m}_k}$ with respect to the lexicographic order. Without loss of generality we can suppose that for every $k \in \boldsymbol{n}$ we have

$$\mathrm{St}_{S_k}(1) \neq \mathrm{St}_{S_k}(2). \tag{1}$$

We will now define $d$ elements of $\widetilde{S}_n$ that together with the generators of $S_1$ will generate $\widetilde{S}_n$. Define the elements $\beta_1, \dots, \beta_d \in \widetilde{S}_n$ as

$$\beta_j = (\alpha_j(n), e, \dots, e)_{\widetilde{m}_{n-1}} \cdot (\alpha_j(n-1), e, \dots, e)_{\widetilde{m}_{n-2}} \cdots (\alpha_j(2), e, \dots, e)_{\widetilde{m}_1}$$

for $j = 1, \dots, d$. Note that the $\alpha_j(k)$'s are in the first place of the $\widetilde{m}_{k-1}$-tuples, which corresponds to the element $(1, \dots, 1)_{\widetilde{m}_{k-1}} \in \{1, \dots, m_k\}^{\widetilde{m}_{k-1}}$.

Let $A = \langle \alpha_1(1), \dots, \alpha_{\mathrm{d}(S_1)}(1), \beta_1, \dots, \beta_d \rangle \leq \widetilde{S}_n$. We claim that $A = \widetilde{S}_n$. We will prove by induction on $k$ that $\widetilde{S}_k \leq A$ for $k = 1, \dots, n$. Trivially $\widetilde{S}_1 = S_1 \leq A$. Supposing by the inductive hypothesis that $\widetilde{S}_k \leq A$, we have to show that we can write any element of $\widetilde{S}_{k+1}$ as a product of the generators in $A$. Because $\widetilde{S}_{k+1} = S_{k+1}^{\widetilde{m}_k} \cdot \widetilde{S}_k$, it will suffice to show that $S_{k+1}^{\widetilde{m}_k} \leq A$.

By the transitivity of $S_k$ there is an element $t \in S_k$ such that $1^t = 2$ and by the inductive hypothesis the element $\sigma = (e, \dots, e, t)_{\widetilde{m}_{k-1}} \in S_k^{\widetilde{m}_{k-1}}$ belongs to $A$. By Lemma 4 it follows that for $j = k, \dots, n$

$$(1, \dots, 1)_{\widetilde{m}_j}^{\sigma} = (1, \dots, 1)_{\widetilde{m}_j} \tag{2}$$

and from the definition of lexicographic order and exponentiation

$$(1, \dots, 1)_{\widetilde{m}_{k-1}}^{\sigma} = (1^e, \dots, 1^e, 1^t)_{\widetilde{m}_{k-1}} = (1, \dots, 1, 2)_{\widetilde{m}_{k-1}}. \tag{3}$$

We remind the reader that the element $(1, \dots, 1, 2)_{\widetilde{m}_{k-1}}$ is the second element in the set $\{1, \dots, m_k\}^{\widetilde{m}_{k-1}}$ with respect to the lexicographic order. Moreover, since $\widetilde{S}_k \leq A$, $\beta'_j = (\alpha_j(n), e, \dots, e)_{\widetilde{m}_{n-1}} \cdots (\alpha_j(k+1), e, \dots, e)_{\widetilde{m}_k}$ belongs to $A$. Set $\gamma_j = [\sigma, \beta'_j]$, then $\gamma_j \in A$. By (2), $(\alpha_j(l), e, \dots, e)_{\widetilde{m}_{l-1}}^{\sigma} = (\alpha_j(l), e, \dots, e)_{\widetilde{m}_{l-1}}$ for $l = k+2, \dots, n$ and, by (3), $(\alpha_j(k+1), e, \dots, e)_{\widetilde{m}_k}^{\sigma} = (e, \alpha_j(k+1), e, \dots, e)_{\widetilde{m}_k}$. Therefore $(\beta'_j)^{\sigma} = (\alpha_j(n), e, \dots, e)_{\widetilde{m}_{n-1}} \cdots (e, \alpha_j(k+1), e, \dots, e)_{\widetilde{m}_k}$ with $\alpha_j(k+1)$ in second position in the last $\widetilde{m}_k$-tuple and so $\gamma_j = ((\beta'_j)^{\sigma})^{-1} \beta'_j = (\alpha_j(k+1), \alpha_j(k+1)^{-1}, e, \dots, e)_{\widetilde{m}_k}$.

By inductive hypothesis $\widetilde{S}_k \leq A$, therefore $A$ is transitive on $\boldsymbol{m}_k^{\widetilde{m}_{k-1}}$. To conclude the proof it is sufficient to show that we can write any element of the form $(\lambda, e, \dots, e)_{\widetilde{m}_k}$ in $S_{k+1}^{\widetilde{m}_k}$ as a word in the $\gamma_j$'s. We can then move the entry $\lambda$ around, using the transitive action of $\widetilde{S}_k$.

As $S_{k+1}$ is perfect it is sufficient to prove that we can write any commutator $([\lambda_1, \lambda_2], e, \dots, e)_{\widetilde{m}_k}$ as a word in the $\gamma_j$'s. By (1) there are $s \in S_k$ and $r \in \boldsymbol{m}_k$, $r \neq 2$, such that $1^s = 1$ and $2^s = r$. By the inductive hypothesis $\mu = (e, \dots, e, s)_{\widetilde{m}_{k-1}}$ belongs to $A$. Let $\lambda_1, \lambda_2 \in S_{k+1}$. Since the $\alpha_j(k+1)$'s generate $S_{k+1}$, there exist two $d$-variables words $w_1$ and $w_2$ such that $\lambda_1 = w_1(\alpha_1(k+1), \dots, \alpha_d(k+1))$ and $\lambda_2 = w_2(\alpha_1(k+1), \dots, \alpha_d(k+1))$. Thus, if we set $\delta_i = w_i(\alpha_1(k+1)^{-1}, \dots, \alpha_d(k+1)^{-1})$ for $i = 1, 2$, the elements $w_1(\gamma_1, \dots, \gamma_d) = (\lambda_1, \delta_1, e, \dots, e)_{\widetilde{m}_k}$ and $w_2(\gamma_1, \dots, \gamma_d) = (\lambda_2, \delta_2, e, \dots, e)_{\widetilde{m}_k}$ belong to $A$. The definition of $\mu$ and an easy calculation now yield

$$\left[(\lambda_1, \delta_1, e, \dots, e)_{\widetilde{m}_k}, (\lambda_2, \delta_2, e, \dots, e)^{\mu}_{\widetilde{m}_k}\right] = ([\lambda_1, \lambda_2], e, \dots, e)_{\widetilde{m}_k}.$$

Thus for every $\lambda \in S_{k+1}$ the $\widetilde{m}_k$-tuple $(\lambda, e, \dots, e)_{\widetilde{m}_k}$ is in $A$. It follows that $S_{k+1}^{\widetilde{m}_k} \leq A$ and $\widetilde{S}_{k+1} = S_{k+1}^{\widetilde{m}_k} \cdot \widetilde{S}_k \leq A$. The result follows by induction. □

We would like to point out that in the previous proof we exhibited an explicit set of $d + \mathrm{d}(S_1)$ generators for $\widetilde{S}_n$. We are now ready for the proof of Theorem 1.

*Proof of Theorem* 1. For every $n \in \mathbb{N}$, Lemma 5 gives us $d + \mathrm{d}(S_1)$ generators of $\widetilde{S}_n$, of the form described at the beginning of the proof of Lemma 5, $\alpha_1(1), \dots, \alpha_{\mathrm{d}(S_1)}(1), \beta_1^{(n)}, \dots, \beta_d^{(n)}$. For $n \in \mathbb{N}$, let $\pi_n$ be the inverse limit projection from $\varprojlim \widetilde{S}_k$ to $\widetilde{S}_n$. Let $a_1(1), \dots, a_{\mathrm{d}(S_1)}(1), b_1, \dots, b_d$ be the unique elements of $\varprojlim \widetilde{S}_k$ such that $\pi_n(a_i(1)) = \alpha_i(1)$ and $\pi_n(b_j) = \beta_j^{(n)}$ for all $i \in \mathbf{d(S_1)}$, $j \in \boldsymbol{d}$, $n \in \mathbb{N}$. Then $a_1(1), \dots, a_{\mathrm{d}(S_1)}(1), b_1, \dots, b_d$ generate $\varprojlim \widetilde{S}_k$ by [9, Proposition 4.1.1]. □

Using [7, Lemma 2] it is possible to improve the previous bound for 2-generated groups with the same method.

**Lemma 6.** *Let $\mathcal{S} = \{S_k\}_{k \in \boldsymbol{n}}$ be a sequence of perfect 2-generated transitive subgroups $S_k \leq \mathrm{Sym}(m_k)$ such that for every $k \in \boldsymbol{n}$ and all $i, j \in \boldsymbol{m}_k$ $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Then $\widetilde{S}_n$ is generated by the generators of $S_1$ together with another suitable element.*

*Proof.* Let $S_k = \langle \alpha_1(k), \alpha_2(k) \rangle$. By [7, Lemma 2], for $k \in \boldsymbol{n}$, there exist $\sigma_k \in S_k$ and $1 \leq r_k \leq m_k$ such that $r_k^{\sigma_k^2} \neq r_k$. Let

$$\beta = (\dots, \alpha_1(n), \dots, \alpha_2(n), \dots)_{\widetilde{m}_{n-1}} \cdot \dots \cdot (\dots, \alpha_1(2), \dots, \alpha_2(2), \dots)_{\widetilde{m}_1}$$

where the element $\alpha_1(2)$ is in position $r_1^{\sigma_1}$, $\alpha_2(2)$ is in position $r_1$, $\alpha_1(k+1)$ is in position $(r_k^{\sigma_k}, \dots, r_k^{\sigma_k})_{\widetilde{m}_{k-1}}$, $\alpha_2(k+1)$ is in position $(r_k, \dots, r_k)_{\widetilde{m}_{k-1}}$ for $k = 2, \dots, n-1$ and the identity in all the unspecified positions. Set

$A = \langle \alpha_1(1), \alpha_2(1), \beta \rangle$ and proceed exactly as in the proof of Lemma 5, with $\beta$ instead of $\beta_i$ and $(\sigma_k, \dots, \sigma_k)_{\widetilde{m}_{k-1}}$ instead of $\sigma$, to show that $A = \widetilde{S}_n$. $\square$

Using Lemma 6 in place of Lemma 5 in the proof of Theorem 1 yields the following corollary.

**Corollary 7.** *Let $\mathcal{S} = \{S_k\}_{k \in \mathbb{N}}$ be a sequence of perfect 2-generated transitive subgroups $S_k \leq \mathrm{Sym}(m_k)$. Suppose that for every $k \in \mathbb{N}$ and all $i, j \in \boldsymbol{m}_k$ we have $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Then the infinitely iterated exponentiation $\varprojlim \widetilde{S}_n$ of the groups $S_k$ satisfies $\mathrm{d}\left(\varprojlim \widetilde{S}_n\right) \leq 3$.*

Again we would like to point out that in Corollary 7 we can find an explicit set of three generators for $\varprojlim \widetilde{S}_n$.

As a consequence of [6], the minimal number of generators of the infinitely iterated exponentiation of a sequence $\mathcal{S}$ of finite non-abelian simple transitive permutation groups is two. However, perfect groups can be "far" from simple and we conjecture that in the case of perfect non-simple groups Lemma 6 is best possible but we do not have an explicit example to confirm this.

## 3. Proof of Theorem 2

We now proceed to the proof of Theorem 2. We will use the following.

**Theorem.** ([4, Theorem 3.1]) *Let $n_1, n_2$ and $n_3$ be integers and let $A \leq \mathrm{Sym}(n_1), B \leq \mathrm{Sym}(n_2)$ and $C \leq \mathrm{Sym}(n_3)$ be permutation groups. Then $A \textcircled{\wr} (B \wr C)$ and $(A \textcircled{\wr} B) \textcircled{\wr} C$ are isomorphic as permutation groups.*

The next lemma is an application of [4, Theorem 3.1] and it will be used in the proof of Theorem 2. Remember that we denote by $\widetilde{H}_n$ the iterated exponentiation of the sequence of permutation groups $\{H_k\}_{k \in \boldsymbol{n}}$.

**Lemma 8.** *Let $\mathcal{S} = \{S_k\}_{k \in \mathbb{N}}$ be a sequence of subgroups $S_k \leq \mathrm{Sym}(m_k)$. Let $\{k_n\}_{n \in \mathbb{N}}$ be an increasing sequence of integers and let $G_n$ be an iterated mixed wreath product of type $(\mathcal{S}, \{k_n\}_{n \in \mathbb{N}})$. Set $k_0 = 0$ and define the permutation groups $\widehat{S}_{k_n}^{(i)}$ for $n \in \mathbb{N}$ and $i \in \boldsymbol{k}_n \smallsetminus \boldsymbol{k}_{n-1}$ as follows: $\widehat{S}_{k_n}^{(k_n)} = S_{k_n}$ and $\widehat{S}_{k_n}^{(i)} = \widehat{S}_{k_n}^{(i+1)} \textcircled{\wr} S_i$. Define $H_n = \widehat{S}_{k_n}^{(k_{n-1}+1)}$ for $n \in \mathbb{N}$. Then $G_{k_n}$ is isomorphic to $\widetilde{H}_n$ as a permutation group, for every $n \in \mathbb{N}$.*

*Proof.* The proof is by induction on $n$. If $n = 1$ and $k_1 = 1$ the claim is trivial. If $k_1 > 1$ repeated applications of [4, Theorem 3.1] yield $G_{k_1} \cong H_1$.

Suppose that $G_{k_{n-1}} \cong \widetilde{H}_{n-1}$. By construction $G_{k_n} \cong S_{k_n} \textcircled{\wr} G_{k_n - 1}$ and $G_i \cong S_i \wr G_{i-1}$ for $i \in \boldsymbol{k}_n \smallsetminus \boldsymbol{k}_{n-1}$. Therefore repeated applications of [4, Theorem 3.1] yield $G_{k_n} \cong (\widehat{S}_{k_n}^{(i+1)} \textcircled{\wr} S_i) \textcircled{\wr} G_{i-1}$ for $i \in \boldsymbol{k}_n \smallsetminus \boldsymbol{k}_{n-1}$. Thus $G_{k_n} \cong \widehat{S}_{k_n}^{(k_{n-1}+1)} \textcircled{\wr} G_{k_{n-1}}$ and, by the inductive hypothesis, we conclude $G_{k_n} \cong H_n \textcircled{\wr} \widetilde{H}_{n-1} \cong \widetilde{H}_n$. The claim follows by induction. $\square$

**Lemma 9.** *Let $A \leq \mathrm{Sym}(m)$ and $B \leq \mathrm{Sym}(n)$ be permutation groups and set $G = A \circledR B$. Suppose that $m, n \geq 2$ and $B$ is transitive. Then there exist $x, y \in \boldsymbol{m}^n$ such that $\mathrm{St}_G(x) \neq \mathrm{St}_G(y)$.*

*Proof.* Consider the elements $x = (1, \dots, 1)_n$ and $y = (2, 1, \dots, 1)_n$ in $\boldsymbol{m}^n$. Because $B$ is transitive there exists $b \in B$ such that $1^b = 2$, so $x^b = x$ and $y^b = (1, 2, 1, \dots, 1)_n \neq y$. So $b$ is in the stabiliser of $x$ but not in the stabiliser of $y$. □

The following lemma follows directly from the definition of the exponentiation of permutation groups.

**Lemma 10.** *Let $A \leq \mathrm{Sym}(m)$ and $B \leq \mathrm{Sym}(n)$ be permutation groups and suppose that $A$ is transitive. Then $A \circledR B$ is transitive.*

Finally we use Lemma 5, Lemma 8, Lemma 9 and Lemma 10 to prove Theorem 2.

*Proof of Theorem 2.* Let $G = \varprojlim G_n$ be an infinitely iterated mixed wreath product of type $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$ and of stride at most $m$. We will use the same setup and notation as in Lemma 8. We have that $G_{k_n}$ is isomorphic to $\widetilde{H}_n$ for every $n \in \mathbb{N}$, hence it is sufficient to show that the sequence $\{H_n\}_{n\in\mathbb{N}}$ satisfies the hypotheses of Lemma 5. It is clear that every $H_n$ is perfect and it can be generated by $md$ elements because it is an iterated wreath product of length at most $m$ made of $d$-generated groups. Since each $S_k$ is transitive, the permutation group $H_n = \widehat{S}_{k_n}^{(k_{n-1}+2)} \circledR S_{k_{n-1}+1}$ is transitive by iterated applications of Lemma 10. Moreover, by Lemma 9, $H_n$ satisfies the hypothesis on the stabilisers in Lemma 5. The proof is completed by applying Lemma 5 and [9, Proposition 4.1.1]. □

If the "inverse" iterated exponentiations $\widehat{S}_n$ in Lemma 8 had uniformly bounded number of generators, it would be possible to prove that infinitely iterated mixed wreath products of arbitrarily large stride are topologically finitely generated. We do not know if this is the case.

*Remark* 1. We can weaken the hypothesis of Theorem 2 in the following ways. Let $\{k_n\}_{n\in\mathbb{N}}$ be an increasing sequence of integers and $\mathcal{S}$ a sequence of finite perfect, at most $d$-generated, transitive permutation groups such that:

- for every $k_n$ satisfying $k_n = k_{n-1} + 1$ there exist elements $i, j \in \boldsymbol{m}_{k_n}$ that have different stabilisers for the action of $S_{k_n}$.
- $m \geq 2$.

The proof of Theorem 2 with these hypotheses remains the same.

## 4. An application

In this section we find explicitly two generators for the infinitely iterated exponentiation of particular sequences $\mathcal{S}$. We start with a lemma.

**Lemma 11.** *Let $\mathcal{S} = \{S_k\}_{k\in\boldsymbol{n}}$, be a sequence of 2-generated perfect transitive subgroups $S_k \le \mathrm{Sym}(m_k)$. Suppose that for every $k \in \boldsymbol{n}$ there exist two generators $a_k, b_k$ of $S_k$ such that:*

- $\mathrm{fix}(a_k)$ *and* $\mathrm{fix}(b_k)$ *are non-empty,*
- $(|a_1|, |b_j|) = 1$ *and* $(|b_1|, |a_j|) = 1$ *for* $j = 2, \dots, n$.

*Then* $\mathrm{d}(\widetilde{S}_n) = 2$.

*Proof.* Let $u_k \in \mathrm{fix}(a_k)$, $v_k \in \mathrm{fix}(b_k)$. In the spirit of Lemma 5 define the following elements of $\boldsymbol{m}_i^{\widetilde{m}_{i-1}}$

$$\underline{u}_i = (u_i, \dots, u_i)_{\widetilde{m}_{i-1}} \quad \text{and} \quad \underline{v}_i = (v_i, \dots, v_i)_{\widetilde{m}_{i-1}}$$

for $i = 2, \dots, n-1$. By the transitivity of $S_k$ there is $\sigma \in S_k$ such that $u_k^\sigma = v_k$ and, by Lemma 4, $\mu = (\sigma, \dots, \sigma)_{\widetilde{m}_k}$ is such that

$$\underline{u}_j^\mu = \underline{u}_j \quad \text{and} \quad \underline{v}_j^{\mu^{-1}} = \underline{v}_j \tag{4}$$

for every $j \ge k+1$ and by definition of exponentiation we have

$$\underline{u}_k^\mu = (u_k^\sigma, \dots, u_k^\sigma)_{\widetilde{m}_k} = \underline{v}_k. \tag{5}$$

For the rest of the proof we will write the position of an element in a tuple below the element itself. We claim that the elements

$$\beta_1 = (e, \dots, e, \underset{\underline{u}_{n-1}}{a_n}, e \dots, e)_{\widetilde{m}_{n-1}} \cdots (e, \dots, e, \underset{\underline{u}_2}{a_3}, e \dots, e)_{\widetilde{m}_2} \cdot \\ \cdot (e, \dots, e, \underset{v_1}{a_2}, e \dots, e)_{\widetilde{m}_1} b_1$$

and

$$\beta_2 = (e, \dots, e, \underset{\underline{v}_{n-1}}{b_n}, e \dots, e)_{\widetilde{m}_{n-1}} \cdots (e, \dots, e, \underset{\underline{v}_2}{b_3}, e \dots, e)_{\widetilde{m}_2} \cdot \\ \cdot (e, \dots, e, \underset{u_1}{b_2}, e \dots, e)_{\widetilde{m}_1} a_1$$

generate the group $\widetilde{S}_n$. Let $A = \langle \beta_1, \beta_2 \rangle$, we will prove by induction that $\widetilde{S}_k \le A$ for $k = 1, \dots, n$. It follows from Lemma 4 and the definition of $u_i$ and $v_i$ that $(e, \dots, e, a_i, e \dots, e)_{\widetilde{m}_i}$ commutes with $(e, \dots, e, a_j, e \dots, e)_{\widetilde{m}_j}$ for $i \ne j$. Set $p = \prod_{i=2}^n |a_i|$ and $q = \prod_{i=2}^n |b_i|$, then $\beta_1^p = b_1^p$ and $\beta_2^q = a_1^q$, so $S_1 \le A$.

By inductive hypothesis the group $\widetilde{S}_k$ is contained in $A$. Our goal is to write any element of $S_{k+1}^{\widetilde{m}_k}$ as a word in $\beta_1$, $\beta_2$. Clearly the elements

$$\beta_1' = (e, \dots, e, \underset{\underline{u}_{n-1}}{a_n}, e \dots, e)_{\widetilde{m}_{n-1}} \cdots (e, \dots, e, \underset{\underline{u}_k}{a_{k+1}}, e \dots, e)_{\widetilde{m}_k}$$

and

$$\beta_2' = (e, \dots, e, \underset{\underline{v}_{n-1}}{b_n}, e \dots, e)_{\widetilde{m}_{n-1}} \cdots (e, \dots, e, \underset{\underline{v}_k}{b_{k+1}}, e \dots, e)_{\widetilde{m}_k}$$

belong to $A$.

Let us now consider the commutators $\gamma_i = [\mu_i, \beta_i']$ for $i = 1, 2$. Following exactly the steps of Lemma 5 we can use (4), (5) (instead of (2) and (3)) and

Lemma 4 to show $S_{k+1}^{\widetilde{m}_k} \leq A$. Therefore $\widetilde{S}_{k+1} = S_{k+1}^{\widetilde{m}_k} \cdot \widetilde{S}_k \leq A$. The result follows by induction. □

Since none of the groups $\widetilde{S}_k$ is cyclic, the proof that $\varprojlim \widetilde{S}_k$ is topologically 2-generated is now the same as the proof of Theorem 1 using Lemma 11 instead of Lemma 5. As in Lemma 5, in the previous lemma we exhibited an explicit set of two generators. We have proved the following.

**Corollary 12.** *Let $\mathcal{S} = \{S_k\}_{k\in\mathbb{N}}$ be a sequence of 2-generated perfect transitive subgroups $S_k \leq \mathrm{Sym}(m_k)$. Suppose that for every $k \in \mathbb{N}$ there exist two generators $a_k, b_k$ of $S_k$ such that:*

- $\mathrm{fix}(a_k)$ *and* $\mathrm{fix}(b_k)$ *are non-empty,*
- $(|a_1|, |b_j|) = 1$ *and* $(|b_1|, |a_j|) = 1$ *for* $j \geq 2$.

*Then the infinitely iterated exponentiation $\varprojlim \widetilde{S}_k$ is topologically 2-generated and we produce explicitly two generators for the group.*

*Remark* 2. Using the Classification of Finite Simple Groups it was shown that all finite non-abelian simple groups besides $Sp_4(2^n)$, $Sp_4(3^n)$, ${}^2B_2(2^{2n+1})$ with possibly finitely many further exceptions can be generated by an involution and an element of order 3; see [5, Corollary 1.3] for further references on $(2,3)$-generation of finite non-abelian simple groups. A sequence of $(2,3)$-generated non-abelian simple groups satisfies the hypotheses of Lemma 11.

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