RATIONAL REPRESENTATION ZETA FUNCTIONS OF CRYSTALLOGRAPHIC GROUPS

By

Liaqat Ali

Royal Holloway and Bedford New College,
University of London

Thesis submitted to

Royal Holloway, University of London

for the degree of

Doctor of Philosophy.

2015

This thesis is dedicated to the bright memory of my father Abdul Majeed.

Declaration

I, Liaqat Ali, hereby declare that this thesis and the work presented in it is entirely my own. Where I have consulted the work of others, this is always clearly stated.

Signed . . . .

(Liaqat Ali)

Abstract

This thesis is concerned with the rational representation zeta functions of finitely generated abelian groups, frieze groups and crystallographic groups of dimensions one and two. Let G be a group. Then the rational representation zeta function of G is the formal Dirichlet series

$$\zeta_G^{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} r_n^{\mathbb{Q}}(G) n^{-s},$$

where $r_n^{\mathbb{Q}}(G)$ denotes the number of (equivalence classes of) n-dimensional irreducible representations of G over $\mathbb{Q}$ with finite image and where s is a complex variable. Here we assume that the group G has the property that $r_n^{\mathbb{Q}}(G)$ is finite for every n. A first step toward computing the rational representation zeta function consists in parameterising the irreducible complex characters of G.

In the case of finitely generated abelian groups, we proceed as follows. Let G be a finitely generated abelian group. Then there exists a one-to-one correspondence between irreducible rational representations of G with finite image and the Galois orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Hom}(\widehat{G}, \mathbb{C}^*)$, where $\mathrm{Hom}(\widehat{G}, \mathbb{C}^*)$ consists of all the continuous irreducible complex characters of the profinite completion $\widehat{G}$. In particular if $G = \mathbb{Z}^d$, the free abelian group of rank d, then the rational representation zeta function takes the form

$$\zeta_{\mathbb{Z}^d}^{\mathbb{Q}}(s) = \prod_p \left(1 + \frac{p^d - 1}{p - 1}(p-1)^{-s} \frac{1}{1 - p^{(d-1)-s}}\right),$$

where the Euler product extends over all primes p similar to the Riemann zeta function.

In the case of crystallographic groups, we use a similar strategy and Clifford theory. Let G be a crystallographic group of dimension two and T the normal subgroup of G consisting of translations. A character of $T \cong \mathbb{Z}^2$ is easy to describe

and has a certain inertia subgroup in G. We parameterise the irreducible complex characters of T. First, from an irreducible complex character of T and its inertia group in G, we compute those irreducible complex characters of G whose restriction to T involves the given character on T. This uses Clifford theory. Then we obtain the (equivalence classes of) irreducible representations over $\mathbb{Q}$ via Galois orbits of complex characters. There are seventeen crystallographic groups of dimensions two up to isomorphism and in this thesis we treated nine such groups. For instance, let

$$G = pm = \langle x, y, m \mid [x, y] = m^2 = 1, x^m = x, y^m = y^{-1} \rangle.$$

Then the rational representation zeta function of G is

$$\begin{aligned}\zeta_G^{\mathbb{Q}}(s) = 2 \cdot \zeta_{\mathbb{Z}}^{\mathbb{Q}}(s) + 2^{-1-s} \cdot \zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + (1 - 2^{-1-s}) \Bigg(\prod_p (1 + 2\frac{(p-1)^{-s}}{1-p^{-s}}) \\ + \frac{1+2^{-s}}{1-2^{-s}} \prod_{p \neq 2} (1 + 2\frac{(p-1)^{-s}}{1-p^{-s}}) \Bigg).\end{aligned}$$

It does not have an Euler product decomposition but it is a finite sum of Euler products. Similar results hold for the other crystallographic groups that we consider in this thesis.

Acknowledgments

First and foremost, I am heartily thankful to my supervisor, Prof. Benjamin Klopsch, whose encouragement, guidance and continuous support from the initial to the final level enabled me to develop an understanding of the subject. I am also grateful to him for providing me an excellent atmosphere for doing research and for always being next to me through all the difficulties I faced during my research. Above all, I am thankful to him for being such a great teacher.

My thanks also go to the many people in the department of Mathematics at Royal Holloway. In particular, I want to say thanks to Dr. Mark Wildon and Dr. Yiftach Barnea for their cooperation in many aspects.

The most importing thing for my research was finance. I want to say thanks to Royal Holloway for partially funding me by offering College Research Scholarship and thanks to my family for my funding, specially my brother Dr. Amanat Ali and my wife Dr. Ayesha Saleem.

I offer my regards and blessings to all of those who supported me in any respect during the completion of my final thesis.

Last but not least, A special thanks to my family. Words cannot express how grateful I am to my mother, my brothers and sisters and my wife for all of the sacrifices that they have made on my behalf. Their prayer for me was what sustained me thus far. Here I cannot miss my late father Abdul Majeed. This thesis is dedicated to the bright memory of my father Abdul Majeed.

Liaqat Ali

Contents

Chapter 1

Introduction

This thesis deals with the study of rational representation zeta functions of finitely generated abelian groups, frieze groups and crystallographic groups of dimensions one and two. We compute the rational representation zeta functions via Galois orbit zeta functions. Let G be a group. The *rational representation zeta function* of G is the formal Dirichlet series

$$\zeta_G^{\mathbb{Q}}(s) = \zeta_G^{\mathbb{Q}\text{-rep}}(s) = \sum_{n=1}^{\infty} r_n^{\mathbb{Q}}(G) n^{-s},$$

where $r_n^{\mathbb{Q}}(G)$ denotes the number of (equivalence classes of) n-dimensional irreducible representations of G over $\mathbb{Q}$ with finite image and where s is a complex variable. Here and throughout the thesis, we consider groups G such that $r_n^{\mathbb{Q}}(G)$ is finite for all n.

The *Galois orbit zeta function* of G is defined as

$$\omega_G^{\mathbb{Q}}(s,t) = \sum_{\chi \in \mathrm{Irr}(\widehat{G})} \chi(1)^{-s} |\chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}|^{-t},$$

where $\mathrm{Irr}(\widehat{G})$ is the set of continuous irreducible complex characters of the profinite completion $\widehat{G}$ and $\chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ is the Galois orbit of $\chi : \widehat{G} \to \overline{\mathbb{Q}} \subseteq \mathbb{C}$; furthermore s, t are complex variables.

The rational representation zeta function of G is closely related to the Galois orbit zeta function of G. To compute the rational representation zeta function as opposed to the Galois orbit zeta function, one needs to keep track of the Schur indices associated to the irreducible complex characters of $\widehat{G}$. In general it is difficult to compute the Schur indices. In certain cases the Schur indices are all equal to one for fairly standard reasons. In that case the rational representation zeta function is obtained from the Galois orbit zeta function by putting $t = s+1$, that is $\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1)$; see Corollary 2.4.13. The advantage of working with Galois orbit zeta functions is that they are simpler to handle than rational representation zeta functions.

For the finitely generated abelian groups, we proceed as follows. Let G be a finitely generated abelian group. Then there exists a one-to-one correspondence between irreducible rational representations of $\widehat{G}$ and the Galois orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Hom}(\widehat{G}, \mathbb{C}^*)$, where $\mathrm{Hom}(\widehat{G}, \mathbb{C}^*)$ consists of all the irreducible complex characters of the group $\widehat{G}$.

In the case of crystallographic groups, we use a similar strategy and Clifford theory. Let G be a crystallographic group and let T be the normal subgroup of G consisting of translations. A character of T is easy to describe and has a certain inertia subgroup in G. We parameterise the irreducible complex characters of T. First, from an irreducible complex character of T and its inertia group in G, we compute those irreducible complex characters of G whose restriction to T involves the given character on T. Then we apply Clifford theory to obtain the (equivalence classes of) irreducible representations over $\mathbb{Q}$ via Galois orbits of complex characters.

1.1 Thesis structure

Chapter 1 is an introduction to the subject of rational representation zeta functions and gives the motivation behind their study. This chapter also includes a short history of zeta functions in number theory and the definitions of subgroup zeta functions and representation zeta functions. We also present the main results.

Chapter 2 contains some basic facts and auxiliary results on representation theory, Galois theory and cyclotomic fields. This chapter also includes an account of Clifford theory, the Schur index and an overview of crystallographic groups.

Chapter 3 contains a detailed computation of the rational representation zeta functions of finitely generated free abelian groups and some other finitely generated abelian groups.

Chapter 4 deals with the computation of the rational representation zeta functions of frieze groups and one-dimensional crystallographic groups.

In **Chapter 5** we compute the rational representation zeta functions of seven crystallographic groups of dimension two. In two cases, where the Schur indices are not all one we only compute the Galois orbit zeta functions.

Finally, **Chapter 6** concludes the thesis. We compute the rational representation zeta functions of natural generalisations of the infinite dihedral group. We collect some questions to be answered by future research and we describe possible future work in the area.

1.2 Zeta functions in number theory and group theory

Before we introduce zeta functions of groups we provide a short history of zeta functions in number theory, based on [13]. During the eighteenth century mathematicians were interested in finding the value of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Daniel Bernoulli suggested $\frac{8}{5}$ as an estimate for its value, but it was Leonhard Euler who first gave the precise value $\frac{\pi^2}{6}$ of this sum. To do this he defined the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s \in \mathbb{R}$ with $s > 1$. The sum of the inverse squares is then equal to the value of $\zeta(s)$ at $s = 2$. He gave a formula for the values of the zeta function at every even positive integer as

$$\zeta(2k) = \frac{2^{2k-1}\pi^{2k}B_{2k}}{(2k)!},$$

where the rational constants B_{2k} are now known as the Bernoulli numbers. Since $B_2 = \frac{1}{6}$, it follows that $\zeta(2) = \frac{\pi^2}{6}$. Euler also discovered the following "Euler product identity": writing

$$\zeta_p(s) = \sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \frac{1}{1-p^{-s}},$$

one has

$$\zeta(s) = \prod_p \zeta_p(s) \quad \text{for } s \in \mathbb{R} \text{ with } s > 1,$$

where the product is formed over all primes p.

Bernhard Riemann subsequently extended the domain of the zeta function from the real numbers to the complex numbers. The resulting function, which allows a meromorphic continuation to the entire complex plane, is now known as the Riemann zeta function $\zeta(s)$. It plays a profound role in the study of prime numbers.

Gustav Dirichlet transfered the concept of zeta function in a new direction. He attached a coefficient $\chi(n)$ to each term n^{-s}. The Dirichlet L-function $L(s,\chi)$ is then defined by the series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character, defined as follows.

Definition 1.2.1. *A* Dirichlet character *is a function* $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$ *that has the following properties:*

1. χ *is completely multiplicative that is* $\chi(1) = 1$ *and* $\chi(n_1)\chi(n_2) = \chi(n_1 n_2)$ *for all* $n_1, n_2 \in \mathbb{Z}$.

2. *There exists a positive integer* m *such that* $\chi(m+n) = \chi(n)$ *for all* $n \in \mathbb{Z}$ *and* $\chi(n) = 0$, *if* $\gcd(n,m) > 1$.

The treatment of both $\zeta(s)$ and $L(s,\chi)$ can be unified by introducing the Hurwitz zeta function $\zeta(s,a)$ defined by the series

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

where a is a fixed real number in the interval $(0,1]$. When $a = 1$ this reduces to the Riemann zeta function. We can also express $L(s,\chi)$ in terms of Hurwitz zeta functions. If χ is a character modulo k, where a character χ modulo k is a group homomorphism from $(\mathbb{Z}/k\mathbb{Z})^*$ to $\mathbb{C}^*$, then χ can be extended to a multiplicative map from $\mathbb{Z}/k\mathbb{Z}$ to $\mathbb{C}$ by setting $\chi(x) = 0$ if $x \notin (\mathbb{Z}/k\mathbb{Z})^*$. We then rearrange the terms in the series for $L(s,\chi)$ according to the residue classes modulo k. That is we write $n = qk + r$, where $1 \leq r \leq k$ and $q \in \mathbb{N}_0$. From this we obtain

$$L(s,\chi) = k^{-s} \sum_{r=1}^{k} \chi(r)\zeta(s, \frac{r}{k}).$$

For more details see [13, Chapter 1] and [5, Chapters 6,12]. All the series above converge for $s \in \mathbb{C}$ with real part $\Re(s) > 1$. The zeta functions described above have number-theoretic applications, for instance regarding the distribution of prime numbers.

Richard Dedekind used zeta functions in a more algebraic setting. The *Dedekind zeta function* of a number field K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} |V_K : \mathfrak{a}|^{-s},$$

where the sum extends over all non-zero ideals $\mathfrak{a}$ of the ring of integers V_K of K and $|V_K : \mathfrak{a}|$ denotes the index of $\mathfrak{a}$ in V_K as an additive group. This definition was extended by Grunewald, Segal and Smith [18] to study zeta functions of non-commutative groups.

Definition 1.2.2 (Subgroup zeta function of a group). *For a finitely generated group G, let $a_m(G)$ denote the number of subgroups of index m in G. Then the* subgroup zeta function *of G is defined as*

$$\zeta_G^{\leq}(s) = \sum_{m=1}^{\infty} a_m(G) m^{-s} = \sum_{H} |G : H|^{-s},$$

where the last sum is formed over all finite index subgroups H of G.

Thus the subgroup zeta function of G is a Dirichlet generating function encoding the numbers of finite index subgroups of G. If the Dirichlet coefficients $a_m(G)$ are polynomially bounded in terms of m this series converges on the complex right half plane $\{s \in \mathbb{C} \mid \Re(s) > \alpha\}$ for some $\alpha \in \mathbb{R}$. For instance, this is the case if the finitely generated group G is nilpotent and furthermore, in this case $\zeta_G^{\leq}(s)$ has an Euler product decomposition

$$\zeta_G^{\leq}(s) = \prod_p \zeta_{G,p}(s),$$

where the product runs all over primes p and

$$\zeta_{G,p}^{\leq}(s) = \sum_{i=0}^{\infty} a_{p^i}(G)p^{-is}.$$

The functions $\zeta_{G,p}^{\leq}(s)$ are called the *local subgroup zeta functions* of G. If $G = \mathbb{Z}$ then $a_m(G) = 1$ for every m and

$$\zeta_G^{\leq}(s) = \sum_{m=1}^{\infty} m^{-s}$$

is the Riemann zeta function $\zeta(s)$, and the decomposition into local factors is

$$\zeta(s) = \prod_p \zeta_p(s),$$

where

$$\zeta_p(s) = \sum_{i=0}^{\infty} p^{-is} = \frac{1}{1-p^{-s}}.$$

More generally, consider $G = \mathbb{Z}^d$, the free abelian group of rank d. Here we have

$$\zeta_{\mathbb{Z}^d}^{\leq}(s) = \zeta(s)\zeta(s-1)\ldots\zeta(s-d+1).$$

For the proof; see [26, Chapter 15]. Things become more challenging when we turn to non-abelian groups. The smallest one is the 'discrete Heisenberg group' of 3×3 upper uni-triangular matrices over the integers

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \leq \mathrm{GL}_3(\mathbb{Z}).$$

It can be shown that

$$\zeta_H^{\trianglelefteq}(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)},$$

see [26, Chapter 15].

Subgroup zeta functions of groups have been studied intensively for the last thirty years; see [12, 13, 24, 26, 29] for more details. We are interested in representation zeta functions of groups. This concept was introduced more recently. Current research tends to focus on representation zeta functions $\zeta_G^{\mathbb{C}\text{-rep}}(s)$ encoding the distribution of irreducible complex representation of suitable groups G. For more details, see [3, 27, 35]. In this thesis we study the less understood representation zeta functions $\zeta_G^{\mathbb{Q}\text{-rep}}(s)$ encoding the distribution of irreducible rational representations of suitable groups G.

1.3 Zeta functions of crystallographic groups

We refer to [16] for a comprehensive account of the theory of crystallographic groups.

A crystallographic group is a group G of transformations of Euclidean space $\mathbb{R}^n$, containing a discrete translation group T isomorphic to $\mathbb{Z}^n$ as a normal subgroup such that the quotient $G/\mathbb{Z}^n$ is a finite group. The quotient $P = G/T$ is called a point group. In [30], McDermott calculated the subgroup zeta functions of the crystallographic groups of dimension two. There are 17 such groups up to isomorphism, and they are also known as plane-crystallographic groups. We provide a list of these groups in term of presentations with generators and relations in Section 2.3. As a corollary McDermott proved the following result.

Theorem 1.3.1. *[30, Theorem 1] There exist two plane-crystallographic groups with non-isomorphic profinite completions but with the same number of subgroups of each finite index.*

As mentioned in the previous section, if the finitely generated group G is nilpotent then the subgroup zeta function admits an Euler product; see [18, Proposition 1.3]. In [30], McDermott discovered a group which is not nilpotent but admits an Euler product. He showed that the groups $p1 = \langle x, y \mid [x, y] = 1\rangle$ and $pg = \langle x, y, t \mid [x, y] = 1, x^t = x^{-1}, y = t^2\rangle$, which are clearly not isomorphic, have the same subgroup zeta functions. Furthermore, from his explicit computation he computed the abscissae of convergence of the subgroup zeta functions of plane-crystallographic groups (for nine out of seventeen groups).

In [11] Du Sautoy, McDermott and Smith studied the analytic continuation of subgroup zeta functions of the crystallographic groups. Explaining these zeta functions as finite sums of Euler products they proved

Theorem 1.3.2. *([11, Theorem 1.1]) Let G be a finite extension of a free abelian group of finite rank. Then $\zeta_G^{\leq}(s)$ can be extended to a meromorphic function on the whole complex plane.*

In this thesis we study the rational representation zeta functions of plane-crystallographic groups. In contrast to Theorem 1.3.2, already for abelian groups one obtains a partial meromorphic continuation of the rational representation zeta functions that does not extend to the entire complex plane.

1.4 Representation zeta functions

We recall that two linear representations of a group G are said to be equivalent if the corresponding G-modules are isomorphic; see Definition 2.1.1.

Definition 1.4.1 (Representation zeta function)**.** *Let G be a group and for $n \in \mathbb{N}$, let $r_n(G)$ denote the number of equivalence classes of irreducible complex representations of G of dimension n. The group G is called* (representation) rigid *if for every $n \in \mathbb{N}$, it admits only finitely many equivalence classes of irreducible complex* *representations of dimension* n. *The* representation zeta function *of a rigid group* G *is defined as*

$$\zeta_G^{\mathbb{C}}(s) = \zeta_G^{\mathbb{C}\text{-}rep}(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s},$$

where s *is a complex variable.*

The representation zeta function is a Dirichlet generating function encoding the distribution of irreducible complex representations of a group G. In [24, 33], this function is denoted by $\zeta_G^{\mathrm{irr}}(s)$. If the sequence $r_n(G)$ grows polynomially then the Dirichlet series converges on a complex right half-plane $\{s \in \mathbb{C} \mid \Re(s) > \alpha\}$, for some $\alpha \in \mathbb{R}$ by [32, Lemma 2.1]. In general no characterisation of representation rigid groups is known but for certain classes of groups there are such characterisations. For instance, a finitely generated profinite group G is rigid if and only if it is FAb, that is, if every open subgroup of G has finite abelianisation.

Finitely generated infinite nilpotent groups are not rigid, as they already have infinitely many representations of dimension one. For certain arithmetic groups Lubotzky and Martin show that they are rigid if they have the congruence subgroup property; see [27]. As shown by Larsen and Lubotzky in [28], the representation zeta functions of such arithmetic groups admit Euler products. Stasinski and Voll in [32] and Voll in [33] studied related representation zeta functions of finitely generated nilpotent groups. Such a group is not rigid, for example, finitely generated torsion-free nilpotent groups ($\mathcal{T}$-groups) are not rigid. A non-trivial $\mathcal{T}$-group has infinitely many representations of dimension one. However $\mathcal{T}$-group are "rigid up to twisting by one-dimensional representations"; see [32]. Here the product $\chi \otimes \rho$ is a "twist" of ρ, where χ is a one-dimensional complex representation and ρ an n-dimensional complex representation of a $\mathcal{T}$-group G. The detailed study of these groups can be found in [32].

The study of representation zeta functions is motivated by the subject of subgroup growth and subgroup zeta functions in which one counts finite index subgroups; see [12, 26]. Furthermore Witten in [34] initiated the idea of using zeta function to study certain moduli spaces. Lubotzby and Martin in [27] use representation zeta functions to study arithmetic groups.

Jaikin-Zapirain in [35] studied representation zeta functions of compact p-adic analytic groups with the property FAb. Jaikin-Zapirain proved the following result.

Theorem 1.4.2. *[35, Theorem 1.1] Let G be a FAb compact p-adic analytic group with $p > 2$. Then there are natural numbers $n_1, n_2, \ldots, n_k$ and rational functions $f_1(p^{-s}), f_2(p^{-s}), \ldots, f_k(p^{-s})$ in p^{-s} with integer coefficients such that*

$$\zeta_G^{\mathbb{C}}(s) = \sum_{i=1}^{k} n_i^{-s} f_i(p^{-s}).$$

Moreover, if G is a FAb p-adic analytic pro-p group, then $\zeta_G^{\mathbb{C}}(s)$ is a rational function in p^{-s}.

Avni, Klopsch, Onn and Voll in [4] studied the representation zeta functions of compact p-adic analytic groups and arithmetic groups, using p-adic integration. For important results about the representation zeta functions of compact p-adic analytic groups; see [1, 3, 4, 10, 35]. In recent years, several results have been obtained concerning the representation growth and representation zeta functions of these types of groups; see [1, 2, 4, 15, 20, 34], [3, Theorems 1.2, 1.3, 1.4] and [35, Theorem 7.5].

This thesis is concerned with the study of rational representation zeta functions of finitely generated abelian groups and crystallographic groups of dimensions one and two. Let $\mathbb{Q}$ be the field of rational numbers and let G be a group. A $\mathbb{Q}$-*rational representation* of G, for short a rational representation of G, is a homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$ from G into the group $\mathrm{GL}(V)$ of nonsingular linear transformations of a finite-dimensional vector space V over $\mathbb{Q}$. The *character* of ρ is the function $\chi_\rho : G \longrightarrow \mathbb{Q}$ defined as $\chi_\rho(a) = \operatorname{Tr} \rho(a)$ for each element a in G, where $\operatorname{Tr} \rho(a)$ is the sum of the complex eigenvalues of the linear transformation $\rho(a)$. The rational representations are related to problems in number theory, for instance; see [6], where Bateman studies the distributation of values of the Euler function. Implicitly he studies the rational representation zeta function of the infinite cyclic group, namely $\sum_{n=1}^{\infty} \varphi(n)^{-s}$; see Theorem 1.5.1.

Now we define rational representation zeta functions.

Definition 1.4.3 (Rational representation zeta function)**.** *Let G be a group and for $n \in \mathbb{N}$, let $r_n^{\mathbb{Q}}(G)$ denote the number of (equivalence classes of) irreducible rational representations of G with finite image and of dimension n. The group G is called rationally (representation) rigid, if $r_n^{\mathbb{Q}}(G) < \infty$ for all $n \in \mathbb{N}$. The* rational representation zeta function *of a rationally rigid group G is defined as*

$$\zeta_G^{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} r_n^{\mathbb{Q}}(G) n^{-s}.$$

We remark that the rational representation zeta function of G as define above is equal to the rational representation zeta function of the profinite completion $\widehat{G}$ of G counting continuous representations. The rational representation zeta functions of G is closely related to the Galois orbit zeta function.

Definition 1.4.4 (Galois orbit zeta function)**.** *Let G be a rationally rigid group and let $\chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ be the Galois orbit of $\chi : \widehat{G} \to \overline{\mathbb{Q}} \subseteq \mathbb{C}$. Then the* Galois orbit zeta function *of G is defined as*

$$\omega_G^{\mathbb{Q}}(s,t) = \sum_{\chi \in \mathrm{Irr}(\widehat{G})} \chi(1)^{-s} |\chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}|^{-t},$$

where $\mathrm{Irr}(\widehat{G})$ *is the set of continuous irreducible complex characters of the profinite completion $\widehat{G}$ of G.*

We compute the rational representation zeta functions of finitely generated abelian groups and groups which are 'close to abelian'. For instance, groups which are extensions of abelian groups by finite groups, called virtually abelian groups, are 'close to abelian'. A natural class to investigate is the class of crystallographic groups. We compute the rational representation zeta functions of two-dimensional crystallographic groups, also known as plane-crystallographic groups or wallpaper groups. Informally speaking, these groups are the groups of symmetries of ornaments, where an ornament is a plane pattern infinitely repeated in two directions. A wallpaper group is made up of an infinite number of translations, rotations, reflections and glide reflections. There are seventeen such groups up to isomorphism, listed in Section 2.3, also see [30] for the presentations of crystallographic groups. We discuss nine groups out of these seventeen two-dimensional crystallographic groups in Chapter 5.

1.5 Main results

The following theorem about the rational representation zeta functions of finitely generated abelian groups is the main result of Chapter 3.

Theorem 1.5.1. *Let $G = \mathbb{Z}^d$ be the free abelian group of rank d. Then the rational representation zeta function of G admits an Euler product and is equal to*

$$\zeta_G^{\mathbb{Q}}(s) = \prod_p \left(1 + \frac{p^d - 1}{p - 1}(p-1)^{-s}\frac{1}{1 - p^{(d-1)-s}}\right),$$

where the product runs over all primes p.

The following theorem about the rational representation zeta functions of the infinite dihedral group is the main result of Chapter 4.

Theorem 1.5.2. *Let $G = C_2 \ltimes \mathbb{Z}$ be an infinite dihedral group. Then the rational representation zeta function of G is equal to*

$$\zeta_G^{\mathbb{Q}}(s) = 2 + \prod_p \left(1 + \frac{(p-1)^{-s}}{1 - p^{-s}}\right),$$

where the product extends to all primes p.

The main results of Chapter 5 consist of the computation of the rational representation zeta functions of plane-crystallographic groups. We compute the rational representation zeta functions of seven crystallographic groups of dimension two.

Theorem 1.5.3. *Let*

$$G = p1 = \langle x, y \mid [x, y] = 1\rangle.$$

Then the rational representation zeta function of G admits an Euler product and is

$$\zeta_G^{\mathbb{Q}}(s) = \prod_p \left(1 + (p-1)^{-s}\frac{p+1}{1-p^{1-s}}\right).$$

Theorem 1.5.4. *Let*

$$G = p2 = \langle x, y, r \mid [x, y] = r^2 = 1, x^r = x^{-1}, y^r = y^{-1}\rangle.$$

Then the rational representation zeta function of G is

$$\zeta_G^{\mathbb{Q}}(s) = 4 + \prod_p \left(1 + (p-1)^{-s}\frac{p+1}{1-p^{1-s}}\right).$$

Theorem 1.5.5. *Let*

$$G = pm = \langle x, y, m \mid [x, y] = m^2 = 1, x^m = x, y^m = y^{-1}\rangle.$$

Then the rational representation zeta function of G is

$$\begin{aligned}\zeta_G^{\mathbb{Q}}(s) = 2 \cdot \zeta_{\mathbb{Z}}^{\mathbb{Q}}(s) + 2^{-1-s} \cdot \zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + (1 - 2^{-1-s})\Bigg(&\prod_p (1 + 2\frac{(p-1)^{-s}}{1-p^{-s}})\\ &+ \frac{1+2^{-s}}{1-2^{-s}}\prod_{p\neq 2}(1 + 2\frac{(p-1)^{-s}}{1-p^{-s}})\Bigg),\end{aligned}$$

where

$$\zeta_{\mathbb{Z}}^{\mathbb{Q}}(s) = \prod_p \left(1 + \frac{(p-1)^{-s}}{1-p^{-s}}\right)$$

and

$$\zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) = \prod_p \left(1 + (p-1)^{-s}\frac{p+1}{1-p^{1-s}}\right).$$

Theorem 1.5.6. *Let*

$$G = pg = \langle x, y, t \mid [x, y] = 1, x^t = x^{-1}, t^2 = y \rangle.$$

Then the rational representation zeta function of G is

$$\zeta_G^{\mathbb{Q}}(s) = 2^{-1-s}\zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + (1 - 2^{-1-s})\Bigg(\prod_p (1 + 2\frac{(p-1)^{-s}}{1-p^{-s}})$$
$$+ \frac{1+2^{-s}}{1-2^{-s}} \prod_{p \neq 2} (1 + 2\frac{(p-1)^{-s}}{1-p^{-s}})\Bigg).$$

Theorem 1.5.7. *Let*

$$G = p2mm = \langle x, y, p, q \mid [x, y] = [p, q] = 1 = p^2 = q^2, x^p = x, x^q = x^{-1},$$
$$y^p = y^{-1}, y^q = y \rangle.$$

Then the rational representation zeta function of G is

$$\zeta_G^{\mathbb{Q}}(s) = 8 + 2^{-1-s} \cdot \zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + (1 - 2^{-1-s})\Bigg(\prod_p (1 + 2\frac{(p-1)^{-s}}{1-p^{-s}})$$
$$+ \frac{1+2^{-s}}{1-2^{-s}} \prod_{p \neq 2} (1 + 2\frac{(p-1)^{-s}}{1-p^{-s}})\Bigg).$$

Theorem 1.5.8. *Let*

$$G = cm = \langle x, y, t \mid [x, y] = 1 = t^2, x^t = xy, y^t = y^{-1} \rangle.$$

Then the rational representation zeta function of G is

$$\zeta_G^{\mathbb{Q}}(s) = 2\zeta_{\mathbb{Z}}^{\mathbb{Q}}(s) + 2^{-1-s}\zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + (2 - 2^{-s})\left(\frac{1+2^{-2s}}{1-2^{-s}}\right) \prod_{p>2} \left(1 + 2\frac{(p-1)^{-s}}{1-p^{-s}}\right)$$
$$- 2\left(\frac{1+2^{-1-s}}{1-2^{-s}}\right) \prod_{p>2} \left(1 + \frac{(p-1)^{-s}}{1-p^{-s}}\right).$$

Theorem 1.5.9. *Let*

$$G = p4 = \langle x, y, r \mid [x, y] = r^4 = 1, x^r = y, y^r = x^{-1} \rangle.$$

Then the rational representation zeta function of G is

$$\begin{aligned}\zeta_G^{\mathbb{Q}}(s) &= 4 + 2^{-s} - 2^k(2-2^{-s}) + 2^{-1-s}\zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + 2^{-1-s}\zeta_{\mathbb{Z}}^{\mathbb{Q}}(s-1) \\ &- 2^{-1-s}\prod_{p\equiv 3 \bmod 4}(1+\frac{(p-1)^{1-s}}{1-p^{1-s}}) + 2^{-1-s}\prod_{p\equiv 1 \bmod 4}(1+\frac{(p-1)^{1-s}}{1-p^{1-s}}) \\ &+ 2^k(2-2^{-s})\prod_{p\equiv 1 \bmod 4}(1+2\frac{(p-1)^{-s}}{1-p^{-s}}).\end{aligned}$$

In two cases, where the Schur indices are not all one we only compute the Galois orbit zeta functions.

Theorem 1.5.10. *Let*

$$\begin{aligned}G = p2mg = \langle x, y, m, t \mid [x,y] = 1 = t^2, m^2 = y, x^t = x, x^m = x^{-1}, \\ y^t = y^{-1}, m^t = m^{-1}\rangle.\end{aligned}$$

Then the Galois orbit zeta function of G is

$$\begin{aligned}\omega_G^{\mathbb{Q}}(s,t) &= 8 + 2^{1-t-s} - 2^{t-2s} + (3\cdot 2^{t-s} - 4\cdot 2^{2t-2s-2} + 5\cdot 2^{-s})\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) \\ &+ 2^{t-2-2s}\omega_{\mathbb{Z}^2}^{\mathbb{Q}}(s,t)(4^{t-s-1} - 2^{t-2-2s})\prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\ &+ (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}),\end{aligned}$$

where

$$\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) = \prod_p\left(1+\frac{(p-1)^{1-t}}{1-p^{1-t}}\right)$$

and

$$\omega_{\mathbb{Z}^2}^{\mathbb{Q}}(s,t) = \prod_p\left(1+(p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\right).$$

Theorem 1.5.11. *Let*

$$G = p2gg = \langle x, y, u, v \mid [x,y] = 1 = (uv)^2, u^2 = x, v^2 = y, x^v = x^{-1}, y^u = y^{-1}\rangle.$$

Then the Galois orbit zeta function of G is

$$\begin{aligned}\omega_G^{\mathbb{Q}}(s,t) &= 4 + 4\cdot 2^{-t} - 2^{1-s} + (2^{t-s+1} + 3\cdot 2^{1-s} - 4\cdot 4^{t-s-1})\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) \\ &+ 2^{t-2-2s}\omega_{\mathbb{Z}^2}^{\mathbb{Q}}(s,t)(4^{t-s-1} - 2^{t-2-2s})\prod_p(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\ &+ (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}),\end{aligned}$$

where

$$\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) = \prod_p \left(1 + \frac{(p-1)^{1-t}}{1-p^{1-t}}\right)$$

and

$$\omega_{\mathbb{Z}^2}^{\mathbb{Q}}(s,t) = \prod_p \left(1 + (p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\right).$$

The main result of Chapter 6 is the computation of the rational representation zeta functions of certain generalisations of the infinite dihedral group.

Theorem 1.5.12. *The group $\mathbb{Z}^d \rtimes C_2$ is given by the presentation*

$$G = \langle x_1, x_2, \ldots, x_d, r \mid [x_i, x_j] = 1 = r^2, x_i^r = x_i^{-1}\rangle.$$

Then the rational representation zeta function of G is

$$\begin{aligned}\zeta_G^{\mathbb{Q}}(s) &= 2\cdot 2^d + 2^{t-s-1}\prod_p\left(1 + \sum_{d_0=1}^{d}\binom{d}{d_0}(p-1)^{d_0-t}\right. \\ &\left.\cdot \sum_{f_1,f_2,\ldots,f_{d_0}=0}^{\infty} p^{f_1+f_2+\cdots+f_{d_0}}\cdot(p^{max\{f_1,f_2,\ldots,f_{d_0}\}})^{-t}\right) - 2^{d+t-s-1},\end{aligned}$$

where

$$\begin{aligned}&\sum_{f_1,f_2,\ldots,f_{d_0}=0}^{\infty} p^{f_1+f_2+\cdots+f_{d_0}}\cdot(p^{max\{f_1,f_2,\ldots,f_{d_0}\}})^{-t} \\ &= 1 + \sum_{d_1=1}^{d_0}\binom{d_0}{d_1}\sum_{m=1}^{d}\sum_{\substack{\delta_1,\delta_2,\ldots,\delta_m\in\mathbb{N} \\ \text{such that} \\ \delta_1+\delta_2+\cdots+\delta_m=d_1}}\binom{d_1}{\delta_1,\delta_2,\ldots,\delta_m}\prod_{k=1}^{m}\frac{p^{(d_1-\sum_{j=1}^{k}\delta_j)-t}}{(1-p^{(d_1-\sum_{j=1}^{k}\delta_j)-t})}.\end{aligned}$$

Chapter 2

Basic facts and auxiliary results

This chapter deals with definitions and some auxiliary results. All representations of groups that we consider factor through a finite quotient of the relevant group. Therefore it suffices to recall facts from the character theory of finite groups, even though we study representation zeta functions of infinite groups.

2.1 Character theory

In this section we summarise some results and definitions from character theory of finite groups. For more details; see [17], [19], [21] and [22].

Definition 2.1.1 (Representation of a group)**.** *A* representation *of a group G is a homomorphism $\rho : G \to \mathrm{GL}(V)$ into the general linear group of a vector space V over a field F. This means that ρ associates to each $g \in G$ a linear transformation $\rho(g) : V \to V$, and that the map ρ satisfies $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$. If V has finite dimension, say n, then the group $\mathrm{GL}(V)$ is isomorphic to $\mathrm{GL}(n, F)$, the group of all non-singular $n \times n$ matrices over the field F. Two representations $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ over F, are said to be* equivalent *if there exists a linear isomorphism $\psi : V_1 \to V_2$ such that $\psi(\rho_1(g)v) = \rho_2(g)\psi(v)$ for all $g \in G$ and $v \in V_1$.*

Definition 2.1.2 (Irreducible representations). *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G. Then a subset U of V is said to be G-invariant if $\rho(g)(u) \in U$ for all $g \in G$ and $u \in U$. Clearly, the null subspace $\{0\}$ and the vector space V itself are G-invariant. The representation ρ is said to be* irreducible *if $\{0\} \neq V$ and these are the only G-invariant subspaces of V.*

Definition 2.1.3 (Completely reducible). *A representation is* completely reducible *if it is a finite direct sum of irreducible representations.*

Definition 2.1.4 (Absolutely irreducible). *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G over a field F. Then ρ is* absolutely irreducible *if it remains irreducible after any field extension of F.*

Definition 2.1.5 (Splitting field). *The field F is a* splitting field *for a group G if every irreducible F-representation of group G is absolutely irreducible.*

Definition 2.1.6 (Character). *Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional representation of a group G over a field $F \subseteq \mathbb{C}$. Then the* character *$\chi : G \to F$ of G afforded by ρ is the function given by $\chi(g) = \operatorname{Tr}\rho(g)$, the trace of $\rho(g)$. The* degree (or dimension) *of χ is $\chi(1) = \dim(V)$. A* linear character *is a character of degree one. The character χ is said to be* faithful *if the underlying representation ρ is* faithful. *This is equivalent to the condition $\chi(g) \neq \chi(1)$ for all $g \in G \setminus \{1\}$.*

Definition 2.1.7 (Irreducible character). *A character χ of an irreducible finite dimension representation over $\mathbb{C}$ is called an* irreducible character. *The collection of all irreducible complex characters of a group G is denoted by* $\mathrm{Irr}(G)$.

Theorem 2.1.8. *The number of irreducible characters of a finite group G is equal to the number of conjugacy classes of that group G.*

For proof; see [22, Chapter 15, Theorem 15.3].

Corollary 2.1.9. *[21, Chapter 2, Corollary 2.6] The group G is abelian if and only if every irreducible character is linear.*

Definition 2.1.10. *Let T be a subgroup of a finite group G and $\chi : T \to \mathbb{C}$ a character. Then the* induced character $\operatorname{Ind}_T^G \chi : G \to \mathbb{C}$ *is the character on G defined by*

$$\operatorname{Ind}_T^G \chi(a) = \frac{1}{|T|} \sum_{g \in G} \chi(g^{-1}ag),$$

where $\chi(a) = 0$ if $a \notin T$. We denote the induced character as χ^G.

Let $\psi : G \to \mathbb{C}$ be a character. Then ψ_T is simply the restriction *of this function to T, which is obtained from ψ by evaluating it using elements of T.*

Theorem 2.1.11. *[21, Chapter 4, Theorem 4.21] Let $G = H \times K$ be a finite group, where H and K are subgroups of the group G. Then the characters $\chi \times \vartheta$ for $\chi \in \operatorname{Irr}(H)$ and $\vartheta \in \operatorname{Irr}(K)$ are exactly the irreducible characters of group G.*

2.2 Galois theory and cyclotomic fields

In this section we are going to summarise some results and definitions from Galois theory and from the theory of cyclotomic fields. For more details see [31].

Definition 2.2.1. *Let E be a field. An extension of field E/F is a subfield F contained in a larger field E. If $f \in f[X]$ such that f decomposes over E into a product of linear factors*

$$f = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_n),$$

then $F(\alpha_1, \ldots, \alpha_n)$ denotes the smallest intermediate field of E/F over which f splits into a product of linear factors. This field is unique up to isomorphism over F and called a splitting field *for f over F.*

Definition 2.2.2. *For $n \in \mathbb{N}$ the nth* cyclotomic field $\mathfrak{K}_n$ *over $\mathbb{Q}$ is the splitting field of $X^n - 1$ over $\mathbb{Q}$ in $\mathbb{C}$.*

Definition 2.2.3 (Galois group)**.** *A* finite Galois extension *is a field extension E/F for which there exists a finite subgroup G of $\operatorname{Aut}(E)$ such that*

$$F = \{\alpha \in E \mid \forall g \in G : g(\alpha) = \alpha\}.$$

If F has characteristic zero, this is equivalent to saying that E is a splitting field of some polynomial over F. The Galois group *of a finite Galois extension E/F is the group*

$$\mathrm{Gal}(E/F) = \{g \in \mathrm{Aut}(E) \mid \forall \alpha \in F : g(\alpha) = \alpha\}.$$

There is a well known result which we will use in our further discussion.

Theorem 2.2.4. *Let $n \in \mathbb{N}$, and let $\mathfrak{K}_n$ be the nth cyclotomic field over $\mathbb{Q}$. Then $\mathfrak{K}_n/\mathbb{Q}$ is a Galois extension of degree $\varphi(n)$, where φ denotes the Euler phi-function, and $\mathfrak{K}_n = \mathbb{Q}(\zeta)$ for a primitive nth root of unity ζ, for example $\zeta = e^{2\pi i/n}$. The Galois group* $\mathrm{Gal}(\mathfrak{K}_n/\mathbb{Q})$ *is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$, the multiplicative group of units in the finite ring $\mathbb{Z}/n\mathbb{Z}$. Its elements are of the form*

$$g_r : \mathfrak{K}_n \to \mathfrak{K}_n, f(\zeta) \to f(\zeta^r), r \in (\mathbb{Z}/n\mathbb{Z})^*.$$

2.3 Crystallographic groups

In this section we define the frieze groups and the crystallographic groups of dimensions one and two.

2.3.1 Frieze groups

A frieze group is a discrete subgroup of the isometry group of the Euclidean plane $\mathbb{R}^2$ whose translation subgroup is infinitely cyclic. These groups are the groups of symmetries of certain types of ornaments infinitely repeated in one direction. Up to equivalence (scaling and shifting of patterns), there are seven such groups. The presentations of these groups are given below.

$$\begin{aligned}
p1 &= \langle x\rangle \cong \mathbb{Z}.\\
p1m1 &= \langle x, z \mid z^2 = 1, x^z = x^{-1}\rangle \cong \mathbb{Z} \rtimes C_2.\\
p11m &= \langle x, \beta \mid \beta^2 = 1, x\beta = \beta x\rangle \cong \mathbb{Z} \times C_2.\\
p11g &= \langle y\rangle \cong \mathbb{Z}.\\
p2 &= \langle x, \alpha \mid \alpha^2 = 1, x^\alpha = x^{-1}\rangle \cong \mathbb{Z} \rtimes C_2.\\
p2mg &= \langle y, \beta \mid \beta^2 = 1, x^\beta = x^{-1}\rangle \cong \mathbb{Z} \rtimes C_2.\\
p2mm &= \langle x, z, \beta \mid z^2 = 1 = \beta^2, x^\beta = x^{-1}, xz = zx\rangle \cong (\mathbb{Z} \rtimes C_2) \times C_2.
\end{aligned}$$

Note: x denotes the horizontal translation, y denotes the glide reflection, z denotes the vertical reflection, α denotes the 180° rotation and β denotes the reflection about the horizontal axis. A reflection in a line followed by a translation parallel to the same line is called a *glide reflection*.

Definition 2.3.1 (Discrete group). *A group G of isometries of Euclidean space $\mathbb{R}^n$ is said to be* discrete *if every orbit is a discrete set in $\mathbb{R}^n$, that is, for every point $p \in \mathbb{R}^n$ there is an open ball centered at p and containing no other point of the same orbit. The cyclic group generated by a single translation is an example of a discrete group.*

Definition 2.3.2 (Fundamental domain). *A* fundamental domain *D for the action of a group G on an n-dimensional Euclidean space $\mathbb{R}^n$ is a closed subset $D \subseteq \mathbb{R}^n$, such that every point of $\mathbb{R}^n$ belongs to the orbit of some point $x \in D$, and no two interior points of D belong to the same orbit.*

Definition 2.3.3 (Crystallographic groups). *An n-dimensional* crystallographic group *is a discrete group of isometries of an n-dimensional Euclidean space $\mathbb{R}^n$, which admits a bounded fundamental domain.*

Theorem 2.3.4. *Any crystallographic group is a group G of transformations of $\mathbb{R}^n$*

containing the translation group $\mathbb{Z}^n$ as a normal subgroup whose quotient $G/\mathbb{Z}^n$ is a finite group. The quotient $G/\mathbb{Z}^n$ is called a point group and is denoted by P.

Our main interest is to study and compute the rational representation zeta functions of two-dimensional crystallographic groups, also known as plane-crystallographic groups or wallpaper groups. These groups are the groups of symmetries of certain types of ornaments infinitely repeated in two different directions. There are seventeen such groups. For more details about crystallographic groups; see [16]. The presentations of these groups are given below.

$$p1 = \langle x, y \mid [x, y] = 1\rangle.$$

$$p2 = \langle x, y, r \mid [x, y] = r^2 = 1, x^r = x^{-1}, y^r = y^{-1}\rangle.$$

$$pm = \langle x, y, m \mid [x, y] = m^2 = 1, x^m = x, y^m = y^{-1}\rangle.$$

$$pg = \langle x, y, t \mid [x, y] = 1, t^2 = y, x^t = x^{-1}\rangle.$$

$$p2mm = \langle x, y, p, q \mid [x, y] = [p, q] = 1 = p^2 = q^2, x^p = x, x^q = x^{-1}, y^p = y^{-1}, y^q = y\rangle.$$

$$p2mg = \langle x, y, m, t \mid [x, y] = t^2 = 1, m^2 = y, x^t = x, x^m = x^{-1}, y^t = y^{-1}, m^t = m^{-1}\rangle.$$

$$p2gg = \langle x, y, u, v \mid [x, y] = 1 = (uv)^2, u^2 = x, v^2 = y, x^v = x^{-1}, y^u = y^{-1}\rangle.$$

$$cm = \langle x, y, t \mid [x, y] = t^2 = 1, x^t = xy, y^t = y^{-1}\rangle.$$

$$c2mm = \langle x, y, m, r \mid [x, y] = r^2 = 1 = m^2, x^m = xy, x^r = x^{-1}, y^m = y^{-1},$$
$$y^r = y^{-1}, r^m = r^{-1}\rangle.$$

$$p4 = \langle x, y, r \mid [x, y] = r^4 = 1, x^r = y, y^r = x^{-1}\rangle.$$

$$p4mm = \langle x, y, r, m \mid [x, y] = r^4 = 1 = m^2, x^r = x^m = y, y^r = x^{-1}, r^m = r^{-1}\rangle.$$

$$p4gm = \langle x, y, r, t \mid [x, y] = r^4 = 1 = t^2, x^r = y = x^t, y^r = x^{-1}, r^t = r^{-1}x^{-1}\rangle.$$

$$p3 = \langle x, y, r \mid [x, y] = r^3 = 1, x^r = x^{-1}y, y^r = x^{-1}\rangle.$$

$$p31m = \langle x, y, r, t \mid [x, y] = r^2 = 1 = t^2 = (tr)^3, x^r = x, x^t = x^{-1}y, y^t = y, y^r = xy^{-1}\rangle.$$

$$p3m1 = \langle x, y, r, m \mid [x, y] = r^3 = 1 = m^2, x^m = r^{-1}, x^r = x^{-1}y, y^r = x^{-1},$$
$$y^m = x^{-1}y, x^m = x^{-1}\rangle.$$

$$p6 = \langle x, y, r \mid [x, y] = r^6 = 1, x^r = y, y^r = x^{-1}y\rangle.$$

$$p6mm = \langle x, y, r, m \mid [x, y] = r^6 = 1 = m^2, x^r = y, x^m = x^{-1}, y^r = x^{-1}y,$$
$$y^m = x^{-1}y, r^m = r^{-1}y\rangle.$$

Lemma 2.3.5. *The set of all translations belonging to a plane crystallographic group G is a subgroup H of type p1. The quotient group G/H is finite and belongs to one of the ten types C_n or D_{2n}, where n=1,2,3,4,6.*

For the proof of the above lemma; see [14, Chapter 5, Lemma 2&3].

2.4 Clifford theory and Schur index

Let $\mathrm{Irr}(G)$ be the set of irreducible characters of a finite group G and $\mathrm{Irr}(T)$ be the set of irreducible characters of a subgroup T of G which is normal in G. Clifford theory can be used to get irreducible characters of G from the irreducible characters of T. We take the irreducible characters of the normal group T and then we induce them to G. Let $\chi \in \mathrm{Irr}(T)$ and $T \trianglelefteq G$ and $g \in G$. We define $\chi^g : T \longrightarrow \mathbb{C}$ by $\chi(h^{g^{-1}}) = \chi(ghg^{-1})$, where $h \in H$. χ^g is conjugated to χ in G.

Let $\mathbb{Q}$ be the field of rational numbers and $\overline{\mathbb{Q}}$ be the field of algebraic numbers, the algebraic closure of $\mathbb{Q}$. The group of automorphism of $\overline{\mathbb{Q}}$, denoted by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is called the absolute Galois group of $\mathbb{Q}$. Let $\mathrm{Irr}(G)$, consists of all the irreducible complex characters of the group G. Let $g \in G$, $\chi \in \mathrm{Irr}(G)$ and $\alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then the right Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Irr}(G)$ is given by $(g)\chi^\alpha = ((g)\chi)^\alpha$.

Definition 2.4.1. *Let T be a normal subgroup of the finite group G and let $\chi \in \mathrm{Irr}(T)$. Then*

$$I_G(\chi) = \{g \in G \mid \chi^g = \chi\}$$

is the inertia group *of χ in G.*

Lemma 2.4.2. *(Frobenius reciprocity) Let T be a subgroup of the finite group G and suppose that ψ is a character of G and that χ is a character of T. Then*

$$[\chi, \psi_T] = [\chi^G, \psi].$$

For a proof; see [10, Chapter 5, Lemma 5.2]. The notation $[,]$ is used to denote an inner product. For characters χ and ψ of a finite group G, the inner product is defined as

$$[\chi, \psi] = \frac{1}{\mid G \mid} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

If $\chi = \sum_{i=1}^{k} n_i \chi_i$ is a character with $\chi_i \in \mathrm{Irr}(G)$, then those χ_i with $n_i > 0$ are called the *irreducible constituents* of χ. In general, if ψ is a character such that

$\chi - \psi$ is also a character or zero, then ψ is called a *constituent* of χ.

In the following G always denotes a finite group.

Theorem 2.4.3. *(Clifford) Let $T \trianglelefteq G$ and let ψ be an irreducible character of G. Let χ be an irreducible constituent of ψ_T and suppose that $\chi = \chi_1, \chi_1, \ldots, \chi_t$ are the distinct conjugate of χ in G. Then*

$$\psi_T = e \sum_{i=1}^{t} \chi_i, \tag{2.4.1}$$

where $e = [\psi_T, \chi]$.

Proof. See [21, Chapter 6, Theorem 6.2] $\square$

Theorem 2.4.4. *[21, Chapter 6, Theorem 6.11] Let $T \trianglelefteq G$, $\chi \in \mathrm{Irr}(T)$ and $H = I_G(\chi)$. Let*

$$\mathfrak{A} = \{\vartheta \in \mathrm{Irr}(H) \mid [\vartheta_T, \chi] \neq 0\}, \mathfrak{B} = \{\psi \in \mathrm{Irr}(G) \mid [\psi_T, \chi] \neq 0\}.$$

Then

1. *If $\vartheta \in \mathfrak{A}$, then ϑ^G is irreducible.*
2. *The map $\vartheta \longmapsto \vartheta^G$ is a bijection of $\mathfrak{A}$ onto $\mathfrak{B}$.*
3. *If $\vartheta^G = \psi$, with $\vartheta \in \mathfrak{A}$, then ϑ is the unique irreducible constituent of ψ_H which lies in $\mathfrak{A}$.*
4. *If $\vartheta^G = \psi$, with $\vartheta \in \mathfrak{A}$, then $[\vartheta_T, \chi] = [\psi_T, \chi]$.*

Theorem 2.4.5. *[21, Chapter 6, Theorem 6.16] Let $T \trianglelefteq G$ and $\chi_1, \chi_2 \in \mathrm{Irr}(T)$ be invariant under the action of G. Assume $\chi_1\chi_2$ is irreducible and that χ_2 extends to $\psi \in \mathrm{Irr}(G)$. Let*

$$\mathfrak{A} = \{\beta \in \mathrm{Irr}(G) \mid [\chi_1^G, \beta] \neq 0\} \text{ and } \mathfrak{B} = \{\gamma \in \mathrm{Irr}(G) \mid [(\chi_1\chi_2)^G, \gamma] \neq 0\}.$$

Then $\beta \longmapsto \beta\psi$ defines a bijection from $\mathfrak{A}$ onto $\mathfrak{B}$.

Corollary 2.4.6. *[21, Chapter 6, Theorem 6.17] Let $T \trianglelefteq G$ and $\psi \in \mathrm{Irr}(G)$ be such that $\psi_T = \chi \in \mathrm{Irr}(T)$. Then the characters $\beta\psi$ for $\beta \in \mathrm{Irr}(G/T)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of χ^G.*

Suppose for every character $\chi \in \mathrm{Irr}(T)$, there exists a character $\vartheta \in \mathrm{Irr}(I_G(\chi))$ such that $\vartheta_T = \chi$. Then the representation zeta function is

$$\begin{aligned}
\zeta_G(s) &= \sum_{\psi \in \mathrm{Irr}(G)} \psi(1)^{-s} && (2.4.2)\\
&= \sum_{\chi \in \mathrm{Irr}(T)} \sum_{\substack{\psi \in \mathrm{Irr}(G)\\ \text{such that } [\psi_T,\chi]\neq 0}} \mid G : I_G(\chi) \mid^{-1} \psi(1)^{-s} && (2.4.3)\\
&= \sum_{\chi \in \mathrm{Irr}(T)} \mid G : I_G(\chi) \mid^{-1} \sum_{\substack{\psi \in \mathrm{Irr}(G)\\ \text{such that } [\psi_T,\chi]\neq 0}} \psi(1)^{-s} && (2.4.4)\\
&= \sum_{\chi \in \mathrm{Irr}(T)} \mid G : I_G(\chi) \mid^{-1} \sum_{\substack{\vartheta \in \mathrm{Irr}(I_G(\chi))\\ \text{such that } [\vartheta_T,\chi]\neq 0}} (\vartheta(1) \cdot \mid G : I_G(\chi) \mid)^{-s} && (2.4.5)\\
&= \sum_{\chi \in \mathrm{Irr}(T)} \mid G : I_G(\chi) \mid^{-1-s} \sum_{\vartheta_1 \in \mathrm{Irr}(I_G(\chi)/T)} (\chi(1) \cdot \vartheta_1(1))^{-s} && (2.4.6)\\
&= \sum_{\chi \in \mathrm{Irr}(T)} \chi(1)^{-s} \cdot \mid G : I_G(\chi) \mid^{-1-s} \sum_{\vartheta_1 \in \mathrm{Irr}(I_G(\chi)/T)} \vartheta_1(1)^{-s}. && (2.4.7)
\end{aligned}$$

Note: The equation 2.4.7 is satisfied, if one of the following sufficient conditions holds:

1. $\mid I_G(\chi) : T \mid$ is always a prime.
2. $I_G(\chi)/T$ is always cyclic.
3. $\chi(1)$ is always a prime power and the character χ extends to a Sylow-p subgroup of $I_G(\chi)/T$ for the same prime p.

Corollary 2.4.7. *[21, Chapter 11, Corollary 11.22] Let $T \trianglelefteq G$ with G/T cyclic and let $\chi \in \mathrm{Irr}(T)$ be invariant in G. Then χ is extendable in G.*

Theorem 2.4.8. *[17, 4.5] Let F be a subfield of a splitting field $E \subseteq \mathbb{C}$ for G and let $\chi \in \mathrm{Irr}(G)$. Then there exists a unique positive integer $m_F(\chi)$ such that the* *character*

$$m_F(\chi) \sum_{\sigma \in \mathrm{Gal}(F(\chi)|F)} \chi^\sigma$$

is afforded by an F-irreducible representation. Furthermore, every F-irreducible character is of this form (for a suitable choice of χ). The extension $F(\chi)/F$ is Galois with abelian Galois group $\mathrm{Gal}(F(\chi)/F)$ *acting in the natural way on $\chi : G \longrightarrow F(\chi)$.*

The integer $m_F(\chi)$ is called the *Schur index* of the character χ over F and is not dependent on E. If $F(\chi) \subseteq L$, is another splitting field, then $m_F(\chi)$ is the same when computed in E or L; see [21, Chapter 9] for more detail.

Theorem 2.4.9. *Let $F \subseteq E \subseteq \mathbb{C}$, where E is a splitting field for G. Let ρ be an irreducible representation of G over F. Then*

1. *The irreducible constituent of ρ^E, that is ρ regarded as a representation over E, all occur with equal multiplicity $m = m_F(\chi)$, where $\chi \in \mathrm{Irr}(G)$ is the character of one such constituent.*

2. *The characters $\chi_i \in \mathrm{Irr}(G)$ afforded by the irreducible constituents of ρ^E constitute a Galois conjugacy class over F, and so the fields $F(\chi_i)$ are all equal. The irreducible constituents of $\rho^{F(\chi_i)}$ occur with multiplicity* 1.

3. *If γ is any irreducible constituent of $\rho^{F(\chi_i)}$ then γ^E has a unique irreducible constituent. Its multiplicity is m.*

4. *$\rho^{F(\chi_i)}$ and γ^E are completely reducible.*

For the proof see [21, Chapter 9, Theorem 9.21]

Corollary 2.4.10. *[21, Chapter 10, Corollary 10.2] Let $\chi \in \mathrm{Irr}(G)$ and $F \subseteq \mathbb{C}$. We have*

1. *$m_F(\chi) = m_{F(\chi)}(\chi)$.*

2. *Let $\chi_1, \chi_2, \ldots, \chi_r$ be the Galois conjugacy class of χ over F. Then $m_F(\chi) \sum_{i=1}^{r} \chi_i$ is the character of an irreducible F-representation of G.*

3. *If ϑ is the character of any F-representation, then $m_F(\chi)$ divides $[\vartheta, \chi]$.*

4. *$m_F(\chi)$ is the smallest integer m, such that $m\chi$ is afforded by an $F(\chi)$-representation.*

5. *$m_F(\chi)$ is the unique integer m, such that $m\chi$ is afforded by an irreducible $F(\chi)$-representation.*

6. *If $F \subseteq E \subseteq \mathbb{C}$, then $m_E(\chi)$ divides $m_F(\chi)$.*

7. *If $F \subseteq E \subseteq \mathbb{C}$ and $|E : F| = n < \infty$, then $m_F(\chi)$ divides $nm_E(\chi)$.*

8. *$m_F(\chi)$ divides $\chi(1)$.*

Theorem 2.4.11. *(Brauer) Let G have exponent n and let $F = \mathbb{Q}(e^{2\pi i/n})$. Then F is a splitting field for G and every $\chi \in \mathrm{Irr}(G)$ is afforded by an F-representation.*

For the proof; see [21, Chapter 10,Theorem 10.3].
With this background informations, we can now state the application that is important for our computation of rational representation zeta functions.

Theorem 2.4.12. *Let G be a group that is rationally representation rigid, that is assume that $r_n^{\mathbb{Q}}(G) < \infty$ for each $n \in \mathbb{N}$. Then the rational representation zeta function of G satisfies*

$$\zeta_G^{\mathbb{Q}}(s) = \sum_{\chi \in \mathrm{Irr}(\widehat{G})} \mid \chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \mid^{-1} \left(m_{\mathbb{Q}}(\chi) \cdot \chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \cdot \chi(1) \right)^{-s}.$$

Proof. Recall that

$$\zeta_G^{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} r_n^{\mathbb{Q}}(G) n^{-s},$$

where $r_n^{\mathbb{Q}}(G)$ is the number of continuous irreducible $\mathbb{Q}$-representations of dimension n of the profinite completion $\widehat{G}$. These continuous representations of $\widehat{G}$ are precisely the representations factoring through a finite quotient of G. The claim now follows from Theorem 2.4.8 as follows.

Each irreducible $\mathbb{Q}$-representation ρ of G, factoring through a finite quotient of G, has a character of the form

$$m_{\mathbb{Q}}(\chi) \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \chi^{\sigma},$$

where $\chi \in \mathrm{Irr}(\widehat{G})$. Evaluating at 1, we obtain the dimension of ρ. Observe that the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$- and $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$-orbit of χ are the same. Also observe that $\chi^{\sigma}(1) = \chi(1)$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. Thus

$$\dim \rho = m_{\mathbb{Q}}(\chi) \cdot \mid \chi^{\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \mid \cdot \chi(1).$$

Summing over all $\chi \in \mathrm{Irr}(\widehat{G})$ and adding a weight factor $\mid \chi^{\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \mid^{-1}$ to make up for overcounting, we arrive at the claimed formula. $\square$

Corollary 2.4.13. *Let G be as in the Theorem 2.4.12 and suppose that $m_{\mathbb{Q}}(\chi) = 1$ for all $\chi \in \mathrm{Irr}(\widehat{G})$. Then*

$$\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1).$$

Proof. It suffices to recall that

$$\omega_G^{\mathbb{Q}}(s,t) = \sum_{\chi \in \mathrm{Irr}(\widehat{G})} \chi(1)^{-s} \mid \chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \mid^{-t},$$

and

$$\zeta_G^{\mathbb{Q}}(s) = \sum_{\chi \in \mathrm{Irr}(\widehat{G})} \mid \chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \mid^{-1} \left(m_{\mathbb{Q}}(\chi) \cdot \chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \cdot \chi(1) \right)^{-s}.$$

By using $m_{\mathbb{Q}}(\chi) = 1$ and $t = s+1$, we get the required result. $\square$

Proposition 2.4.14. *Let A be a finite abelian group, and let $d \in \mathbb{N}$. Then there is a one-to-one correspondence between*

1. *irreducible rational representations of A of degree d,*

2. *Galois orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $A^{\vee} = \mathrm{Hom}(A, \mathbb{C}^*)$ of length d.*

The action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ factors through the finite Galois group $\mathrm{Gal}(\mathfrak{K}_n/\mathbb{Q})$, if A has the exponent n.

The dual $A^\vee = \mathrm{Hom}(A, \mathbb{C}^*)$ consists of all the irreducible complex characters of the group A, which correspond to one-dimensional irreducible representations over $\mathbb{C}$.

Let A be a finite abelian group and ρ be an irreducible $\mathbb{Q}$-representation of A. Let $\mathbb{Q} \subseteq \mathfrak{K}_n$, where $\mathfrak{K}_n$, the nth cyclotomic field, is a splitting field for A. Then from Corollary 2.4.10(2) we have

$$\rho = m \sum_i \chi_i,$$

where $m = m_{\mathbb{Q}}(\chi_1)$, the summation is taken over a Galois orbit of an irreducible component χ_1 of ρ over $\mathfrak{K}_n$. Since A is abelian then χ_1 is one-dimensional and

$$\mathrm{degree}(\rho) = \rho(1) = m \cdot \rho_1 \cdot \chi_1(1),$$

where ρ_1 denotes the size of the Galois orbit of χ_1. Corollary 2.4.10(8) says $m \mid \chi_1(1)$ implies $m = 1$, as $\chi_1(1) = 1$. Hence the degree of ρ is equal to the length of the orbit.

Lemma 2.4.15. *Let T be a subgroup of G and suppose T has a complement in G. Let $\chi \in \mathrm{Irr}(T)$ and suppose $\chi^G = \psi \in \mathrm{Irr}(G)$. Then $m_F(\psi)$ divides $\chi(1)$ for every subfield $F \subseteq \mathbb{C}$.*

Proof. See [21, Chapter 10, Lemma 10.8]. □

From the above lemma we infer the following corollary.

Corollary 2.4.16. *The Schur index $m_F(\psi)$ is one if the character χ has degree one.*

Definition 2.4.17. *[21, Chapter 10, Definition 10.5] Let F be a subfield of $\mathbb{C}$. Then (H, X, ϑ) is an F-triple provided*

1. *H is a group, $X \trianglelefteq H$, $X = C_H(X)$.*

2. *$\vartheta \in \mathrm{Irr}(H)$ is faithful.*

3. *the irreducible (linear) constituents of ϑ_X are Galois conjugate over $F(\vartheta)$.*

Corollary 2.4.18. *[21, Chapter 10, Corollary 10.6] Let (H, X, ϑ) be an F-triple and let λ be a linear constituent of ϑ_X. Then*

1. *λ is faithful and X is cyclic;*

2. *$I_H(\lambda) = X$, $\lambda^H = \vartheta$ and $F(\vartheta) \subseteq F(\lambda)$;*

3. *ϑ_X is afforded by an irreducible $F(\vartheta)$-representation;*

4. *$H/X \cong \mathrm{Gal}(F(\lambda)/F(\vartheta))$.*

Definition 2.4.19. *A section of a group G is the group of the form H/X, where H is a subgroup of G and X is a normal subgroup of H.*

Theorem 2.4.20. *[21, Chapter 10, Theorem 10.7] Let $\chi \in \mathrm{Irr}(G)$ and $F \subseteq \mathbb{C}$. Assume p^a divides $m_F(\chi)$ for some prime p. Then there exists an F-triple (H, X, ϑ) such that*

1. *H is a section of G;*

2. *$p^a \mid m_F(\vartheta)$;*

3. *H/X is a p-group;*

4. *$p \nmid |F(\chi, \vartheta) : F(\chi)|$.*

2.5 Chinese remainder theorem and solution of congruences

In this section we review the basic notions of the specific part of number theory that we need. We use [5] as our main reference; see also [7] and [9].

Theorem 2.5.1. *Suppose $n_1, n_2, \dots, n_r$ are pairwise relatively prime. Then the system of congruences*

$$\begin{aligned} x &\equiv a_1 \bmod n_1, \\ x &\equiv a_2 \bmod n_2, \\ &\vdots \\ x &\equiv a_r \bmod n_r, \end{aligned} \tag{2.5.1}$$

has the unique solution mod $n_1 n_2 \dots n_r$.

Proof. See [5, Chapter 5]. □

Corollary 2.5.2. *Let $a_1, a_2, \dots, a_r$ be positive integers such that $\gcd(a_i, a_j) = 1$ for $i \neq j$, then*

$$\operatorname{lcm}(a_1, a_2, \dots, a_r) = a_1 a_2 \cdots a_r.$$

Lemma 2.5.3. *Let n_1, n_2 be positive integers such that $\gcd(n_1, n_2) = 1$. If $(n_1, n_2)|(a_1 - a_2)$ then the system*

$$\begin{aligned} x &\equiv a_1 \bmod n_1, \\ x &\equiv a_2 \bmod n_2, \end{aligned} \tag{2.5.2}$$

has a unique solution modulo $l = \operatorname{lcm}(n_1, n_2)$.

Lemma 2.5.4. *The congruences $a \equiv 1 \bmod n_1$ and $a \equiv -1 \bmod n_2$ have a common solution if and only if $g = \gcd(n_1, n_2) \in \{1, 2\}$. Moreover, any such solution is unique modulo $l = \operatorname{lcm}(n_1, n_2)$.*

Proof. Write $n_1 = g\tilde{n_1}$ and $n_2 = g\tilde{n_2}$. Then $a \equiv 1 \bmod n_1$ and $a \equiv -1 \bmod n_2$ implies $1 \equiv a \equiv -1 \bmod g$. Hence $g \in \{1, 2\}$ is a necessary condition for the solubility of the pair of congruences.

If $g = 1$, the Chinese remainder theorem yields a solution.

Now suppose that $g = 2$, Then we have two cases either $2 \nmid \tilde{n_1}$ or $2 \nmid \tilde{n_2}$.

<u>case 1. $2 \nmid \tilde{n_1}$</u>

If $2 \nmid \tilde{n_1}$ then $\gcd(\tilde{n_1}, 2\tilde{n_2}) = 1$. We need to find out x such that

$$1 + 2\tilde{n_1}x \equiv -1 \bmod 2\tilde{n_2}$$

$$\text{equivalently } 2\tilde{n_1}x \equiv -2 \bmod 2\tilde{n_2}$$

$$\text{equivalently } \tilde{n_1}x \equiv -1 \bmod \tilde{n_2}.$$

Since $\gcd(\tilde{n_1}, \tilde{n_2}) = 1$, there is such an x. Put $a = 1 + 2\tilde{n_1}x$, then

$$a \equiv 1 \bmod n_1 = 2\tilde{n_1}$$

$$\text{and } a \equiv -1 \bmod n_2 = 2\tilde{n_2}.$$

Moreover, a is uniquely determined modulo $l = \mathrm{lcm}(n_1, n_2)$.

case 2. $2 \mid \tilde{n_1}$

If $2 \mid \tilde{n_1}$, then $2 \nmid \tilde{n_2}$. A similar argument as in case 1 applies. $\square$

Chapter 3

Finitely generated abelian groups

3.1 Finitely generated abelian groups

In this chapter we compute the rational representation zeta functions of some of the finitely-generated abelian groups.

Note: For an abelian group G, the Schur index of all $\chi \in \mathrm{Irr}(G)$ is one; see Corollary 2.4.16. So throughout Chapter 3, we compute the rational representation zeta functions because

$$\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1),$$

where $\zeta_G^{\mathbb{Q}}(s)$, denotes the rational representation zeta function of group G and $\omega_G^{\mathbb{Q}}(s,t)$, denotes the Galois orbit zeta function of the group G; see Corollary 2.4.13.

The main theorem which deals with finitely generated torsion-free abelian groups is

Theorem 3.1.1. *Let $G = \mathbb{Z}^d$ be the free abelian group of rank d. Then the rational representation zeta function of G admits an Euler product and is equal to*

$$\zeta_G^{\mathbb{Q}}(s) = \prod_p \left(1 + \frac{p^d - 1}{p-1}(p-1)^{-s}\frac{1}{1 - p^{(d-1)-s}}\right).$$

Before proving Theorem 3.1.1 we need some preliminary results.

Proposition 3.1.2. *Let $G = \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_d\mathbb{Z}$ be a finite abelian group with $n_1 \mid n_2 \mid \cdots \mid n_d = n$. Then there are*

- *a group isomorphism $\Phi\colon \mathcal{G} = \mathrm{Gal}(\mathfrak{K}_n/\mathbb{Q}) \to (\mathbb{Z}/n\mathbb{Z})^*$,*
- *a bijection $\Psi\colon G^\vee = \mathrm{Hom}(G, \mathbb{C}^*) \to G$*

providing an equivariance between the Galois action of $\mathcal{G}$ on $G^\vee$, given by

$$G^\vee \times \mathcal{G} \to G^\vee, \quad (\chi, \alpha) \mapsto \chi^\alpha, \text{ where } \chi^\alpha(r) = (\chi(r))^\alpha \text{ for } r \in G,$$

and the diagonal action of $(\mathbb{Z}/n\mathbb{Z})^*$ on G, given by

$$G \times (\mathbb{Z}/n\mathbb{Z})^* \to G, \quad ((\overline{e_1}, \overline{e_2}, \ldots, \overline{e_d}), \overline{j}) \mapsto (\overline{e_1 j}, \overline{e_2 j}, \ldots, \overline{e_d j}).$$

This means that the following diagram commutes:

$$\begin{tikzcd}
G^\vee \times \mathcal{G} \arrow[r, "\textit{Galois action}"] \arrow[d, "\Psi \times \Phi"'] & G^\vee \arrow[d, "\Psi"] \\
G \times (\mathbb{Z}/n\mathbb{Z})^* \arrow[r, "\textit{diagonal action}"] & G
\end{tikzcd} \tag{CD}$$

Proof. Choosing a primitive nth root of 1, say $\xi = e^{2\pi i/n}$, we obtain by Theorem 2.2.4 an isomorphism $\Phi\colon \mathcal{G} \to (\mathbb{Z}/n\mathbb{Z})^*$ such that $\alpha_j \in \mathcal{G}$ mapping to $\Phi(\alpha_j) = \overline{j} \in (\mathbb{Z}/n\mathbb{Z})^*$ acts as $\xi^{\alpha_j} = \xi^j$. The dual $G^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$ consists of all group homomorphism $\chi\colon G \to \mathbb{C}^*$ from G into the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers. Since G has exponent n, for every $\chi \in G^\vee$ the image $\chi(r)$ of an element $r \in G$ of order $m \mid n$ lies in the subgroup of $\langle \xi^{n/m} \rangle$ of $\mathbb{C}^*$. Put $\xi_i = \xi^{n/n_i}$ for $i \in \{1, \ldots, d\}$. Choosing generators $r_1, r_2, \ldots, r_d$ of G such that $G = \langle r_1 \rangle \times \cdots \times \langle r_d \rangle$ with $\langle r_i \rangle \cong \mathbb{Z}/n_i\mathbb{Z}$ for $i \in \{1, \ldots, d\}$, we obtain two bijections:

$$\Psi_1\colon G^\vee \to \langle \xi_1 \rangle \times \cdots \times \langle \xi_d \rangle, \quad \chi \mapsto (\chi(r_1), \ldots, \chi(r_d))$$

and

$$\Psi_2\colon \langle \xi_1 \rangle \times \cdots \times \langle \xi_d \rangle \to G, \quad (\xi_1^{e_1}, \ldots, \xi_d^{e_d}) \mapsto (\overline{e_1}, \ldots, \overline{e_d}).$$

Table 3.1: Length and number of orbits

length of orbits	number of orbits of this length
1	1
$p-1$	$\frac{p^d-1}{p-1}$
$(p-1)p$	$p^{d-1}\frac{p^d-1}{p-1}$
.	.
.	.
.	.
$(p-1)p^{k-1}$	$p^{(k-1)(d-1)}\frac{p^d-1}{p-1}$

The composition of Ψ_1 and Ψ_2 yields a bijection $\Psi\colon G^\vee \to G$. We need to check that Φ and Ψ, defined as above, set up the desired equivariance. This can be verified via the following computation, resulting in the commutative diagram (CD):

$$
\begin{tikzcd}
(\chi, \alpha_j) \arrow[r, "\text{Galois action}"] \arrow[d, "\Psi_1 \times \mathrm{id}"] & \chi^{\alpha_j} \arrow[d, "\Psi_1"] \\
(\chi(r_1), \ldots, \chi(r_d), \alpha_j) \arrow[d, equal] & (\chi(r_1)^{\alpha_j}, \ldots, (\chi(r_d)^{\alpha_j}) \arrow[d, equal] \\
(\xi_1^{e_1}, \ldots, \xi_d^{e_d}, \alpha_j) \arrow[d, "\Psi_2 \times \Phi"] & ((\xi_1^{e_1})^{\alpha_j}, \ldots, (\xi_d^{e_d})^{\alpha_j}) = (\xi_1^{e_1 j}, \ldots, \xi_d^{e_d j}) \arrow[d, "\Psi_2"] \\
((\overline{e_1}, \ldots, \overline{e_d}), \bar{j}) \arrow[r, "\text{diagonal action}"] & (\overline{e_1 j}, \ldots, \overline{e_d j})
\end{tikzcd}
$$

Here $(\chi, \alpha_j) \in G^\vee \times \mathcal{G}$ and the computations along the two different paths yield the result $(\overline{e_1 j}, \ldots, \overline{e_d j}) \in G$. □

Proposition 3.1.3. *Let p be a prime, and let $k \in \mathbb{N}$. Consider the diagonal action of $(\mathbb{Z}/p^k\mathbb{Z})^*$ on the homocyclic p-group $(\mathbb{Z}/p^k\mathbb{Z})^d$ given by $(\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}) \cdot \bar{x} = (\bar{a_1}\bar{x}, \bar{a_2}\bar{x}, \ldots, \bar{a_d}\bar{x})$ for $(\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}) \in (\mathbb{Z}/p^k\mathbb{Z})^d$ and $\bar{x} \in (\mathbb{Z}/p^k\mathbb{Z})^*$. Then the number of orbits of any given length is as in Table* 3.1.

Proof. We prove the values given in Table 3.1 are correct by using induction on k. Induction base: $k = 1$.

Table 3.2: Table for induction base

length of orbits	number of orbits of this length
1	1
$p-1$	$\frac{p^d-1}{p-1}$

We consider the action of $(\mathbb{Z}/p\mathbb{Z})^*$ on $(\mathbb{Z}/p\mathbb{Z})^d$. Let

$$\bar{a} = (\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}) \in (\mathbb{Z}/p\mathbb{Z})^d.$$

There are two possibilities for $\bar{a}$: either $\bar{a} = (\bar{0}, \bar{0}, \ldots, \bar{0})$ or $\bar{a} \neq (\bar{0}, \bar{0}, \ldots, \bar{0})$. If $\bar{a} = (\bar{0}, \bar{0}, \ldots, \bar{0})$ then for any $\bar{x} \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $\bar{a}\bar{x} = \bar{0}$ hence the orbit of $\bar{a} = (\bar{0}, \bar{0}, \ldots, \bar{0})$ has length 1.

Now suppose that $\bar{a} \neq (\bar{0}, \bar{0}, \ldots, \bar{0})$. Then we may assume without loss of generality that $\bar{a_1} \neq \bar{0}$. This implies that $a_1, 2a_1, \ldots, (p-1)a_1$ are distinct, hence

$$\bar{a}, 2\bar{a}, \ldots, (p-1)\bar{a}$$

are distinct and form an orbit of length $p-1$. The total number of such orbits of length $p-1$ is equal to $\frac{p^d-1}{p-1}$. The situation is summarised in Table 3.2.

Induction step: $k \geq 2$.

We consider the action of $(\mathbb{Z}/p^k\mathbb{Z})^*$ on $(\mathbb{Z}/p^k\mathbb{Z})^d$. Let $\bar{a} = (\bar{a_1}, \ldots, \bar{a_d}) \in (\mathbb{Z}/p^k\mathbb{Z})^d$. First suppose that $\bar{a} \equiv (\bar{0}, \bar{0}, \ldots, \bar{0})$ modulo p. Then $\bar{a} \in (p\mathbb{Z}/p^k\mathbb{Z})^d$. As $(p\mathbb{Z}/p^k\mathbb{Z})^d \cong (\mathbb{Z}/p^{k-1}\mathbb{Z})^d$ we can use induction to determine the length of the orbit of $\bar{a}$. Counting the number of orbits of these types of elements according to their length results in the first k rows of Table 3.3. Now suppose that $\bar{a} \not\equiv (\bar{0}, \bar{0}, \ldots, \bar{0})$ modulo p. Similarly as before, we may assume without loss of generality that $\bar{a_1} \not\equiv \bar{0}$ modulo p, and hence a unit in $(\mathbb{Z}/p^k\mathbb{Z})^*$. Looking at the first coordinate, we conclude that $\bar{a} \cdot \bar{x} \neq \bar{a} \cdot \bar{y}$ for $\bar{x}, \bar{y} \in (\mathbb{Z}/p^k\mathbb{Z})^*$ with $\bar{x} \neq \bar{y}$. This means that the orbit of $\bar{a}$ has length $|(\mathbb{Z}/p^k\mathbb{Z})^*| = (p-1)p^{k-1}$. There are $|(\mathbb{Z}/p^k\mathbb{Z})^d| - |(\mathbb{Z}/p^{k-1}\mathbb{Z})^d|$ elements of this kind, falling into orbits of length $(p-1)p^{k-1}$. Hence there are

$$\frac{p^{kd} - p^{(k-1)d}}{(p-1)p^{k-1}} = p^{(k-1)(d-1)} \frac{p^d - 1}{p - 1}$$

Table 3.3: Table for induction step

length of orbits	number of orbits of this length
1	1
$p-1$	$\frac{p^d-1}{p-1}$
$(p-1)p$	$p^{d-1}\frac{p^d-1}{p-1}$
.	.
.	.
.	.
$(p-1)p^{k-2}$	$p^{(k-2)(d-1)}\frac{p^d-1}{p-1}$
$(p-1)p^{k-1}$	$p^{(k-1)(d-1)}\frac{p^d-1}{p-1}$

Table 3.4: Degree and number of irreducible rational representations

Degree	number of Irreducible rational representations of this degree
1	1
$p-1$	$\frac{p^d-1}{p-1}$
$(p-1)p$	$p^{d-1}\frac{p^d-1}{p-1}$
.	.
.	.
.	.
$(p-1)p^{k-2}$	$p^{(k-2)(d-1)}\frac{p^d-1}{p-1}$
$(p-1)p^{k-1}$	$p^{(k-1)(d-1)}\frac{p^d-1}{p-1}$

orbits of this kind. The situation is summarised in Table 3.3. □

Proposition 3.1.4. *Let p be the prime and let $k \in \mathbb{N}$. Consider the finite homocyclic p-group $G = (\mathbb{Z}/p^k\mathbb{Z})^d$. Then the number of (equivalence classes of) irreducible rational representations of G of any given degree is given in Table* 3.4.

Proof. For an abelian group, there is a one-to-one correspondence between the Galois orbits of length d and the irreducible rational representations of degree d; see Proposition 2.4.14. Using Propositions 3.1.2 and 3.1.3 we get the required result. □

Proposition 3.1.5. *Let $n \in \mathbb{N}$ with prime factorisation $n = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$. Consider* *the diagonal action of* $(\mathbb{Z}/n\mathbb{Z})^*$ *on the homocyclic group* $(\mathbb{Z}/n\mathbb{Z})^d$ *given by*

$$(\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}) \cdot \bar{x} = (\bar{a_1}\bar{x}, \bar{a_2}\bar{x}, \ldots, \bar{a_d}\bar{x})$$

for $(\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}) \in (\mathbb{Z}/n\mathbb{Z})^d, \bar{x} \in (\mathbb{Z}/n\mathbb{Z})^*$. *Then the natural maps*

$$(\mathbb{Z}/n\mathbb{Z})^* \xrightarrow{\cong} \prod_{i=1}^{r} (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^*, x + n\mathbb{Z} \mapsto (x + p_i^{e_i}\mathbb{Z})_{i=1}^r$$

$$(\mathbb{Z}/n\mathbb{Z})^d \xrightarrow{\cong} \prod_{i=1}^{r} (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^d, (a_1 + n\mathbb{Z}, \ldots, a_d + n\mathbb{Z}) \mapsto (a_1 + p_i^{e_i}\mathbb{Z}, \ldots, a_d + p_i^{e_i}\mathbb{Z})_{i=1}^r$$

provide an equivalence between

1. *the diagonal action of* $(\mathbb{Z}/n\mathbb{Z})^*$ *on* $(\mathbb{Z}/n\mathbb{Z})^d$.
2. *the product action of* $\prod (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^*$ *on* $\prod (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^d$

with each factor $(\mathbb{Z}/p_i^{e_i}\mathbb{Z})^*$ *acting diagonally on* $(\mathbb{Z}/p_i^{e_i}\mathbb{Z})^d$.

Proof. This is a consequence of the Chinese Remainder Theorem. □

We are now ready to prove Theorem 3.1.1 which we restate.

Theorem 3.1.6. *Let* $G = \mathbb{Z}^d$ *be the free abelian group of rank* d. *Then the rational representation zeta function of* G *admits an Euler product and is equal to*

$$\zeta_G^{\mathbb{Q}}(s) = \prod_p \left(1 + \frac{p^d - 1}{p - 1}(p-1)^{-s} \frac{1}{1 - p^{(d-1)-s}}\right).$$

Proof. By Proposition 3.1.5 we know that for every $n \in \mathbb{N}$ there is an equivalence between the diagonal action of $(\mathbb{Z}/n\mathbb{Z})^*$ on $(\mathbb{Z}/n\mathbb{Z})^d$ and the product action of $\prod (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^*$ on $\prod (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^d$, where $n = p_1^{e_1} \cdots p_r^{e_r}$ is the prime factorisation of n. Proposition 3.1.4 give us the possible lengths of orbits and the number of orbits of any given length. Combining Propositions 3.1.2 and 3.1.3 we get

$$\begin{aligned}\zeta^{\mathbb{Q}}_{(\mathbb{Z}/n\mathbb{Z})^d}(s) &= \prod_{p^e|n}\Bigg(1+\frac{p^d-1}{p-1}(p-1)^{-s}+p^{d-1}\frac{p^d-1}{p-1}(p(p-1))^{-s}+\dots\\ &+p^{(k-1)(d-1)}\frac{p^d-1}{p-1}((p-1)p^{k-1})^{-s}\Bigg)\\ &=\prod_{p^e|n}\Bigg(1+\frac{p^d-1}{p-1}(p-1)^{-s}(1+p^{(d-1)-s}+p^{2(d-1)-2s}\\ &+\dots+p^{(k-1)(d-1)-(k-1)s})\Bigg)\end{aligned}$$

The rational representation zeta function of $G=\mathbb{Z}^d$ can be obtained as the limit of the rational representation zeta function of its finite quotients $(\mathbb{Z}/n\mathbb{Z})^d$. This is because any homomorphism with finite image factors through one of the congruence quotients $(\mathbb{Z}/n\mathbb{Z})^d$. The limit is taken by regarding $\mathbb{N}$ as a directed set with $m \preceq n$ if $m|n$ and taking the limit of the formal Dirichlet series coefficient-wise.

$$\begin{aligned}\zeta^{\mathbb{Q}}_{\mathbb{Z}^d}(s) &= \lim_{n\to\infty}\zeta^{\mathbb{Q}}_{(\mathbb{Z}/n\mathbb{Z})^d}(s)\\ &=\prod_{p}\left(1+\frac{p^d-1}{p-1}(p-1)^{-s}\left(\frac{1}{1-p^{(d-1)-s}}\right)\right).\end{aligned}$$

□

Corollary 3.1.7. $\zeta^{\mathbb{Q}}_{\mathbb{Z}}(s)=\prod_p(1+\frac{(p-1)^{-s}}{1-p^{-s}})=\sum_{m=1}^{\infty}\varphi(m)^{-s}$, *where* φ *denotes the Euler* φ*-function.*

The Dirichlet function $\sum_{m=1}^{\infty}\varphi(m)^{-s}$ has been studied by number theorists; for instance; see [6]. From their result we infer the following corollary.

Corollary 3.1.8. *The rational representation zeta function* $\zeta^{\mathbb{Q}}_{\mathbb{Z}}(s)$ *has abscissa of convergence* 1 *and admits meromorphic continuation to an extended half plan* $\{s\in\mathbb{C}|Re(s)>1-\epsilon\}$ *for some* $\epsilon>0$. *The rational representation growth of the infinite cyclic group* $\mathbb{Z}$ *is asymptotically given by*

$$\sum_{n=1}^{N} r^{\mathbb{Q}}_n(\mathbb{Z})\sim c\cdot N$$

as $N \to \infty$, *where* $c = \frac{\zeta(2)\zeta(3)}{\zeta(6)}$.

For a more detailed study; see [6,25]

3.2 Further examples

In this section we discuss some other groups.

Theorem 3.2.1. *Let* $G = T \times C_2$ *be a group. Then the Galois orbit zeta function of* G *is*

$$\omega_G^{\mathbb{Q}}(s,t) = 2 \cdot \omega_T^{\mathbb{Q}}(s,t).$$

Proof. Let $G = T \times \langle r \rangle$, where $r^2 = 1$. Consider $\chi \in \mathrm{Irr}(\widehat{G})$, where $\mathrm{Irr}(\widehat{G})$ is the set of irreducible continuous complex characters of G. The character $\chi \in \mathrm{Irr}(\widehat{G})$ decomposes as

$$\chi = \chi_1 \times \chi_2$$

by using Theorem 2.1.11, where $\chi_1 \in \mathrm{Irr}(\widehat{T})$ and $\chi_2 \in \mathrm{Irr}(\langle r \rangle)$. We have two possibilities for χ_2, either $\chi_2(r) = 1$ or $\chi_2(r) = -1$. In any case

$$\chi(1) = \chi_1(1) \cdot \chi_2(1) = \chi_1(1)$$

and

$$\chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = \{\chi_1^{\alpha} \times \chi_2 \mid \alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$$

has the same size as $\chi_1^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$. Hence

$$\begin{aligned}
\omega_G^{\mathbb{Q}}(s,t) &= \sum_{\chi \in \mathrm{Irr}(\widehat{G})} \chi(1)^{-s} \mid \chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \mid^{-t} \\
&= 2 \sum_{\chi \in \mathrm{Irr}(\widehat{T})} \chi_1(1)^{-s} \mid \chi_1^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \mid^{-t} \\
&= 2 \cdot \omega_T^{\mathbb{Q}}(s,t).
\end{aligned}$$

$\square$

Theorem 3.2.2. *Let $G = \mathbb{Z}^d \times C_2$. Then the rational representation zeta function of G is*

$$\zeta_G^{\mathbb{Q}}(s) = 2\zeta_{\mathbb{Z}^d}^{\mathbb{Q}}(s).$$

Proof. By using Theorem 3.2.1 we have

$$\omega_G^{\mathbb{Q}}(s,t) = 2 \cdot \omega_{\mathbb{Z}^d}^{\mathbb{Q}}(s,t).$$

Since G is abelian, the Schur index is one. Hence

$$\zeta_G^{\mathbb{Q}}(s) = 2\zeta_{\mathbb{Z}^d}^{\mathbb{Q}}(s).$$

$\square$

Theorem 3.2.3. *Let $G = \mathbb{Z}^d \times C_p$ be a group, where $p > 2$. Then the rational representation zeta function of G is*

$$\zeta_{\mathbb{Z}^d \times C_p}^{\mathbb{Q}}(s) = \left(1 + (p-1)^{-s}\left(\frac{p^{d+1}-1}{p-1} + p^{(d-s)}\frac{p^d-1}{p-1}\left(\frac{1}{1-p^{(d-1)-s}}\right)\right)\right)$$
$$\cdot \prod_{q \neq p}\left(1 + (q-1)^{-s}\frac{q^d-1}{q-1}\frac{1}{1-q^{(d-1)-s}}\right),$$

where the product runs over all primes q not equal to p.

Proof. In order to compute the representation zeta function of $G = \mathbb{Z}^d \times C_p$, we consider its finite quotient

$$\frac{\mathbb{Z}^d \times C_p}{n(\mathbb{Z}^d \times C_p)},$$

and then use limit to complete our calculation. For this we consider two cases:

1. $p \nmid n$

2. $p \mid n$

$$\underline{1. \quad p \nmid n}$$

We know

$$\frac{\mathbb{Z}^d \times C_p}{n(\mathbb{Z}^d \times C_p)} \cong \mathbb{Z}^d/n\mathbb{Z}^d,$$

since $nC_p = C_p$. In this case the representation zeta function of

$$\frac{\mathbb{Z}^d \times C_p}{n(\mathbb{Z}^d \times C_p)}$$

will be the same as $\mathbb{Z}^d/n\mathbb{Z}^d = (\mathbb{Z}/n\mathbb{Z})^d$, because of Euler product. It is enough to look at the special case where n is a prime power $n = q^k \neq p$. Hence it is enough to calculate the representation zeta function of $(\mathbb{Z}/q^k\mathbb{Z})^d$. By theorem 3.1.6 we have

$$\zeta^{\mathbb{Q}}_{\mathbb{Z}^d}(s) = \prod_{q \neq p} \left(1 + \frac{q^d - 1}{q - 1}(q-1)^{-s} \frac{1}{1 - q^{(d-1)-s}}\right).$$

<u>2. $p \mid n$</u>

As before, it is enough to look at the special case where n is a prime power $n = p^k$. In order to compute the representation zeta function of $(\mathbb{Z}/p^k\mathbb{Z})^d \times \mathbb{Z}/p\mathbb{Z}$, we shall consider the diagonal action of $(\mathbb{Z}/p^k\mathbb{Z})^*$ on $(\mathbb{Z}/p^k\mathbb{Z})^d \times (\mathbb{Z}/p\mathbb{Z})$. We prove it by using induction on k.

Induction base: $k = 1$.

We consider the action of $(\mathbb{Z}/p\mathbb{Z})^*$ on $(\mathbb{Z}/p\mathbb{Z})^d \times (\mathbb{Z}/p\mathbb{Z})$

$$\bar{a} = (\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}, \bar{b}).$$

There are two possibilities for $\bar{a}$: either $\bar{a} = (\bar{0}, \bar{0}, \ldots, \bar{0})$ or $\bar{a} \neq (\bar{0}, \bar{0}, \ldots, \bar{0})$. If $\bar{a} = (\bar{0}, \bar{0}, \ldots, \bar{0})$ then for any $\bar{x} \in (\mathbb{Z}/p\mathbb{Z})^*$ we have $\bar{a} \cdot \bar{x} = \bar{0}$. Hence the orbit of $\bar{a}$ has length 1. Now suppose that $\bar{a} \neq (\bar{0}, \bar{0}, \ldots, \bar{0})$. Then we may assume without loss of generality $\bar{a_1} \neq \bar{0}$. This implies that $a_1, 2a_1, \ldots, (p-1)a_1$ are distinct, hence

$$\bar{a}, 2\bar{a}, \ldots, (p-1)\bar{a}$$

are distinct and form an orbit of length $p - 1$. The total number of such orbits of length $p - 1$ is equal to $\frac{p^{d+1}-1}{p-1}$. The situation is summarised in table 3.5.

Induction step: $k \geq 2$

Since $(\mathbb{Z}/p^{k-1}\mathbb{Z})^d \times (\mathbb{Z}/p\mathbb{Z})$ is isomorphic to a subgroup of $(\mathbb{Z}/p^k\mathbb{Z})^d \times (\mathbb{Z}/p\mathbb{Z})$. Define an inclusion map

$$\alpha : (\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}, \bar{b}) \longrightarrow (p\bar{a_1}, p\bar{a_2}, \ldots, p\bar{a_d}, \bar{b})$$

Table 3.5: Table for induction base

length of orbits	number of orbits
1	1
$p-1$	$\frac{p^{d+1}-1}{p-1}$

The inclusion respects the $(\mathbb{Z}/p^k\mathbb{Z})^*$-action. Let $\bar{a} = (\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}, \bar{b}) \in (\mathbb{Z}/p^{k-1}\mathbb{Z})^d \times (\mathbb{Z}/p\mathbb{Z})$ then $\lambda \in (\mathbb{Z}/p^k\mathbb{Z})^*$ maps onto $\bar{\lambda} \in (\mathbb{Z}/p^{k-1}\mathbb{Z})^*$ as $\alpha(\bar{\lambda}\bar{a}) = \lambda\alpha(\bar{a})$. This implies $(\mathbb{Z}/p^k\mathbb{Z})^*$-orbits on the subgroup $\alpha((\mathbb{Z}/p^{k-1}\mathbb{Z})^d \times \mathbb{Z}/p\mathbb{Z})$ correspond to $(\mathbb{Z}/p^{k-1}\mathbb{Z})^*$-orbits on $(\mathbb{Z}/p^{k-1}\mathbb{Z})^d \times \mathbb{Z}/p\mathbb{Z}$ which we understand by induction. By induction we understand orbits of elements $\bar{a} = (\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}, \bar{b}) \in (\mathbb{Z}/p^k\mathbb{Z})^d \times (\mathbb{Z}/p\mathbb{Z})$ where $(\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}) \equiv (\bar{0}, \bar{0}, \ldots, \bar{0}) \bmod p$. We need to deal with $(\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}, \bar{b})$ where $(\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d}) \not\equiv (\bar{0}, \bar{0}, \ldots, \bar{0}) \bmod p$. As before, we may assume without loss of generality that $\bar{a_1} \not\equiv \bar{0} \bmod p$, that is one of $(\bar{a_1}, \bar{a_2}, \ldots, \bar{a_d})$ is a unit in $(\mathbb{Z}/p^k\mathbb{Z})^*$, and multiplying these elements by $(\mathbb{Z}/p^k\mathbb{Z})^*$ results in an orbit of length

$$|(\mathbb{Z}/p^k\mathbb{Z})^*| = (p-1)p^{k-1}.$$

There are

$$|(\mathbb{Z}/p^k)^d \times (\mathbb{Z}/p\mathbb{Z})| - |(\mathbb{Z}/p^{k-1})^d \times (\mathbb{Z}/p\mathbb{Z})|,$$

elements of this kind, falling into orbits of equal length $(p-1)p^{k-1}$. Hence there are

$$\frac{p^{dk+1} - p^{d(k-1)+1}}{(p-1)p^{k-1}} = p^{(d-1)(k-1)+1}\frac{p^d-1}{p-1}$$

orbits of this kind. The situation is summarised in table 3.6. Since the Schur index is 1, the rational representation zeta function is (by taking limit, as in Theorem

Table 3.6: Table for induction step

length of orbits	number of orbits
1	1
$p-1$	$\frac{p^{d+1}-1}{p-1}$
$(p-1)p$	$p^d\frac{p^d-1}{p-1}$
.	.
.	.
.	.
$(p-1)p^{k-1}$	$p^{(k-1)(d-1)+1}\frac{p^d-1}{p-1}$

3.1.6)

$$\begin{aligned}
\zeta^{\mathbb{Q}}_{1(\mathbb{Z}^d\times C_p)}(s) &= 1+\frac{p^{(d+1)}-1}{(p-1)}(p-1)^{-s}+p^d\frac{p^d-1}{p-1}(p-1)^{-s}p^{-s} \\
&+p^{(2d-1)}\frac{p^d-1}{p-1}(p-1)^{-s}p^{-2s}+\dots \\
&= 1+(p-1)^{-s}\left(\frac{p^{d+1}-1}{p-1}+p^{(d-s)}\frac{p^d-1}{p-1}+p^{(2d-2s-1)}\frac{p^d-1}{p-1}+\dots\right) \\
&= 1+(p-1)^{-s}\Bigg(\frac{p^{d+1}-1}{p-1}+p^{(d-s)}\frac{p^d-1}{p-1} \\
&\cdot\left(1+p^{(d-1)-s}+p^{(2d-2)-2s}+\dots\right)\Bigg) \\
&= 1+(p-1)^{-s}\left(\frac{p^{d+1}-1}{p-1}+p^{(d-s)}\frac{p^d-1}{p-1}(\frac{1}{1-p^{(d-1)-s}})\right)
\end{aligned}$$

From 1 and 2, we get the rational representation zeta function of G as

$$\begin{aligned}
\zeta^{\mathbb{Q}}_{\mathbb{Z}^d\times C_p}(s) = &\left(1+(p-1)^{-s}\left(\frac{p^{d+1}-1}{p-1}+p^{(d-s)}\frac{p^d-1}{p-1}(\frac{1}{1-p^{(d-1)-s}})\right)\right) \\
&\cdot\prod_{q\neq p}\left(1+(q-1)^{-s}\frac{q^d-1}{q-1}\frac{1}{1-q^{(d-1)-s}}\right).
\end{aligned}$$

□

Corollary 3.2.4. *Let $G=\mathbb{Z}\times\mathbb{Z}$ be the infinite abelian group. Then the rational representation zeta function of G is*

$$\zeta^{\mathbb{Q}}_G(s)=\prod_p\left(1+(p+1)(p-1)^{-s}\frac{1}{1-p^{1-s}}\right).$$

Proof. The group $G = \mathbb{Z} \times \mathbb{Z}$ is isomorphic to the group $\mathbb{Z}^2$. Since the group $\mathbb{Z}^2$ is the special case of the group $\mathbb{Z}^d$ when $d = 2$. By using Theorem 3.1.6 we get the representation zeta function of G over $\mathbb{Q}$ as

$$\zeta_G^{\mathbb{Q}}(s) = \prod_p \left(1 + (p+1)(p-1)^{-s} \frac{1}{1 - p^{1-s}}\right).$$

□

Chapter 4

Frieze groups and one-dimensional crystallographic groups

4.1 Frieze groups

In this section we compute the rational representation zeta functions of frieze groups. These are groups of symmetries of certain types of ornaments infinitely repeated in one direction. There are seven such groups. The presentations of these groups are given below.

$$\begin{aligned}
p1 &= \langle x \rangle \cong \mathbb{Z}.\\
p1m1 &= \langle x, z \mid z^2 = 1, x^z = x^{-1} \rangle \cong \mathbb{Z} \rtimes C_2.\\
p11m &= \langle x, \beta \mid \beta^2 = 1, x\beta = \beta x \rangle \cong \mathbb{Z} \times C_2.\\
p11g &= \langle y \rangle \cong \mathbb{Z}.\\
p2 &= \langle x, \alpha \mid \alpha^2 = 1, x^\alpha = x^{-1} \rangle \cong \mathbb{Z} \rtimes C_2.\\
p2mg &= \langle y, \beta \mid \beta^2 = 1, x^\beta = x^{-1} \rangle \cong \mathbb{Z} \rtimes C_2.\\
p2mm &= \langle x, z, \beta \mid z^2 = 1 = \beta^2, x^\beta = x^{-1}, xz = zx \rangle \cong (\mathbb{Z} \times C_2) \rtimes C_2.
\end{aligned}$$

Note: x denotes the horizontal translation, y denotes the glide reflection, z denotes the vertical reflection, α denotes the $180°$ rotation and β denotes the reflection about the horizontal axis.

4.1.1 The group p1

The group

$$G = p1 = \langle x \rangle$$

is isomorphic to $\mathbb{Z}$, since the group $\mathbb{Z}$ is a special case of the group $\mathbb{Z}^d$ when $d = 1$. By Theorem 3.1.6, the rational representation zeta function of G is

$$\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1) = \prod_p \left(1 + \frac{(p-1)^{-s}}{1 - p^{-s}}\right).$$

4.1.2 The group p1m1

The group

$$G = p1m1 = \langle x, z \mid z^2 = 1, x^z = x^{-1} \rangle$$

is isomorphic to the infinite dihedral group. The group G consists of a translation group T, which is normal in G and generated by x. The quotient group G/T, generated by zT is the point group of G and is a cyclic group of order 2.

We want to compute the rational representation zeta function $\zeta_G^{\mathbb{Q}}(s)$ via the Galois orbit zeta function of G. To do this we shall use Clifford theory. First we compute, from an irreducible complex character of T and its inertia group in G, those irreducible complex characters of G, whose restriction to T involves the given character on T. Then we apply Galois theory to obtain the (equivalence classes of) irreducible representations over $\mathbb{Q}$ via Galois orbits of complex characters.

We claim that the irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{m,l} : T \longrightarrow \mathbb{C}^*.$$

where $m \in \mathbb{N}$ with $1 \leq l \leq m, \gcd(m, l) = 1$ and

$$\chi_{m,l}(x^a) = \xi_{m,l}^a = e^{2\pi i(la/m)},$$

where $\xi_{m,l} = e^{2\pi il/m}$.

Indeed, since T is abelian, each character is obtained by choosing a primitive mth root of unity $\xi_{m,l} = e^{2\pi il/m}$. Note that all these characters are one-dimensional.

Consider the action of G on $\mathrm{Irr}(T)$ given by $\chi^g(x) = \chi(x^{g^{-1}})$ where $\chi \in \mathrm{Irr}(T), g \in G, x \in T$. We need to understand the orbits of the character χ and it stabiliser. The stabiliser is called the *inertia group* and is defined as

$$I_G(\chi) = \{g \in G \mid \chi^g = \chi\}.$$

Since $I_G(\chi)$ is the stabiliser of χ in the action of G on $\mathrm{Irr}(T)$, it follows that it is a subgroup of G and $T \subseteq I_G(\chi) \subseteq G$. In our case $|G : T| = 2$. The size of the orbit of χ is $\mid G : I_G(\chi) \mid$. So either $I_G(\chi) = T$ when $z \notin I_G(\chi)$ or $I_G(\chi) = G$ when $z \in I_G(\chi)$. Correspondingly, $I_G(\chi)/T$ is cyclic and isomorphic to either C_2 or 1.

Now consider $\psi \in \mathrm{Irr}(G)$ factoring over a finite quotient of G. We can cover all possible ψ by considering two cases: $\psi(1) = 1$ and $\psi(1) > 1$.

Case(1) : $\psi \in \mathrm{Irr}(G)$ with $\psi(1) = 1$

To compute the linear character of G it is enough to consider the abelianisation of G:

$$\begin{aligned} G/[G, G] &= \langle x, z \mid x^2 = z^2 = 1\rangle \\ &\cong C_2 \times C_2. \end{aligned}$$

This means that there are 4 linear characters of G. Observe that, since $G/[G, G]$ has exponent 2, the corresponding representations are all defined over $\mathbb{Q}$. Thus we obtain 4 one-dimensional irreducible characters over $\mathbb{Q}$. The contribution to the Galois orbit zeta function is

$$\omega_1(s, t) = 4 \cdot 1^{-s} \cdot 1^{-t} = 4.$$

$\underline{\textbf{Case(2)} : \ \psi \in \mathrm{Irr}(G) \text{ with } \ \psi(1) > 1}$

In this case ψ cannot restrict to an irreducible character of the abelian group T. This means that

$$\psi = \mathrm{Ind}_T^G(\chi) = \mathrm{Ind}_T^G(\chi^z),$$

where $\chi = \chi_{m,l}$ and $\chi^z = \chi_{m,-l}$ are such that $\chi \neq \chi^z$.

We want to compute the Galois orbits of such ψ. We observe that ψ is uniquely determined by $\psi_T = \chi + \chi^z$. So it suffices to describe the Galois orbit in

$$\{(\chi + \chi^z) \mid \chi \in \mathrm{Irr}(T), I_G(\chi) = T\}.$$

The character $\chi + \chi^z$ is uniquely determined by the set $\{\chi, \chi^z\}$, which in turn is uniquely determined by $\{\chi(x), \chi^z(x)\}$. Therefore we need to describe the Galois orbits in

$$\{\{e^{2\pi i l/m}, e^{2\pi i(m-l)/m}\} \mid m \in \mathbb{N}, m > 2, 1 \leq l \leq m, \gcd(m,l) = 1\},$$

equivalently the orbits in

$$\{\{\xi_{m,l}, \xi_{m,l}^{-1}\} \mid m \in \mathbb{N}, m > 2, 1 \leq l \leq m, \gcd(m,l) = 1\},$$

under the action of $\mathrm{Gal}(\mathfrak{K}_m/\mathbb{Q})$, where $\mathfrak{K}_m = \mathbb{Q}(\xi_{m,l})$. The Galois group satisfies $\mathrm{Gal}(\mathfrak{K}_m/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$. Consider $\alpha \in \mathrm{Gal}(\mathfrak{K}_m/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/m\mathbb{Z})^*$. Its action on $\{\xi_{m,l}, \xi_{m,l}^{-1}\}$ is given by

$$\{\xi_{m,l}, \xi_{m,l}^{-1}\}^\alpha = \{\xi_{m,l}^a, \xi_{m,l}^{-a}\}.$$

We find the lengths and the numbers of such orbits. Since $|\,\mathrm{Gal}(\mathfrak{K}_m/\mathbb{Q})| = |(\mathbb{Z}/m\mathbb{Z})^*| = \varphi(m)$, the length of an orbit is

$$\varphi(m)/\operatorname{Stab}_{\mathrm{Gal}(\mathfrak{K}_m/\mathbb{Q})}\{\xi_{m,l}, \xi_{m,l}^{-1}\}.$$

We have to determine all $a \in (\mathbb{Z}/l\mathbb{Z})^*$ such that

$$\{\xi_{m,l}, \xi_{m,l}^{-1})^\alpha = \{\xi_{m,l}^a, \xi_{m,l}^{-a}\} = \{\xi_{m,l}, \xi_{m,l}^{-1}\},$$

equivalently

$$a \equiv 1 \bmod m$$
$$\text{or } a \equiv -1 \bmod m.$$

Since $m > 2$, as per Lemma 2.5.4, the stabiliser of $\{\xi_{m,l}, \xi_{m,l}^{-1}\}$ has size 2, hence the length of an orbit is $\varphi(m)/2$. For m fixed, there are $\varphi(m)$ choices for l, giving $\varphi(m)/2$ possible choices for $\{\xi_{m,l}, \xi_{m,l}^{-1}\}$. Every $\psi \in \mathrm{Irr}(G)$ corresponds to $\psi_T = \chi + \chi^z$, and thus to $\{\xi_{m,l}, \xi_{m,l}^{-1}\}$, and has $\psi(1) = 2$. Its Galois orbit has length $\varphi(m)/2$.
Hence the Galois orbit zeta function of G over $\mathbb{Q}$ is

$$\begin{aligned}
\omega_G^{\mathbb{Q}}(s,t) &= \omega_1(s,t) + \sum_{m=3}^{\infty} \frac{\varphi(m)}{2} \cdot 2^{-s} (\frac{\varphi(m)}{2})^{-t} \\
&= 4 + 2^{t-s-1} \sum_{m=3}^{\infty} \varphi(m)^{1-t}
\end{aligned}$$

“Since $\varphi(1) = \varphi(2) = 1$”

$$\begin{aligned}
&= 4 - 2^{t-s} + 2^{t-s-1} \sum_{m=1}^{\infty} \varphi(m)^{1-t} \\
&= 4 - 2^{t-s} + 2^{t-s-1} \omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t).
\end{aligned}$$

Since $G = \langle z \rangle \ltimes T$ is a semi direct product, according to Lemma 2.4.15, the Schur index is one. Hence

$$\begin{aligned}
\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1) &= 4 - 2 + 2^0 \zeta_{\mathbb{Z}}^{\mathbb{Q}}(s) \\
&= 2 + \zeta_{\mathbb{Z}}^{\mathbb{Q}}(s).
\end{aligned}$$

4.1.3 The group p11m

The group $p11m$ is given by the presentation

$$G = p11m = \langle x, \beta \mid \beta^2 = 1, x\beta = \beta x \rangle,$$

which is isomorphic to $\mathbb{Z} \times C_2$. The irreducible representations of C_2 are all defined over $\mathbb{Q}$, so the irreducible representations of $\mathbb{Z} \times C_2$ over $\mathbb{Q}$ are the tensor products of irreducible representations of $\mathbb{Z}$ over $\mathbb{Q}$, and irreducible representations of C_2 over $\mathbb{Q}$. Since G is abelian, the Schur index of G is 1. By using Theorem 3.2.1, the representation zeta function of G over $\mathbb{Q}$ is

$$\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1) = 2\omega_{\mathbb{Z}}^{\mathbb{Q}}(s, s+1) = 2\zeta_{\mathbb{Z}}^{\mathbb{Q}}(s).$$

4.1.4 The group p2mm

The group $p2mm$ is given by the presentation

$$G = p2mm = \langle x, z, \beta \mid z^2 = 1 = \beta^2, x^\beta = x^{-1}, xz = zx \rangle,$$

which is isomorphic to $(C_2 \times \mathbb{Z}) \rtimes C_2$. The group G contains a subgroup T, which is normal in G and generated by x and z. The quotient group G/T, generated by βT, is a cyclic group of order 2.

We want to compute the rational representation zeta function $\zeta_G^{\mathbb{Q}}(s)$ of G. As in Section 4.1.2, the irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{k_1,n_1,k_2,n_2} : T \longrightarrow \mathbb{C}^*.$$

where $n_1 \in \mathbb{N}$, $n_2 \in \{1, 2\}$, $k_1 \in \mathbb{N}, k_2 \in \{1, 2\}$ with $1 \le k_i \le n_i, \gcd(k_i, n_i) = 1$ and

$$\chi_{k_1,n_1,k_2,n_2}(x^a y^b) = \xi_{n_1,k_1}^a \xi_{n_2,k_2}^b = e^{2\pi i (k_1 a/n_1 + k_2 b/n_2)},$$

where $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$, $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$.

Consider the action of G on $\mathrm{Irr}(T)$ given by $\chi^g(z) = \chi(z^{g^{-1}})$, where $\chi \in \mathrm{Irr}(T)$, $g \in G, z \in T$. We need to understand the orbit of the character χ and its stabiliser. The stabiliser is a subgroup of G and

$$T \subseteq I_G(\chi) \subseteq G.$$

In our case $|G : T| = 2$. The size of the orbit of χ is $|G : I_G(\chi)|$. So either $I_G(\chi) = T$ when $\beta \notin I_G(\chi)$, or $I_G(\chi) = G$ when $\beta \in I_G(\chi)$. Correspondingly, $I_G(\chi)/T$ is cyclic and isomorphic to either C_2 or 1.

Now consider $\psi \in \mathrm{Irr}(G)$ factoring over a finite quotient of G. We can cover all possible ψ by considering two cases: $\psi(1) = 1$ and $\psi(1) > 1$.

$$\underline{\textbf{Case(1)}: \quad \psi \in \mathrm{Irr}(G) \text{ with } \psi(1) = 1}$$

To compute the linear characters of G it is enough to consider the abelianisation of G:

$$\begin{aligned} G/[G,G] &= \langle x, z, \beta \mid [x,z] = [x,\beta] = [z,\beta] = 1, x^2 = z^2 = \beta^2 = 1\rangle \\ &\cong C_2 \times C_2 \times C_2. \end{aligned}$$

This means that there are 8 linear characters of G. Observe that, since $G/[G,G]$ has exponent 2, the corresponding representations are all defined over $\mathbb{Q}$. Thus we obtain 8 one-dimensional irreducible characters over $\mathbb{Q}$. The contribution to the Galois orbit zeta function is

$$\omega_1(s,t) = 8 \cdot 1^{-s} \cdot 1^{-t} = 8.$$

$$\underline{\textbf{Case(2)}: \quad \psi \in \mathrm{Irr}(G) \text{ with } \psi(1) > 1}$$

Using a similar strategy as in Section 4.1.2 Case(2), we get

$$\begin{aligned} \zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1) &= 8 + 2\sum_{n_1 \geq 3}^{\infty} \varphi(n_1)^{-s} \\ &= 4 + 2\sum_{n_1=1}^{\infty} \varphi(n)^{-s} \\ &= 4 + 2\zeta_{\mathbb{Z}}^{\mathbb{Q}}(s) \\ &= 2\zeta_{\mathbb{Z}\rtimes C_2}^{\mathbb{Q}}(s) \end{aligned}$$

4.2 One-dimensional crystallographic groups

In this section we compute the rational representation zeta function of crystallographic groups of dimension one.

4.2.1 p1

The group $G = p1 = \langle x \rangle$ is isomorphic to $\mathbb{Z}$. From Section 4.1.1, we have

$$\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1) = \prod_p \left(1 + \frac{(p-1)^{-s}}{1 - p^{-s}}\right).$$

4.2.2 p1m

The group

$$G = p1m = \langle x, z \mid z^2 = 1, x^z = x^{-1} \rangle,$$

is isomorphic to the infinite dihedral group. From Section 4.1.2, we have

$$\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1) = 2 + \zeta_{\mathbb{Z}}^{\mathbb{Q}}(s).$$

Chapter 5

Two-dimensional crystallographic groups

In this chapter we compute the rational representation zeta functions of seven crystallographic groups of dimension two. In two cases where the Schur indices are not all one we compute only the Galois orbit zeta functions.

5.1 The group p1

The group $p1$ is given by the presentation

$$G = p1 = \langle x, y \mid [x, y] = 1 \rangle.$$

The translation group of G is the whole group generated by x and y because the point group of G is the trivial group. We want to compute the rational representation zeta function $\zeta_G^{\mathbb{Q}}(s)$ of G. By Corollary 3.2.4, the representation zeta function of G over $\mathbb{Q}$ is

$$\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1) = \prod_p \left(1 + (p+1)(p-1)^{-s} \frac{1}{1 - p^{1-s}}\right).$$

Remark 5.1.1.

$$\begin{aligned} \zeta_{p1}^{\mathbb{Q}}(s) &= \prod_p \left(1 + (p-1)^{-s} \frac{p+1}{1-p^{1-s}}\right) \\ &\approx \prod_p (1 + p^{1-s}) \end{aligned}$$

The abscissa of convergence at $\Re(s) = 2$. And

$$\begin{aligned}
\zeta_G^{\mathbb{Q}}(s) &= \prod_p \left(1 + (p-1)^{-s}\frac{p+1}{1-p^{1-s}}\right) \\
&= \prod_p \left(\frac{1-p^{1-s}+p(p-1)^{-s}+(p-1)^{-s}}{1-p^{1-s}}\right) \\
&= \underbrace{\prod_p \frac{1}{1-p^{1-s}}}\underbrace{\prod_p \left(1-p^{1-s}+p(p-1)^{-s}+(p-1)^{-s}\right)} \\
&= \zeta(s-1)\prod_p \left(1+(p-1)^{-s}+p(p-1)^{-s}-p^{1-s}\right),
\end{aligned}$$

which admits a meromorphic continuation to $\Re(s) \geq 1$ and has a simple pole at $s = 2$. Note that meromorphic continuation is not on the entire complex plane; see [6, Section 2] and [25] for more details.

5.2 The group p2

The group $p2$ is given by the presentation

$$G = \langle x, y, r \mid [x,y] = r^2 = 1, x^r = x^{-1}, y^r = y^{-1}\rangle.$$

The group G contains a translation group T, which is normal in G and generated by x and y. The quotient group G/T, generated by rT, is the point group of G and it is a cyclic group of order 2.

We want to compute the rational representation zeta function $\zeta_G^{\mathbb{Q}}(s)$ via the Galois orbit zeta function of G. To do this we shall use Clifford theory. First, from an irreducible complex character of T and its inertia group in G, we compute those irreducible complex characters of G, whose restriction to T involves the given character on T. Then we apply Galois theory to obtain the (equivalence classes of) irreducible representations over $\mathbb{Q}$ via the Galois orbits of complex characters.

We claim that the irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{k_1,n_1,k_2,n_2} : T \longrightarrow \mathbb{C}^*.$$

where $n_1, n_2 \in \mathbb{N}$, $k_1, k_2 \in \mathbb{N}$ with $1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1$ and

$$\chi_{k_1,n_1,k_2,n_2}(x^a y^b) = \xi_{n_1,k_1}^a \xi_{n_2,k_2}^b = e^{2\pi i(k_1 a/n_1 + k_2 b/n_2)},$$

where $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$, $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$.

Indeed, since T is abelian, each character is obtained by choosing a primitive n_1th root of unity $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$ and a primitive n_2th root of unity $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$ as the images of x and y. Note that all these characters are one-dimensional.

Consider the action of G on $\mathrm{Irr}(T)$ given by $\chi^g(z) = \chi(z^{g^{-1}})$, where $\chi \in \mathrm{Irr}(T), g \in G, z \in T$. We need to understand the orbit of the character χ and its stabiliser. The stabiliser is called the inertia group and is defined as

$$I_G(\chi) = \{g \in G \mid \chi^g = \chi\}.$$

Since $I_G(\chi)$ is the stabiliser of χ in the action of G on $\mathrm{Irr}(T)$, it follows that it is a subgroup of G and

$$T \subseteq I_G(\chi) \subseteq G.$$

In our case $|G : T| = 2$. Also $|G : I_G(\chi)|$ is the size of the orbit of χ. So either $I_G(\chi) = T$ when $r \notin I_G(\chi)$, or $I_G(\chi) = G$ when $r \in I_G(\chi)$. Correspondingly, $I_G(\chi)/T$ is cyclic and isomorphic to either C_2 or 1.

Now consider $\psi \in \mathrm{Irr}(G)$ factoring over a finite quotient of G. We can cover all possible ψ by considering two cases: $\psi(1) = 1$ and $\psi(1) > 1$.

Case 1 : $\psi \in \mathrm{Irr}(G)$ with $\psi(1) = 1$

To compute the linear characters of G it is enough to consider the abelianisation of G:

$$\begin{aligned} G/[G,G] &= \langle x, y, r \mid [x,y] = [x,r] = [y,r] = 1, x^2 = y^2 = r^2 = 1 \rangle \\ &\cong C_2 \times C_2 \times C_2. \end{aligned}$$

This means that there are 8 linear characters of G. Observe that, since $G/[G,G]$ has exponent 2, the corresponding representations are all defined over $\mathbb{Q}$. Thus we obtain 8 one-dimensional irreducible characters over $\mathbb{Q}$. The contribution to the Galois orbit zeta function is

$$\omega_1(s,t) = 8 \cdot 1^{-s} \cdot 1^{-t} = 8.$$

$$\underline{\textbf{Case 2}: \quad \psi \in \mathrm{Irr}(G) \text{ with } \psi(1) > 1}$$

In this case, ψ cannot restrict to an irreducible character of the abelian group T. This means

$$\psi = \mathrm{Ind}_T^G(\chi) = \mathrm{Ind}_T^G(\chi^r),$$

where $\chi = \chi_{k_1,n_1,k_2,n_2}$ and $\chi^r = \chi_{n_1-k_1,n_1,n_2-k_2,n_2}$ are such that $\chi \neq \chi^r$, i.e $I_G(\chi) = T$.

We want to compute the Galois orbits of such ψ. We observe that ψ is uniquely determined by $\psi_T = \chi \oplus \chi^r$. Hence it suffices to describe the Galois orbits in

$$\{(\chi \oplus \chi^r) \mid \chi \in \mathrm{Irr}(T), I_G(\chi) = T\}.$$

The character $\chi \oplus \chi^r$ is uniquely determined by the set $\{\chi, \chi^r\}$, which in turn is uniquely determined by $\{(\chi(x), \chi(y)), (\chi^r(x), \chi^r(y))\}$. Therefore we need to describe the Galois orbits in

$$\{\{(e^{2\pi i k_1/n_1}, e^{2\pi i k_2/n_2}), (e^{2\pi i (n_1-k_1)/n_1}, e^{2\pi i (n_2-k_2)/n_2})\} \mid n_1, n_2 \in \mathbb{N},$$
$$n_1 > 2 \text{ or } n_2 > 2, k_1, k_2 \in \mathbb{N}, 1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1\},$$

equivalently the orbits in

$$\{(\xi_{n_1,k_1}, \xi_{n_2,k_2}), (\xi_{n_1,k_1}^{-1}, \xi_{n_2,k_2}^{-1}) \mid n_1, n_2 \in \mathbb{N}, n_1 > 2 \text{ or } n_2 > 2,$$
$$k_1, k_2 \in \mathbb{N}, 1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1\},$$

under the action of $\mathrm{Gal}(K_l/\mathbb{Q})$, where $l = \mathrm{lcm}(n_1, n_2)$ and $K_l = \mathbb{Q}(\xi_{n_1,k_1}, \xi_{n_2,k_2})$. The Galois group satisfies $\mathrm{Gal}(K_l/\mathbb{Q}) \cong (\mathbb{Z}/l\mathbb{Z})^*$. Consider that $\alpha \in \mathrm{Gal}(K_l/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$. Its action on

$$(\xi_{n_1,k_1}, \xi_{n_2,k_2}), (\xi_{n_1,k_1}^{-1}, \xi_{n_2,k_2}^{-1})$$

is given by

$$\{(\xi_{n_1,k_1},\xi_{n_2,k_2}),(\xi_{n_1,k_1}^{-1},\xi_{n_2,k_2}^{-1})\}^{\alpha}=\{(\xi_{n_1,k_1}^{a},\xi_{n_2,k_2}^{a}),(\xi_{n_1,k_1}^{-a},\xi_{n_2,k_2}^{-a})\}.$$

We find the lengths and the numbers of such orbits. Since $|\operatorname{Gal}(K_l/\mathbb{Q})|=|(\mathbb{Z}/l\mathbb{Z})^*|=\varphi(l)$, the length of an orbit is

$$\varphi(l)/\mid \operatorname{Stab}_{\operatorname{Gal}(K_l/\mathbb{Q})}\{(\xi_{n_1,k_1},\xi_{n_2,k_2}),(\xi_{n_1,k_1}^{-1},\xi_{n_2,k_2}^{-1})\}\mid .$$

We have to determine all $a\in(\mathbb{Z}/l\mathbb{Z})^*$ such that

$$\begin{aligned}(\xi_{n_1,k_1},\xi_{n_2,k_2})^{\alpha}&=(\xi_{n_1,k_1}^{a},\xi_{n_2,k_2}^{a})=(\xi_{n_1,k_1},\xi_{n_2,k_2})\\ \text{or } (\xi_{n_1,k_1},\xi_{n_2,k_2})^{\alpha}&=(\xi_{n_1,k_1}^{a},\xi_{n_2,k_2}^{a})=(\xi_{n_1,k_1}^{-1},\xi_{n_2,k_2}^{-1})\end{aligned}$$

equivalently

$$a\equiv 1 \bmod l$$
$$\text{or } a\equiv -1 \bmod l,$$

where $l=\operatorname{lcm}(n_1,n_2)$. Since $n_1>2$ or $n_2>2$, as per Lemma 2.5.4, the stabiliser of

$$\{(\xi_{n_1,k_1},\xi_{n_2,k_2}),(\xi_{n_1,k_1}^{-1},\xi_{n_2,k_2}^{-1})\}$$

has size 2. Hence the length of an orbit is $\varphi(l)/2$. Moreover the total number of such orbits is

$$\frac{\varphi(n_1)\varphi(n_2)}{2}/\frac{\varphi(l)}{2},$$

as for n_1,n_2 fixed, and there are $\varphi(n_1)$ choices for k_1 and $\varphi(n_2)$ choices for k_2, giving

$$\frac{\varphi(n_1)\varphi(n_2)}{2}$$

possible choices for

$$\{(\xi_{n_1,k_1},\xi_{n_2,k_2}),(\xi_{n_1,k_1}^{-1},\xi_{n_2,k_2}^{-1})\}.$$

Every $\psi\in\operatorname{Irr}(G)$ corresponds to $\psi_T=\chi\oplus\chi^r$, and thus to
$\{(\xi_{n_1,k_1},\xi_{n_2,k_2}),(\xi_{n_1,k_1}^{-1},\xi_{n_2,k_2}^{-1})\}$, and has $\psi(1)=2$. Its Galois orbit has length $\varphi(l)/2$,

where $l = \mathrm{lcm}(n_1, n_2)$. Hence the Galois orbit zeta function of G over $\mathbb{Q}$ is

$$\begin{aligned}\omega_G^{\mathbb{Q}}(s,t) &= \omega_1(s,t) + \sum_{\substack{n_1,n_2=1\\ \max\{n_1,n_2\}\geq 3}}^{\infty} \frac{\varphi(n_1)\varphi(n_2)}{2} \cdot 2^{-s}(\frac{\varphi(l)}{2})^{-t} \\ &= 8 + 2^{t-s-1} \sum_{n_1,n_2=1}^{\infty} \varphi(n_1)\varphi(n_2)\varphi(l)^{-t} - 4 \cdot 2^{t-s-1} \\ &= 8 + 2^{t-s-1}\eta(t) - 4 \cdot 2^{t-s-1},\end{aligned} \tag{5.2.1}$$

where

$$\eta(t) = \sum_{n_1,n_2=1}^{\infty} \varphi(n_1)\varphi(n_2)\varphi(l)^{-t}.$$

To calculate $\eta(t)$, consider $n_1 = \prod_i p_i^{e_i}$ and $n_2 = \prod_i p_i^{f_i}$, factorised into primes. Then $l = \mathrm{lcm}(n_1, n_2) = \prod_i p_i^{\max\{e_i,f_i\}}$. Hence

$$\eta(t) = \prod_{p\in\mathbb{P}} \sum_{e,f=0}^{\infty} \varphi(p^e)\varphi(p^f)\varphi(p^{\max\{e,f\}})^{-t}.$$

There are four cases covering the possible values for e and f.

Case(1): If $e = f = 0$ then the double sum has value 1.

Case(2): If $e \geq 1$ and $f = 0$ then the double sum is

$$\begin{aligned}\eta_1(t) &= \sum_{e=1}^{\infty}((p-1)p^{e-1})^{-t}((p-1)p^{e-1}) = (p-1)^{1-t}\sum_{e=1}^{\infty}(p^{e-1})^{1-t} \\ &= (p-1)^{1-t}\frac{1}{1-p^{1-t}}.\end{aligned}$$

Case(3): If $e = 0$ and $f \geq 1$ then the double sum is

$$\begin{aligned}\eta_2(t) &= \sum_{f=1}^{\infty}((p-1)p^{f-1})^{-t}((f-1)p^{f-1}) = (p-1)^{1-t}\sum_{e=1}^{\infty}(p^{f-1})^{1-t} \\ &= (p-1)^{1-t}\frac{1}{1-p^{1-t}}.\end{aligned}$$

Case(4): If $e \geq 1$ and $f \geq 1$ then we have the following sub-cases

1. If $e = f$ then the double sum is

$$\begin{aligned}\eta_3(t) &= \sum_{e=1}^{\infty}(p-1)p^{e-1} \cdot (p-1)p^{e-1} \cdot ((p-1)p^{e-1})^{-t} \\ &= (p-1)^{2-t}\sum_{e=1}^{\infty}(p^{e-1})^{2-t} = \frac{(p-1)^{2-t}}{1-p^{2-t}}.\end{aligned}$$

2. If $e < f$ then the double sum is

$$\begin{aligned}
\eta_4(t) &= \sum_{e=1}^{\infty} \sum_{f=e+1}^{\infty} (p-1)p^{e-1} \cdot (p-1)p^{f-1} \cdot ((p-1)p^{f-1})^{-t} \\
&= (p-1)^{2-t} \sum_{e=1}^{\infty} p^{e-1} \sum_{f=e+1}^{\infty} (p^{f-1})^{1-t} \\
&= (p-1)^{2-t} \sum_{e=1}^{\infty} p^{e-1} \frac{p^{e(1-t)}}{1-p^{1-t}} \\
&= (p-1)^{2-t} \frac{p^{-1}}{(1-p^{1-t})} \sum_{e=1}^{\infty} p^{e(2-t)} \\
&= (p-1)^{2-t} \frac{p^{-1}}{(1-p^{1-t})} \frac{p^{2-t}}{1-p^{2-t}} \\
&= (p-1)^{2-t} \frac{p^{1-t}}{(1-p^{1-t})(1-p^{2-t})}.
\end{aligned}$$

3. If $e > f$ then, similarly, the double sum is equal to $\eta_5(t) = \eta_4(t)$.

Hence

$$\begin{aligned}
\eta(t) &= \prod_p (1 + \eta_1(t) + \eta_2(t) + \eta_3(t) + \eta_4(t) + \eta_5(t)) \\
&= \prod_p \left(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}} + \frac{(p-1)^{2-t}}{1-p^{2-t}} + 2\frac{p^{1-t}(p-1)^{2-t}}{(1-p^{1-t})(1-p^{2-t})}\right) \\
&= \prod_p \left(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}} + \frac{(p-1)^{2-t}}{1-p^{2-t}}(1 + 2\frac{p^{1-t}}{1-p^{1-t}})\right) \\
&= \prod_p \left(1 + (p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\right).
\end{aligned}$$

By using the value of $\eta(t)$ in equation 5.2.1 we get

$$\omega_G^{\mathbb{Q}}(s,t) = 8 + 2^{t-s-1} \prod_p \left(1 + (p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\right) - 4 \cdot 2^{t-s-1}.$$

Since $G = \langle r \rangle \ltimes T$, is a semi direct product, according to Lemma 2.4.15, the Schur index is equal to one. So the rational representation zeta function is

$$\begin{aligned}
\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1) &= 8 + \prod_p \left(1 + (p-1)^{-s}\frac{p+1}{1-p^{1-s}}\right) - 4 \\
&= 4 + \zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s).
\end{aligned}$$

Remark 5.2.1. The analytic properties of $\zeta_G^{\mathbb{Q}}(s)$ and $\zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s)$ are very closely related.

5.3 The group pm

The group pm is give by the presentation

$$G = \langle x, y, m \mid [x, y] = m^2 = 1, x^m = x, y^m = y^{-1} \rangle.$$

The group G contains a translation group T, which is normal in G and generated by x and y. The quotient group G/T, generated by mT is the point group of G and it is a cyclic group of order 2.

We want to compute the rational representation zeta function $\zeta_G^{\mathbb{Q}}(s)$ of G via $\omega_G^{\mathbb{Q}}(s,t)$. We use a similar strategy as in Section 5.2. The irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{k_1,n_1,k_2,n_2} : T \longrightarrow \mathbb{C}^*,$$

where $n_1, n_2 \in \mathbb{N}$, $k_1, k_2 \in \mathbb{N}$ with $1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1$ and

$$\chi_{k_1,n_1,k_2,n_2}(x^a y^b) = \xi_{n_1,k_1}^a \xi_{n_2,k_2}^b = e^{2\pi i (k_1 a/n_1 + k_2 b/n_2)},$$

where $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$, $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$.

Consider the action of G on $\mathrm{Irr}(T)$ given by $\chi^g(z) = \chi(z^{g^{-1}})$, where $\chi \in \mathrm{Irr}(T), g \in G, z \in T$. We need to understand the orbit of the character χ and its stabiliser. Since $|G : T| = 2$. The size of the orbit of χ is $|G : I_G(\chi)|$. It follows, either $I_G(\chi) = T$ when $m \notin I_G(\chi)$, or $I_G(\chi) = G$ when $m \in I_G(\chi)$. Correspondingly, $I_G(\chi)/T$ is cyclic and isomorphic to either C_2 or 1.

Now consider $\psi \in \mathrm{Irr}(G)$ factoring over a finite quotient of G. We can cover all possible ψ by considering two cases: $\psi(1) = 1$ and $\psi(1) > 1$.

Case(1) $\psi \in \mathrm{Irr}(G)$ with $\psi(1) = 1$

To compute the linear characters of G it is enough to consider the abelianisation of G.

$$\begin{aligned} G/[G,G] &= \langle x, y, m \mid [x,y] = [x,m] = [y,m] = 1, y^2 = m^2 = 1 \rangle \\ &\cong \mathbb{Z} \times C_2 \times C_2. \end{aligned}$$

The contribution to the Galois orbit zeta function is

$$\omega_1(s,t) = 4\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t),$$

where 4 comes from the irreducible representation of $C_2 \times C_2$. The irreducible representations of $C_2 \times C_2$ are all defined over $\mathbb{Q}$, so that the irreducible representations of $\mathbb{Z} \times C_2 \times C_2$ over $\mathbb{Q}$ are tensor products of irreducible representations of $\mathbb{Z}$ over $\mathbb{Q}$, and irreducible representations of $C_2 \times C_2$ over $\mathbb{Q}$.

$$\underline{\textbf{Case(2)}: \quad \psi \in \mathrm{Irr}(G) \text{ with } \psi(1) > 1}$$

In this case ψ cannot restrict to an irreducible character of the abelian group T. This means that

$$\psi = \mathrm{Ind}_T^G(\chi) = \mathrm{Ind}_T^G(\chi^m)$$

where $\chi = \chi_{k_1,n_1,k_2,n_2}$ and $\chi^m = \chi_{k_1,n_1,n_2-k_2,n_2}$ are such that $\chi \neq \chi^m$, i.e $I_G(\chi) = T$.

We want to compute the Galois orbits of such ψ. We observe that ψ is uniquely determined by $\psi_T = \chi \oplus \chi^m$. Hence it suffices to describe the Galois orbits in

$$\{(\chi \oplus \chi^m) \mid \chi \in \mathrm{Irr}(T), I_G(\chi) = T\}.$$

The character $\chi \oplus \chi^m$ is uniquely determined by the set $\{\chi, \chi^m\}$, which in turn is uniquely determined by $\{(\chi(x), \chi(y)), (\chi^m(x), \chi^m(y))\}$. Therefore, we need to describe the Galois orbits in

$$\{\{(e^{2\pi i k_1/n_1}, e^{2\pi i k_2/n_2}), (e^{2\pi i k_1/n_1}, e^{2\pi i(n_2-k_2)/n_2})\} \mid n_1, n_2 \in \mathbb{N},$$
$$n_2 > 2, k_1, k_2 \in \mathbb{N}, 1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1\},$$

equivalently the orbits in

$$\{(\xi_{n_1,k_1}, \xi_{n_2,k_2}), (\xi_{n_1,k_1}, \xi_{n_2,k_2}^{-1}) \mid n_1, n_2 \in \mathbb{N}, n_2 > 2,$$
$$k_1, k_2 \in \mathbb{N}\ 1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1\}$$

under the action of $\mathrm{Gal}(K_l/\mathbb{Q})$, where $l = \mathrm{lcm}(n_1, n_2)$ and $K_l = \mathbb{Q}(\xi_{n_1,k_1}, \xi_{n_2,k_2})$. The Galois group satisfies $\mathrm{Gal}(K_l/\mathbb{Q}) \cong (\mathbb{Z}/l\mathbb{Z})^*$. Consider that $\alpha \in \mathrm{Gal}(K_l/\mathbb{Q})$

corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$. Its action on $(\xi_{n_1,k_1}, \xi_{n_2,k_2}), (\xi_{n_1,k_1}, \xi_{n_2,k_2}^{-1})$ is given by

$$\{(\xi_{n_1,k_1}, \xi_{n_2,k_2}), (\xi_{n_1,k_1}, \xi_{n_2,k_2}^{-1})\}^\alpha = \{(\xi_{n_1,k_1}^a, \xi_{n_2,k_2}^a), (\xi_{n_1,k_1}^a, \xi_{n_2,k_2}^{-a})\}.$$

We find the lengths and the numbers of such orbits. The length of an orbit is

$$\varphi(l)/ \mid \mathrm{Stab}_{\mathrm{Gal}(K_l/\mathbb{Q})}\{(\xi_{n_1,k_1}, \xi_{n_2,k_2}), (\xi_{n_1,k_1}, \xi_{n_2,k_2}^{-1})\} \mid .$$

Consider $\xi_1 = \xi_{n_1,k_1}$, a primitive n_1th root of unity and $\xi_2 = \xi_{n_2,k_2}$, a primitive n_2th root of unity. We have to determine all $a \in (\mathbb{Z}/l\mathbb{Z})^*$ such that

$$\begin{aligned} (\xi_1, \xi_2)^\alpha &= (\xi_1^a, \xi_2^a) = (\xi_1, \xi_2) \\ \text{or } (\xi_1, \xi_2)^\alpha &= (\xi_1^a, \xi_2^a) = (\xi_1, \xi_2^{-1}), \end{aligned}$$

equivalently

$$\begin{aligned} &a \equiv 1 \bmod l \\ \text{or } &a \equiv 1 \bmod n_1 \text{ and } a \equiv -1 \bmod n_2. \end{aligned}$$

As per Lemma 2.5.4 and using condition $n_2 > 2$, we have

$$|\,\mathrm{Stab}\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1})\}| = \begin{cases} 1 & \text{if } \gcd(n_1, n_2) \geq 3, \\ 2 & \text{if } \gcd(n_1, n_2) \in \{1, 2\}. \end{cases}$$

Hence the length of the corresponding orbit is

$$\begin{cases} \varphi(l) & \text{if } \gcd(n_1, n_2) \geq 3, \\ \varphi(l)/2 & \text{if } \gcd(n_1, n_2) \in \{1, 2\}, \end{cases}$$

where $l = \mathrm{lcm}(n_1, n_2)$.

For fixed n_1, n_2 there are

$$\frac{\varphi(n_1)\varphi(n_2)}{2}$$

choices for $\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1})\}$. Since $\varphi(l)\varphi(g) = \varphi(n_1)\varphi(n_2)$, the total number of orbits is

$$\begin{cases} \frac{\varphi(n_1)\varphi(n_2)}{2\varphi(l)} = \varphi(g)/2 & \text{if } \gcd(n_1, n_2) \geq 3, \\ \frac{\varphi(n_1)\varphi(n_2)}{\varphi(l)} = \varphi(g) = 1 & \text{if } \gcd(n_1, n_2) \in \{1, 2\}, \end{cases}$$

where $g = \gcd(n_1, n_2)$. Every $\psi \in \mathrm{Irr}(G)$ corresponds to $\psi_T = \chi \oplus \chi^m$ and thus to $\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1})\}$ and has $\psi(1) = 2$. Its Galois orbit has length

$$\begin{cases} \varphi(l) & \text{if } \gcd(n_1, n_2) \geq 3 \\ \varphi(l)/2 & \text{if } \gcd(n_1, n_2) \in \{1, 2\}. \end{cases}$$

Since $G = \langle m \rangle \ltimes T$ is a semi direct product, according to Lemma 2.4.15, the Schur index is one. The Galois orbit of ψ leads to an irreducible character over $\mathbb{Q}$ of degree

$$\begin{cases} 2\varphi(l) & \text{if } \gcd(n_1, n_2) \geq 3 \\ 2\varphi(l)/2 = \varphi(l) & \text{if } \gcd(n_1, n_2) \in \{1, 2\}, \end{cases}$$

where $l = \mathrm{lcm}(n_1, n_2)$.

Hence the Galois orbit zeta function is

$$\omega_G^{\mathbb{Q}}(s,t) = \omega_1(s,t) + \gamma_1(s,t) + \gamma_2(s,t), \tag{5.3.1}$$

where

$$\gamma_1(s,t) = \sum_{\substack{n_1=1, n_2=3 \\ g=\gcd(n_1,n_2)\geq 3 \\ l=\mathrm{lcm}(n_1,n_2)}}^{\infty} \varphi(g)/2 \cdot 2^{-s} \varphi(l)^{-t+1} = 2^{-1-s} \sum_{\substack{n_1=1, n_2=3 \\ g=\gcd(n_1,n_2)\geq 3 \\ l=\mathrm{lcm}(n_1,n_2)}}^{\infty} \varphi(g) \cdot \varphi(l)^{-t+1},$$

and

$$\begin{aligned} \gamma_2(s,t) &= \sum_{\substack{n_1=1, n_2=3 \\ g=\gcd(n_1,n_2)\in\{1,2\} \\ l=\mathrm{lcm}(n_1,n_2)}}^{\infty} \varphi(g) \cdot 2^{-s} (\varphi(l)/2)^{-t+1} \\ &= 2^{t-1-s} \sum_{\substack{n_1=1, n_2=3 \\ g=\gcd(n_1,n_2)\in\{1,2\} \\ l=\mathrm{lcm}(n_1,n_2)}}^{\infty} \varphi(g) \varphi(l)^{-t+1}. \end{aligned}$$

Note: In the definition of the Galois orbit zeta function, we sum over characters, not orbits. Hence we use $-t+1$ rather than $-t$.

We compute $\gamma_1(s,t)$ and $\gamma_2(s,t)$ separately.

$$
\begin{aligned}
\gamma_1(s,t) &= 2^{-1-s} \sum_{\substack{n_1=1,n_2=3\\ g=\gcd(n_1,n_2)\geq 3\\ l=\operatorname{lcm}(n_1,n_2)}}^{\infty} \varphi(g)\cdot\varphi(l)^{-t+1} \\
&= 2^{-1-s} \sum_{n_1=1,n_2=1}^{\infty} \varphi(g)\cdot\varphi(l)^{-t+1} - 2^{-1-s} \sum_{\substack{n_1=1,n_2=1\\ g=1}}^{\infty} \varphi(l)^{-t+1} \\
&\quad - 2^{-1-s} \sum_{\substack{n_1=1,n_2=1\\ g=2}}^{\infty} \varphi(l)^{-t+1} \\
&= 2^{-1-s}\eta(s,t) - 2^{-1-s}\eta_a(s,t) - 2^{-1-s}\eta_b(s,t),
\end{aligned}
\tag{5.3.2}
$$

where

$$\eta(t) = \sum_{n_1,n_2=1}^{\infty} \varphi(g)\cdot\varphi(l)^{-t+1},$$

$$\eta_a(t) = \sum_{\substack{n_1,n_2=1\\ g=1}}^{\infty} \varphi(l)^{-t+1},$$

and

$$\eta_b(t) = \sum_{\substack{n_1,n_2=1\\ g=2}}^{\infty} \varphi(l)^{-t+1}.$$

We compute $\eta(t)$, $\eta_a(t)$ and $\eta_b(t)$ separately. To compute $\eta(t)$, consider $n_1 = \prod_i p_i^{e_i}$ and $n_2 = \prod_i p_i^{f_i}$, factorised into primes. Then $g = \gcd(n_1,n_2) = \prod_i p_i^{\min\{e_i,f_i\}}$ and $l = \operatorname{lcm}(n_1,n_2) = \prod_i p_i^{\max\{e_i,f_i\}}$. Hence

$$\varphi(g) = \prod_i \begin{cases} 1 & \text{if } \min\{e_i,f_i\} = 0 \\ (p_i-1)\cdot p_i^{\min\{e_i,f_i\}-1} & \text{if } \min\{e_i,f_i\} > 0, \end{cases}$$

and

$$\varphi(l) = \prod_i \begin{cases} 1 & \text{if } \max\{e_i,f_i\} = 0, \\ (p_i-1)\cdot p_i^{\max\{e_i,f_i\}-1} & \text{if } \max\{e_i,f_i\} > 0. \end{cases}$$

Using these values we get

$$
\begin{aligned}
\eta(t) = \prod_p \sum_{e=0}^{\infty}\sum_{f=0}^{\infty} \Bigg[&\left\{ \begin{matrix} 1 & \text{if } \min\{e,f\} = 0 \\ (p-1)\cdot p^{\min\{e,f\}-1} & \text{if } \min\{e,f\} > 0 \end{matrix} \right\} \cdot \\
&\left\{ \begin{matrix} 1 & \text{if } \max\{e,f\} = 0 \\ (p-1)\cdot p^{\max\{e,f\}-1} & \text{if } \max\{e,f\} > 0 \end{matrix} \right\}^{-t+1} \Bigg].
\end{aligned}
$$

There are four cases covering the possible values for e and f.

Case(1): If $e = f = 0$ then the double sum has value 1.

Case(2): If $e \geq 1$ and $f = 0$ then the double sum is equal to

$$\eta_1(t) = \sum_{e=1}^{\infty} ((p-1)p^{e-1})^{-s} = (p-1)^{-t+1} \frac{1}{1-p^{-t+1}}.$$

Case(3): If $e = 0$ and $f \geq 1$ then the double sum is equal to

$$\eta_2(t) = \sum_{f=1}^{\infty} ((p-1)p^{f-1})^{-t+1} = (p-1)^{-s} \frac{1}{1-p^{-t+1}}.$$

Case(4): If $e \geq 1$ and $f \geq 1$ then we have the following sub-cases

1. If $e = f$ then the double sum is equal to

$$\eta_3(t) = \sum_{e=1}^{\infty} (p-1)p^{e-1}((p-1)p^{e-1})^{-t+1} = \sum_{e=1}^{\infty} ((p-1)p^{e-1})^{2-t} = \frac{(p-1)^{2-t}}{1-p^{2-t}}.$$

2. If $e < f$ then the double sum is equal to

$$\begin{aligned}
\eta_4(t) &= \sum_{e=1}^{\infty} \sum_{f=e+1}^{\infty} (p-1)p^{e-1}((p-1)p^{f-1})^{1-t} \\
&= (p-1)^{2-t} \sum_{e=1}^{\infty} p^{e-1} \sum_{f=e+1}^{\infty} (p^{f-1})^{1-t} \\
&= (p-1)^{2-t} \sum_{e=1}^{\infty} p^{e-1} \frac{p^{e(1-t)}}{1-p^{1-t}} \\
&= (p-1)^{2-t} \frac{p^{-1}}{(1-p^{1-t})} \sum_{e=1}^{\infty} p^{e(2-t)} \\
&= (p-1)^{2-t} \frac{p^{-1}}{(1-p^{1-t})} \frac{p^{2-t}}{1-p^{2-t}} \\
&= (p-1)^{2-t} \frac{p^{1-t}}{(1-p^{1-t})(1-p^{2-t})}.
\end{aligned}$$

3. If $e > f$ then, similarly, the double sum is equal to $\eta_5(t) = \eta_4(t)$.

Hence

$$\begin{aligned}
\eta(t) &= \prod_p (1 + \eta_1(t) + \eta_2(t) + \eta_3(t) + \eta_4(t) + \eta_5(t)) \\
&= \prod_p \left(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}} + \frac{(p-1)^{2-t}}{1-p^{2-t}} + 2\frac{p^{1-t}(p-1)^{2-t}}{(1-p^{1-t})(1-p^{2-t})}\right) \\
&= \prod_p \left(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}} + \frac{(p-1)^{2-t}}{1-p^{2-t}}(1 + 2\frac{p^{1-t}}{1-p^{1-t}})\right) \\
&= \prod_p \left(1 + (p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\right).
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
\eta_a(t) &= \prod_p \left(1 + \sum_{e=1}^{\infty}((p-1)\cdot p^{e-1})^{1-t} + \sum_{f=1}^{\infty}((p-1)\cdot p^{f-1})^{1-t}\right) \\
&= \prod_p \left(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\eta_b(t) = (1 + 2\cdot 2^{1-t} + 2\cdot 2^{2-2t} + 2\cdot 2^{3-3t} + \cdots)\cdot \prod_{p\neq 2}\Bigg(1 + \sum_{e=1}^{\infty}((p-1)\cdot p^{e-1})^{1-t} \\
+ \sum_{f=1}^{\infty}((p-1)\cdot p^{f-1})^{1-t}\Bigg) \\
= \frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}\left(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}\right).
\end{aligned}$$

Using the values in the above equation 5.3.2 we get

$$\begin{aligned}
\gamma_1(s,t) = 2^{-1-s}\prod_p\left(1 + (p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\right) - 2^{-1-s}\prod_p(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
- 2^{-1-s}\cdot\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}).
\end{aligned}$$

Similarly we can compute $\gamma_2(s,t)$.

$$
\begin{aligned}
\gamma_2(s,t) &= 2^{t-s-1} \sum_{\substack{n_1\geq 1, n_2\geq 3\\ g=\gcd(n_1,n_2)\in\{1,2\}\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t} = 2^{t-s-1} \sum_{\substack{n_1\geq 1, n_2\geq 3\\ g=\gcd(n_1,n_2)\in\{1,2\}\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(l)^{1-t}\\
&= 2^{t-s-1} \sum_{\substack{n_1\geq 1, n_2\geq 1\\ g=1}} \varphi(l)^{1-t} - 2^{t-s-1} \sum_{\substack{n_1\geq 1, n_2=1\\ g=1}} \varphi(l)^{1-t} - 2^{t-s-1} \sum_{\substack{n_1\geq 1, n_2=2\\ g=1\\ n_1 \text{ is odd}}} \varphi(l)^{1-t}\\
&+ 2^{t-s-1} \sum_{\substack{n_1\geq 1, n_2\geq 1\\ g=2}} \varphi(l)^{1-t} - 2^{t-s-1} \sum_{\substack{n_1\geq 1, n_2=2\\ g=2\\ n_1 \text{ is even}}} \varphi(l)^{1-t}\\
&= 2^{t-s-1} \Bigg(\prod_p (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - \prod_p (1 + \frac{(p-1)^{1-t}}{1-p^{1-t}})\\
&+ \frac{1+2^{1-t}}{1-2^{1-t}} \prod_{p\neq 2} (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - \prod_p (1 + \frac{(p-1)^{1-t}}{1-p^{1-t}}) \Bigg)\\
&= 2^{t-s-1} \Bigg(\prod_p (1 + 2\frac{(p-1)^{-s}}{1-p^{-s}}) - 2\prod_p (1 + \frac{(p-1)^{1-t}}{1-p^{1-t}}\\
&+ \frac{1+2^{1-t}}{1-2^{1-t}} \prod_{p\neq 2} (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \Bigg).
\end{aligned}
$$

Using $\zeta_1(s,t)$, $\gamma_1(s,t)$ and $\gamma_2(s,t)$ in the above equation 5.3.1 we get the Galois orbit zeta function

$$
\begin{aligned}
\omega_G^{\mathbb{Q}}(s,t) &= 4\cdot\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) + 2^{-1-s} \prod_p \left(1 + (p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\right)\\
&- 2^{-1-s} \prod_p (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - 2^{-1-s}\frac{1+2^{1-t}}{1-2^{1-t}} \prod_{p\neq 2} (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}})\\
&+ 2^{t-s-1} \Bigg(\prod_p (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) + \frac{1+2^{1-t}}{1-2^{1-t}} \prod_{p\neq 2} (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}})\\
&- 2\prod_p (1 + \frac{(p-1)^{1-t}}{1-p^{1-t}} \Bigg)\\
&= 4\cdot\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{t-s} \prod_p (1 + \frac{(p-1)^{1-t}}{1-p^{1-t}} + 2^{-1-s} \prod_p \left(1 + (p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\right)\\
&+ (2^{t-s-1} - 2^{-1-s}) \prod_p (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}})\\
&+ (2^{t-s-1} - 2^{-1-s}) \frac{1+2^{1-t}}{1-2^{1-t}} \prod_{p\neq 2} (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}).
\end{aligned}
$$

Since $G = \langle m \rangle \ltimes T$, is a semi direct product, according to Lemma 2.4.15, the Schur index is equal to one. So the rational representation zeta function is

$$\begin{aligned}
\zeta_G^{\mathbb{Q}}(s) &= \omega_G^{\mathbb{Q}}(s,t) = \omega_G^{\mathbb{Q}}(s,s+1) \\
&= 4\cdot \zeta_G^{\mathbb{Q}}(s) - 2\prod_p (1+\frac{(p-1)^{-s}}{1-p^{-s}}) + 2^{-1-s}\prod_p \left(1+(p-1)^{-s}\frac{p+1}{1-p^{1-s}}\right) \\
&+ (1-2^{-1-s})\prod_p (1+2\frac{(p-1)^{-s}}{1-p^{-s}}) \\
&+ (1-2^{-1-s})\frac{1+2^{-s}}{1-2^{-s}}\prod_{p\neq 2}(1+2\frac{(p-1)^{-s}}{1-p^{-s}}) \\
&= 2\cdot \zeta_{\mathbb{Z}}^{\mathbb{Q}}(s) + 2^{-1-s}\cdot \zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + (1-2^{-1-s})\Bigg(\prod_p (1+2\frac{(p-1)^{-s}}{1-p^{-s}}) \\
&+ \frac{1+2^{-s}}{1-2^{-s}}\prod_{p\neq 2}(1+2\frac{(p-1)^{-s}}{1-p^{-s}})\Bigg).
\end{aligned}$$

5.4 The group pg

The group pg is given by the presentation

$$G = \langle x, y, t \mid [x,y] = 1, x^t = x^{-1}, t^2 = y\rangle.$$

The group G contains a translation group T, which is normal in G and generated by x and y. The quotient group G/T, generated by tT, is the point group of G.

We want to compute the rational representation zeta function $\zeta_G^{\mathbb{Q}}(s)$ of G. As in Section 5.2, the irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{k_1,n_1,k_2,n_2} : T \longrightarrow \mathbb{C}^*,$$

where $n_1, n_2 \in \mathbb{N}$, $k_1, k_2 \in \mathbb{N}$ with $1 \le k_i \le n_i, \gcd(k_i, n_i) = 1$ and

$$\chi_{k_1,n_1,k_2,n_2}(x^a y^b) = \xi_{n_1,k_1}^a \xi_{n_2,k_2}^b = e^{2\pi i(k_1 a/n_1 + k_2 b/n_2)},$$

where $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$, $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$.

Consider the action of G on $\mathrm{Irr}(T)$ given by $\chi^g(z) = \chi(z^{g^{-1}})$, where $\chi \in \mathrm{Irr}(T)$, $g \in G, z \in T$. We need to understand the orbit of the character χ and its stabiliser.

The size of the orbit of χ is $|G : I_G(\chi)|$. So either $I_G(\chi) = T$ when $t \notin I_G(\chi)$ or $I_G(\chi) = G$ when $t \in I_G(\chi)$.

Now consider $\psi \in \mathrm{Irr}(G)$, factoring over a finite quotient of G. We can cover all possible ψ by considering two cases $\psi(1) = 1$ and $\psi(1) > 1$.

<u>**Case(1)** : $\quad \psi \in \mathrm{Irr}(G)$ with $\psi(1) = 1$</u>

To compute the linear characters of G it is enough to consider the abelianisation of G:

$$\begin{aligned} G/[G,G] &= \langle x, y, t \mid [x,y] = [x,t] = [y,t] = 1, t^2 = y, x^2 = 1 \rangle \\ &\cong \mathbb{Z} \times C_2. \end{aligned}$$

The irreducible representations of C_2 are all defined over $\mathbb{Q}$, so that the irreducible representations of $\mathbb{Z} \times C_2$ over $\mathbb{Q}$ are tensor products of irreducible representations of $\mathbb{Z}$ over $\mathbb{Q}$, and irreducible representations of C_2 over $\mathbb{Q}$. Hence the contribution to the Galois orbit zeta function is

$$\omega_1(s,t) = 2 \cdot \omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t).$$

<u>**Case(2)**. $\quad \psi \in \mathrm{Irr}(G)$ with $\psi(1) > 1$</u>

In this case ψ cannot restrict to an irreducible character of the abelian group T. This means that

$$\psi = \mathrm{Ind}_T^G(\chi) = \mathrm{Ind}_T^G(\chi^t)$$

where $\chi = \chi_{k_1,n_1,k_2,n_2}$ and $\chi^t = \chi_{n_1-k_1,n_1,k_2,n_2}$ are such that $\chi \neq \chi^t$, i.e $I_G(\chi) = T$. We need to compute the Galois orbits of such ψ. We observe that ψ is uniquely determined by $\psi_T = \chi \oplus \chi^t$. Hence it suffices to describe the Galois orbits in

$$\{(\chi \oplus \chi^t) \mid \chi \in \mathrm{Irr}(T), I_G(\chi) = T\}.$$

The character $\chi \oplus \chi^t$ is uniquely determined by the set $\{\chi, \chi^t\}$, which in turn is uniquely determined by $\{(\chi(x), \chi(y)), (\chi^t(x), \chi^t(y))\}$. Therefore we need to describe the Galois orbits in

$$\{\{(e^{2\pi i k_1/n_1}, e^{2\pi i k_2/n_2}), (e^{2\pi i(n_1-k_1)/n_1}, e^{2\pi i(k_2)/n_2})\} \mid n_1, n_2 \in \mathbb{N}, k_1, k_2 \in \mathbb{N},$$
$$n_1 > 2, 1 \le k_i \le n_i, \gcd(k_i, n_i) = 1\}$$

equivalently the orbits in

$$\{(\xi_{n_1,k_1}, \xi_{n_2,k_2}), (\xi_{n_1,k_1}^{-1}, \xi_{n_2,k_2}) \mid n_1, n_2 \in \mathbb{N}, n_1 > 2, k_1, k_2 \in \mathbb{N},$$
$$1 \le k_i \le n_i, \gcd(k_i, n_i) = 1\}.$$

under the action of $\mathrm{Gal}(K_l/\mathbb{Q})$, where $l = \mathrm{lcm}(n_1, n_2)$ and $K_l = \mathbb{Q}(\xi_{n_1,k_1}, \xi_{n_2,k_2})$. The Galois group satisfies $\mathrm{Gal}(k_l/\mathbb{Q}) \cong (\mathbb{Z}/l\mathbb{Z})^*$. Consider that $\alpha \in \mathrm{Gal}(K_l/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$. Its action on $(\xi_{n_1,k_1}, \xi_{n_2,k_2}), (\xi_{n_1,k_1}^{-1}, \xi_{n_2,k_2})$ is given by

$$\{(\xi_{n_1,k_1}, \xi_{n_2,k_2}), (\xi_{n_1,k_1}^{-1}, \xi_{n_2,k_2})\}^{\alpha} = \{(\xi_{n_1,k_1}^{a}, \xi_{n_2,k_2}^{a}), (\xi_{n_1,k_1}^{-a}, \xi_{n_2,k_2}^{a})\}.$$

We want to determine the lengths and the numbers of such orbits. Consider $\xi_1 = \xi_{n_1,k_1}$, a primitive n_1th root of unity and $\xi_2 = \xi_{n_2,k_2}$, a primitive n_2th root of unity. Using the same method as in Section 5.3 we get

$$|\,\mathrm{Stab}\{(\xi_1, \xi_2), (\xi_1^{-1}, \xi_2)\}| = \begin{cases} 1 & \text{if } \gcd(n_1, n_2) \ge 3, \\ 2 & \text{if } \gcd(n_1, n_2) \in \{1, 2\}. \end{cases}$$

Hence the length of the corresponding orbit is

$$\begin{cases} \varphi(l) & \text{if } g \ge 3, \\ \varphi(l)/2 & \text{if } g \in \{1, 2\}, \end{cases}$$

where $l = \mathrm{lcm}(n_1, n_2)$ and $g = \gcd(n_1, n_2)$. For fixed n_1, n_2 there are $\varphi(n_1)\varphi(n_2)/2$ choices for $\{(\xi_1, \xi_2), (\xi_1^{-1}, \xi_2)\}$. The total number of orbits is

$$\begin{cases} \varphi(n_1)\varphi(n_2)/2\varphi(l) = \varphi(g)/2 & \text{if } g \ge 3, \\ \varphi(n_1)\varphi(n_2)/\varphi(l) = \varphi(g) = 1 & \text{if } g \in \{1, 2\}. \end{cases}$$

Hence the Galois orbit zeta function is

$$\omega_G^{\mathbb{Q}}(s,t) = \omega_1(s,t) + \sum_{\substack{n_1=3,n_2=1\\ g=\gcd(n_1,n_2)\geq 3\\ l=\mathrm{lcm}(n_1,n_2)}}^{\infty} \varphi(g)/2 \cdot 2^{-s}\varphi(l)^{1-t}$$

$$+ \sum_{\substack{n_1=3,n_2=1\\ g=\gcd(n_1,n_2)\in\{1,2\}\\ l=\mathrm{lcm}(n_1,n_2)}}^{\infty} \varphi(g)\cdot 2^{-s}(\varphi(l)/2)^{1-t}$$

$$= 2\cdot\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) + 2^{-1-s}\sum_{\substack{n_1=3,n_2=1\\ g\geq 3}}^{\infty}\varphi(g)\varphi(l)^{1-t} + 2^{t-s-1}\sum_{\substack{n_1=3,n_2=1\\ g\in\{1,2\}}}^{\infty}\varphi(g)\varphi(l)^{1-t}$$

using the same calculation as in Section 5.3 we get

$$= 2\cdot\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{t-s}\prod_p(1+\frac{(p-1)^{1-t}}{1-p^{1-t}} + 2^{-1-s}\prod_p\left(1+(p-1)^{1-t}\frac{(p+1)}{1-p^{2-t}}\right)$$

$$+ (2^{t-s-1}-2^{-1-s})\prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}})$$

$$+ (2^{t-s-1}-2^{-1-s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}})$$

$$= 2\cdot\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{t-s}\prod_p(1+\frac{(p-1)^{1-t}}{1-p^{1-t}} + 2^{-1-s}\prod_p\left(1+(p-1)^{1-t}\frac{(p+1)}{1-p^{2-t}}\right)$$

$$+ (2^{t-s-1}-2^{-1-s})\left(\prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) + \frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}})\right).$$

Note: Lemma 2.4.15, does not apply directly to show the Schur index is one. We prove the Schur index of the group G of type pg is one. For a contradiction, suppose there exist $\psi\in\mathrm{Irr}(G)$ with $m_{\mathbb{Q}}(\psi)\neq 1$. As per Corollary 2.4.10(8), $m_{\mathbb{Q}}(\psi)|\psi(1)$, and $\psi(1)\leq 2$, we must have $m_{\mathbb{Q}}(\psi)=\psi(1)=2$ and $\psi=\mathrm{Ind}_T^G(\chi)$, $\chi\in\mathrm{Irr}(T)$ with $I_G(\chi)=T$. As per Theorem 2.4.18, there exists a $\mathbb{Q}$-triple (H,X,ϑ) such that

1. H is a section of G
2. $2|m_{\mathbb{Q}}(\vartheta)$
3. H/X is a 2-group

The section H is a quotient of a non-abelian finite index subgroup of G. Any non-abelian finite index subgroup of G is isomorphic to G. Hence H is a quotient of a group isomorphic to G. So, if we take H to be a finite quotient of G then H must be non abelian, otherwise the Schur index is one. It gives $H = X \rtimes C_2$, dihedral group. Hence, according to Lemma 2.4.15, $m_{\mathbb{Q}}(\vartheta) = 1$.

Hence the Schur index of G is one. The rational representation zeta function of G is

$$\begin{aligned}
\zeta_G^{\mathbb{Q}}(s) &= \omega_G^{\mathbb{Q}}(s,t) = \omega_G^{\mathbb{Q}}(s,s+1)\\
&= 2\cdot\zeta_{\mathbb{Z}}^{\mathbb{Q}}(s) - 2\prod_p (1+\frac{(p-1)^{-s}}{1-p^{-s}}) + 2^{-1-s}\prod_p\left(1+(p-1)^{-s}\frac{(p+1)}{1-p^{1-s}}\right)\\
&+(1-2^{-1-s})\Big(\prod_p(1+2\frac{(p-1)^{-s}}{1-p^{-s}}) + \frac{1+2^{-s}}{1-2^{-s}}\prod_{p\neq 2}(1+2\frac{(p-1)^{-s}}{1-p^{-s}})\Big)\\
&= 2^{-1-s}\zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + (1-2^{-1-s})\Big(\prod_p(1+2\frac{(p-1)^{-s}}{1-p^{-s}})\\
&+\frac{1+2^{-s}}{1-2^{-s}}\prod_{p\neq 2}(1+2\frac{(p-1)^{-s}}{1-p^{-s}})\Big).
\end{aligned}$$

5.5 The group p2mm

The group $p2mm$ is given by the presentation

$$G = \langle x,y,p,q \mid [x,y]=[p,q]=1=p^2=q^2, x^p=x, x^q=x^{-1}, y^p=y^{-1}, y^q=y\rangle.$$

The group G contains a translation group T, which is generated by x and y. The quotient group G/T, generated by pT and qT, is called the point group of G and it is a Klein 4-group.

We want to compute the rational representation zeta function $\zeta_G^{\mathbb{Q}}(s)$ of G. As in section 5.2, the irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{k_1,n_1,k_2,n_2} : T \longrightarrow \mathbb{C}^*,$$

where $n_1, n_2 \in \mathbb{N}$, $k_1, k_2 \in \mathbb{N}$ with $1 \le k_i \le n_i, \gcd(k_i,n_i) = 1$ and

$$\chi_{k_1,n_1,k_2,n_2}(x^a y^b) = \xi_{n_1,k_1}^a \xi_{n_2,k_2}^b = e^{2\pi i(k_1 a/n_1 + k_2 b/n_2)},$$

where $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$, $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$.

Consider $\xi_{n_1,k_1} = \xi_1$, a primitive n_1th root of unity and $\xi_{n_2,k_2} = \xi_2$, a primitive n_2th root of unity.

The possibilities for inertia groups are

Case(1): If $\xi_1 \in \{1, -1\}$ and $\xi_2 \in \{1, -1\}$ then $I_G(\chi) = G$.

Case(2): If $\xi_1 \in \{1, -1\}$ and $\xi_2 \notin \{1, -1\}$ then $I_G(\chi) = \langle q \rangle T$.

Case(3): $\xi_1 \notin \{1, -1\}$ and $\xi_2 \in \{1, -1\}$ then $I_G(\chi) = \langle p \rangle T$.

Case(4): If $\xi_1 \notin \{1, -1\}$ and $\xi_2 \notin \{1, -1\}$ then $I_G(\chi) = T$.

We discuss these possibilities separately.

Case(1) : $\quad I_G(\chi) = G$

To compute the linear character of G it is enough to consider the abelinisation of G.

$$\begin{aligned} G/[G,G] &= \langle x, y, p, q \mid [x,y] = [x,p] = [y,p] = [x,q] = [y,q] = [p,q] = 1, \\ &\quad x^2 = y^2 = p^2 = q^2 = 1 \rangle \\ &\cong C_2 \times C_2 \times C_2 \times C_2 \end{aligned}$$

which gives precisely 16 one-dimensional representations of G (4 choices of the character χ, each of them extends 4 different ways to the linear character of G). The linear character of G defined over $\mathbb{Q}$, gives the 16 one-dimensional irreducible characters over $\mathbb{Q}$. The contribution to the Galois orbit zeta function is

$$\omega_1(s,t) = 16 \cdot 1^{-s} \cdot 1^{-t} = 16$$

Case(2) : $\quad I_G(\chi) = \langle q \rangle T$

In this case we extend the given characters of T to the inertia group and then induce it to G as shown in the following diagram.

$$\begin{array}{cccc}
G & \bullet & \vartheta_1 = \mathrm{Ind}_{\langle q\rangle T}^{G}(\psi_1) & \vartheta_2 = \mathrm{Ind}_{\langle q\rangle T}^{G}(\psi_2) \\
 & \Big| & & \\
I_G(\chi) = \langle q\rangle T & \bullet & \psi_1 \oplus \psi_1^p & \psi_2 \oplus \psi_2^p \\
 & \Big| & & \\
T & \bullet & \chi \neq \chi^p &
\end{array}$$

The characters ψ_1 and ψ_2 are the extension of χ, and ψ_1^p and ψ_2^p are the extension of χ^p, since $\psi_T(x) = \chi(x) = \xi_1$, $\psi_T(y) = \chi(y) = \xi_2$ and $\psi(p) = \pm 1$.

We want to compute the Galois orbits of the characters ϑ_1 and ϑ_2. The characters ϑ_1 and ϑ_2 are uniquely determined by $\vartheta_{1T} = \psi_1 \oplus \psi_1^p$ and $\vartheta_{2T} = \psi_2 \oplus \psi_2^p$. It is suffices to describe the Galois orbits of

$$\{(\psi_1 \oplus \psi_1^p) \mid \psi_1 \in I_G(\chi)\}$$

and

$$\{(\psi_2 \oplus \psi_2^p) \mid \psi_2 \in I_G(\chi)\}$$

equivalently

$$\{(\xi_1, \xi_2, 1), (\xi_1, \xi_2^{-1}, 1)\}$$

and

$$\{(\xi_1, \xi_2, -1), (\xi_1, \xi_2^{-1}, -1)\},$$

under the action of $\mathrm{Gal}(K_l/\mathbb{Q})$, where $l = \mathrm{lcm}(n_1, n_2)$, $K_l = \mathbb{Q}(\xi_1, \xi_2)$ and $\psi(x) = \xi_1$, $\psi(y) = \xi_2$, $\psi(p) = \pm 1$, since $\psi_T = \chi$. Consider that $\alpha \in \mathrm{Gal}(K_l/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$. Its action on $\{(\xi_1, \xi_2, 1), (\xi_1, \xi_2^{-1}, 1)\}$ given by

$$\{(\xi_1, \xi_2, 1), (\xi_1, \xi_2^{-1}, 1)\}^{\alpha} = \{(\xi_1^a, \xi_2^a, 1), (\xi_1^a, \xi_2^{-a}, 1)\}.$$

We find the lengths and numbers of such orbits. The length of an orbit is

$$\varphi(l)/|\operatorname{Stab}_{\mathrm{Gal}(K_l/\mathbb{Q})}\{(\xi_1, \xi_2, 1), (\xi_1, \xi_2^{-1}, 1)\}|.$$

As per Lemma 2.5.4, the stabiliser of $\{(\xi_1, \xi_2, 1), (\xi_1, \xi_2^{-1}, 1)\}$ has size 2. Hence the length of an orbit is

$$\varphi(l)/2 = \varphi(n_2)/2$$

as in this case $n_1 \in \{1, 2\}$. For fixed n_1 and n_2, there are $\varphi(n_2)/2$ choices for $\{(\xi_1, \xi_2, 1), (\xi_1, \xi_2^{-1}, 1)\}$. So the number of orbits is

$$\frac{\varphi(n_2)/2}{\varphi(n_2)/2} = 1.$$

According to Lemma 2.4.15, the Schur index is one. The Galois orbit of ϑ leads to an irreducible character over $\mathbb{Q}$ of degree $2 \cdot \varphi(n_2)/2 = \varphi(n_2)$, as $\vartheta_1(1)$ has degree 2. The contribution to the Galois orbit zeta function is

$$\begin{aligned}\omega_2(s,t) &= 2\sum_{n_2=3}^{\infty} 2^{-s}(\varphi(n_2)/2)^{1-t} = 2 \cdot 2^{t-s-1} \sum_{n_2=1}^{\infty} \varphi(n_2)^{1-t} - 4 \cdot 2^{t-s-1} \\ &= 2^{t-s}\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{t-s+1}.\end{aligned}$$

<u>**Case(3)** : $\quad I_G(\chi) = \langle p \rangle T$</u>

Similar to Case 2.

<u>**Case(4)** : $\quad I_G(\chi) = T$</u>

In this case $\vartheta \in \mathrm{Irr}(G)$ cannot restrict to an irreducible character of the abelian group T. This means that

$$\vartheta = \mathrm{Ind}_T^G(\chi) = \mathrm{Ind}_T^G(\chi^p) = \mathrm{Ind}_T^G(\chi^q) = \mathrm{Ind}_T^G(\chi^{pq}), \vartheta(1) = 4$$

where $\chi \neq \chi^p \neq \chi^q \neq \chi^{pq}$. We compute the Galois orbits of ϑ, which is uniquely determined by $\vartheta_T = \chi \oplus \chi^p \oplus \chi^q \oplus \chi^{pq}$. It suffices to describe the Galois orbits of

$$\{\chi \oplus \chi^p \oplus \chi^q \oplus \chi^{pq} \mid \chi \in \mathrm{Irr}(T), I_G(\chi) = T\},$$

in terms of ξ_1 and ξ_2

$$\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2), (\xi_1^{-1}, \xi_2^{-1})\},$$

under the action of $\mathrm{Gal}(K_l/\mathbb{Q}) \cong (\mathbb{Z}/l\mathbb{Z})^*$, where $l = \mathrm{lcm}(n_1, n_2)$. Consider that $\alpha \in \mathrm{Gal}(K_l/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$ and its action on

$$\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2), (\xi_1^{-1}, \xi_2^{-1})\}$$

given by

$$\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2), (\xi_1^{-1}, \xi_2^{-1})\}^{\alpha} = \{(\xi_1^a, \xi_2^a), (\xi_1^a, \xi_2^{-a}), (\xi_1^{-a}, \xi_2^a), (\xi_1^{-a}, \xi_2^{-a})\}.$$

We have to determine all $a \in (\mathbb{Z}/l\mathbb{Z})^*$ such that

$$\{(\xi_1^a, \xi_2^a), (\xi_1^a, \xi_2^{-a}), (\xi_1^{-a}, \xi_2^a), (\xi_1^{-a}, \xi_2^{-a})\} = \{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2), (\xi_1^{-1}, \xi_2^{-1})\},$$

equivalently

$$\begin{aligned} (\xi_1^a, \xi_2^a) &= (\xi_1, \xi_2) \\ \text{or } (\xi_1^a, \xi_2^a) &= (\xi_1, \xi_2^{-1}) \\ \text{or } (\xi_1^a, \xi_2^a) &= (\xi_1^{-1}, \xi_2) \\ \text{or } (\xi_1^a, \xi_2^a) &= (\xi_1^{-1}, \xi_2^{-1}), \end{aligned}$$

equivalently, we determine all a such that

$$\begin{aligned} & a \equiv 1 \bmod l \\ & \text{or } a \equiv 1 \bmod n_1 \text{ and } a \equiv -1 \bmod n_2 \\ & \text{or } a \equiv -1 \bmod n_1 \text{ and } a \equiv 1 \bmod n_2 \\ & \text{or } a \equiv -1 \bmod l \end{aligned}$$

The stabiliser in this case is $1 \leq |\operatorname{Stab}\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2), (\xi_1^{-1}, \xi_2^{-1})\}| \leq 4$. As per Lemma 2.5.4 and using conditions $n_1 \geq 3, n_2 \geq 3$, we have

$$|\operatorname{Stab}\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2), (\xi_1^{-1}, \xi_2^{-1})\}| = \begin{cases} 2 & \text{if } \gcd(n_1, n_2) \geq 3 \\ 4 & \text{if } \gcd(n_1, n_2) \in \{1, 2\} \end{cases}$$

Hence the length of the corresponding orbit is

$$\begin{cases} \varphi(\mathrm{lcm}(n_1, n_2))/2 & \text{if } \gcd(n_1, n_2) \geq 3, \\ \varphi(\mathrm{lcm}(n_1, n_2))/4 & \text{if } \gcd(n_1, n_2) \in \{1, 2\}. \end{cases}$$

For fixed n_1, n_2 there are $\varphi(n_1) \cdot \varphi(n_2)/4$ choices for

$$\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2), (\xi_1^{-1}, \xi_2^{-1})\}.$$

The total number of orbits is

$$\begin{cases} \frac{\varphi(n_1)\varphi(n_2)/4}{\varphi(l)/2} = \varphi(\gcd(n_1, n_2))/2 & \text{if } \gcd(n_1, n_2) \geq 3, \\ \frac{\varphi(n_1)\varphi(n_2)/4}{\varphi(l)/4} = \varphi(\gcd(n_1, n_2)) = 1 & \text{if } \gcd(n_1, n_2) \in \{1, 2\}. \end{cases}$$

Using Lemma 2.4.15, the Galois orbit of ϑ leads to an irreducible character over $\mathbb{Q}$ of degree

$$\begin{cases} 4\varphi(\operatorname{lcm}(n_1, n_2))/2 = 2\varphi(\operatorname{lcm}(n_1, n_2)) & \text{if } \gcd(n_1, n_2) \geq 3, \\ 4\varphi(\operatorname{lcm}(n_1, n_2))/4 = \varphi(\operatorname{lcm}(n_1, n_2)) & \text{if } \gcd(n_1, n_2) \in \{1, 2\}. \end{cases}$$

The contribution to the Galois orbit zeta function is

$$\begin{aligned} \omega_3(s,t) = & \sum_{\substack{n_1 \geq 3, n_2 \geq 3 \\ g = \gcd(n_1, n_2) \geq 3 \\ l = \operatorname{lcm}(n_1, n_2)}} \varphi(g)/2 \cdot 4^{-s} (\varphi(l)/2)^{1-t} \\ & + \sum_{\substack{n_1 \geq 3, n_2 \geq 3 \\ g = \gcd(n_1, n_2) \in \{1,2\} \\ l = \operatorname{lcm}(n_1, n_2)}} \varphi(g) \cdot 4^{-s} (\varphi(l)/4)^{1-t} \\ & = 2^{t-2-2s} \sum_{\substack{n_1 \geq 3, n_2 \geq 3 \\ g = \gcd(n_1, n_2) \geq 3 \\ l = \operatorname{lcm}(n_1, n_2)}} \varphi(g)\varphi(l)^{1-t} + 4^{t-s-1} \sum_{\substack{n_1 \geq 3, n_2 \geq 3 \\ g = \gcd(n_1, n_2) \in \{1,2\} \\ l = \operatorname{lcm}(n_1, n_2)}} \varphi(g)\varphi(l)^{1-t} \\ & = 2^{t-2-2s} \eta_1(s,t) + 4^{t-s-1} \eta_2(s,t), \end{aligned}$$

where

$$\eta_1(s,t) = \sum_{\substack{n_1 \geq 3, n_2 \geq 3 \\ g = \gcd(n_1, n_2) \geq 3 \\ l = \operatorname{lcm}(n_1, n_2)}} \varphi(g)\varphi(l)^{1-t}$$

and

$$\eta_2(s,t) = \sum_{\substack{n_1 \geq 3, n_2 \geq 3 \\ g = \gcd(n_1, n_2) \in \{1,2\} \\ l = \operatorname{lcm}(n_1, n_2)}} \varphi(g)\varphi(l)^{1-t}.$$

We compute $\eta_1(s,t)$ and $\eta_2(s,t)$ separately.

$$\eta_1(s,t) = \sum_{\substack{n_1\geq 3, n_2\geq 3\\ g=\gcd(n_1,n_2)\geq 3\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}$$

$$= \sum_{n_1,n_2\geq 1} \varphi(g)\varphi(l)^{1-t} - \sum_{\substack{n_1,n_2\geq 1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t} - \sum_{\substack{n_1,n_2\geq 1\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}$$

Using the same calculation as in Section 5.3 we get

$$= \omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - \prod_p (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - \frac{1+2^{1-t}}{1-2^{1-t}} \prod_{p\neq 2} (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}})$$

Now we compute $\eta_2(s,t)$

$$\eta_2(s,t) = \sum_{\substack{n_1\geq 3, n_2\geq 3\\ g=\gcd(n_1,n_2)\in\{1,2\}\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}$$

$$= \sum_{\substack{n_1\geq 3, n_2\geq 3\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t} + \sum_{\substack{n_1\geq 3, n_2\geq 3\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}$$

$$= \sum_{\substack{n_1,n_2\geq 1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(l)^{1-t} - \sum_{\substack{n_1=1,n_2\geq 1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(l)^{1-t} - \sum_{\substack{n_1=2,n_2\geq 1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)\\ n_2 \text{ is odd}}} \varphi(l)^{1-t}$$

$$- \sum_{\substack{n_1\geq 1,n_2=1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(l)^{1-t} - \sum_{\substack{n_1\geq 1,n_2=2\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)\\ n_1 \text{ is odd}}} \varphi(l)^{1-t} + \sum_{\substack{n_1,n_2\geq 1\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(l)^{1-t}$$

$$- \sum_{\substack{n_1=2,n_2\geq 1\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)\\ n_2 \text{ is even}}} \varphi(l)^{1-t} - \sum_{\substack{n_1\geq 1,n_2=2\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)\\ n_1 \text{ is even}}} \varphi(l)^{1-t}$$

Using the same calculation as in Section 5.3 we get

$$= \prod_p (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) + \frac{1+2^{1-t}}{1-2^{1-t}} \prod_{p\neq 2} (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - 4\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t)$$

Using $\eta_1(s,t)$ and $\eta_2(s,t)$ we get

$$\begin{aligned}
\omega_5(s,t) &= 2^{t-2-2s}\Big(\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - \prod_p (1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - \frac{1+2^{1-t}}{1-2^{1-t}} \\
&\cdot \prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}})\Big) + 4^{t-s-1}\Big(\prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) + \frac{1+2^{1-t}}{1-2^{1-t}} \\
&\cdot \prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - 4\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t)\Big) \\
&= 2^{t-2-2s}\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - 4\cdot 4^{t-s-1}\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t) + (4^{t-s-1} - 2^{t-2-2s}) \\
&\cdot \prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) + (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}})
\end{aligned}$$

Hence the Galois orbit zeta function of G is

$$\begin{aligned}
\omega^{\mathbb{Q}}_G(s,t) &= \omega_1(s,t) + 2\cdot\omega_2(s,t) + \omega_3(s,t) \\
&= 16 + 2\cdot 2^{t-s}\cdot\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t) - 2\cdot 2^{t-s+1} + 2^{t-2-2s}\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - 4^{t-s}\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t) \\
&+ (4^{t-s-1} - 2^{t-2-2s})\prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
&+ (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
&= 16 - 2\cdot 2^{t-s+1} + 2^{t-2-2s}\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) + (4^{t-s-1} - 2^{t-2-2s}) \\
&\cdot \prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) + (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}).
\end{aligned}$$

According to Lemma 2.4.15, the Schur index of G is one. Hence the rational representation zeta function of G is

$$\begin{aligned}
\zeta^{\mathbb{Q}}_G(s) &= \omega^{\mathbb{Q}}_G(s,t) = \omega^{\mathbb{Q}}_G(s,s+1) \\
&= 8 + 2^{-1-s}\cdot\zeta^{\mathbb{Q}}_{\mathbb{Z}^2}(s) + (1-2^{-1-s})\Big(\prod_p(1+2\frac{(p-1)^{-s}}{1-p^{-s}}) \\
&+ \frac{1+2^{-s}}{1-2^{-s}}\prod_{p\neq 2}(1+2\frac{(p-1)^{-s}}{1-p^{-s}})\Big).
\end{aligned}$$

5.6 The group p2mg

The group $p2mg$ is give by the presentation

$$G = \langle x,y,m,t \mid [x,y] = 1 = t^2, m^2 = y, x^t = x, x^m = x^{-1}, y^t = y^{-1}, m^t = m^{-1}\rangle.$$

The group G consists of a translation group T, which is generated by x and y. The quotient group G/T generated by mT and tT is called the point group of G.

We want to compute the Galois orbit zeta function $\omega_G^{\mathbb{Q}}(s,t)$ of G. As before, the irreducible complex characters of T corresponding to the representations with finite image are as follows:

$$\chi_{k_1,n_1,k_2,n_2} : T \longrightarrow \mathbb{C}^*,$$

where $n_1, n_2 \in \mathbb{N}$, $k_1, k_2 \in \mathbb{N}$ with $1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1$, and

$$\chi_{k_1,n_1,k_2,n_2}(x^a y^b) = \xi_{n_1,k_1}^a \xi_{n_2,k_2}^b = e^{2\pi i(k_1 a/n_1 + k_2 b/n_2)},$$

where $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$, $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$.

For simplicity we denote

$$(\chi(x), \chi(y)) = (e^{2\pi i k_1/n_1}, e^{2\pi i k_2/n_2}) = (\xi_{n_1,k_1}, \xi_{n_2,k_2}) = (\xi_1, \xi_2).$$

The possibilities for inertia groups are

Case(1): If $\xi_1 \in \{1, -1\}$ and $\xi_2 \in \{1, -1\}$ then $I_G(\chi) = G$.

Case(2): If $\xi_1 \in \{1, -1\}$ and $\xi_2 \notin \{1, -1\}$ then $I_G(\chi) = \langle m \rangle T$.

Case(3): If $\xi_1 \notin \{1, -1\}$ and $\xi_2 \in \{1, -1\}$ then $I_G(\chi) = \langle t \rangle T$.

Case(4): If $\xi_1 \notin \{1, -1\}$ and $\xi_2 \notin \{1, -1\}$ then $I_G(\chi) = T$.

Now consider $\vartheta \in \mathrm{Irr}(G)$. We can cover all possible ϑ by considering the above four cases.

$$\underline{\textbf{Case(1)}: \quad I_G(\chi) = G}$$

To calculate the linear character of G it is enough to consider the abelinisation of G.

$$\begin{aligned} G/[G,G] &= \langle x, m, t \mid [x,m] = [x,t] = [m,t] = 1, x^2 = t^2 = 1 = m^2 \rangle \\ &\cong C_2 \times C_2 \times C_2. \end{aligned}$$

Since $I_G(\chi)/T$ is not cyclic, we cannot extend a character of T to G. We split this case as

$$\underline{\textbf{Case(a)}: \quad \vartheta(1) = 1}$$

In this case $\vartheta_T = \chi = (\pm 1, 1)$, which gives 8 linear characters defined over $\mathbb{Q}$. The contribution to the Galois orbit zeta function is

$$\omega_1(s,t) = 8 \cdot 1^{-s} \cdot 1^{-t} = 8.$$

<u>**Case(b)** : $\quad \vartheta(1) \neq 1$</u>

In this case $\chi = (\pm 1, -1)$, χ does not extend to a linear character of the group G. We extend the given characters of T to the subgroup $H = \langle x, y, m\rangle$ and then induce to G as shown in the following diagram.

$$\begin{array}{lcll}
G & \bullet & \vartheta = \mathrm{Ind}_H^T(\widehat{\chi}) & \\
 & | & & \\
H & \bullet & \widehat{\chi}(x) = \pm 1 & \widehat{\chi}(m) = \pm \iota \\
 & | & & \\
T & \bullet & \chi(x) = \pm 1 & \chi(y) = -1
\end{array}$$

We get 4 linear characters of $\widehat{\chi}$ of H, parametrised by

$$(\widehat{\chi}(x), \widehat{\chi}(m)) \in \{(1, \iota), (1, -\iota), (-1, \iota), (-1, -\iota)\}.$$

The character $\vartheta = \mathrm{Ind}_H^T(\widehat{\chi})$ is determined by $\vartheta_T = \widehat{\chi} \oplus \widehat{\chi}^t$, parametrised by

$$\{(1, \iota), (1, -\iota)\}, \{(-1, \iota), (-1, -\iota)\}.$$

The Galois orbit has size 1. The contribution to the Galois orbit zeta function is

$$\omega_2(s,t) = 2 \cdot 2^{-s} \cdot 1^{-t} = 2 \cdot 2^{-s}.$$

<u>**Case(2)** : $\quad I_G(\chi) = \langle m\rangle T$</u>

In this case we extend the given characters of T to the inertia group and then induce it to G as shown in the following diagram.

$$\begin{array}{lcll}
G & \bullet & \mathrm{Ind}_{\langle m\rangle T}^G(\widehat{\chi}) = \mathrm{Ind}_{\langle m\rangle T}^G(\widehat{\chi}^t) & \mathrm{Ind}_{\langle m\rangle T}^G(\epsilon\widehat{\chi}) = \mathrm{Ind}_{\langle m\rangle T}^G(\epsilon\widehat{\chi}^t) \\
 & | & & \\
I_G(\chi) = \langle m\rangle T & \bullet & \widehat{\chi} \quad \epsilon\widehat{\chi} & \widehat{\chi}^t \quad \epsilon\widehat{\chi}^t \\
 & | & & \\
T & \bullet & \chi \neq \chi^t &
\end{array}$$

The characters $\widehat{\chi}, \epsilon\widehat{\chi}$ and $\widehat{\chi}^t, \epsilon\widehat{\chi}^t$ are the extension of χ and χ^t respectively.

We want to compute the Galois orbits of ϑ. The character ϑ is determined by $\vartheta_{\langle m\rangle T} = \widehat{\chi} \oplus \widehat{\chi}^t$. The linear characters $\widehat{\chi}$ and $\widehat{\chi}^t$ of $\langle m\rangle T$ are parameterised by

$$(\widehat{\chi}(x), \widehat{\chi}(m)) = (\xi_1, \xi_2^{1/2})$$

and

$$(\widehat{\chi}^t(x), \widehat{\chi}^t(m)) = (\xi_1, \xi_2^{-1/2}).$$

The character ϑ parameterised by

$$\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_2^{-1/2})\}.$$

We find the Galois orbits of

$$\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_2^{-1/2})\}$$

under the action of $\mathrm{Gal}(K_b/\mathbb{Q})$, where K_b is the field which contains K_{n_2} as shown in the following figure:

$$\begin{tikzcd}
K_b & \bullet \arrow[d, dash] & \mathbb{Q}(\xi_2^{1/2}) \\
K_{n_2} & \bullet \arrow[d, dash] & \mathbb{Q}(\xi_1, \xi_2) = \mathbb{Q}(\xi_2) \\
\mathbb{Q} & \bullet & \mathbb{Q}(\xi_1)
\end{tikzcd}$$

where $\xi_2^{1/2}$ is primitive bth roots of unity with $b \in \{n_2, 2n_2\}$. Observe that

$$\varphi(b) = \begin{cases} \varphi(n_2) & \text{if } n_2 \equiv 1 \bmod 2 \\ 2\varphi(n_2) & \text{if } n_2 \equiv 0 \bmod 2 \end{cases}$$

Consider that $\alpha \in \mathrm{Gal}(K_b/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/b\mathbb{Z})^*$ and its action on $\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_2^{-1/2})\}$ is given by

$$\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_2^{-1/2})\}^\alpha = \{(\xi_1^a, \xi_2^{a/2}), (\xi_1^a, \xi_2^{-a/2})\}.$$

The length of an orbit is

$$\varphi(b)/|\operatorname{Stab}_{\mathrm{Gal}(K_b/\mathbb{Q})}\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_2^{-1/2})\}|.$$

Since $\xi_2^{1/2}$ and $\xi_2^{-1/2}$ are primitive bth roots of unity with $b \geq 3$. Hence

$$|\operatorname{Stab}\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_2^{-1/2})\}| = \begin{cases} 2 & \text{if } n_2 \equiv 1 \bmod 2 \\ 2 & \text{if } n_2 \equiv 0 \bmod 2 \end{cases}$$

Hence the lengths of the corresponding orbits are

$$\begin{cases} \varphi(n_2)/2 & \text{if } n_2 \equiv 1 \bmod 2 \\ 2\varphi(n_2)/2 = \varphi(n_2) & \text{if } n_2 \equiv 0 \bmod 2 \end{cases}$$

For fixed n_1, n_2 there are $\varphi(n_1)\varphi(n_2)$ choices for (ξ_1, ξ_2) and $2\varphi(n_1)\varphi(n_2) = 2\varphi(n_2)$ choices for $(\xi_1, \xi_2^{1/2})$ gives $\varphi(n_2)$ choices for $\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_2^{-1/2})\}$. So the total number of orbits are

$$\begin{cases} \frac{\varphi(n_2)}{\varphi(n_2)/2} = 2 & \text{if } n_2 \equiv 1 \bmod 2 \\ \frac{\varphi(n_2)}{\varphi(n_2)} = 1 & \text{if } n_2 \equiv 0 \bmod 2 \end{cases}$$

Hence the contribution to the Galois orbit zeta function is

$$\begin{aligned}
\omega_3(s,t) &= 2 \sum_{\substack{n_2 \geq 3 \\ n_2 \equiv 1 \bmod 2}} 2 \cdot 2^{-s}(\varphi(n_2)/2)^{-t+1} + 2 \sum_{\substack{n_2 \geq 4 \\ n_2 \equiv 0 \bmod 2}} 1 \cdot 2^{-s}\varphi(n_2)^{-t+1} \\
&= 2^{t-s+1} \sum_{\substack{n_2 \geq 3 \\ n_2 \equiv 1 \bmod 2}} \varphi(n_2)^{-t+1} + 2^{1-s} \sum_{\substack{n_2 \geq 4 \\ n_2 \equiv 0 \bmod 2}} \varphi(n_2)^{-t+1} \\
&= 2^{t-s+1} \sum_{\substack{n_2 \geq 1 \\ n_2 \equiv 1 \bmod 2}} \varphi(n_2)^{-t+1} + 2^{1-s} \sum_{\substack{n_2 \geq 2 \\ n_2 \equiv 0 \bmod 2}} \varphi(n_2)^{-t+1} - 2^{t-s+1} - 2^{1-s} \\
&= (2^{t-s+1} + 2^{1-s})\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{-s}(2^{1+t} + 2).
\end{aligned}$$

Note: In defining the Galois orbit zeta function we sum over characters, not orbits. So we used $1 - t$ in place of $-t$.

Case(3) : $\underline{I_G(\chi) = \langle t \rangle T}$

Similarly

G	•	$\operatorname{Ind}_{\langle t \rangle T}^{G}(\widehat{\chi}) = \operatorname{Ind}_{\langle t \rangle T}^{G}(\widehat{\chi}^m)$	$\operatorname{Ind}_{\langle t \rangle T}^{G}(\epsilon\widehat{\chi}) = \operatorname{Ind}_{\langle t \rangle T}^{G}(\epsilon\widehat{\chi}^m)$
$I_G(\chi) = \langle t \rangle T$	•	$\widehat{\chi} \quad \epsilon\widehat{\chi}$	$\widehat{\chi}^m \quad \epsilon\widehat{\chi}^m$
T	•	$\chi \neq \chi^m$	

The characters $\widehat{\chi}, \epsilon\widehat{\chi}$ and $\widehat{\chi}^m, \epsilon\widehat{\chi}^m$ are the extension of χ and χ^m respectively.

We want to compute the Galois orbits of ϑ. The character ϑ is determined by $\vartheta_{\langle t\rangle T} = \widehat{\chi} \oplus \widehat{\chi}^m$. The characters $\widehat{\chi}$ and $\widehat{\chi}^m$ parameterised by

$$(\widehat{\chi}(x), \widehat{\chi}(y), \widehat{\chi}(t)) = (\xi_1, \xi_2, \pm 1)$$

and

$$(\widehat{\chi}^m(x), \widehat{\chi}^m(y), \widehat{\chi}^m(t)) = (\xi_1^{-1}, \xi_2, \pm\xi_2,)$$

where $\widehat{\chi}(x) = \xi_1$, $\widehat{\chi}(y) = \xi_2 = \pm 1$ and $\widehat{\chi}(t) = \pm 1$. Correspondingly, we have a Galois action by the Galois group of $K_{n_1} = \mathbb{Q}(\xi_1)/\mathbb{Q}$. We find the lengths and the numbers of such orbits. The length of an orbit is

$$\varphi(n_1)/|\operatorname{Stab}\{(\xi_1, \xi_2, \pm 1), (\xi_1^{-1}, \xi_2, \pm\xi_2)\}|.$$

$$|\operatorname{Stab}\{(\xi_1, \xi_2, \pm 1), (\xi_1^{-1}, \xi_2, \pm\xi_2)\}| = \begin{cases} 2 & \text{if } \xi_2 = 1 \\ 1 & \text{if } \xi_2 = -1 \end{cases}$$

Hence the length of the corresponding orbits is

$$\begin{cases} \varphi(n_1)/2 & \text{if } \xi_2 = 1 \\ \varphi(n_1) & \text{if } \xi_2 = -1 \end{cases}$$

For fixed n_1, n_2 there are $\varphi(n_1)$ choices for (ξ_1, ξ_2) and $2\varphi(n_1))$ choices for (ξ_1, ξ_2, ξ_3), $\xi_3 = \widehat{\chi} \in \{\pm 1\}$. So the total number of orbits are

$$\begin{cases} \frac{\varphi(n_1)}{\varphi(n_1)/2} = 2 & n_2 = 1 \\ \frac{\varphi(n_1)}{\varphi(n_1)} = 1 & n_2 = 2 \end{cases}$$

The contribution to the Galois orbit zeta function is

$$\begin{aligned}
\omega_4(s,t) &= 2\sum_{n_1=3}^{\infty} 2 \cdot 2^{-s}(\varphi(n_1)/2)^{-t+1} + \sum_{n_1=3}^{\infty} 1 \cdot 2^{-s}\varphi(n_1)^{-t+1} \\
&= 2^{t-s}\sum_{n=3}^{\infty} \varphi(n)^{-t+1} + 2^{-s}\sum_{n=3}^{\infty} \varphi(n)^{-t+1} \\
&= 2^{t-s}\sum_{n=1}^{\infty} \varphi(n)^{-t+1} + 2^{-s}\sum_{n=1}^{\infty} \varphi(n)^{-t+1} - 2^{t-2s} \\
&= (2^{t-s} + 2^{-s})\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{t-2s}
\end{aligned}$$

$$\underline{\textbf{Case(4)}: \quad I_G(\chi) = T}$$

In this case $\vartheta \in \mathrm{Irr}(G)$ cannot restrict to an irreducible character of the abelian group T. This means that

$$\vartheta = \mathrm{Ind}_T^G(\chi) = \mathrm{Ind}_T^G(\chi^m) = \mathrm{Ind}_T^G(\chi^t) = \mathrm{Ind}_T^G(\chi^{mt}), \vartheta(1) = 4.$$

We want to compute the Galois orbits of ϑ, which is uniquely determined by $\vartheta_T = \chi \oplus \chi^m \oplus \chi^t \oplus \chi^{mt}$. It suffices to describe the Galois orbits of

$$\{\chi \oplus \chi^m \oplus \chi^t \oplus \chi^{mt} \mid \chi \in \mathrm{Irr}(T), I_G(\chi) = T\},$$

in terms of ξ_1 and ξ_2

$$\{(\xi_1, \xi_2), (\xi_1^{-1}, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2^{-1})\},$$

under the action of $\mathrm{Gal}(K_l/\mathbb{Q})$, where $l = \mathrm{lcm}(n_1, n_2)$. Consider that $\alpha \in \mathrm{Gal}(K_l|\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$ and its action on $\{(\xi_1, \xi_2), (\xi_1^{-1}, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2^{-1})\}$ given by

$$\{(\xi_1, \xi_2), (\xi_1^{-1}, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2^{-1})\}^\alpha = \{(\xi_1^a, \xi_2^a), (\xi_1^{-a}, \xi_2^a), (\xi_1^a, \xi_2^{-a}), (\xi_1^{-a}, \xi_2^{-a})\}.$$

We have to find all a such that

$$\{(\xi_1^a, \xi_2^a), (\xi_1^{-a}, \xi_2^a), (\xi_1^a, \xi_2^{-a}), (\xi_1^{-a}, \xi_2^{-a})\} = \{(\xi_1, \xi_2), (\xi_1^{-1}, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2^{-1})\}.$$

Equivalently

$$\begin{aligned} (\xi_1^a, \xi_2^a) &= (\xi_1, \xi_2) \\ \text{or } (\xi_1^a, \xi_2^a) &= (\xi_1^{-1}, \xi_2) \\ \text{or } (\xi_1^a, \xi_2^a) &= (\xi_1, \xi_2^{-1}) \\ \text{or } (\xi_1^a, \xi_2^a) &= (\xi_1^{-1}, \xi_2^{-1}) \end{aligned}$$

By using the same computation as in Section 5.5, we get

$$|\,\mathrm{Stab}\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2), (\xi_1^{-1}, \xi_2^{-1})\}| = \begin{cases} 2 & \text{if } \gcd(n_1, n_2) \geq 3 \\ 4 & \text{if } \gcd(n_1, n_2) \in \{1, 2\} \end{cases}$$

Hence the length of the corresponding orbit is

$$\begin{cases} \varphi(l)/2 & \text{if } \gcd(n_1,n_2) \geq 3, \\ \varphi(l)/4 & \text{if } \gcd(n_1,n_2) \in \{1,2\}. \end{cases}$$

For fixed n_1, n_2 there are $\varphi(n_1) \cdot \varphi(n_2)/4$ choices for

$$\{(\xi_1,\xi_2),(\xi_1,\xi_2^{-1}),(\xi_1^{-1},\xi_2),(\xi_1^{-1},\xi_2^{-1})\}.$$

The total number of orbits is

$$\begin{cases} \frac{\varphi(n_1)\varphi(n_2)/4}{\varphi(l)/2} = \varphi(g)/2 & \text{if } \gcd(n_1,n_2) \geq 3, \\ \frac{\varphi(n_1)\varphi(n_2)/4}{\varphi(l)/4} = \varphi(g) = 1 & \text{if } \gcd(n_1,n_2) \in \{1,2\}. \end{cases}$$

The contribution to the Galois orbit zeta function is

$$\begin{aligned} \omega_5(s,t) = & \sum_{\substack{n_1\geq 3, n_2 \geq 3 \\ g=\gcd(n_1,n_2)\geq 3 \\ l=\operatorname{lcm}(n_1,n_2)}} \varphi(g)/2 \cdot 4^{-s}(\varphi(l)/2)^{1-t} \\ & + \sum_{\substack{n_1\geq 3, n_2 \geq 3 \\ g=\gcd(n_1,n_2)\in\{1,2\} \\ l=\operatorname{lcm}(n_1,n_2)}} \varphi(g) \cdot 4^{-s}(\varphi(l)/4)^{1-t} \\ & = 2^{t-2-2s} \sum_{\substack{n_1\geq 3, n_2 \geq 3 \\ g=\gcd(n_1,n_2)\geq 3 \\ l=\operatorname{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t} + 4^{t-s-1} \sum_{\substack{n_1\geq 3, n_2 \geq 3 \\ g=\gcd(n_1,n_2)\in\{1,2\} \\ l=\operatorname{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t} \\ & = 2^{t-2-2s}\eta_1(t) + 4^{t-s-1}\eta_2(t), \end{aligned}$$

where

$$\eta_1(t) = \sum_{\substack{n_1\geq 3, n_2 \geq 3 \\ g=\gcd(n_1,n_2)\geq 3 \\ l=\operatorname{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}$$

and

$$\eta_2(t) = \sum_{\substack{n_1\geq 3, n_2 \geq 3 \\ g=\gcd(n_1,n_2)\in\{1,2\} \\ l=\operatorname{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}.$$

We compute $\eta_1(t)$ and $\eta_2(t)$ separately.

$$\eta_1(t) = \sum_{\substack{n_1\geq 3, n_2\geq 3\\ g=\gcd(n_1,n_2)\geq 3\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}$$

$$= \sum_{n_1,n_2\geq 1} \varphi(g)\varphi(l)^{1-t} - \sum_{\substack{n_1,n_2\geq 1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t} - \sum_{\substack{n_1,n_2\geq 1\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}$$

Using the same calculation as in Section 5.3 we get

$$= \omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - \prod_p (1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - \frac{1+2^{1-t}}{1-2^{1-t}} \prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}).$$

Now we compute $\eta_2(t)$

$$\eta_2(t) = \sum_{\substack{n_1\geq 3, n_2\geq 3\\ g=\gcd(n_1,n_2)\in\{1,2\}\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}$$

$$= \sum_{\substack{n_1\geq 3, n_2\geq 3\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t} + \sum_{\substack{n_1\geq 3, n_2\geq 3\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(g)\varphi(l)^{1-t}$$

$$= \sum_{\substack{n_1,n_2\geq 1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(l)^{1-t} - \sum_{\substack{n_1=1,n_2\geq 1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(l)^{1-t} - \sum_{\substack{n_1=2,n_2\geq 1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)\\ n_2 \text{ is odd}}} \varphi(l)^{1-t}$$

$$- \sum_{\substack{n_1\geq 1,n_2=1\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(l)^{1-t} - \sum_{\substack{n_1\geq 1,n_2=2\\ g=\gcd(n_1,n_2)=1\\ l=\mathrm{lcm}(n_1,n_2)\\ n_1 \text{ is odd}}} \varphi(l)^{1-t} + \sum_{\substack{n_1,n_2\geq 1\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)}} \varphi(l)^{1-t}$$

$$- \sum_{\substack{n_1=2,n_2\geq 1\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)\\ n_2 \text{ is even}}} \varphi(l)^{1-t} - \sum_{\substack{n_1\geq 1,n_2=2\\ g=\gcd(n_1,n_2)=2\\ l=\mathrm{lcm}(n_1,n_2)\\ n_1 \text{ is even}}} \varphi(l)^{1-t}$$

Using the same calculation as in Section 5.3 we get

$$= \prod_p (1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) + \frac{1+2^{1-t}}{1-2^{1-t}} \prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - 4\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t).$$

Using $\eta_1(t)$ and $\eta_2(t)$ we get

$$
\begin{aligned}
\omega_5(s,t) = 2^{t-2-2s}\Bigg(&\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - \prod_p (1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - \frac{1+2^{1-t}}{1-2^{1-t}} \\
&\cdot \prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}})\Bigg) + 4^{t-s-1}\Bigg(\prod_p (1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
&+ \frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - 4\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t)\Bigg) \\
&= 2^{t-2-2s}\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - 4\cdot 4^{t-s-1}\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t) + (4^{t-s-1} - 2^{t-2-2s}) \\
&\cdot \prod_p (1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) + (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}).
\end{aligned}
$$

Hence the Galois orbit zeta function is

$$
\begin{aligned}
\omega^{\mathbb{Q}}_G(s,t) &= \omega_1(s,t) + \omega_2(s,t) + \omega_3(s,t) + \omega_4(s,t) + \omega_5(s,t) \\
&= 8 + 2\cdot 2^{-s} + (2^{t-s+1} + 2^{2-s})\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t) - 2^{-s}(2^{1+t}+2) \\
&+ (2^{t-s} + 2^{-s})\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t) - 2^{t-2s} + 2^{t-2-2s}\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - 4\cdot 4^{t-s-1}\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t) \\
&+ (4^{t-s-1} - 2^{t-2-2s})\prod_p (1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
&+ (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
&= 8 + 2^{1-t-s} - 2^{t-2s} + (3\cdot 2^{t-s} - 4\cdot 2^{2t-2s-2} + 5\cdot 2^{-s})\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t) \\
&+ 2^{t-2-2s}\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t)(4^{t-s-1} - 2^{t-2-2s})\prod_p (1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
&+ (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}).
\end{aligned}
$$

5.7 The group p2gg

The group $p2gg$ is given by the presentation

$$G = \langle x, y, u, v \mid [x,y] = 1 = (uv)^2, u^2 = x, v^2 = y, x^v = x^{-1}, y^u = y^{-1}\rangle.$$

The group G consists of a translation group T, which is generated by x and y. The quotient group G/T generated by uT and vT is called the point group of G.

We want to compute the Galois orbit zeta function $\omega_G^{\mathbb{Q}}(s,t)$ of G. As before, the irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{k_1,n_1,k_2,n_2} : T \longrightarrow \mathbb{C}^*$$

where $n_1, n_2 \in \mathbb{N}$ $k_1, k_2 \in \mathbb{N}$ with $1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1$ and

$$\chi_{k_1,n_1,k_2,n_2}(x^a y^b) = \xi_{n_1,k_1}^a \xi_{n_2,k_2}^b = e^{2\pi i(k_1 a/n_1 + k_2 b/n_2)},$$

where $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$, $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$.

Consider $\xi_{n_1,k_1} = \xi_1$, a primitive n_1th root of unity and $\xi_{n_2,k_2} = \xi_2$, a primitive n_2th root of unity as the images of x and y.

The possibilities for inertia groups are:

Case(1): If $\xi_1 \in \{1,-1\}$ and $\xi_2 \in \{1,-1\}$ then $I_G(\chi) = G$.

Case(2): If $\xi_1 \in \{1,-1\}$ and $\xi_2 \notin \{1,-1\}$ then $I_G(\chi) = \langle v \rangle T$.

Case(3): If $\xi_1 \notin \{1,-1\}$ and $\xi_2 \in \{1,-1\}$ then $I_G(\chi) = \langle u \rangle T$.

Case(4): If $\xi_1 \notin \{1,-1\}$ and $\xi_2 \notin \{1,-1\}$ then $I_G(\chi) = T$.

Now consider $\vartheta \in \mathrm{Irr}(G)$ such that ϑ_T involves χ. We can cover all possible ϑ by considering the above four cases.

<u>**Case(1)** : $\quad I_G(\chi) = G$</u>

To compute the linear characters of G it is enough to consider the abelianisation of G.

$$\begin{aligned} G/[G,G] = \langle x, y, u, v \mid & [x,y] = [x,u] = [y,u] = [x,v] = [y,v] = [u,v] = 1, u^2 = x, \\ & v^2 = y, x^2 = 1 = y^2 = (uv)^2 \rangle \\ & = \langle u, v \mid u^4 = v^4 = (uv)^2 = 1 \rangle \\ & \cong C_4 \times C_2 \end{aligned}$$

Since $I_G(\chi)/T$ is not cyclic, we cannot extend a character of T to G. Consider $\vartheta \in \mathrm{Irr}(G)$. As $\xi_1 = \xi_2 \in \{1, 2\}$, we split this case as

<u>**Case(a)** : $\quad \vartheta(1) = 1$</u>

In this case we have $\vartheta_T = \chi = (\pm 1, \pm 1)$, which gives 8 linear characters, 4 of which are defined over $\mathbb{Q}$ and 4 which are defined over $\mathbb{Q}(\iota)$. The contribution to the Galois orbit zeta function is

$$\omega_1(s,t) = 4 \cdot 1^{-s} \cdot 1^{-t} + 4 \cdot 1^{-s} \cdot 2^{-t} = 4 + 4 \cdot 2^{-t}.$$

<u>**Case(b)** : $\vartheta(1) \neq 1$</u>

In this case

$$\chi = (1, -1) \text{ or } (-1, 1).$$

We extend the given characters of T to subgroup H and then induce it to G as shown in the following diagram.

$$\begin{tikzcd}[row sep=large]
G & \bullet \arrow[d, no head] & \vartheta = \mathrm{Ind}_H^T(\widehat{\chi}) & \\
H & \bullet \arrow[d, no head] & \widehat{\chi}(x) = 1 & \widehat{\chi}^u(v) \pm \iota \\
T & \bullet & \chi(x) = 1 & \chi(v) = -1
\end{tikzcd}$$

We get 4 linear characters of $\widehat{\chi}$ of H, parameterised by

$$(1, \pm\iota), (\pm\iota, 1).$$

The character $\vartheta = \mathrm{Ind}_H^T(\widehat{\chi})$ is determined by $\vartheta_T = \widehat{\chi} \oplus \widehat{\chi}^u$, parameterised by

$$\{(1, \iota), (1, -\iota)\} \text{ or } \{(\iota, 1), (-\iota, 1)\}.$$

The Galois orbit has size 1. The contribution to the Galois orbit zeta function is

$$\omega_2(s,t) = 2 \cdot 2^{-s} \cdot 1^{-t} = 2 \cdot 2^{-s}.$$

<u>**Case(2)** : $I_G(\chi) = \langle v \rangle T$</u>

In this case we extend the characters of T to inertia group and then induce it to G, which explained as following

$$\begin{array}{cccc}
G & \bullet & \mathrm{Ind}_{\langle v\rangle T}^{G}(\widehat{\chi}) = \mathrm{Ind}_{\langle v\rangle T}^{G}(\widehat{\chi}^u) & \mathrm{Ind}_{\langle v\rangle T}^{G}(\epsilon\widehat{\chi}) = \mathrm{Ind}_{\langle v\rangle T}^{G}(\epsilon\widehat{\chi}^u) \\
 & | & & \\
I_G(\chi) = \langle v\rangle T & \bullet & \widehat{\chi} \quad \epsilon\widehat{\chi} & \widehat{\chi}^u \quad \epsilon\widehat{\chi}^u \\
 & | & & \\
T & \bullet & \chi \neq \chi^u &
\end{array}$$

where $\vartheta = \mathrm{Ind}_{\langle v\rangle T}^{G}(\widehat{\chi})$ has $\vartheta(1) = 2$.

The characters $\widehat{\chi}, \epsilon\widehat{\chi}$ and $\widehat{\chi}^u, \epsilon\widehat{\chi}^u$ are the extensions of χ and χ^u respectively.

We want to compute the Galois orbits of ϑ. The character ϑ is determined by $\vartheta_{\langle v\rangle T} = \widehat{\chi} \oplus \widehat{\chi}^u$. The characters $\widehat{\chi}$ and $\widehat{\chi}^u$ are determined by

$$(\widehat{\chi}(x), \widehat{\chi}(v)) = (\xi_1, \xi_2^{1/2})$$

$$(\widehat{\chi}^u(x), \widehat{\chi}^u(v)) = (\xi_1, \xi_1\xi_2^{-1/2}),$$

where $\xi_2^{1/2}$ is a primitive bth roots of unity. We want to compute the Galois orbit of ϑ, as ϑ determined by $\vartheta_{\langle v\rangle T} = \widehat{\chi} \oplus \widehat{\chi}^u$. It suffices to describe the Galois orbit of

$$\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_1\xi_2^{-1/2})\}$$

under the action of $\mathrm{Gal}(K_b/\mathbb{Q})$, where K_b is the field that contains K_{n_2}.

$$\begin{array}{ccc}
K_b & \bullet & \mathbb{Q}(\xi_2^{1/2}) \\
 & | & \\
K_{n_2} & \bullet & \mathbb{Q}(\xi_1, \xi_2) = \mathbb{Q}(\xi_2) \\
 & | & \\
\mathbb{Q} & \bullet & \mathbb{Q}(\xi_1)
\end{array}$$

Observe

$$\varphi(b) = \begin{cases} \varphi(n_2) & \text{if } n_2 \text{ is odd} \\ 2\varphi(n_2) & \text{if } n_2 \text{ is even} \end{cases}$$

Consider that $\alpha \in \mathrm{Gal}(K_l/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$ and its action on $\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_1\xi_2^{-1/2})\}$ given by

$$\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_1\xi_2^{-1/2})\}^\alpha = \{(\xi_1^a, \xi_2^{a/2}), (\xi_1^a, \xi_1^a\xi_2^{-a/2})\}.$$

The length of an orbit is

$$\varphi(b)/|\operatorname{Stab}\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_1\xi_2^{-1/2})\}|.$$

We find the stabiliser of $\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_1\xi_2^{-1/2})\}$. For this we have two cases: $n_1 = 1$ and $n_1 = 2$.

<u>**Case(a)** : $n_1 = 1$</u>

In this case $\xi_1 = 1$ and the stabiliser of $\{(1, \xi_2^{1/2}), (1, \xi_2^{-1/2})\}$ has size 2, because both $\xi_2^{1/2}$ and $\xi_2^{-1/2}$ are primitive bth root of unity. Hence the length of the corresponding orbit is

$$\begin{cases} \varphi(n_2)/2 & \text{if } n_2 \equiv 1 \bmod 2 \\ 2\varphi(n_2)/2 = \varphi(n_2) & \text{if } n_2 \equiv 0 \bmod 2 \end{cases}$$

For fixed $n_1 = 1, n_2 \geq 3$ there are $\varphi(n_1)\varphi(n_2) = \varphi(n_2)$ choices for (ξ_1, ξ_2) and $\varphi(n_1)\varphi(n_2) \cdot 2 = 2\varphi(n_2)$ choices for $(\xi_1, \xi_2^{1/2})$ and gives $\varphi(n_2)$ choices for $\{(\xi_1, \xi_2^{1/2}), (\xi_1, \xi_1\xi_2^{-1/2})\}$. So the total number of orbits are

$$\begin{cases} \frac{\varphi(n_2)}{\varphi(n_2)/2} = 2 & \text{if } n_2 \equiv 1 \bmod 2 \\ \frac{\varphi(n_2)}{\varphi(n_2)} = 1 & \text{if } n_2 \equiv 0 \bmod 2 \end{cases}$$

Hence the contribution to the Galois orbit zeta function is

$$\begin{aligned}
\omega_3(s,t) &= \sum_{\substack{n_2 \geq 3 \\ n_2 \equiv 1 \bmod 2}} 2 \cdot 2^{-s}(\varphi(n_2)/2)^{-t+1} + \sum_{\substack{n_2 \geq 4 \\ n_2 \equiv 0 \bmod 2}} 1 \cdot 2^{-s}\varphi(n_2)^{-t+1} \\
&= 2^{t-s} \sum_{\substack{n_2 \geq 3 \\ n_2 \equiv 1 \bmod 2}} \varphi(n_2)^{-t+1} + 2^{-s} \sum_{\substack{n_2 \geq 4 \\ n_2 \equiv 0 \bmod 2}} \varphi(n_2)^{-t+1} \\
&= 2^{t-s} \sum_{\substack{n_2 \geq 1 \\ n_2 \equiv 1 \bmod 2}} \varphi(n_2)^{-t+1} + 2^{-s} \sum_{\substack{n_2 \geq 2 \\ n_2 \equiv 0 \bmod 2}} \varphi(n_2)^{-t+1} - 2^{t-s} - 2^{-s} \\
&= (2^{t-s} + 2^{1-s})\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{-s}(2^t + 1).
\end{aligned}$$

Case(b) : $n_1 = 2$

In this case $\xi_1 = -1$ and

$$|\operatorname{Stab}\{(-1, \xi_2^{1/2}), (-1, \xi_2^{-1/2})\}| = \begin{cases} 2 & \text{if } n_2 \equiv 0 \bmod 2 \\ 1 & \text{if } n_2 \equiv 1 \bmod 2 \end{cases}$$

Hence the length of the corresponding orbit is

$$\begin{cases} 2\varphi(n_2)/2 = \varphi(n_2) & \text{if } n_2 \equiv 0 \bmod 2 \\ \varphi(n_2) & \text{if } n_2 \equiv 1 \bmod 2 \end{cases}$$

For fixed $n_1 = 2, n_2 \geq 3$ there are $\varphi(n_1)\varphi(n_2) = \varphi(n_2)$ choices for (ξ_1, ξ_2) and $\varphi(n_1)\varphi(n_2) \cdot 2 = 2\varphi(n_2)$ choices for $(\xi_1, \xi_2^{1/2})$ gives $\varphi(n_2)$ choices for $\{(-1, \xi_2^{1/2}), (-1, \xi_2^{-1/2})\}$. So the total number of orbits is 1. The contribution to the Galois orbit zeta function is

$$\begin{aligned} \omega_4(s,t) &= \sum_{n_2=3}^{\infty} 1 \cdot 2^{-s} \cdot \varphi(n_2)^{1-t} = 2^{-s} \sum_{n_2=1}^{\infty} 1 \cdot 2^{-s} \cdot \varphi(n_2)^{1-t} - 2 \cdot 2^{-s} \\ &= 2^{-s}\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{1-s}. \end{aligned}$$

Case(3) : $I_G(\chi) = \langle u \rangle T$

This case is the same as case 2.

Case(4) : $I_G(\chi) = T$

In this case $\vartheta \in \operatorname{Irr}(G)$ cannot restrict to an irreducible character of the abelian group T. This means that

$$\vartheta = \operatorname{Ind}_T^G(\chi) = \operatorname{Ind}_T^G(\chi^u) = \operatorname{Ind}_T^G(\chi^v) = \operatorname{Ind}_T^G(\chi^{uv}), \text{where } \vartheta(1) = 4.$$

We want to compute the Galois orbits of ϑ, which are uniquely determined by $\vartheta_T = \chi \oplus \chi^u \oplus \chi^v \oplus \chi^{uv}$. It suffices to describe the Galois orbits of

$$\{\chi \oplus \chi^u \oplus \chi^v \oplus \chi^{uv} \mid \chi \in \operatorname{Irr}(T), I_G(\chi) = T\},$$

in terms of ξ_1 and ξ_2

$$\{(\xi_1, \xi_2), (\xi_1, \xi_2^{-1}), (\xi_1^{-1}, \xi_2), (\xi_1^{-1}, \xi_2^{-1})\},$$

under the action of $\mathrm{Gal}(K_l/\mathbb{Q}) \cong (\mathbb{Z}/l\mathbb{Z})^*$, where $l = \mathrm{lcm}(n_1, n_2)$. Consider that $\alpha \in \mathrm{Gal}(K_l/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$ and its action on

$$\{(\xi_1,\xi_2),(\xi_1,\xi_2^{-1}),(\xi_1^{-1},\xi_2),(\xi_1^{-1},\xi_2^{-1})\}$$

given by

$$\{(\xi_1,\xi_2),(\xi_1,\xi_2^{-1}),(\xi_1^{-1},\xi_2),(\xi_1^{-1},\xi_2^{-1})\}^\alpha = \{(\xi_1^a,\xi_2^a),(\xi_1^a,\xi_2^{-a}),(\xi_1^{-a},\xi_2^a),(\xi_1^{-a},\xi_2^{-a})\}.$$

We have to find all a such that

$$\{(\xi_1^a,\xi_2^a),(\xi_1^a,\xi_2^{-a}),(\xi_1^{-a},\xi_2^a),(\xi_1^{-a},\xi_2^{-a})\} = \{(\xi_1,\xi_2),(\xi_1,\xi_2^{-1}),(\xi_1^{-1},\xi_2),(\xi_1^{-1},\xi_2^{-1})\}.$$

Equivalently

$$\begin{aligned} (\xi_1^a,\xi_2^a) &= (\xi_1,\xi_2) \\ \text{or } (\xi_1^a,\xi_2^a) &= (\xi_1,\xi_2^{-1}) \\ \text{or } (\xi_1^a,\xi_2^a) &= (\xi_1^{-1},\xi_2) \\ \text{or } (\xi_1^a,\xi_2^a) &= (\xi_1^{-1},\xi_2^{-1}). \end{aligned}$$

Using the same calculation done in Section 5.6, we get the contribution toward zeta function, which is

$$\begin{aligned} \omega_5(s,t) = {} & 2^{t-2-2s}\Bigg(\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - \prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - \frac{1+2^{1-t}}{1-2^{1-t}} \\ & \cdot\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}})\Bigg) + 4^{t-s-1}\Bigg(\prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\ & + \frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) - 4\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t)\Bigg) \\ & = 2^{t-2-2s}\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) - 4\cdot 4^{t-s-1}\omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t) + (4^{t-s-1} - 2^{t-2-2s}) \\ & \cdot\prod_p(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}) + (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}). \end{aligned}$$

Hence the Galois orbit zeta function is

$$\begin{aligned}
\omega_G^{\mathbb{Q}}(s,t) &= \omega_1(s,t) + \omega_2(s,t) + 2\omega_3(s,t) + 2\omega_4(s,t) + \omega_5(s,t) \\
&= 4 + 4\cdot 2^{-t} + 2\cdot 2^{-s} + (2^{t-s+1} + 2^{2-s})\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{-s}(2^{1+t}+2) \\
&+ 2^{1-s}\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) - 2^{2-s} + 2^{t-2-2s}\omega_{\mathbb{Z}^2}^{\mathbb{Q}}(s,t) - 4\cdot 4^{t-s-1}\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) \\
&+ (4^{t-s-1} - 2^{t-2-2s})\prod_p (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
&+ (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
&= 4 + 4\cdot 2^{-t} - 2^{1-s} + (2^{t-s+1} + 3\cdot 2^{1-s} - 4\cdot 4^{t-s-1})\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t) \\
&+ 2^{t-2-2s}\omega_{\mathbb{Z}^2}^{\mathbb{Q}}(s,t)(4^{t-s-1} - 2^{t-2-2s})\prod_p (1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}) \\
&+ (4^{t-s-1} - 2^{t-2-2s})\frac{1+2^{1-t}}{1-2^{1-t}}\prod_{p\neq 2}(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}}).
\end{aligned}$$

5.8 The group cm

The group cm is given by the presentation

$$G = \langle x, y, t \mid [x,y] = 1 = t^2, x^t = xy, y^t = y^{-1}\rangle.$$

The group G consists of a translation group T, which is normal in G and generated by x and y. The quotient group G/T, generated by tT, is the point group of G and is a cyclic group of order 2.

We want to compute the rational representation zeta function $\zeta_G^{\mathbb{Q}}(s)$ of G. As in Section 5.2, the irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{k_1,n_1,k_2,n_2} : T \longrightarrow \mathbb{C}^*,$$

where $n_1, n_2 \in \mathbb{N}$, $k_1, k_2 \in \mathbb{N}$ with $1 \le k_i \le n_i, \gcd(k_i, n_i) = 1$ and

$$\chi_{k_1,n_1,k_2,n_2}(x^a y^b) = \xi_{n_1,k_1}^a \xi_{n_2,k_2}^b = e^{2\pi i(k_1 a/n_1 + k_2 b/n_2)},$$

where $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$, $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$. Consider $\xi_{n_1,k_1} = \xi_1$, a primitive n_1th root of unity and $\xi_{n_2,k_2} = \xi_2$, a primitive n_2th root of unity.

Consider the action of G on $\mathrm{Irr}(T)$ given by $\chi^g(z) = \chi(z^{g^{-1}})$, where $\chi \in \mathrm{Irr}(T)$, $g \in G, z \in T$. We need to understand the orbits of χ and its stabiliser. The size of the orbit of χ is $|G : I_G(\chi)|$. So either $I_G(\chi) = T$ when $t \notin I_G(\chi)$ or $I_G(\chi) = G$ when $t \in I_G(\chi)$. In our case $I_G(\chi)/T$ is cyclic and isomorphic to either C_2 or 1.

Now consider $\psi \in \mathrm{Irr}(G)$ factoring over a finite quotient of G. We can cover all possible ψ by considering two cases: $\psi(1) = 1$ and $\psi(1) > 1$.

$$\underline{\textbf{Case(1)}: \quad \psi \in \mathrm{Irr}(G), \ \ \psi(1) = 1}$$

To compute the linear character of G, it is enough to consider the abelianisation of G.

$$G/[G,G] \cong \mathbb{Z} \times C_2.$$

As in Section 5.4, the contribution to the Galois orbit zeta function is

$$\omega_1(s,t) = \omega^{\mathbb{Q}}_{\mathbb{Z}\times C_2}(s,t) = 2 \cdot \omega^{\mathbb{Q}}_{\mathbb{Z}}(s,t).$$

$$\underline{\textbf{Case(2)}: \quad \psi \in \mathrm{Irr}(G), \ \ \psi(1) > 1}$$

In this case ψ cannot restrict to an irreducible character of the abelian group T. This means that

$$\psi = \mathrm{Ind}_T^G(\chi) = \mathrm{Ind}_T^G(\chi^t)$$

where χ and χ^t, are such that $\chi \neq \chi^t$, i.e $I_G(\chi) = T$.

We compute the Galois orbits of the character ψ. We observed that the character ψ is uniquely determined by $\psi_T = \chi \oplus \chi^t$. So it suffices to describe the Galois orbits in

$$\{(\chi \oplus \chi^t) \mid \chi \in \mathrm{Irr}(T), I_G(\chi) = T\}.$$

The characters $\chi \oplus \chi^t$ are uniquely determined by the set $\{\chi, \chi^t\}$, which in turn is uniquely determined by $\{(\chi(x), \chi(y)), (\chi^t(x), \chi^t(y))\}$. Therefore we need to describe the orbits in

$$\{\chi, \chi^t\},$$

under the action of $\mathrm{Gal}(K_l/\mathbb{Q})$, where $l = \mathrm{lcm}(n_1, n_2)$ and $K_l = \mathbb{Q}(e^{2\pi i/l})$, the lth cyclotomic field. Consider $l = \prod_{i=1}^r p_i^{e_i}$ is the prime factorisation of l. Then

$$\mathrm{Gal}(K_l/\mathbb{Q}) = \prod_{i=1}^r \mathrm{Gal}(K_{p_i^{e_i}}/\mathbb{Q}) \cong \prod_{i=1}^r (\mathbb{Z}/p_i^{e_i}\mathbb{Z})$$

and

$$\mathrm{Irr}\left((\mathbb{Z}/l\mathbb{Z})^2\right) = \prod_{i=1}^r \mathrm{Irr}\left((\mathbb{Z}/p_i^{e_i}\mathbb{Z})^2\right).$$

The Galois orbit size of the character ψ are

$$\begin{aligned} &\mid \mathrm{Gal}(K_l/\mathbb{Q}) : \mathrm{Stab}_{\mathrm{Gal}(K_l/\mathbb{Q})}(\chi, \chi^t) \mid \\ &= \begin{cases} \varphi(l) = \prod_{i=1}^r \varphi(p_i^{e_i}) & \text{if for all } \alpha \in \mathrm{Gal}(K_l/\mathbb{Q}) : \chi^\alpha \neq \chi^t \\ \varphi(l)/2 = \prod_{i=1}^r \varphi(p_i^{e_i}) & \text{if there exists } \alpha \in \mathrm{Gal}(K_l/\mathbb{Q}) : \chi^\alpha = \chi^t \end{cases} \end{aligned}$$

Consider that $\alpha \in \mathrm{Gal}(K_l/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$. Its action on $(\xi_1, \xi_2), (\xi_1\xi_2, \xi_2^{-1})$ given by

$$\{(\xi_1, \xi_2), (\xi_1\xi_2, \xi_2^{-1})\}^\alpha = \{(\xi_1^a, \xi_2^a), ((\xi_1\xi_2)^a, \xi_2^{-a})\}.$$

The case $\chi^\alpha = \chi^t$, in terms of ξ_1 and ξ_2 is

$$\chi^\alpha = \chi^t \iff \begin{cases} \xi^a = \xi_1\xi_2 \text{ and } \xi_2^a = \xi_2^{-1} & \\ \xi_2 = \xi_1^{a-1} \text{ and } \xi_1^{a^2-1} = 1 & [l = n_1, n_2 \mid n_1] \\ n_2 \mid n_1 = l \text{ and } a^2 - 1 \equiv 0 \bmod l \text{ and } \xi_2 = \xi_1^{a-1} & [\xi_2 \neq 1 \Rightarrow a \not\equiv 1 \bmod l] \end{cases}$$

The contribution to the Galois orbit zeta function is

$$\omega_2(s,t) = \frac{1}{2} \sum_{\substack{\chi\in\mathrm{Irr}(T)\\ \chi^t\neq\chi}} \psi(1)^{-s} \mid \chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \mid^{-t} = 2^{-1-s}\Bigg[\sum_{l=1}^{\infty} \sum_{\substack{\chi\in\mathrm{Irr}(T)\\ \chi^t\neq\chi}} \mid \chi^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \mid^{-t}\Bigg]$$

$$= 2^{-1-s}\Bigg[\prod_p \Big(1+\sum_{e=1}^{\infty} \mid (\mathbb{Z}/p^e\mathbb{Z})^2\backslash(p\mathbb{Z}/p^e\mathbb{Z})^2 \mid \varphi(p^e)^{-t}\Big)$$

$$- \Big(1+1+2\cdot 2\cdot 2^{-t}+\sum_{e=3}^{\infty} 2^{e-1}\cdot 4(2^{e-1})^{-t}\prod_{p>2}(1+\sum_{e=1}^{\infty} 2\varphi(p^e)\varphi(p^e)^{-t})\Big)$$

$$+ 2^t\Big(1+1+2\cdot 2\cdot 2^{-t}+\sum_{e=3}^{\infty} 2^{e-1}\cdot 4(2^{e-1})^{-t}\prod_{p>2}(1+\sum_{e=1}^{\infty} 2\varphi(p^e)\varphi(p^e)^{-t})\Big)$$

$$- 2^t\Big(1+1+2\cdot 1\cdot 2^{-t}+\sum_{e=3}^{\infty} 2^{e-1}\cdot 1\cdot (2^{e-1})^{-t}\prod_{p>2}(1+\sum_{e=1}^{\infty} \varphi(p^e)\varphi(p^e)^{-t})\Big)\Bigg]$$

$$= 2^{-1-s}\prod_p \Big(1+\sum_{e=1}^{\infty}(p^2-1)p^{2(e-1)}((p-1)p^{e-2})\Big)$$

$$- 2^{-1-s}\Big(2+4\cdot 2^{-t}+4\sum_{e=3}^{\infty}(2^{1-t})^{e-1}\Big)\prod_{p>2}\Big(1+2\sum_{e=1}^{\infty}((p-1)p^{e-1})^{1-t}\Big)$$

$$+ 2^{t-1-s}\Big(2+4\cdot 2^{-t}+4\sum_{e=3}^{\infty}(2^{1-t})^{e-1}\Big)\prod_{p>2}\Big(1+2\sum_{e=1}^{\infty}((p-1)p^{e-1})^{1-t}\Big)$$

$$- 2^{t-1-s}\Big(2+2\cdot 2^{-t}+4\sum_{e=3}^{\infty}(2^{1-t})^{e-1}\Big)\prod_{p>2}\Big(1+\sum_{e=1}^{\infty}((p-1)p^{e-1})^{1-t}\Big)$$

$$= 2^{-1-s}\prod_p \Big(1+(p^2-1)(p-1)^{-t}\frac{1}{1-p^{2-t}}\Big)$$

$$- 2^{-1-s}\Big(-2-4\cdot 2^{-t}+\frac{4}{1-2^{1-t}}\Big)\prod_{p>2}\Big(1+2(p-1)^{1-t}\frac{1}{1-p^{1-t}}\Big)$$

$$+ 2^{t-1-s}\Big(-2-4\cdot 2^{-t}+\frac{4}{1-2^{1-t}}\Big)\prod_{p>2}\Big(1+2(p-1)^{1-t}\frac{1}{1-p^{1-t}}\Big)$$

$$- 2^{t-1-s}\Big(1+\frac{1}{1-2^{1-t}}\Big)\prod_{p>2}\Big(1+(p-1)^{1-t}\frac{1}{1-p^{1-t}}\Big)$$

$$= 2^{1-s}\prod_p\Big(1+(p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\Big) - 2^{-s}\Big(\frac{1+2^{2-2t}}{1-2^{1-t}}\Big)\prod_{p>2}\Big(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}\Big)$$

$$+ 2^{t-s}\Big(\frac{1+2^{2-2t}}{1-2^{1-t}}\Big)\prod_{p>2}\Big(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}\Big)$$

$$- 2^{t-s}\Big(\frac{1+2^{-t}}{1-2^{1-t}}\Big)\prod_{p>2}\Big(1+\frac{(p-1)^{1-t}}{1-p^{1-t}}\Big)$$

$$= 2^{-1-s}\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t) + (2^{t-s}-2^{-s})\Big(\frac{1+2^{2-2t}}{1-2^{1-t}}\Big)\prod_{p>2}\Big(1+2\frac{(p-1)^{1-t}}{1-p^{1-t}}\Big)$$

$$- 2^{t-s}\Big(\frac{1+2^{-t}}{1-2^{1-t}}\Big)\prod_{p>2}\Big(1+\frac{(p-1)^{1-t}}{1-p^{1-t}}\Big).$$

According to Lemma 2.4.15, the Schur index of G is one. Hence the rational representation zeta function is

$$\begin{aligned}\zeta_G^{\mathbb{Q}}(s) &= \omega_G^{\mathbb{Q}}(s,s+1) = \omega_G^{\mathbb{Q}}(s,t) = \omega_1(s,t) + \omega_2(s,t)\\ &= 2\zeta_{\mathbb{Z}}^{\mathbb{Q}}(s) + 2^{-1-s}\zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + (2-2^{-s})\Big(\frac{1+2^{-s}}{1-2^{-s}}\Big)\prod_{p>2}\Big(1+2\frac{(p-1)^{-s}}{1-p^{-s}}\Big)\\ &- 2\Big(\frac{1+2^{-1-s}}{1-2^{-s}}\Big)\prod_{p>2}\Big(1+\frac{(p-1)^{-s}}{1-p^{-s}}\Big).\end{aligned}$$

5.9 The group p4

The group $p4$ is given by the presentation

$$G = \langle x, y, r \mid [x,y] = r^4 = 1, x^r = y, y^r = x^{-1}\rangle.$$

The group G consists of a translation group T, generated by x and y. The quotient group G/T, generated by rT, is called the point group and is a cyclic group of order 4.

As in Section 5.2, the irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{k_1,n_1,k_2,n_2} : T \longrightarrow \mathbb{C}^*,$$

where $n_1, n_2 \in \mathbb{N}$, $k_1, k_2 \in \mathbb{N}$ with $1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1$ and

$$\chi_{k_1,n_1,k_2,n_2}(x^a y^b) = \xi_{n_1,k_1}^a \xi_{n_2,k_2}^b = e^{2\pi i(k_1 a/n_1 + k_2 b/n_2)},$$

where $\xi_{n_1,k_1} = e^{2\pi i k_1/n_1}$, $\xi_{n_2,k_2} = e^{2\pi i k_2/n_2}$.

Consider $\xi_{n_1,k_1} = \xi_1$, a primitive n_1th root of unity and $\xi_{n_2,k_2} = \xi_2$, a primitive n_2th root of unity. Consider the action of G on $\mathrm{Irr}(T)$ given by $\chi^g(z) = \chi(z^{g^{-1}})$, where $\chi \in \mathrm{Irr}(T), g \in G, z \in T$. We need to understand the orbits of χ and its stabiliser. In our case $|G : T| = 4$. Also we know that $|G : I_G(\chi)|$ is the size of the orbit of χ.

The possibilities for inertia groups are

Case(1): If $\xi_1 \in \{1,-1\}$ and $\xi_2 \in \{1,-1\}$ then $I_G(\chi) = G$.

Case(2): If $(\xi_1, \xi_2) \in \{1,-1\}$ or $(\xi_1, \xi_2) \in \{-1,1\}$ then $I_G(\chi) = \langle r^2 \rangle T$.

Case(3): If $\xi_1 \notin \{1,-1\}$ and $\xi_2 \notin \{1,-1\}$ then $I_G(\chi) = T$.

Case(1) : $\quad I_G(\chi) = G$

To compute the linear character of G, it is enough to consider the abelianisation of G.

$$G/[G,G] \cong C_4 \times C_2.$$

The eight one-dimensional complex representations of $G/[G,G]$, give us four Galois orbits of length 1 representing $\{(1,1),(-1,1),(1,-1),(-1,-1)\}$ and two Galois orbits of length 2 representing $[\{(i,1),(-i,1)\},\{(i,-1),(-i,1)\}]$. This means 4 irreducible representations of G over $\mathbb{Q}$ of dimension 1, and 2 irreducible representations of G over $\mathbb{Q}$ of dimension 2. Hence the Galois orbit zeta function is

$$\omega_1(s,t) = 4 + 2 \cdot 2^{-t+1}.$$

Case(2) : $\quad I_G(\chi) = \langle r^2 \rangle T$

In this case $I_G(\chi)/T \cong C_2$ is cyclic, and we know that χ extends to $H = \langle r^2 \rangle T$, resulting in two irreducible characters: $\vartheta_1, \vartheta_2 \in \mathrm{Irr}(H)$. To get the irreducible characters of G above χ, we induce

$$\psi = \mathrm{Ind}_H^G(\vartheta_1) = \mathrm{Ind}_H^G(\vartheta_2),\ \psi(1) = 2 \ \text{ for } \ I_G(\vartheta) = H,$$

which is explained in the following diagram

$$\begin{array}{cccc}
G & \bullet & \psi_1 = \mathrm{Ind}_H^G(\vartheta_1) & \psi_2 = \mathrm{Ind}_H^G(\vartheta_2) \\
 & | & & \\
I_G(\chi) = H & \bullet & \vartheta_1 \oplus \vartheta_1^r & \vartheta_2 \oplus \vartheta_2^r \\
 & | & & \\
T & \bullet & \chi \neq \chi^r &
\end{array}$$

We compute the Galois orbits of ψ. The characters $\vartheta_1 \oplus \vartheta_1^r$ and $\vartheta_2 \oplus \vartheta_2^r$ are uniquely determined by the set $\{\vartheta_1, \vartheta_1^r\}$ and $\{\vartheta_2, \vartheta_2^r\}$. We observe that ψ is uniquely determined by $\psi_{1T} = \vartheta_1 \oplus \vartheta_1^r$ and $\psi_{2T} = \vartheta_2 \oplus \vartheta_2^r$. So it is suffices to describe the Galois orbits in

$$\{(\vartheta_1 \oplus \vartheta_1^r) \mid \vartheta_1 \in \mathrm{Irr}(H),\ I_G(\chi) = H\}$$

and

$$\{(\vartheta_2 \oplus \vartheta_2^r) \mid \vartheta_2 \in \mathrm{Irr}(H),\ I_G(\chi) = H\}.$$

Therefore we need to describe the orbit of

$$\{(1, -1, 1), (-1, 1, -1)\},$$

which give us 2 orbits. Hence the Galois orbit zeta function in this case is

$$\omega_2(s, t) = 2 \cdot 2^{-s} \cdot 1^{-t} = 2 \cdot 2^{-s}.$$

$$\underline{\textbf{Case(3)}: \quad I_G(\chi) = T}$$

In this case ψ cannot restrict to an irreducible character of the abelian group T. This means that

$$\psi = \mathrm{Ind}_T^G(\chi) = \mathrm{Ind}_T^G(\chi^r) = \mathrm{Ind}_T^G(\chi^{r^2}) = \mathrm{Ind}_T^G(\chi^{r^3}).$$

We want to compute the Galois orbits of such ψ. The characters of $\psi_T = \chi \oplus \chi^r \oplus \chi^{r^2} \oplus \chi^{r^3}$ are determined uniquely by the set $\{\chi, \chi^r, \chi^{r^2}, \chi^{r^3}\}$. It suffices to describe the Galois orbits in

$$\{(\chi \oplus \chi^r \oplus \chi^{r^2} \oplus \chi^{r^3}) \mid \chi \in \mathrm{Irr}(T),\ I_G(\chi) = T\}.$$

In terms of ξ_1 and ξ_2 we can consider

$$\{(\xi_1, \xi_2), (\xi_2, \xi_1^{-1}), (\xi_1^{-1}, \xi_2^{-1}), (\xi_2^{-1}, \xi_1)\}$$

under the action of $\mathrm{Gal}(\mathbb{Q}(\xi_1, \xi_2)/\mathbb{Q})$. Consider that $\alpha \in \mathrm{Gal}(\mathbb{Q}(\xi_1, \xi_2)/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$, where $l = \mathrm{lcm}(n_1, n_2)$ and its action on

$$\{(\xi_1, \xi_2), (\xi_2, \xi_1^{-1}), (\xi_1^{-1}, \xi_2^{-1}), (\xi_2^{-1}, \xi_1)\}$$

is given by

$$\{(\xi_1,\xi_2),(\xi_2,\xi_1^{-1}),(\xi_1^{-1},\xi_2^{-1}),(\xi_2^{-1},\xi_1)\}^\alpha = \{(\xi_1,\xi_2)^a,(\xi_2,\xi_1^{-1})^a,(\xi_1^{-1},\xi_2^{-1})^a,(\xi_2^{-1},\xi_1)^a\}$$

We have to determine all a such that

$$\begin{aligned} (\xi_1,\xi_2) &= (\xi_1^a,\xi_2^a) \\ \text{or } (\xi_2,\xi_1^{-1}) &= (\xi_1^a,\xi_2^a) \\ \text{or } (\xi_1^{-1},\xi_2^{-1}) &= (\xi_1^a,\xi_2^a) \\ \text{or } (\xi_2^{-1},\xi_1) &= (\xi_1^a,\xi_2^a) \end{aligned}$$

equivalently

$$\begin{aligned} & a \equiv 1 \bmod n_1 \text{ and } a \equiv 1 \bmod n_2 \iff a \equiv 1 \bmod l \\ & \text{or } \xi_2 = \xi_1^a \text{ and } a^2 \equiv 1 \bmod n_2 \text{ when } l = n_1 = n_2 \\ & \text{or } a \equiv -1 \bmod n_1 \text{ and } a \equiv -1 \bmod n_2 \iff a \equiv -1 \bmod l \\ & \text{or } \xi_2 = \xi_1^{-a} \text{ and } a^2 \equiv 1 \bmod n_2 \text{ when } l = n_1 = n_2 \end{aligned}$$

Consider different cases

Case(a) : $n_1 \neq n_2,\ n_1 \geq 3$ or $n_2 \geq 3$

In this case there are precisely two solutions to the above set of conditions unless $l = 2$. These are $a \equiv 1 \bmod n_1$ and $a \equiv -1 \bmod n_2$. Hence, under no extra conditions, there are precisely two solutions that give us a Galois orbit of ψ and has size $\varphi(l)/2$. The total number of such orbits are

$$\frac{\varphi(n_1)\varphi(n_2)}{4} / \frac{\varphi(l)}{2} = \frac{\varphi(g)}{2}.$$

Hence the Galois orbit zeta function in this case is

$$\begin{aligned} \omega_3(s,t) &= \sum_{\substack{n_1,n_2 \\ \max\{n_1,n_2\}\geq 3 \\ \text{and } n_1\neq n_2}} \frac{\varphi(g)}{2} \cdot 4^{-s} \cdot \left(\frac{\varphi(l)}{2}\right)^{-t+1} \\ &= 2^{t-2-2s} \sum_{\substack{n_1,n_2 \\ \max\{n_1,n_2\}\geq 3 \\ \text{and } n_1\neq n_2}} \varphi(g)\varphi(l)^{-t+1} \end{aligned}$$

here $g = \gcd(n_1, n_2)$ and $l = \mathrm{lcm}(n_1, n_2)$.

Case(b) : $n_1 = n_2,\ n_1 \geq 3$ and $n_2 \geq 3$

For this case we have $a^2 \equiv -1 \bmod l$. We find a which satisfies $a^2 \equiv -1 \bmod l$. For -1 to be a square modulo l, either $l = 2p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ or $l = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ where each prime p is congruent to 1 modulo 4. If this is the case then there are 2^k choices for the square root of -1, where k is the number of distinct prime factors of l, and 0 choices for the square root of -1 if $4|l$ or $p|l$ where p is congruent to 3 modulo 4.

Case(b.1) : $n_1 = n_2 = l \geq 3$ and $4|l$ or $p|l$ where $p \equiv 3 \bmod 4$

When using same computation as in Case(a), specialised to $n_1 = n_2 = l \geq 3$, we get a contribution to the Galois orbit zeta function of

$$\begin{aligned}\omega_4(s,t) &= \sum_{\substack{l=3\\ 4|l \text{ or } p|l\\ \text{with } p\equiv 3 \bmod 4}} \frac{\varphi(l)}{2}\cdot 4^{-s}\cdot(\varphi(l)/2)^{-t+1}\\ &= 2^{t-2-2s}\sum_{\substack{l=3\\ 4|l \text{ or } p|l\\ \text{with } p\equiv 3 \bmod 4}} \varphi(l)^{2-t}\end{aligned}$$

Case(b.2) : $n_1 = n_2 = l \geq 3$ and $l = 2p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ or $l = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ where $p_i \equiv 1 \bmod 4$

For this case, fix (ξ_1, ξ_2) then $\xi_2 = \xi_1^b$ for a unique $b \in (\mathbb{Z}/l\mathbb{Z})^*$
If $b^2 \not\equiv -1 \bmod l$, then we have 2 solutions as before, this will be the case for $\varphi(l)(\varphi(l) - 2^k)$ choices for (ξ_1, ξ_2). In this case the Galois orbit of size $\varphi(l)/2$, (using the same argument as before) and the number of orbits are

$$\frac{\varphi(l)(\varphi(l)-2^k)}{4}\Big/\frac{\varphi(l)}{2} = \frac{\varphi(l)-2^k}{2}.$$

The contribution to the Galois orbit zeta function is

$$\begin{aligned}\omega_5(s,t) &= \sum_{\substack{l=3\\ 4\nmid l \text{ and } p\nmid l\\ \text{for } p\equiv 1 \bmod 4}} \frac{\varphi(l)-2^k}{2}\cdot 4^{-s}\cdot(\varphi(l)/2)^{-t+1}\\ &= 2^{t-2-2s}\sum_{\substack{l=3\\ 4\nmid l \text{ and } p\nmid l\\ \text{for } p\equiv 1 \bmod 4}} \varphi(l)^{2-t} - 2^k\cdot\varphi(l)^{-t+1}\end{aligned}$$

If $b^2 \equiv -1 \bmod l$, then we have 4 solutions and this will be the case for the $2^k\varphi(l)$ choices of (ξ_2, ξ_2). In this case, as before, the Galois orbit has size $\varphi(l)/4$ and the number of such orbits are

$$\frac{\varphi(l)2^k}{4} / \frac{\varphi(l)}{4} = 2^k.$$

The contribution to the Galois orbit zeta function is

$$\omega_6(s,t) = \sum_{\substack{l=3 \\ 4\nmid l \text{ and } p\nmid l \\ \text{for } p\equiv 1 \bmod 4}} 2^k \cdot 4^{-s} \cdot (\varphi(l)/4)^{-t+1} = 2^k \cdot 4^{t-s-1} \sum_{\substack{l=3 \\ 4\nmid l \text{ and } p\nmid l \\ \text{for } p\equiv 1 \bmod 4}} \varphi(l)^{-s}$$

The Galois orbit zeta function of G is

$$\begin{aligned}
\omega_G^{\mathbb{Q}}(s,t) &= \omega_1(s,t)+\omega_2(s,t)+\omega_3(s,t)+\omega_4(s,t)+\omega_5(s,t)+\omega_6(s,t)\\
&= 4+2\cdot 2^{-t+1}+2\cdot 2^{-s}+2^{t-2-2s}\sum_{\substack{n_1,n_2\\ \max\{n_1,n_2\}\geq 3\\ \text{and } n_1\neq n_2}}\varphi(g)\varphi(l)^{-t+1}\\
&+2^{t-2-2s}\sum_{\substack{l=3\\ 4|l \text{ or } p|l\\ \text{with } p\equiv 3 \bmod 4}}\varphi(l)^{2-t}+2^{t-2-2s}\sum_{\substack{l=3\\ 4\nmid l \text{ and } p\nmid l\\ \text{for } p\equiv 1 \bmod 4}}\{\varphi(l)^{2-t}-2^k\cdot\varphi(l)^{-t+1}\}\\
&+2^k\cdot 4^{t-s-1}\sum_{\substack{l=3\\ 4\nmid l \text{ and } p\nmid l\\ \text{for } p\equiv 1 \bmod 4}}\varphi(l)^{-t+1}\\
&= 4+2\cdot 2^{-t+1}+2^{1-s}+2^{t-2-2s}\sum_{\substack{n_1,n_2\\ \max\{n_1,n_2\}\geq 3\\ \text{and } n_1\neq n_2}}\varphi(g)\varphi(l)^{-t+1}\\
&+2^{t-2-2s}\sum_{\substack{l=3\\ 4|l \text{ or } p|l\\ \text{with } p\equiv 3 \bmod 4}}\varphi(l)^{2-t}+2^{t-2-2s}\sum_{\substack{l=3\\ 4\nmid l \text{ and } p\nmid l\\ \text{for } p\equiv 1 \bmod 4}}\varphi(l)^{2-t}\\
&+(4^{t-s-1}-2^{t-2-2s})\sum_{\substack{l=3\\ 4\nmid l \text{ and } p\nmid l\\ \text{for } p\equiv 1 \bmod 4}}2^k\cdot\varphi(l)^{-t+1}\\
&= 4+2\cdot 2^{-t+1}+2^{1-s}+2^{t-2-2s}\sum_{n_1,n_2=1}^{\infty}\varphi(g)\varphi(l)^{-t+1}-2^{t-1-2s}\\
&+2^{t-2-2s}\sum_{l=1}^{\infty}\varphi(l)^{2-t}-2^{t-2-2s}\sum_{\substack{l=1\\ 4\nmid l \text{ or } p\nmid l\\ \text{with } p\equiv 3 \bmod 4}}^{\infty}\varphi(l)^{2-t}+2^{t-2-2s}\\
&\cdot\sum_{\substack{l=1\\ 4\nmid l \text{ and } p\nmid l\\ \text{for } p\equiv 1 \bmod 4}}^{\infty}\varphi(l)^{2-t}-2^{t-1-2s}+(4^{t-1-s}-2^{t-2-2s})\sum_{\substack{l=1\\ 4\nmid l \text{ and } p\nmid l\\ \text{for } p\equiv 1 \bmod 4}}^{\infty}2^k\varphi(l)^{-t+1}\\
&-2^{k+1}(4^{t-1-s}-2^{t-2-2s})
\end{aligned}$$

Using the same calculation as in Section 5.3, we get

$$
\begin{aligned}
\omega_G^{\mathbb{Q}}(s,t) &= 4 + 2\cdot 2^{-t+1} + 2^{1-s} - 2^{t-2s} - 2^{k+1}(4^{t-s-1} - 2^{t-1-2s}) \\
&+ 2^{t-2-2s}\prod_p (1 + (p-1)^{1-t}\frac{p+1}{1-p^{2-t}}) + 2^{t-2-2s}\prod_p (1 + \frac{(p-1)^{2-t}}{1-p^{2-t}}) \\
&- 2^{t-2-2s}\prod_{p\equiv 3 \bmod 4}(1 + \frac{(p-1)^{2-t}}{1-p^{2-t}}) + 2^{t-2-2s}\prod_{p\equiv 1 \bmod 4}(1 + \frac{(p-1)^{2-t}}{1-p^{2-t}}) \\
&+ 2^k(4^{t-s-1} - 2^{t-2-2s})\prod_{p\equiv 1 \bmod 4}(1 + 2\frac{(p-1)^{-s}}{1-p^{-s}}).
\end{aligned}
$$

According to Lemma 2.4.15, the Schur index of G is one. Hence the rational representation zeta function is

$$
\begin{aligned}
\zeta_G^{\mathbb{Q}}(s) &= \omega_G^{\mathbb{Q}}(s, s+1) \\
&= 4 + 2^{-s} - 2^k(2 - 2^{-s}) + 2^{-1-s}\prod_p (1 + (p-1)^{-s}\frac{p+1}{1-p^{1-s}}) \\
&+ 2^{-1-s}\prod_p (1 + \frac{(p-1)^{1-s}}{1-p^{1-s}}) - 2^{-1-s}\prod_{p\equiv 3 \bmod 4}(1 + \frac{(p-1)^{1-s}}{1-p^{1-s}}) \\
&+ 2^{-1-s}\prod_{p\equiv 1 \bmod 4}(1 + \frac{(p-1)^{1-s}}{1-p^{1-s}}) + 2^k(2 - 2^{-s})\prod_{p\equiv 1 \bmod 4}(1 + 2\frac{(p-1)^{-s}}{1-p^{-s}}) \\
&= 4 + 2^{-s} - 2^k(2 - 2^{-s}) + 2^{-1-s}\zeta_{\mathbb{Z}^2}^{\mathbb{Q}}(s) + 2^{-1-s}\zeta_{\mathbb{Z}}^{\mathbb{Q}}(s-1) \\
&- 2^{-1-s}\prod_{p\equiv 3 \bmod 4}(1 + \frac{(p-1)^{1-s}}{1-p^{1-s}}) + 2^{-1-s}\prod_{p\equiv 1 \bmod 4}(1 + \frac{(p-1)^{1-s}}{1-p^{1-s}}) \\
&+ 2^k(2 - 2^{-s})\prod_{p\equiv 1 \bmod 4}(1 + 2\frac{(p-1)^{-s}}{1-p^{-s}}).
\end{aligned}
$$

Chapter 6

Generalisations of the infinite dihedral group

We have computed the rational representation zeta functions of finitely generated abelian groups and crystallographic groups of dimensions one and two. We conclude this thesis by computing the rational representation zeta functions of natural generalisations of the infinite dihedral group. We also collect some questions to be answered by future research in the area of the rational representation zeta functions of crystallographic groups.

6.1 Natural generalisations of the infinite dihedral group

We consider the group $G \cong \mathbb{Z}^d \rtimes C_2$ given by the presentation

$$G = \langle x_1, x_2, \ldots, x_d, r \mid [x_i, x_j] = 1 = r^2, x_i^r = x_i^{-1} \text{ for all } i \neq j \rangle.$$

The group G contains a subgroup T, which is normal in G and generated by $\{x_1, x_2, \ldots, x_d\}$. The quotient group G/T, generated by rT, is a cyclic group of order 2.

We want to compute the rational representation zeta function $\zeta_G^{\mathbb{Q}}(s)$ of G. We use a similar strategy as in Section 5.2 of Chapter 5. As in Section 5.2, the irreducible complex characters of T corresponding to representations with finite image are as follows:

$$\chi_{(k_1,k_2,\ldots,k_l),(n_1,n_2,\ldots,n_l)} : T \longrightarrow \mathbb{C}^*,$$

where $n_i \in \mathbb{N}$, $k_i \in \mathbb{N}$ with $1 \leq k_i \leq n_i, \gcd(k_i, n_i) = 1$ and

$$\chi_{(k_1,k_2,\ldots,k_d),(n_1,n_2,\ldots,n_d)}\left(\prod_{j=1}^{d} \chi_j^{a_j}\right) = \prod_{j=1}^{d} \xi_{n_j,k_j}^{a_j} = e^{2\pi i \sum k_j a_j / n_j},$$

where $\xi_{n_j,k_j} = e^{2\pi i k_j/n_j}$ and $j \in \{1, 2, \ldots, d\}$.

Indeed, since T is abelian, each character is obtained by choosing primitive n_jth roots of unity $\xi_{n_j,k_j} = e^{2\pi i k_j/n_j} = \xi_j$ as the images of the generators x_j. Note that all these characters are one-dimensional.

Consider the action of G on $\mathrm{Irr}(T)$ given by $\chi^g(x) = \chi(x^{g^{-1}})$, where $\chi \in \mathrm{Irr}(T)$, $g \in G$ and $x \in T$. We need to understand the orbit of χ and its stabiliser. The stabiliser is called the inertia group and is defined as

$$I_G(\chi) = \{g \in G \mid \chi^g = \chi\}.$$

Since $I_G(\chi)$ is the stabiliser of χ in the action of G on $\mathrm{Irr}(T)$, it follows that it is a subgroup of G and

$$T \subseteq I_G(\chi) \subseteq G.$$

In our case $|G : T| = 2$. The size of the orbit of χ is $|G : I_G(\chi)|$. So either $I_G(\chi) = T$ when $r \notin I_G(\chi)$, or $I_G(\chi) = G$ when $r \in I_G(\chi)$. Correspondingly, $I_G(\chi)/T$ is cyclic and isomorphic to either C_2 or 1.

Now consider $\psi \in \mathrm{Irr}(G)$ factoring over a finite quotient of G. We can cover all possible ψ by considering two cases: $\psi(1) = 1$ and $\psi(1) > 1$.

Case(1) : $\psi \in \mathrm{Irr}(G)$ with $\psi(1) = 1$

To calculate the linear characters of G it is enough to consider the abelianisation of G:

$$G/[G, G] = \langle x_1, x_2, \ldots, x_d, r \mid [x_i, r] = [x_i, x_j] = 1, x_i^2 = r^2 = 1 \text{ for all } i \neq j\rangle$$

$$\cong \underbrace{C_2 \times C_2 \times \cdots \times C_2}_{d \text{ factors}} \times C_2.$$

This means that there are 2^{d+1} linear characters of G. Observe that, since $G/[G,G]$ has exponent 2, the corresponding representations are all defined over $\mathbb{Q}$. Thus we obtain 2^{d+1} one-dimensional irreducible characters over $\mathbb{Q}$. The contribution to the zeta function is

$$\eta_1(s,t) = 2 \cdot 2^d.$$

$$\underline{\textbf{Case(2)}: \quad \psi \in \mathrm{Irr}(G) \text{ with } \psi(1) > 1}$$

In this case ψ cannot restrict to an irreducible character of the abelian group T. This means that

$$\psi = \mathrm{Ind}_T^G(\chi) = \mathrm{Ind}_T^G(\chi^r),$$

where $\chi = \chi_{(k_1,k_2,\ldots,k_d),(n_1,n_2,\ldots,n_d)}$ and $\chi^r = \chi_{(n_1-k_1,n_2-k_2,\ldots,n_d-k_d),(n_1,n_2,\ldots,n_d)}$ are such that $\chi \neq \chi^r$, i.e $I_G(\chi) = T$.

We want to compute the Galois orbits of such ψ. We observe that ψ is uniquely determined by $\psi_T = \chi \oplus \chi^r$. So it suffices to describe the Galois orbits in

$$\{(\chi \oplus \chi^r) \mid \chi \in \mathrm{Irr}(T), I_G(\chi) = T\}.$$

The character $\chi \oplus \chi^r$ is uniquely determined by the set $\{\chi, \chi^r\}$, parameterised by

$$\{(\xi_1, \xi_2, \ldots, \xi_d), (\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_d^{-1})\},$$

where ξ_i primitive n_ith roots of unity and $\max\{n_1, n_2, \ldots, n_d\} \geq 3$. Therefore we need to describe the Galois orbits of such two-elements sets

$$\{(\xi_1, \xi_2, \ldots, \xi_d), (\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_d^{-1})\}$$

under the action of $\mathrm{Gal}(K_l/\mathbb{Q})$, where $l = \mathrm{lcm}(n_1, n_2, \ldots, n_d)$ and $K_l = \mathbb{Q}(\xi_1, \xi_2, \ldots, \xi_d)$. The Galois group satisfies $\mathrm{Gal}(K_l \mid \mathbb{Q}) \cong (\mathbb{Z}/l\mathbb{Z})^*$. Consider that $\alpha \in \mathrm{Gal}(K_l/\mathbb{Q})$ corresponds to $a \in (\mathbb{Z}/l\mathbb{Z})^*$. Its action on $\{(\xi_1, \xi_2, \ldots, \xi_d), (\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_d^{-1})\}$ is given by

$$\{(\xi_1, \xi_2, \ldots, \xi_d), (\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_d^{-1})\}^\alpha = \{(\xi_1, \xi_2, \ldots, \xi_d), (\xi_1^{-a}, \xi_2^{-a}, \ldots, \xi_d^{-a})\}.$$

We find the lengths and the numbers of such orbits. Since $|\operatorname{Gal}(K_l/\mathbb{Q})| = |(\mathbb{Z}/l\mathbb{Z})^*| = \varphi(l)$, the length of an orbit is

$$\varphi(l)/|\operatorname{Stab}_{\operatorname{Gal}(K_l/\mathbb{Q})}\{(\xi_1, \xi_2, \ldots, \xi_d), (\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_d^{-1})\}|.$$

We have to determine all $a \in (\mathbb{Z}/l\mathbb{Z})^*$ such that

$$\begin{aligned} (\xi_1^a, \xi_2^a, \ldots, \xi_d^a) &= (\xi_1, \xi_2, \ldots, \xi_d) \\ \text{or } (\xi_1^a, \xi_2^a, \ldots, \xi_d^a) &= (\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_d^{-1}) \end{aligned}$$

equivalently

$$\begin{gathered} a \equiv 1 \bmod l \\ \text{or } a \equiv -1 \bmod l, \end{gathered}$$

where $l = \operatorname{lcm}(n_1, n_2, \ldots, n_d)$. Since $\max\{n_1, n_2, \ldots, n_d\} > 2$, clearly the stabiliser of

$\{(\xi_1, \xi_2, \ldots, \xi_d), (\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_d^{-1})\}$ has size 2. Hence the length of an orbit is $\varphi(l)/2$. Moreover the total number of such orbits for fixed $(n_1, n_2, \ldots, n_d)$ is

$$\frac{\varphi(n_1)\varphi(n_2)\cdots\varphi(n_d)}{2} / \frac{\varphi(l)}{2},$$

because there are

$$\frac{\varphi(n_1)\varphi(n_2)\cdots\varphi(n_d)}{2}$$

possible choices for $\{(\xi_1, \xi_2, \ldots, \xi_d), (\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_d^{-1})\}$. Every $\psi \in \operatorname{Irr}(G)$ corresponds to $\psi_T = \chi \oplus \chi^r$, and thus to $\{(\xi_1, \xi_2, \ldots, \xi_d), (\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_d^{-1})\}$, and has $\psi(1) = 2$. Its Galois orbit has length $\varphi(l)/2$, where $l = \operatorname{lcm}(n_1, n_2, \ldots, n_d)$.

Thus the overall contribution to the Galois orbit zeta function is

$$\begin{aligned}
\eta_2(s,t) = & \sum_{\substack{n_1,n_2,\ldots,n_d=1\\ \max\{n_1,n_2,\ldots,n_d\}\geq 3}}^{\infty} \frac{\varphi(n_1)\varphi(n_2)\cdots\varphi(n_d)}{2} \cdot 2^{-s} \cdot \left(\frac{\varphi(\mathrm{lcm}(n_1,n_2,\ldots,n_d))}{2}\right)^{-t} \\
&= 2^{t-s-1}\left(\sum_{n_1,n_2,\ldots,n_d=1}^{\infty} \varphi(n_1)\varphi(n_2)\cdots\varphi(n_d)\cdot\varphi(\mathrm{lcm}(n_1,n_2,\ldots,n_d))^{-t} - 2^d\right) \\
&= 2^{t-s-1} \sum_{n_1,n_2,\ldots,n_d=1}^{\infty} \varphi(n_1)\varphi(n_2)\cdots\varphi(n_d)\cdot\varphi(\mathrm{lcm}(n_1,n_2,\ldots,n_d))^{-t} \\
&- 2^{d+t-s-1} \\
&= 2^{t-s-1}\eta_3(t) - 2^{d+t-s-1},
\end{aligned}$$

where

$$\eta_3(t) = \sum_{n_1,n_2,\ldots,n_d=1}^{\infty} \varphi(n_1)\varphi(n_2)\cdots\varphi(n_d)\cdot\varphi(\mathrm{lcm}(n_1,n_2,\ldots,n_d))^{-t}.$$

Now we compute $\eta_3(t)$ as an Euler product over all primes p:

$$\begin{aligned}
\eta_3(t) = & \sum_{n_1,n_2,\ldots,n_d=1}^{\infty} \varphi(n_1)\varphi(n_2)\cdots\varphi(n_d)\cdot\varphi(\mathrm{lcm}(n_1,n_2,\ldots,n_d))^{-t} \\
&= \prod_p \left(\sum_{e_1,e_2,\ldots,e_d=0}^{\infty} \varphi(p^{e_1})\varphi(p^{e_2})\cdots\varphi(p^{e_d})\cdot\varphi(p^{\max\{e_1,e_2,\ldots,e_d\}})^{-t}\right) \\
&= \prod_p \left(1+\sum_{d_0=1}^{d}\binom{d}{d_0}\sum_{f_1,f_2,\ldots,f_{d_0}=1}^{\infty} \varphi(p^{f_1})\varphi(p^{f_2})\cdots\varphi(p^{f_{d_0}})\cdot\varphi(p^{\max\{f_1,f_2,\ldots,f_{d_0}\}})^{-t}\right) \\
&= \prod_p \left(1+\sum_{d_0=1}^{d}\binom{d}{d_0}(p-1)^{d_0-t}\sum_{f_1,f_2,\ldots,f_{d_0}=0}^{\infty} p^{f_1+f_2+\cdots+f_{d_0}}\cdot(p^{\max\{f_1,f_2,\ldots,f_{d_0}\}})^{-t}\right).
\end{aligned}$$

Here the third equality is obtained as follows.

For $e_1 = e_2 = \cdots = e_d = 0$, we obtain a summand 1. The other summands we arrange according to $\#\{i \mid e_d \neq 0\}$; we call this parameter $d_0 \in \{1,2,\ldots,d\}$. There are $\binom{d}{d_0}$ ways of choosing a d_0-element index subset of $\{1,2,\ldots,d\}$. Suppose this is $\{i_1,i_2,\ldots,i_{d_0}\}$. Then the sum over all parameters $f_1 = e_{i_1}, f_2 = e_{i_2},\ldots,f_{d_0} = e_{i_{d_0}} \geq 1$ and $e_j = 0$ for $j \notin \{i_1,i_2,\ldots,i_{d_0}\}$ contributes

$$\sum_{f_1,f_2,\ldots,f_{d_0}=1}^{\infty} \varphi(p^{f_1})\varphi(p^{f_2})\cdots\varphi(p^{f_{d_0}})\cdot\varphi(p^{\max\{f_1,f_2,\ldots,f_{d_0}\}})^{-t}.$$

Hence

$$\eta_2(s,t) = 2^{t-s-1}\prod_p \Bigg(1+\sum_{d_0=1}^{d}\binom{d}{d_0}(p-1)^{d_0-t}\sum_{f_1,f_2,\ldots,f_{d_0}=0}^{\infty} p^{f_1+f_2+\cdots+f_{d_0}} \cdot (p^{\max\{f_1,f_2,\ldots,f_{d_0}\}})^{-t}\Bigg) - 2^{d+t-s-1}$$

Hence the Galois orbit zeta function is

$$\begin{aligned}\omega(s,t) &= \eta_1(s,t)+\eta_2(s,t)\\ &= 2\cdot 2^d + 2^{t-s-1}\prod_p \Bigg(1+\sum_{d_0=1}^{d}\binom{d}{d_0}(p-1)^{d_0-t}\\ &\quad \cdot \sum_{f_1,f_2,\ldots,f_{d_0}=0}^{\infty} p^{f_1+f_2+\cdots+f_{d_0}} \cdot (p^{\max\{f_1,f_2,\ldots,f_{d_0}\}})^{-t}\Bigg) - 2^{d+t-s-1},\end{aligned}$$

where

$$\begin{aligned}&\sum_{f_1,f_2,\ldots,f_{d_0}=0}^{\infty} p^{f_1+f_2+\cdots+f_{d_0}} \cdot (p^{\max\{f_1,f_2,\ldots,f_{d_0}\}})^{-t}\\ &= 1+\sum_{d_1=1}^{d_0}\binom{d_0}{d_1}\sum_{m=1}^{d_1}\sum_{\substack{\delta_1,\delta_2,\ldots,\delta_m\in\mathbb{N}\\ \text{such that}\\ \delta_1+\delta_2+\cdots+\delta_m=d_1}}\binom{d_1}{\delta_1,\delta_2,\ldots,\delta_m}\prod_{k=0}^{m-1}\frac{p^{(d_1-\sum_{j=1}^{k}\delta_j)-t}}{(1-p^{(d_1-\sum_{j=1}^{k}\delta_j)-t})}.\end{aligned}$$

The last equality is satisfied as follows.

Similar to the computation of η_3, we introduce a parameter $d_1 \in \{1,2,\ldots,d_0\}$ to work out the partial sum corresponding to $f_1,f_2,\ldots,f_{d_1} \geq 1$ and $f_{d_1+1},\ldots,f_{d_0} = 0$. Furthermore we subdivide the sum according to parameters $\delta_1,\delta_2,\ldots,\delta_m$ with $\delta_1+\delta_2+\cdots+\delta_m = d_1$. These parameters are to record to what extent the parameters $f_1,f_2,\ldots,f_{d_1}$ are equal. The index set $I = \{1,2,\ldots,d_1\}$ can be partitioned into disjoint union $I = I_1 \cup I_2 \cup \cdots \cup I_m$ of subsets I_j of prescribed sizes $\mid I_j \mid= \delta_j$ in $\binom{d_1}{\delta_1,\delta_2,\ldots,\delta_m}$ ways. The contribution

$$\prod_{k=0}^{m-1}\frac{p^{(d_1-\sum_{j=1}^{k}\delta_j)-t}}{(1-p^{(d_1-\sum_{j=1}^{k}\delta_j)-t})}$$

account for all $f_1,f_2,\ldots,f_{d_1}$ such that

1. for each $j \in \{1,2,\ldots,m\}$, f_i is constant, equal to F_j, for $i \in I_j$;

2. $1 \leq F_1 < F_2 < \cdots < F_m$.

Note: According to Lemma 2.4.15, the Schur index of every character of G is one, hence

$$\zeta_G^{\mathbb{Q}}(s) = \omega_G^{\mathbb{Q}}(s, s+1).$$

6.2 Examples

1. $d = 1$

   In this case the Galois orbit zeta function is

$$\begin{aligned}
\omega(s,t) &= 4 + 2^{t-s-1} \prod_p \left(1 + (p-1)^{1-t} \sum_{f=0}^{\infty} p^{f(1-t)}\right) - 2^{t-s} \\
&= 4 + 2^{t-s-1} \prod_p \left(1 + \frac{(p-1)^{1-t}}{1-p^{1-t}}\right) - 2^{t-s} \\
&= 4 - 2^{t-s} + 2^{t-s-1}\omega_{\mathbb{Z}}^{\mathbb{Q}}(s,t).
\end{aligned}$$

   Since, the Schur indices are one,

$$\zeta_G^{\mathbb{Q}}(s) = \omega(s, s+1) = 2 + \zeta_{\mathbb{Z}}^{\mathbb{Q}}(s).$$

   This confirms the direct computation in Section 4.1.2.

2. $d = 2$

In this case the Galois orbit zeta function is

$$
\begin{aligned}
\omega(s,t) = 8 + 2^{t-s-1}\prod_p \Bigg(1 + 2(p-1)^{1-t}\sum_{f=0}^{\infty} p^{f(1-t)} \\
&+ (p-1)^{2-t}\sum_{f_1,f_2=0}^{\infty} p^{f_1+f_2}(p^{\max\{f_1,f_2\}})^{-t}\Bigg) - 2^{t-s+1} \\
&= 8 + 2^{t-s-1}\prod_p \Bigg(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}} + (p-1)^{2-t} \\
&\Bigg(1 + 2\frac{p^{1-t}}{1-p^{1-t}} + \frac{p^{2-t}}{1-p^{2-t}} + 2\frac{p^{2-t}p^{1-t}}{(1-p^{1-t})(1-p^{2-t})}\Bigg)\Bigg) - 2^{t-s+1} \\
&= 8 + 2^{t-s-1}\prod_p \Bigg(1 + 2\frac{(p-1)^{1-t}}{1-p^{1-t}} + (p-1)^{2-t}\frac{1+p^{1-t}}{(1-p^{1-t})(1-p^{2-t})}\Bigg) \\
&- 2^{t-s+1} \\
&= 8 + 2^{t-s-1}\prod_p \Bigg(1 + (p-1)^{1-t}\Bigg(\frac{2}{1-p^{1-t}} + \frac{(p-1)(1+p^{1-t})}{(1-p^{1-t})(1-p^{2-t})}\Bigg)\Bigg) \\
&- 2^{t-s+1} \\
&= 8 + 2^{t-s-1}\prod_p \Bigg(1 + (p-1)^{1-t}\frac{p+1}{1-p^{2-t}}\Bigg) - 2^{t-s+1} \\
&= 8 - 2^{t-s+1} + 2^{t-s-1}\omega^{\mathbb{Q}}_{\mathbb{Z}^2}(s,t)
\end{aligned}
$$

Again, the Schur indices are equal to one. Hence

$$\zeta^{\mathbb{Q}}_G(s) = \omega^{\mathbb{Q}}_G(s,s+1) = 4 + \zeta^{\mathbb{Q}}_{\mathbb{Z}^2}(s).$$

This confirms the direct computation in Section 5.2.

6.3 Open problems

In this section we have formulated some questions to be answered by future research.

The group pm and pg are given by the presentations

$$
\begin{aligned}
pm &= \langle x, y, m \mid [x,y] = m^2 = 1, x^m = x, y^m = y^{-1}\rangle, \\
pg &= \langle x, y, t \mid [x,y] = 1, t^2 = y, x^t = x^{-1}\rangle.
\end{aligned}
$$

As shown in Section 5.3, we computed the rational representation zeta function of the group pm, and in Section 5.4 we computed the rational representation zeta function of the group pg. A problem then arises:

Problem 6.3.1. *Find natural generalisations of the groups pm and pg and compute the corresponding rational representation zeta functions.*

In Section 5.6, we computed the Galois orbit zeta function of the group $p2mg$. The presentation of $p2mg$ is given by

$$G = \langle x, y, m, t \mid [x, y] = 1 = t^2, m^2 = y, x^t = x, x^m = x^{-1}, y^t = y^{-1}, m^t = m^{-1} \rangle.$$

The Galois orbit zeta function does not directly give the rational representation zeta function of the group G of type $p2mg$. A natural problem arises:

Problem 6.3.2. *Compute the rational representation zeta function of the group G of type $p2mg$.*

Similarly, in Section 5.7, we computed the Galois orbit zeta function of the group $p2gg$. The presentation of $p2gg$ is given by

$$G = \langle x, y, u, v \mid [x, y] = 1 = (uv)^2, u^2 = x, v^2 = y, x^v = x^{-1}, y^u = y^{-1} \rangle.$$

The Galois orbit zeta function does not directly give the rational representation zeta function of the group G of type $p2gg$. Again, a problem arises:

Problem 6.3.3. *Compute the rational representation zeta function of the group G of type $p2gg$.*

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