On spectral constructions for Salem graphs

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Declaration

I, Lee Gumbrell, hereby declare that this thesis and the work presented in it is entirely my own. Where I have consulted the work of others, this is always clearly stated.

Signed .... (Lee Gumbrell)
Date:

For my parents.

Abstract

In recent years, people have begun studying Salem numbers by looking at the spectrum of the adjacency matrix of a graph. In this thesis we classify infinitely many new infinite families of Salem graphs using results about graph spectra. Our first method is to define a notion of how close a Salem graph is to being cyclotomic, the m-Salem graphs, and classify the whole family of 1-Salem graphs. The second method uses the Courant-Weyl inequalities in a novel way, partitioning the edges of a graph into two sets and considering the graphs they form. We exhaustively work through all possibilities to find even more families of Salem graphs. We also study when some of these graphs produce trivial Salem numbers, using a new extension of Hoffman and Smith's subdivision theorem.

Acknowledgement

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Part I

Background

Chapter 1

Introduction

We begin by defining just enough basic graph theory in Section 1.1 so that we may properly define a Salem graph in Section 1.2. After this we will work through the remaining important definitions from graph theory and note some important results about graph spectra in Section 1.3.

The books [32], [2], [18] and [23] all provide an excellent introduction to basic graph theory and graph spectra, and almost everything in this opening chapter can be read about in more detail in any of these.

Firstly we mention an important piece of notation used throughout this thesis: all non-integral real numbers will be written to three decimal places and treated as symbols in place of the full number they represent. Table A.1 in Appendix A.1 gives a complete list of these symbols and their full number. The reason for doing this is that it is much easier to see that $0.382 < 0.586$ than it is to see that $\left(3-\sqrt{5}\right)/2 < 2-\sqrt{2}$, and we will be comparing many such values in Chapter 3.

1.1 Some basic graph theory

In order to make this thesis as self-contained as possible, we shall start from the very beginning. Define a **graph** G to be a finite non-empty set $V = V(G)$ of **vertices** v along with a set $E = E(G)$ of **edges**, defined to be unordered pairs of distinct vertices from $V(G)$. Note that "unordered" means that our graphs will not have any direction to the edges (like you might find in a network, perhaps) and "distinct" means that our graphs will not contain any loops (an edge from a vertex to itself). We also specify that a graph contains at most one edge between any two particular vertices, and use the term **multigraph** to describe a graph that breaks this rule. Such edges will be called **multiple edges**. Two vertices with an edge connecting them will be called

adjacent (and **non-adjacent** if they are not) and the edges are said to be **incident** to the vertices. The number of vertices $|V|$ will often be n, but not always.

A **subgraph** H of a graph G is a graph whose vertex and edge sets are (not necessarily non-trivial) subsets of the vertex and edge sets of G; that is, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $V(G) = \{v_1, \ldots, v_n\}$, then for some $0 \leq m \leq n$ we define the **induced subgraph** $G \setminus \{v_1, \ldots, v_m\}$ to be the maximal subgraph on the remaining vertices $v_{m+1}, \ldots, v_n$. This can be thought of as G with the vertices $v_1, \ldots, v_m$ removed, along with all the of edges incident to them, and we include the possibility that an induced subgraph may be the graph itself. We will mostly think about induced subgraphs and so simply refer to them as subgraphs, specifying when we mean a non-induced subgraph. Also, if we are only removing one vertex v we will use the shorter notation $G \setminus v$ rather than $G \setminus \{v\}$. If H is a subgraph of G, then G is a **supergraph** of H.

Define the **adjacency matrix** $A = A(G)$ to be the $n \times n$ matrix with a row and column for each vertex $v_1, \ldots, v_n$ from V whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, define the **characteristic polynomial** $\chi_G(x)$ of G to be the polynomial $\det(xI_n - A(G))$, where I_n is the $n \times n$ identity matrix. We then let the **eigenvalues** $\lambda_1, \ldots, \lambda_n$ of G (and of A) be the n solutions to the equation $\chi_G(x) = 0$. We will also call these n numbers the **spectrum** of G. The fact that A is both real and symmetric tells us that each $\lambda_i \in \mathbb{R}$ and that A is a Hermitian matrix. An **eigenvector $\boldsymbol{x}$** will be one that satisfies the equation $A\boldsymbol{x} = \lambda \boldsymbol{x}$. We will always treat the n eigenvalues in decreasing order as below:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$

We also define the largest eigenvalue λ_1 to be the **index** and often refer to the smallest eigenvalue λ_n as the **least eigenvalue**. A graph is said to be **integral** if $\lambda_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. Every graph has a unique adjacency matrix and can be recovered from it, but the same cannot be said for recovering a graph from the spectrum. Most mathematical programs offer a way of calculating the spectrum given the adjacency matrix; most of the calculations here were performed using Pari/GP and the commands `charpoly` and `polroots`. Adjacency matrices were constructed in a spreadsheet allowing the user to simply enter the 1's above the diagonal.

In a graph a **walk** is a sequence of vertices where each vertex is adjacent to the two either side of it in the sequence, and it is **closed** if the first and last vertices in the sequence are the same vertex. A **cycle** is a closed walk where the m vertices in the sequence are all distinct and $m \geq 3$. A graph is **connected** if every pair of points are joined by a path, else it is **disconnected** and the disconnected parts are referred to as the **components** of the graph.

We now make a very important definition, partitioning the set of all graphs into two very different groups. A **bipartition** of the vertices of a graph is a partition of the vertices into two sets V_1 and V_2 such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, the empty set. We call a graph **bipartite** if the vertices can be bipartitioned into sets V_1 and V_2 such that no two vertices in the same set are adjacent to each other. If there are no such sets V_1 and V_2 then we call the graph **non-bipartite**. We present some alternative ways of thinking about this in the form of the following theorem.

Theorem 1.1.1 (see [32], Theorem 2.4 and [1], Proposition 2.3.3)**.** *For a graph G the following are equivalent:*

(i) G is bipartite;

(ii) G contains no cycles of odd length;

(iii) the spectrum of G is symmetric about 0; that is if λ is an eigenvalue then $-\lambda$ must also be, with the same multiplicity.

A graph is called **cyclotomic** if all of its eigenvalues lie in the interval $[-2, 2]$. A complete description of all cyclotomic graphs was provided by Smith in [51]; there are the two infinite families $\tilde{A}_n$ and $\tilde{D}_n$ and the three sporadic graphs $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ (we call graphs **sporadic** if they do not belong to a defined infinite family).

Lemma 1.1.2 (see [51])**.** *The connected cyclotomic graphs are precisely the induced subgraphs of the graphs $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{A}_n\,(n \geq 2)$ and $\tilde{D}_n\,(n \geq 4)$, shown in Figure 1.1.*

Note that each graph in Figure 1.1 has one more vertex than the subscript in its name. Removing a certain vertex from each of these graphs results in the commonly used graphs E_6, E_7, E_8, A_n and D_n, and standard definitions of these can be found in [23], for example.

1.2 Salem numbers and Salem graphs

We now understand enough about graph theory to define a Salem graph, although we will motivate this definition first. To an extent, this motivation can be either number theoretical or graph theoretical, but we shall favour the former and simply mention the

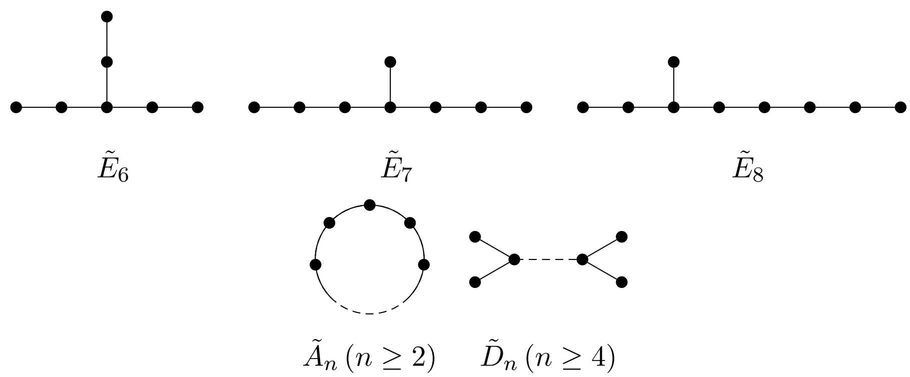


Figure 1.1: The maximal connected cyclotomic graphs.

latter. For this number theoretical approach we need the definition of Salem numbers, from where our Salem graphs take their name. For the vast majority of this thesis we will work almost solely with Salem graphs, only considering the related Salem numbers in Chapter 4.

Let us begin by defining these such numbers. We call a real algebraic integer τ a **Salem number** if $\tau > 1$ and all of its other Galois conjugates lie in the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$, with at least one having modulus exactly 1. It is not known how close to 1 Salem numbers appear, although the smallest known Salem number is 1.176 (the larger real root of the polynomial $L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$).

Graph eigenvalues are totally real algebraic integers, and it was shown by Estes in [26] that all totally real algebraic integers appear as graph eigenvalues. To get Salem numbers from graphs we use one of the transformations $z \mapsto z + 1/z$ or $z \mapsto \sqrt{z} + 1/\sqrt{z}$ to map the union of the unit circle and the real line to the real line. The unit circle is mapped to the interval $[-2, 2]$ and a real $\tau > 1$ is mapped to a $\lambda > 2$. We define the following **reciprocal polynomials** for graphs G on n vertices:

$$\begin{array}{rcll} R_G(z) & = & z^{n/2}\chi_G\left(\sqrt{z} + \frac{1}{\sqrt{z}}\right) & \text{for } G \text{ bipartite} \\ R_G(z) & = & z^n\chi_G\left(z + \frac{1}{z}\right) & \text{for } G \text{ non-bipartite.} \end{array}$$

In the bipartite case, Theorem 1.1.1(iii) can be used to show that $R_G(z)$ is a polynomial in z. The following definition will now seem very natural:

Definition 1.2.1. *A bipartite graph G is called a* **Salem graph** *if the largest eigenvalue λ_1 is greater than 2, and the remaining $n-1$ eigenvalues are ≤ 2. Then $\lambda_n = -\lambda_1 < -2$ by Theorem 1.1.1. A bipartite Salem graph is called* **trivial** *if $\lambda_1^2 \in \mathbb{Z}$. The associated number $\tau(G)$ is the larger root of $\sqrt{z} + 1/\sqrt{z} = \lambda_1$; this is a Salem number unless G is trivial (in which case it is a quadratic Pisot number).*

A non-bipartite graph G is called a **Salem graph** *if the largest eigenvalue $\lambda_1 > 2$ and the remaining $n-1$ eigenvalues are in the interval $[-2, 2]$. A non-bipartite Salem graph is called* **trivial** *if $\lambda_1 \in \mathbb{Z}$. The associated number $\tau(G)$ is the larger root of $z + 1/z = \lambda_1$; this is a Salem number unless G is trivial.*

Note how these Salem graphs appear very similar in their spectra to the cyclotomic graphs in Figure 1.1, as nearly all of their eigenvalues fall in the interval $[-2, 2]$. From a graph theoretical point of view, we are studying the non-cyclotomic graphs with the fewest number of eigenvalues outside this cyclotomic interval. If a graph is non-bipartite and has one eigenvalue outside $[-2, 2]$ then it must be positive by Frobenius theory (see [18], Theorem 0.3, for example). If a graph is bipartite then the symmetry of bipartite eigenvalues in Theorem 1.1.1 tells us that if any eigenvalues are outside the interval $[-2, 2]$ there must be an even number. This simple change to the spectrum motivates this problem from a graph theoretical perspective, and we note the nice coincidence that these graphs also produce Salem numbers. Whilst the cyclotomic graphs form a very neat family, Salem graphs are much harder to classify despite this small change.

Salem graphs first appeared in this form in [41], and certain special cases have appeared in other papers (see [9], [44], [39], [35], [38], [36], [43], [40], [42] and [37]). The paper [41] also gave some partial descriptions of some families of Salem graphs, along with a complete description of Salem trees and a proof that all limit points of sets of Salem numbers from graphs are Pisot numbers (a related family of real algebraic integers). Graphs with an index just outside the cyclotomic interval have also been studied in [6] and [16] (also see [21]), but not always with the conditions on λ_2 or λ_n. In this thesis we classify infinitely many more infinite families of Salem graphs (along with a number of sporadic graphs), increasing the knowledge of such graphs greatly.

1.3 Graph theory and graph spectra

We now make the remaining definitions we will require from graph theory and state a number of theorems that we will use throughout the thesis.

1.3.1 Some more basic graph theory

We call a graph a **tree** if it contains no cycles, and a graph whose connected components are trees is called a **forest**. Two graphs G and H are called **isomorphic** if there exists a one-to-one correspondence between their vertex sets $V(G)$ and $V(H)$ which preserves adjacency (or alternatively, if $A(G) = PA(H)P^T$ for some permutation matrix P).

The **degree** $d(v)$ of a vertex v is the number of edges incident to it and a graph is called **regular** of degree r if $d(v_i) = r$ for all $i = 1, \ldots, n$. The **complement** $\overline{G}$ of a graph G has the same vertex set V but two vertices in $\overline{G}$ are adjacent if and only if they are non-adjacent in G.

The **union** of two disjoint graphs G and H is denoted $G \cup H$ and defined, as expected, to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We write $G \cup G$ as $2G$. The **join** $G \triangledown H$ of two disjoint graphs G and H is the graph obtained from $G \cup H$ by joining every vertex of G to every vertex in H. Note that $\overline{G \triangledown H} = \overline{G} \cup \overline{H}$. In [32] and [2] the same operation is defined with the notation $G + H$, which in [23] is used to denote a different operation (the sum). In Chapter 3 we use $G + H$ for a new definition so mention it here only to flag up the difference.

We will now define some standard graphs that we will use frequently. The **complete graph** K_n is the graph on n vertices where every distinct pair of vertices are adjacent, and is the unique regular graph of degree $n-1$. We also call these **cliques**, usually when referring to a subgraph. For simplicity, we define the **complete bipartite graph** $K_{n,m}$ and, for $k \geq 3$, the **complete multipartite graph** $K_{n_1,\ldots,n_k}$ to be the graphs isomorphic to $\overline{K_n \cup K_m}$ and $\overline{K_{n_1} \cup \ldots \cup K_{n_k}}$, respectively (where $n, m, n_1, \ldots, n_k \geq 1$); we define them like this as it helps to think of them as a complete graph with the edges of smaller cliques removed.

We call a graph a **star** if it is isomorphic to $K_{1,n}$ for some n, and a **cocktail party graph** if it is isomorphic to $K_{2,\ldots,2}$. A pair of edges are called **independent** if neither edge is incident to the same vertex, so we can think of the cocktail party graph as a K_{2n} with n independent edges removed; it is also the unique regular graph of degree $2n-2$. We define a **generalized cocktail party graph (GCP)** as a graph isomorphic to a complete graph but with some independent edges removed; that is a graph where all vertices have degree $n-1$ or $n-2$. We will use $GCP(n,m)$ to denote the GCP on n vertices with m edges removed, where $0 \leq m \leq \lfloor n/2 \rfloor$. In a $GCP(n,m)$, we will refer to a vertex of degree $n-1$ as being "of maximal degree".

Also define a **cycle** C_n to be the unique (for each $n \geq 3$) connected regular graph on n vertices each of degree 2, and a **path** P_n to be either the connected graph on $n \geq 2$ vertices with two vertices of degree 1 and the remaining $n-2$ of degree 2, or the graph K_1. Note that the graph $\tilde{A}_n$ in Figure 1.1 is isomorphic to C_{n+1}, that $A_n = P_n$ and a path is simply any of the connected subgraphs of a cycle. A **pendent path** in a graph is the union of the graph and a path on n vertices along with an edge joining one of the degree 1 vertices to the desired vertex of the graph. If the path in question is simply a K_1 then we call it **single vertex pendent path** and the edge a **pendent edge**. Also, an **isolated vertex** in a graph is a disconnected K_1.

Finally we note a result about the coefficients of the characteristic polynomial of a graph.

Lemma 1.3.1 (see [2], Proposition 2.3)**.** *Let the characteristic polynomial of a graph G be*

$$\chi_G(x) = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + c_3x^{n-3} + \ldots + c_n,$$

then the following hold:

(i) $c_0 = 1$*;*

(ii) $c_1 = 0$*;*

(iii) $-c_2 = |E(G)|$*, the number of edges in G;*

(iv) $-c_3$ *is twice the number of 3-cycles in G.*

1.3.2 Line graphs and generalised line graphs

Throughout the thesis we will spend a great deal of time studying both line graphs and generalised line graphs, so we devote this section to their definition and some key results.

The **line graph** of a (multi-)graph G is denoted $L(G)$ and defined to be the graph whose vertices are the edges of G, with two vertices in $L(G)$ adjacent whenever the corresponding edges in G have exactly one vertex in common. Now let G be an n-vertex graph and let $a_1, \ldots, a_n$ be non-negative integers. Define a **pendent 2-cycle** to be a single vertex pendent path connected to a vertex v of a graph where there are two edges between v and our new vertex instead of one; such a vertex is sometimes called a **leaf**. The **generalized line graph** $L(G; a_1, \ldots, a_n)$ is the graph $L(\hat{G})$, where $\hat{G}$ is the multigraph $G(a_1, \ldots, a_n)$ obtained from G by adding a_i pendent 2-cycles at vertex v_i $(i = 1, \ldots, n)$. The **root graph** of a line graph $L(G)$ is simply the (multi-)graph G itself.

Clearly, if $a_i = 0$ for all $i = 1, \ldots, n$ then our generalised line graph is simply a line graph, so line graphs form a subset of the generalised line graphs and for simplicity we shall usually just talk about generalised line graphs, referring to line graphs only when we mean just them. There are a number of alternative ways of thinking about generalised line graphs and the most commonly used here will be the following characterisation.

Theorem 1.3.2 (see [23], Theorems 2.3.1 and 2.1.1)**.** *A connected graph is a generalized line graph if and only if its edges can be partitioned into GCPs such that*

(i) two GCPs have at most one common vertex;

(ii) each vertex is no more than two GCPs;

(iii) if two GCPs have a common vertex, then it is of maximal degree in both of them.

Also, a graph is a line graph if and only if its edges can be partitioned in such a way that every edge is in a clique and no vertex is in more than two cliques.

This theorem allows us to see generalised line graphs as collections of GCPs sharing certain vertices. In some generalised line graphs the partitioning of the edges will be very easy to see, but in others not so. For all but seven generalised line graphs, the partitioning of the edges is unique; that is, there are only seven pairs of root graphs that result in the same generalised line graph (up to isomorphism).

Lemma 1.3.3 (see [7] or [23], Theorem 2.3.4)**.** *With the exception of the seven pairs of graphs G_i and H_i in Figure 1.2 ($i = 1, \ldots, 7$), if two connected generalised line graphs are isomorphic then their root multigraphs are also isomorphic.*

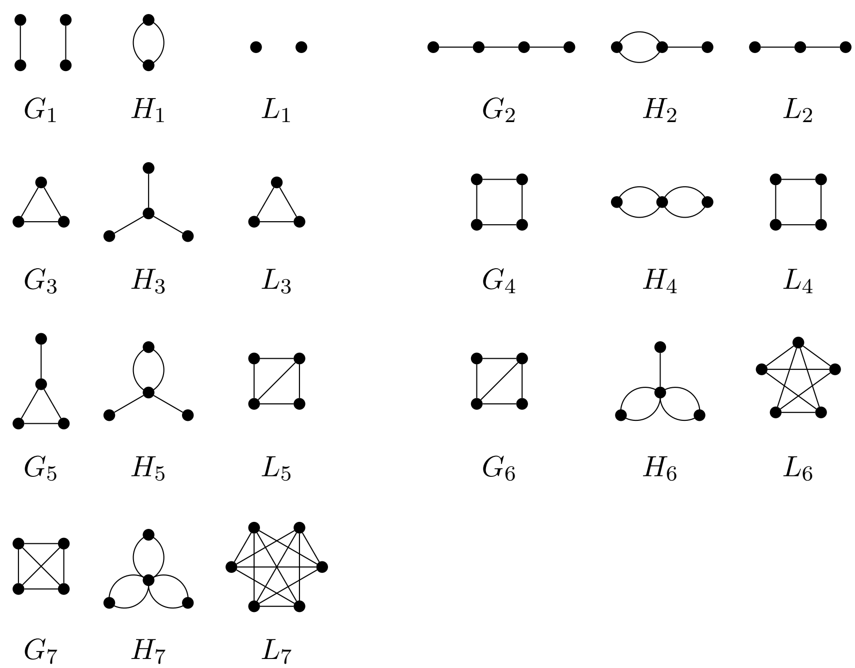


Figure 1.2: The seven pairs of (multi-)graphs G_i and H_i from Lemma 1.3.3, along with the generalised line graphs $L_i = L(G_i) = L(H_i)$ for which they are the root graphs (for $i = 1, \ldots, 7$). Note that L_3 is the only graph that does not have a multigraph for a root graph, and the equivalent result about line graphs involves only the pair G_3 and H_3.

An alternative description comes in the form of minimal forbidden subgraphs, the smallest possible set of induced subgraphs that forbid any supergraph from being a generalised line graph. We will not describe all of the graphs here, instead depicting them when we need to.

Theorem 1.3.4 (see [19] or [23], Theorem 2.3.18). *A graph G is a generalised line graph if and only if it does not contain any of the* 31 *graphs in Figure 2.4 of [23] as an induced subgraph.*

The following two results are key in understanding why generalised line graphs are quite so important in the search for non-bipartite Salem graphs.

Theorem 1.3.5 (see [28], Lemma 8.6.2). *Line graphs and generalized line graphs have all their eigenvalues in the interval* $[-2, \infty)$*.*

Theorem 1.3.6 (see [8]). *A graph G has $\lambda_n(G) \geq -2$ if and only if G is a generalised line graph, or is in the finite family of graphs represented in the root system of E_8.*

This second result, by Cameron, Goethals, Seidel and Shult, tells us that clearly all non-bipartite Salem graphs will either be generalised line graphs or in this other finite family of graphs. We shall refer to the graphs represented in the root system of E_8 that are *not* also generalised line graphs as the **exceptional graphs** (the same terminology is used in [23]). They are a very large but finite family of graphs, meaning that any work involving them tends to require a more computational approach. We shall mostly be interested in the infinite family of generalised line graphs.

1.3.3 Results on graph spectra

In this subsection we observe some results about graph spectra in general, and begin with a collection of useful results.

Lemma 1.3.7. *(i)* (see [5], Proposition 1.3.6) *The spectrum of a disconnected graph is equal to the union of the spectra of the connected components.*

(ii) (see [3], Theorem 8.2.5(v)) *Let $\delta(G)$ be the minimum of the degrees of the vertices of G and let $\Delta(G)$ be their maximum. Then $\delta(G) \leq \lambda_1(G) \leq \Delta(G)$.*

(iii) (see [2], Proposition 3.1) *If G is a regular graph with degree r then $\lambda_1 = r$.*

The following theorem collects some results about the eigenvalues of certain graphs.

Theorem 1.3.8 (see [25] or [17]). *For a non-trivial connected graph G we have $\lambda_2(G) > 0$ unless $G = K_n$ for $n \geq 2$ (which has $\lambda_2(K_n) = -1$) or $G = K_{n_1,\ldots,n_k}$ with $\max\{n_i\} \geq 2$ (which has $\lambda_2(K_{n_1,\ldots,n_k}) = 0$).*

Furthermore, the spectrum of K_n is $n-1^{(1)}, -1^{(n-1)}$ and, for n even, the spectrum of $GCP\,(n, n/2)$ is $n-2^{(1)}, 0^{(n/2)}, -2^{(n/2-1)}$, where the numbers in brackets in the superscript tell us the multiplicity of that eigenvalue.

We now move on to some very important results about the spectra of graph. Arguably the two most important results will be interlacing and the Perron-Frobenius theorem; both will appear frequently throughout this thesis.

Theorem 1.3.9 (Interlacing; see [11] or [28], Theorem 9.1.1)**.** *Let G be an n-vertex graph with vertex set $V(G)$ and eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Also, let H be the induced subgraph on $G \setminus v$, for some vertex v of G. Then the eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1}$ of H interlace with those of G; that is*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \lambda_n.$$

We will use interlacing to tell us things about individual eigenvalues but also about the whole spectrum of a graph in relation to the spectrum of a subgraph. To make the interlacing very clear in these cases we will often write the spectra in boxes to see how they affect each other, as can be seen in Figure 1.3.

λ_1	λ_2	$\cdots$	λ_{n-1}	λ_n
μ_1	μ_2	$\cdots$	μ_{n-2}	μ_{n-1}

Figure 1.3: An example of how the eigenvalues of a graph and an induced subgraph interlace using boxes. The value of any particular box is constrained by the values of the pair directly above it or below it.

Note that interlacing explains why it is possible to know all of the cyclotomic graphs in Figure 1.1: we have a bound on the largest and smallest eigenvalues meaning that once any supergraph has an eigenvalue outside the $[-2, 2]$ interval, every supergraph containing it must also have such an eigenvalue. It also tells us that any subgraph of a Salem graph must either be Salem or cyclotomic.

Theorem 1.3.10 (Perron-Frobenius; see [28], Theorem 8.8.1)**.** *Let $A(G)$ be the adjacency matrix of a connected graph G, then:*

(i) *$\lambda_1(G)$ is a simple eigenvalue of $A(G)$ (meaning that $\lambda_1(G) > \lambda_2(G)$) and, if $\boldsymbol{z}$ is an eigenvector for λ_1, then none of the entries of $\boldsymbol{z}$ are zero and they all have the same sign;*

(ii) for another graph H, if $A(G) - A(H)$ is non-negative then $\lambda_1(H) \leq \lambda_1(G)$, with equality if and only if G and H are isomorphic.

Note that part (ii) of Theorem 1.3.10 applies to both induced and non-induced subgraphs H. If H has fewer vertices than G, let $|V(H)| = n-m$ (for some $0 < m < n$) and consider the graph $H \cup mK_1$. By Lemma 1.3.7(i) these extra isolated vertices only add 0's to the spectrum of H, so $\lambda_1(H) = \lambda_1(H \cup mK_1)$. Then, taking $A(H \cup mK_1)$ in part (ii) above gives a non-negative matrix and hence $\lambda_1(H) < \lambda_1(G)$.

An **internal path** of a graph G is a sequence of vertices $v_1, \ldots, v_k$ of G such that all vertices are distinct (except possibly v_1 and v_k), v_i is adjacent to v_{i+1} (for $i = 1, \ldots, k-1$), v_1 and v_k have degree at least 3, and all of $v_2, \ldots, v_{k-1}$ have degree 2. We also define a **subdivision** of an internal path to be the same path, but with one more degree 2 vertex in the sequence. A subdivided graph G' is isomorphic to G except with one more vertex and one more edge on the subdivided internal path, and we say that G' and G are **topologically equivalent** (note that this last definition also holds for the same operation on non-internal paths). Hoffman and Smith proved the following result.

Theorem 1.3.11 (Subdivision; see [33] or [22], Theorem 3.2.3)**.** *Let G be a graph with an internal path, and let G' be the graph obtained by subdividing an edge on that path. If G is not equal to the graph $\tilde{D}_n$ in Figure 1.1 then $\lambda_1(G') < \lambda_1(G)$.*

Later on in Section 4.1 we will revisit this theorem, generalise it and extend it slightly.

We call a graph **planar** if it can be drawn on a plane such that no two edges intersect, and these graphs have been characterised by the following result.

Theorem 1.3.12 (see [34] or [32], Theorem 11.13)**.** *A graph is planar if and only if it does not contain a subgraph topologically equivalent to K_5 or $K_{3,3}$.*

The following lemma allows us our first, simple construction for non-bipartite Salem graphs.

Lemma 1.3.13 (see [48] or [22], Corollary 9.1.12)**.** *For a graph G on n vertices we have*

$$\lambda_2(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Corollary 1.3.14. *Let G be a generalised line graph or exceptional graph and also non-cyclotomic (that is, not an induced subgraph of any of the graphs in Figure 1.1). If G has 7 or fewer vertices then it is a Salem graph.*

Proof. As G is a generalised line graph or exceptional graph and non-cyclotomic, we must have that $\lambda_1 > 2$ by Lemma 1.1.2 and that $\lambda_n \geq -2$ by Theorem 1.3.5. For $n \leq 7$, Lemma 1.3.13 tells us that $\lambda_2 \leq 2$, satisfying all the conditions of Definition 1.2.1. $\square$

In this thesis we shall spend the majority of our time looking for constructions for Salem graphs. The construction above in Corollary 1.3.14 produces a small, finite number of graphs, but we will instead look to classify as many infinite families as possible. For this we will use two techniques: in Chapter 2 we will define the m-Salem graphs and classify all of the graphs that are 1-Salem; these graphs tell us much more about the underlying structure of Salem graphs than was previously known. In Chapter 3 we will define a new way of adding graphs and construct families using the Courant-Weyl inequalities. This construction is a very powerful one, giving us infinitely many infinite families as well as another proof of the classification of the 1-Salem graphs. Chapter 4 deals with the question of when these families give trivial Salem graphs and Chapter 5 collects some smaller related results.

Part II

Spectral constructions for Salem graphs

Chapter 2

The m-Salem graphs and the HAM lemma

The majority of the work in this chapter comes from the author's contribution to a joint paper with James McKee (see [31]). McKee's contribution is discussed briefly in Section 2.4. An early online version ([13]) also contained contributions from Jonathan Cooley, which we will touch on slightly.

To begin with we will define an m-Salem graph and motivate this definition. The main result of this chapter is that we can completely classify the first family of these graphs – the 1-Salem graphs. To do this we will consider bipartite and non-bipartite Salem graphs separately (recall from Definition 1.2.1 that they have differing spectral structures). The result for bipartite 1-Salem graphs is found in Section 2.2. As non-bipartite Salem graphs have $\lambda_n \geq -2$ we can use Theorem 1.3.6 to split the non-bipartite graphs further into the generalised line graphs or the finite family of exceptional graphs. The work on these graphs is found in Sections 2.3 and 2.4, respectively. Finally, in Section 2.5 we will discuss m-Salem graphs further.

2.1 Salem graphs: A new perspective

A way of thinking about Salem graphs is that, in a manner of speaking, they are spectrally very similar to the cyclotomic graphs (in Figure 1.1). By this we mean that almost all of their eigenvalues are in the interval $[-2, 2]$, except for one in the non-bipartite case or two in the bipartite case. Interlacing (Theorem 1.3.9) tells us that every graph contains a cyclotomic graph as an induced subgraph, so a natural question is to ask is what is the smallest number of vertices that need to be removed to induce such a graph. We therefore make the following definition:

Definition 2.1.1. *A connected Salem graph G is called an m-*Salem graph *if m is minimal such that there exists a set of m vertices $v_1, \ldots, v_m$ for which the induced graph $G \setminus \{v_1, \ldots, v_m\}$ is cyclotomic. If an m-Salem graph G is a trivial Salem graph, then naturally we refer to G as a trivial m-Salem graph.*

To show that this is an interesting definition we will show that there are m-Salem graphs for every m. We firstly make the simple remark that the complete graph K_n is a trivial $(n-3)$-Salem graph (for each $n \geq 4$): we know from Theorem 1.3.8 that the eigenvalues of K_n are $n-1^{(1)}$ and $-1^{(n-1)}$ and deleting any $n-3$ vertices leaves K_3, which is cyclotomic. However, these are all trivial Salem graphs, so it would be nice to see that there are *non-trivial* m-Salem graphs for every m. Proposition 2.1.3 below proves exactly that. We will use the following result of Cvetković, recalling that $G \triangledown H$ denotes the join of the graphs G and H.

Theorem 2.1.2 (see [14] or [18], Theorem 2.7)**.** *The characteristic polynomial of the join of two graphs G_1 and G_2 (on n_1 and n_2 vertices, respectively) is given by*

$$\begin{aligned} \chi_{G_1 \triangledown G_2}(x) \;\; = \;\; & (-1)^{n_2} \chi_{G_1}(x) \chi_{\overline{G}_2}(-x-1) \\ & + (-1)^{n_1} \chi_{G_2}(x) \chi_{\overline{G}_1}(-x-1) \\ & - (-1)^{n_1+n_2} \chi_{\overline{G}_1}(-x-1) \chi_{\overline{G}_2}(-x-1) . \end{aligned}$$

There are a number of ways of proving the proposition below and the method here uses a variety of the results from Section 1.3, allowing us to get more familiar with them.

Proposition 2.1.3. *Let $G = (K_{n-1} \cup K_1) \triangledown K_1$, a complete graph on n vertices with a single vertex pendent path attached to one of its vertices. Then for each $n \geq 4$, G is a non-trivial $(n-3)$-Salem graph.*

Proof. Since G is non-bipartite, we know from Definition 1.2.1 that we need $\lambda_n \geq -2$, $\lambda_2 \leq 2$ and $\lambda_1 > 2$, but also $\lambda_1 \notin \mathbb{Z}$. Interlacing G with K_n tells us that the other $n-2$ eigenvalues here must all be equal to -1, as can be seen in Figure 2.1 below.

λ_1	λ_2	-1	$\cdots$	-1	λ_n
$n-1$	-1	$\cdots$	-1	-1	

Figure 2.1: Interlacing of the spectra of $G = (K_{n-1} \cup K_1) \triangledown K_1$ and K_n.

We can partition the edges easily into two cliques (a K_n and a K_2), so Theorem 1.3.2 tells us that G is a line graph and we have $\lambda_n \geq -2$ by Theorem 1.3.5. Let the vertex on our pendent path on G be called x. Next, take a new vertex y ($y \notin V(G)$) and attach it by a pendent edge to x and call this graph H (H is a complete graph on n vertices with a two vertex pendent path attached to one of its vertices). Lemma 1.3.7(ii) tells us that $\lambda_1(H) \leq \Delta(H) = n$ and since $H \setminus y = G$, we have $\lambda_1(G) < \lambda_1(H)$ by Theorem 1.3.10. Furthermore, $G \setminus x = K_n$, so $\lambda_1(K_n) < \lambda_1(G)$ and we know very well that $\lambda_1(K_n) = n - 1$. Bringing these facts together we get

$$n - 1 = \lambda_1(K_n) < \lambda_1(G) < \lambda_1(H) \leq n$$

which gives $\lambda_1(G) \notin \mathbb{Z}$, as it is trapped strictly between two consecutive integers. Since $n \geq 4$ we also have that $\lambda_1 > 2$, as required.

The final step is to show that $\lambda_2 \leq 2$. To do this we will calculate the characteristic polynomial $\chi_G(x)$. Since $G = (K_{n-1} \cup K_1) \triangledown K_1$ we can use Theorem 2.1.2, with $G_1 = K_{n-1} \cup K_1$ and $G_2 = K_1$. We then have $\overline{G}_1 = K_{1,n-1}$ and $\overline{G}_2 = K_1$ and the characteristic polynomials

$$\begin{array}{rclrcl} \chi_{G_1}(x) &=& (x-n+2)\,x\,(x+1)^{n-2}, & \qquad \chi_{G_2}(x) &=& x, \\ \chi_{\overline{G}_1}(x) &=& \left(x^2-n+1\right)x^{n-2}, & \qquad \chi_{\overline{G}_2}(x) &=& x. \end{array}$$

Note that there are a number of different ways of calculating the characteristic polynomial of the star $\overline{G}_1$, for example by observing that $K_{1,n-1} = ((n-1)\,K_1) \triangledown K_1$ and using Theorem 2.1.2. Using the expression in Theorem 2.1.2 with the characteristic polynomials above we find that

$$\chi_G(x) = \left(x^3 - (n-2)\,x - nx + n - 2\right)(x+1)^{n-2}.$$

Finally, we note that $\chi_G(0) > 0$ and $\chi_G(2) < 0$ for $n \geq 4$ meaning there is a root between 0 and 2. The only eigenvalue unaccounted for is λ_2 so we must have $\lambda_2 \leq 2$. Hence G is a non-trivial $(n-3)$-Salem graph as required. $\square$

We will use the idea of trapping the index strictly between two consecutive integers to show non-triviality again in Chapter 4. We will also return to the idea of studying the characteristic polynomial to tell us things about the spectrum in Chapter 5. Different methods will be used there to calculate the characteristic polynomial than here; both methods are interesting and allow us to use a greater variety of results about the spectrum. In Corollary 3.2.2 in the next chapter we will in fact strengthen this result to there being *infinitely* many non-trivial m-Salem graphs for each m. The proof uses a result that is much more powerful. For now we can content ourselves in the knowledge that there are interesting m-Salem graphs for each m.

Taking m to be minimal means that we can can uniquely partition the set of all Salem graphs into m-Salem graphs, for $m = 1, 2, \ldots$. Furthermore, interlacing gives us the following lemma as an immediate consequence of the definition.

Lemma 2.1.4. *A connected subgraph of an m-Salem graph is either cyclotomic or is m'-Salem for some $m' \leq m$. For every m-Salem graph G, there is a chain of induced subgraphs $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m = G$ such that each G_i is i-Salem and G_0 is cyclotomic.*

It is then easy to see, for example, that all 4-Salem graphs contain a 3-Salem graph as an induced subgraph. A more important observation however, is that *all* Salem graphs contain a 1-Salem graph as an induced subgraph. In proving what these 1-Salem graphs are, as we will, we can provide a more detailed description of the necessary substructure of a Salem graph than has been known before. Theorem 2.1.5 below collects the results of the following three sections in which we classify all the 1-Salem graphs.

Theorem 2.1.5. *(i) All bipartite 1-Salem graphs are described by Theorem 2.2.1.*

(ii) Every 1-Salem graph that is a generalised line graph is described by Theorem 2.3.2; there are 25 infinite families and 6 sporadic examples.

(iii) There are 377 non-bipartite 1-Salem graphs that are not generalised line graphs: see Theorem 2.4.1.

Clearly all graphs are either bipartite or non-bipartite, and we have uniquely partitioned the non-bipartite graphs further by considering generalised line graphs and the exceptional graphs separately; Theorem 2.1.5 covers all possible types of graph and hence gives a complete classification.

2.2 All 1-Salem bipartite graphs

We now begin our proof of Theorem 2.1.5, starting with part (i); the bipartite 1-Salem graphs. This result is essentially the same as Theorem 8 in [41], but now using the language of m-Salem graphs.

Theorem 2.2.1 (see [41])**.** *Let $H_1, \ldots, H_s$ be a finite set of connected cyclotomic bipartite graphs (so excluding odd cycles from the connected cyclotomic graphs described in Lemma 1.1.2), and for each i ($i = 1, \ldots, s$) let S_i be a non-empty subset of the vertices of H_i such that all the vertices of S_i fall in the same subset of the bipartition of H_i. Form the graph G by taking the union of all the H_i along with a new vertex v, and with edges joining v to each vertex in each S_i. Then*

(i) *unless G is cyclotomic, it is* 1*-Salem;*

(ii) *all connected bipartite* 1*-Salem graphs arise in this way.*

Proof. The second part is clear: if G is a bipartite 1-Salem graph, then there exists a vertex v whose deletion leaves a cyclotomic induced subgraph, and the components of this give the H_i. The first part is a consequence of interlacing: G has at most one eigenvalue greater than 2, and being bipartite we are done. □

This description is fairly general, but so are the graphs that arise from it. An example of this construction with $s = 3$ is given below in Figure 2.2 below. Of course there are many different graphs that can be formed using just these three graphs H_i alone by changing which vertices are in the subsets S_i, for $i = 1, 2, 3$.

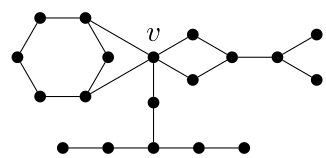


Figure 2.2: An example of a graph from Theorem 2.2.1. Here $H_1 = C_6$, $H_2 = \tilde{D}_5$ and $H_3 = E_6$.

We can be more explicit and describe precisely which combinations of H_i and S_i result in G being cyclotomic, simply by running through the possibilities for cyclotomic G given by Lemma 1.1.2. Table 2.1 below shows exactly which maximal cyclotomic graphs can arise for each choice of H_i ($i = 1, \dots, s$); for non-maximal ones we simply take the appropriate subgraphs of the H_i. The subsets S_i that give the graphs G are easy to spot. As the degree of v is at least s, we see that no cyclotomic graphs arise when $s \geq 5$.

2.3 All 1-Salem generalised line graphs

We now turn our attention to the non-bipartite 1-Salem graphs, and more specifically, to those that are generalised line graphs. In this case we are able to explicitly describe all of the graphs that appear here. We use the so-called "HAM Lemma" of McKee and Smyth, although calling it a lemma is perhaps underplaying its importance in this section.

Lemma 2.3.1 ([41], Proposition 3.2)**.** *Let G be a connected graph with $\lambda_1 > 2$ and $\lambda_2 \leq 2$, then the vertices V of G can be partitioned as $V = M \cup A \cup H$ where*

s	$H_1, \dots, H_s$	G	s	$H_1, \dots, H_s$	G
1	$H_1 = E_6$	$\tilde{E}_6$	2	$H_1 = K_1,\ H_2 = P_5$	$\tilde{E}_6$
	$H_1 = P_7$	$\tilde{E}_7$		$H_1 = K_1,\ H_2 = D_6$	$\tilde{E}_7$
	$H_1 = E_7$	$\tilde{E}_7$		$H_1 = K_2,\ H_2 = P_5$	$\tilde{E}_7$
	$H_1 = P_8$	$\tilde{E}_8$		$H_1 = K_1,\ H_2 = P_7$	$\tilde{E}_8$
	$H_1 = D_8$	$\tilde{E}_8$		$H_1 = H_2 = P_4$	$\tilde{E}_8$
	$H_1 = E_8$	$\tilde{E}_8$		$H_1 = P_3,\ H_2 = D_5$	$\tilde{E}_8$
	$H_1 = P_n$	$\tilde{A}_n$		$H_1 = P_2,\ H_2 = E_6$	$\tilde{E}_8$
	$H_1 = D_n$	$\tilde{D}_n$		$H_1 = K_1,\ H_2 = E_7$	$\tilde{E}_8$
				$H_1 = D_{n_1},\ H_2 = D_{n_2}$	$\tilde{D}_n$
3	$H_1 = H_2 = H_3 = K_2$	$\tilde{E}_6$	4	$H_1 = H_2 = H_3 = H_4 = K_1$	$\tilde{D}_4$
	$H_1 = K_1,\ H_2 = H_3 = P_3$	$\tilde{E}_7$			
	$H_1 = K_1,\ H_2 = K_2,\ H_3 = P_5$	$\tilde{E}_8$			
	$H_1 = H_2 = K_1,\ H_3 = D_{n-2}$	$\tilde{D}_n$			

Table 2.1: The cyclotomic graphs that may arise in Theorem 2.2.1, arranged by the number of components s.

(i) *the induced subgraph $G|_M$ (by which we mean $G \setminus \{A, H\}$) is minimal subject to $\lambda_1(G|_M) > 2$;*

(ii) *the set A consists of all vertices of $G|_{A\cup H}$ adjacent to some vertex of M;*

(iii) *the induced subgraph $G|_H$ is cyclotomic.*

This result and the structure of generalised line graphs in Theorem 1.3.2 allow us to classify the 1-Salem generalized line graphs quite easily. A basic summary of the proof of our classification in Theorem 2.3.2 is that we grow the graphs starting with the vertices in the minimal graphs M, then include the adjacent vertices in $M \cup A$, then the cyclotomic parts in $M \cup A \cup H$.

Theorem 2.3.2. *The 1-Salem generalized line graphs are precisely the non-cyclotomic subgraphs of the graphs in Figures A.1, A.2 and A.3.*

For convenience, these graphs can be found in Appendix A.2 but we shall make a few quick notational points here: as is traditional, a dashed edge indicates a path between the endpoints of the dashed edge, having an arbitrary number of vertices (perhaps even none, or perhaps with a lower bound shown); the parameter attached to a dashed edge gives the number of vertices on this path. In Figures A.1 and A.2 dotted edges and vertices are used to indicate edges and vertices that may or may not be there. Here the dotted edges and vertices always form a **snake's tongue** shape, and we will use the notation $\hat{a}$ to indicate a path of length a (i.e., having a edges) with two extra vertices in the shape of a snake's tongue on the loose end. Thus, for example, $G_{10}(1, 1)$ and $G_{10}(1, \hat{1})$ are shown in Figure 2.3. The 1-Salem generalised line graphs of Theorem 2.3.2 are presented as 25 infinite families and six sporadic graphs, but there are in fact 60 non-isomorphic infinite families when we consider all the possible options of paths with snake's tongues attached.

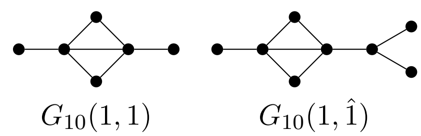


Figure 2.3: An illustration of the hat convention.

Also, as a simple corollary to Theorem 2.3.2 we can easily note which of the graphs in Figures A.1–A.3 are line graphs (rather than generalised line graphs) based on the characterization in Theorem 1.3.2. These are the graphs where the edges can be partitioned into cliques rather than GCPs and there are 12 infinite families and 3 sporadic graphs.

We will establish a few basic points before starting to prove the theorem over the next three subsections, one for each of the parts of Lemma 2.3.1.

In [41, Theorem 3.4] McKee and Smyth observed a construction for non-bipartite Salem graphs using the line graphs. The following trivial extension of that result can be used to show that all the graphs in Figures A.1–A.3 are Salem graphs.

Lemma 2.3.3. *Suppose that G is a non-cyclotomic non-bipartite graph containing a vertex v such that the induced subgraph $G \setminus v$ is cyclotomic. Also suppose that G is in the family of graphs with least eigenvalue greater than -2, then G is a Salem graph.*

To apply Lemma 2.3.3 to the graphs in Figures A.1–A.3, we note that all the graphs in the figures are generalised line graphs (using the characterisation in Theorem 1.3.2, for example), and in each case one readily spots a vertex v whose deletion leaves a cyclotomic induced subgraph.

We also note a simple lemma that will be referred to a number of times as we prove Theorem 2.3.2.

Lemma 2.3.4. *A 1-Salem graph G does not contain an induced K_5. Also, if it contains an induced K_4, then only one of the four vertices in that K_4 may be attached to another vertex in G and the vertex we remove to make G cyclotomic must be this distinguished vertex in K_4.*

Proof. The first sentence is clear, since if we remove any one of the vertices of K_5 we obtain a K_4, which is not cyclotomic. For the second sentence, removing a vertex v of K_4 leaves a K_3 so if any of the other vertices of the K_4 were attached to any vertices of G, so would the K_3 be after removing v, and no connected supergraph of K_3 is cyclotomic. □

2.3.1 M — the minimal graphs

We begin by considering what our minimal graphs may look like.

Proposition 2.3.5. *All Salem generalized line graphs must contain a K_3.*

Proof. We know from Theorem 1.3.2 that generalised line graphs are built from GCPs and it is easy to see that the only GCPs that do not contain a K_3 are

$$\begin{aligned} GCP\,(1,0) &= K_1 \\ GCP\,(2,0) &= K_2 \\ GCP\,(2,1) &= 2K_1 \\ GCP\,(3,1) &= P_3 \\ GCP\,(4,2) &= C_4 \end{aligned}$$

(if this is not clear, then we note that it can also be observed as a simple corollary of Lemma 5.1.1). These graphs are certainly all cyclotomic and, moreover, all the generalised line graphs that can be made using them (observing the rules in Theorem 1.3.2) will be subgraphs of either $\tilde{A}_n$ or $\tilde{D}_n$. However, by definition Salem graphs are non-cyclotomic, so a Salem generalised line graph must include at least one GCP that contains a K_3. □

This result was also proved by the author by using three of the minimal forbidden subgraphs mentioned in Theorem 1.3.4 and by considering odd cycles of length greater than 3 (see Proposition 15 of [13]), and again by thinking about how a four vertex subgraph of a generalised line graph may look, but the above proof is the shortest and clearest. As a corollary to this we get that the minimal graphs we are after must be the three ways of attaching a single vertex to a K_3.

Corollary 2.3.6. *The minimal graphs with respect to the property of being a Salem generalized line graph are the three graphs in Figure 2.4.*

We now know what our minimal graphs look like but before proceeding we make one more observation. For $G \setminus v$ to be cyclotomic we must have $v \in M$; that is, the

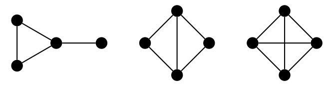


Figure 2.4: The three minimal graphs in Corollary 2.3.6.

vertex we are removing to induce a cyclotomic graph must be one of the vertices of the minimal graph M. Certainly, to make the minimal graphs cyclotomic we must remove one of their vertices. Further still, no matter what vertices and edges we add to our minimal graphs we will still need to remove one of their vertices to induce a cyclotomic graph. The consequence of this is that the vertex we remove to induce a cyclotomic graph cannot be in A or H.

2.3.2 A — the adjacent vertices

Growing our graph from $G|_M$ to $G|_{M\cup A}$ is the most difficult part, however the restriction to 1-Salem generalized line graphs reduces this to a finite search, after some work. By Theorem 1.3.2 we have a highly-constrained structure: we cannot simply add vertices and edges anywhere. First let us consider the ways we can partition M_1, M_2 and M_3 into GCPs; this will reveal where we can add vertices. The generalised line graph M_1 can be seen uniquely as a K_3 and a K_2; it then has three vertices of maximal degree that are not already in two GCPs. And M_3 can be seen uniquely as a K_4, but by Lemma 2.3.4 we know that we can attach further vertices to only one of its original vertices. The graph M_2, however, is one of the seven graphs that has two non-isomorphic root multigraphs (see Lemma 1.3.3) so we need to consider both versions. Let $M_{2,1}$ be M_2 partitioned as a K_3 with two K_2's attached and both joined at their other ends; here we only have one vertex that is of maximal degree and not already in two GCPs. Let $M_{2,2}$ be M_2 partitioned as a $GCP\,(4,1)$ where we then have two vertices of maximal degree.

The set of vertices A are those in the graph $G|_{A\cup H}$ that are adjacent to a vertex in M. By Theorem 1.3.2, to grow from $G|_M$ to $G|_{M\cup A}$ we can:

- expand a GCP to a larger one that contains the original one (taking care of the degrees of vertices);
- attach a new GCP to a vertex of maximal degree that is only in one GCP (attaching only at vertices of maximal degree);
- do both of the above.

Once we have done this we can then also add an edge between any two vertices of A of maximal degree in their GCPs (in effect, adding a K_2) or take two maximal degree vertices of A that are in separate GCPs and only in one GCP each and "merge" them together (connect two disconnected GCPs by making them share an available maximal degree vertex).

Lemma 2.3.4 helps here as we know that we need not consider any GCPs that contain a K_5. In fact, by studying the GCPs G that have this property and looking at the induced graph $G \setminus v$ for each $v \in V(G)$ we find the fairly short list of GCPs that we can attach or expand to in Table 2.2 below.

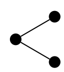
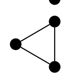
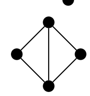
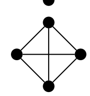
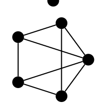
$GCP\,(2,0) = K_2$		two vertices of maximal degree
$GCP\,(3,1)$		one vertex of maximal degree
$GCP\,(3,0) = K_3$		three vertices of maximal degree
$GCP\,(4,1)$		two vertices of maximal degree
$GCP\,(4,0) = K_4$		four vertices of maximal degree
$GCP\,(5,2)$		one vertex of maximal degree

Table 2.2: The GCPs that we can attach or expand to in A.

Note that $GCP\,(4,2) = C_4$ is not included; when partitioned as $GCP\,(4,2)$ it has no vertices of maximal degree (recall from the definition of a GCP that this means it has no vertices of degree $n-1$), so cannot be attached to anything and when partitioned as four K_2's it has no vertices that are not already in two GCPs. Also, $GCP\,(3,1)$ and $GCP\,(5,2)$ are two of the seven graphs in Lemma 1.3.3 that have non-isomorphic root multigraphs. However, when $GCP\,(5,2)$ is partitioned as two K_3's and two K_2's it has no vertices that are not already in two GCPs and when $GCP\,(3,1)$ is partitioned as two K_2's one of its vertices will not be in A.

The process of going from $G|_M$ to $G|_{M\cup A}$ is then a finite one; we only have so many GCPs in the minimal graphs M_1, $M_{2,1}$, $M_{2,2}$ and M_3 to expand and only so many ways to attach these six GCPs to them. We also have a small number of cases where we can add in extra edges between vertices of A or merge them. In working through all these combinations we discard a number of graphs that are not 1-Salem as they require more than one vertex to be removed to make them cyclotomic.

The list of 1-Salem generalised line graphs $G|_{M\cup A}$ has been omitted for reasons of space. There are 224 that are distinct as graphs, although many of these graphs can arise in more than one way as $G|_{M\cup A}$. The largest has 11 vertices. Initially these were all found by hand by the author; the list was then checked by James McKee using a computer search.

2.3.3 H — the cyclotomic parts

We now look at the set H in Lemma 2.3.1. We can reduce our choices by observing that the only cyclotomic graphs that are also generalized line graphs are $\tilde{D}_n$ and $\tilde{A}_n = C_{n+1}$. However, we will show that $G|_H$ cannot contain cycles and can only contain subgraphs of $\tilde{D}_n$. For $n \geq 5$ we note that C_n can be partitioned uniquely as n lots of K_2 with each vertex in two GCPs, so we cannot simply attach vertices to it. The other option is then to expand a GCP to a larger one so that we now have vertices only in one GCP to attach to other things. The smallest case is to expand one of the n K_2's to a K_3. Clearly, this new vertex must be in A rather than H as the graph is no longer cyclotomic. However, in order to make the graph cyclotomic we must remove a vertex but all of the vertices are in A or H and we know that the vertex we remove must be in M. Cycles of length 3 can be seen as both three K_2's or one K_3 but by similar reasoning on the choice of vertex we are removing, we can exclude this case too. A similar argument holds again for cycles of length 4 for both ways of partitioning its edges.

The graph $\tilde{D}_n$ can be uniquely partitioned as $(n-4)$ K_2's with a $GCP\,(2,0)$ (or snake's tongue) at either end. However, in this graph each vertex of maximal degree within its GCP is already in two GCPs. If we remove one or both of the snake's tongues we are left with graphs we can work with – a path or a path with a snake's tongue on the end – each with at least one vertex of maximal degree that is only in one GCP. With this in mind we can think of $G|_H$ simply consisting of paths of any length, possibly with a snake's tongue on one end.

The final step in growing these graphs is to go from $G|_{M\cup A}$ to $G|_{M\cup A\cup H} = G$. To any vertices in A of maximal degree and only in one GCP we can attach a single path of arbitrary length (remembering that Lemma 2.3.4 tells us that we can only attach things to one vertex of a K_4). If we so choose, we can join any two of these pendent paths together (equivalently, attach a path of arbitrary length to two different elements of A of maximal degree that are each only in one GCP). Furthermore, on the end of any pendent paths we can include a snake's tongue.

These graphs are then precisely the graphs in Figures A.1–A.3, and Lemma 2.3.3 tells us that they are all 1-Salem, completing the proof of Theorem 2.3.2.

2.4 All 1-Salem exceptional graphs

To prove the final part of Theorem 2.1.5 we need to consider the exceptional graphs. Theorem 1.3.6 tells us that these make up the remaining graphs with $\lambda_n \geq -2$ and are found in the root system of E_8. We also know that there is only a finite number of these graphs, however that number is quite large, certainly too large to search for the 1-Salem graphs by hand. In order to complete the classification of the 1-Salem graphs and make the result in the paper [31] more substantial, James McKee performed a computer search for the remaining graphs. A much more detailed account of the process is given in [31]. By looking for (not necessarily connected) cyclotomic exceptional graphs he was able to find that they had at most 10 vertices, meaning that the 1-Salem graphs would have no more than 11 vertices. He then went on to find all the 1-Salem graphs. This result was confirmed by Jonathan Cooley, who has an unpublished list of all Salem graphs having up to 12 vertices. Their findings are summarised by Theorem 2.4.1 below.

Theorem 2.4.1. *There are 377 non-bipartite connected 1-Salem graphs that are not generalised line graphs. The numbers of vertices for these graphs range between 6 and 11; the number of graphs for each of these numbers of vertices is shown in Table 2.3.*

Number of vertices	6	7	8	9	10	11
Number of graphs	10	43	111	153	58	2

Table 2.3: The number of non-bipartite connected 1-Salem graphs that are not generalised line graphs by the number of vertices.

We can see from Table 2.3 that the largest examples of 1-Salem exceptional graphs have 11 vertices; these two graphs are shown in Figure 2.5, and the vertex one must remove to make them cyclotomic is readily spotted.

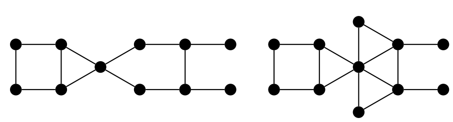


Figure 2.5: The two 11-vertex 1-Salem graphs that are not generalised line graphs.

This completes our proof of Theorem 2.1.5, and hence all the 1-Salem graphs are classified.

2.5 Some remarks on m-Salem graphs

We will now discuss some results that arise from the work we have just done, and consider the possibility of m-Salem graphs for $m \geq 2$. The first is an easy corollary of [41, Theorem 7.2] about Salem trees, using the new language of m-Salem graphs. The result actually goes into much more detail about the structure of the trees, showing that they are of one of two types. As we are proving a weaker result, we will give an alternative proof for the proposition below.

Proposition 2.5.1 ([41], Theorem 7.2)**.** *All Salem trees are either* 1*-Salem or* 2*-Salem.*

Proof. Assume we had a 3-Salem tree G, and consider the three vertices we need to remove to induce a cyclotomic graph. Since trees cannot contain cycles, the smallest subtree containing these three vertices will either be topologically equivalent to a P_3, or a $K_{1,3}$ (along with one more vertex). In the case that the three vertices are topologically equivalent to a path, removing the vertex that sits in the walk between the other two disconnects the graph into two 1-Salem graphs. In the case that the three vertices are topologically equivalent to a $K_{1,3}$, removing the central vertex of this star disconnects the graph into three 1-Salem graphs.

Therefore, in either case, when this vertex is removed the spectrum of G will interlace a (disconnected) subgraph with either two or three eigenvalues strictly greater than 2, a contradiction to G being Salem. Finally, Lemma 2.1.4 tells us that there are no m-Salem trees for any $m \geq 3$ either. $\square$

Another observation is that all 1-Salem graphs are planar (although those in Theorem 2.3.2 are not drawn in a such a way), but m-Salem graphs for $m \geq 2$ may not be. We know from Theorem 1.3.12 that a graph G is not planar if and only if it contains an induced subgraph topologically equivalent to either K_5 or $K_{3,3}$. Removing one vertex from a graph topologically equivalent to a K_5 leaves a graph with at least four vertices of degree 3 while the connected cyclotomic graphs we hope to induce have at most two such vertices. Removing a vertex from a graph topologically equivalent to a $K_{3,3}$ leaves a connected graph with more than one cycle, which again cannot be cyclotomic. The graphs in Proposition 2.1.3 provide (non-trivial) examples of m-Salem graphs that are not planar.

We proved in Proposition 2.3.5 that all Salem generalized line graphs must contain a K_3 as a way to find the minimal graphs M for the HAM Lemma. Some graph theorists are interested in triangle-free graphs and we can use this result to look at the triangle-free non-bipartite Salem graphs. Let G be a triangle-free non-bipartite Salem graph, then by Theorem 1.3.6 and Proposition 2.3.5 G must be an exceptional graph.

This of course means that there are only finitely many such graphs. Again Jonathan Cooley was able to search for these graphs computationally and find the complete list, establishing the following Proposition.

Proposition 2.5.2. *There are exactly* 25 *non-bipartite triangle-free Salem graphs. All have at most* 10 *vertices. The numbers of each size are given in Table 2.4.*

Number of vertices	6	7	8	9	10
Number of graphs	1	4	7	8	5

Table 2.4: The number of non-bipartite triangle-free Salem graphs by the number of vertices.

The 10-vertex examples include the Petersen graph and this is the only one that is a trivial Salem graph (moreover it is an integral graph). Furthermore, all but one of the 25 graphs are planar.

A natural question to ask at this point is can we continue and grow the 2-Salem graphs? The second sentence of Lemma 2.1.4 certainly gives rise to a naive growing algorithm for constructing all small m-Salem graphs: we simply take the 1-Salem graphs and consider all the possible places to attach a new vertex. The problem here, however, is that the graphs we have grown will not necessarily be Salem; they may have two eigenvalues greater than 2. We knew our 1-Salem graphs were Salem by Lemma 2.3.3, but we cannot extend that further.

We can extend Lemma 2.3.4 to 2-Salem graphs by observing that a 2-Salem graph cannot contain an induced K_6 and if it contains a K_5 then at most two of its vertices may have vertices from another GCP adjacent to them. In fact, if G is an m-Salem graph then it may not contain an induced K_{m+4} and if it contains an induced K_{m+3}, then only m of the vertices in that K_{m+3} may be adjacent to vertices from another GCP.

So whilst an m-Salem graph cannot contain an induced K_{m+4}, it may contain a $GCP\,(m+4, b)$ (for $1 \leq b \leq \lfloor m/2 \rfloor + 2$). We can use this to calculate the largest possible degree of a vertex in an m-Salem generalised line graph, knowing that each vertex can be in at most two GCPs. The graphs $GCP\,(m+4, b)$ will have $\Delta = m+3$ and so a graph containing two of them sharing the same vertex will have $\Delta = 2\,(m+3)$. Lemma 1.3.7(ii) then gives a simple bound on the index of any m-Salem generalised line graph to be $2\,(m+3)$. Graph G_{31} is an example of such a graph and has $\Delta = 8$ but $\lambda_1 = 4$, showing that this bound need not be very sharp.

Chapter 3

Constructions using the Courant-Weyl inequalities

We are looking to construct families of Salem graphs, and in this chapter we will show that the Courant-Weyl inequalities provide a fruitful picking ground of possibilities. We use them here in a novel way, along with many other facts and results about graph spectra.

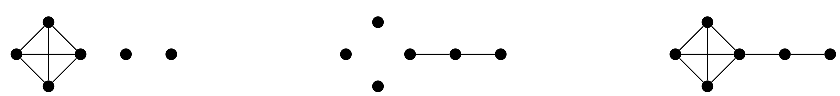
3.1 Introduction

In this first section we will define a method for adding graphs that we will use throughout the chapter. After that we will consider how this can be used to construct Salem graphs, before moving on to look at these constructions in greater detail in the sections that follow.

3.1.1 A different way of adding graphs

Consider a graph G with vertices V, edges $E(G)$ and a fixed labelling of the vertices. Now consider another graph H on the same vertex set V with the same labelling but edges $E(H)$ where $E(G) \cap E(H) = \emptyset$; that is, the graphs have no edges in common. When we add the two adjacency matrices $A(G)$ and $A(H)$, the resulting matrix is the adjacency matrix of a new graph; the fixed labelling of the vertices and empty intersection of the edge sets means that this new graph has no multiple edges so it is a simple graph (clearly there can be no loops). In a slight abuse of common notation, we define $G + H$ to be the graph with adjacency matrix $A(G + H) = A(G) + A(H)$. A way of thinking about this is that we are taking a graph and placing another graph (with an independent edge set) on top of it, almost as if the graphs were Lego bricks and the vertices were the studs on top; we are constricting the way we can put one on top of the other (of course the analogy ends when trying to find a Lego brick the exact length of our edges!)

Clearly, if $H = \overline{G}$ then the new graph $G + H$ will be K_n, where $|V| = n$. We can still add graphs together in this way even if the two graphs G and H do not share the exact same vertices by adding in the appropriate amount of isolated vertices to each graph. This idea will be used frequently in this section so we will consider the small example in Figure 3.1. Take the graphs K_4 and P_3 and say we wish to form a graph that is a K_4 with a 2-vertex pendent path attached to one of its vertices. We then consider a fixed embedding of 6 vertices and on these vertices take $G = K_4 \cup 2K_1$ and $H = P_3 \cup 3K_1$ in such a way that $G + H$ is the desired graph. Lemma 1.3.7(i) tells us that the spectrum of the union of disjoint graphs is simply the union of the spectra, so adding extra isolated vertices like this just adds 0's to the spectrum.



$G = K_4 \cup 2K_1$ $\qquad$ $H = P_3 \cup 3K_1$ $\qquad$ $G + H$

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A(H) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad A(G+H) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Figure 3.1: An example of the way we are adding graphs with a fixed vertex embedding, with the corresponding adjacency matrices below.

This is clearly different to other graph products, as it is subject to the exact positioning of the vertices and the edges. The construction is similar to that of the "rooted product" mentioned by Godsil and McKay in [27]; take our graph G to be the labeled graph on n vertices and our rooted graphs to be the s non-trivial components of H we are attaching along with $n - s$ isolated vertices. In this case we do not need the fixed embedding of all the vertices (or G to have extra isolated vertices) but the polynomial they find does not allow us to bound the eigenvalues as easily as we will do here.

3.1.2 The seven ways of summing to 2

The following inequalities are known as the Courant-Weyl inequalities and are the focus of the work in this chapter, as they will provide a number of new ways to construct Salem graphs.

Theorem 3.1.1 (Courant-Weyl inequalities; see [49], Theorem 34.2.1)**.** *For two Hermitian $n \times n$ matrices A and B we have*

$$\begin{array}{rcll} \lambda_i(A+B) & \leq & \lambda_{i-j+1}(A) + \lambda_j(B) & (i \geq j), \\ \lambda_i(A+B) & \geq & \lambda_{i-j+n}(A) + \lambda_j(B) & (i \leq j). \end{array}$$

For two graphs G and H as before, let $A = A(G)$ and $B = A(H)$. As the adjacency matrix of a graph is Hermitian, the Courant-Weyl inequalities then allow us to bound certain eigenvalues of the graph $G + H$ by the eigenvalues of the graphs G and H; this is the key to finding Salem graphs here. We are interested in graphs with the second largest eigenvalue less than or equal to 2, so using the first inequality with $i = 2$ and $j = 1$ we get

$$\lambda_2(A+B) \leq \lambda_2(A) + \lambda_1(B),$$

which, under our new notation, is

$$\lambda_2(G+H) \leq \lambda_2(G) + \lambda_1(H). \tag{3.1}$$

We will refer back to this equation regularly throughout this chapter. Let us consider the example in Figure 3.1 again. The graph G has $\lambda_2(G) = 0$ and H has $\lambda_1(H) = 1.414$, so we get that $\lambda_2(G+H)$ must be bounded by the sum of these two, 1.414. By calculating the spectrum we see that in fact $\lambda_2(G+H) = 1.117$ (all the while remembering our table in Appendix A.1).

The Courant-Weyl inequalities have appeared numerous times in previous papers, but mostly with $A = A(G)$, $B = A\left(\overline{G}\right)$ giving $A + B = A(K_n)$. The simple spectrum of K_n (see Theorem 1.3.8) means that this gives many interesting bounds on the eigenvalues by looking at properties of the complement. We will also use the inequalities in this way. However, the author is unaware of any other examples of them being used to add disconnected graphs as we are here, a method that is also very effective for bounding eigenvalues.

The next step is to consider the ways we can bound the right hand side of Equation (3.1) by 2, so let us partition the different ways of adding two ordered numbers together to make 2. There are seven ways of doing this, and they are exhaustive.

(i)	$(0-\alpha)$	$+$	$(2+\alpha)$	for $\alpha > 0$
(ii)	0	$+$	2	
(iii)	$(0+\epsilon)$	$+$	$(2-\epsilon)$	for $\epsilon \in (0,1)$
(iv)	1	$+$	1	
(v)	$(1+\epsilon)$	$+$	$(1-\epsilon)$	for $\epsilon \in (0,1)$
(vi)	2	$+$	0	
(vii)	$(2+\alpha)$	$+$	$(0-\alpha)$	for $\alpha > 0$

If we take the left hand column to be an upper bound for $\lambda_2(G)$ and the right hand column to be an upper bound for $\lambda_1(H)$ then Equation 3.1 tells us that the new graph $G+H$ will have $\lambda_2(G+H) \leq 2$. We must take some care in how we bound these graphs; the partitions above all add up to 2, but we are happy provided the overall bound on $G+H$ is less than or equal to 2. Interlacing tells us that each time we increase the bound on the second largest eigenvalue of G we get a larger family of graphs that includes the ones bounded by the previous value. Similarly as we decrease the bound on the index of H we get a subset of the previous family.

We will now work through each of these options and, using some simple facts about the largest and second largest eigenvalue of a graph, discuss the possibility of constructing Salem graphs in such a way. At each step we will think about what the largest possible families of both G and H are, to ensure we have considered every possible outcome.

(i) $(0-\alpha)+(2+\alpha)$ for $\alpha > 0$: This corresponds to a G with $\lambda_2(G) < 0$ and $\lambda_1(H) \leq 2-\lambda_2(G)$. The only such G is the complete graph K_n which has $\lambda_2 = -1$ (see Theorem 1.3.8). However, this graph, by definition, has every possible edge so the only option for H is $H = nK_1$ and the resulting graph is just K_n again. (Of course, for $n > 2$, the complete graphs K_n are a family of trivial non-bipartite Salem graphs, but there are more exciting constructions to come.)

(ii) 0+2: It turns out that this choice of graphs G and H gives us the most general and interesting construction for Salem graphs. Here we have the graphs with $\lambda_2(G) \leq 0$ and $\lambda_1(H) \leq 2$. The graphs G are known to be the complete multipartite graphs $K_{n_1,\ldots,n_k} \cup mK_1$ where $\max\{n_i\} \geq 2$ and $m \geq 0$, along with the disconnected graphs $K_n \cup mK_1$ for $m \geq 1$ (by Theorem 1.3.8 and interlacing). The graphs H are the cyclotomic graphs in Figure 1.1. The resulting graphs will be explored much further in Section 3.2.

(iii) $(0+\epsilon)+(2-\epsilon)$ for $\epsilon \in (0,1)$: Here we get the graphs with $\lambda_2(G) < 1$ and H with $\lambda_1(H) \leq 2-\lambda_2(G)$, the non-maximal cyclotomics. This construction will be looked at in more detail in Section 3.4.

(iv) $1+1$: This will be the graphs G with $\lambda_2(G) \leq 1$ and H with $\lambda_1(H) \leq 1$. The only graphs H with this property are K_2 and K_1, so H must then be the disjoint union of s copies of K_2 and t copies of K_1. Some things are also known about the graphs G with $\lambda_2(G) \leq 1$ so we will also study this construction later, in Section 3.3.

(v) $(1+\epsilon)+(1-\epsilon)$ for $\epsilon \in (0,1)$: Our set H here must have $\lambda_1(H) < 1$. Any graph that contains at least one edge must contain an induced K_2, so has $\lambda_1 \geq 1$ by interlacing. We are then left with a graph with no edges, which is mK_1 and has $\lambda_1 = 0$ for all $m \geq 1$. Adding isolated vertices to a graph does not really give us anything new, so this construction produces no interesting new graphs.

(vi) $2+0$: As mentioned above, the only graph H with $\lambda_1(H) = 0$ is $H = mK_1$, so again we find no new graphs (not to mention that the graphs G with $\lambda_2(G) \leq 2$ are the ones we are seeking in the first place).

(vii) $(2+\alpha)+(0-\alpha)$ for $\alpha > 0$: Clearly, there are no graphs H with $\lambda_1(H) < 0$, since $\lambda_1(K_1) = 0$, so this construction yields no new graphs either. Alternatively, interlacing tells us that $\lambda_2(G+H)$ cannot be less than the second largest eigenvalue of any subgraph, so starting with $\lambda_2(G) > 2$ will not work either.

Before continuing, we shall make a quick definition that will make it much easier to discuss the graphs we will construct in the rest of this chapter.

Definition 3.1.2. *We will say a graph, or a family of graphs, is in the set $\mathcal{P}_{a+b}$ if it can be shown to be Salem using the Courant-Weyl inequalities as in Equation 3.1 with $\lambda_2(G) \leq a$ and $\lambda_1(H) \leq b$ for some $a, b \in \mathbb{R}$. This will not necessarily be unique as a certain graph may be in more than one set, depending on how we choose the graphs G and H.*

So to summarise the above information: there are three sets that we will study in the further sections: the graphs that arise from adding complete multipartite graphs and the cyclotomics in $\mathcal{P}_{0+2}$; small $\lambda_2(G)$ and the non-maximal cyclotomics in $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ (for some $0 < \epsilon < 1$); and the graphs in $\mathcal{P}_{1+1}$ from $\lambda_2(G) \leq 1$ and $H = sK_2 \cup tK_1$. We will study the sets in the order that generally produces the most interesting graphs first; that is, $\mathcal{P}_{0+2}$ first in Section 3.2, then $\mathcal{P}_{1+1}$ in Section 3.3 and finally $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ in Section 3.4. In each section we will consider bipartite graphs and non-bipartite graphs separately.

For bipartite graphs we need both graphs G and H to be bipartite as well as the graph $G+H$. Clearly, if a graph is non-bipartite then there are no ways we can add edges to it to make it bipartite. By the symmetry of eigenvalues in bipartite graphs (Theorem 1.1.1), having $\lambda_2(G+H) \leq 2$ forces $\lambda_{n-1}(G+H) \geq -2$, so provided the resulting graph $G+H$ is not cyclotomic, it must be Salem. Over the following three sections we will be able to classify every possible bipartite graph that can arise from this construction.

For non-bipartite graphs we will need to make sure the smallest eigenvalue is greater than or equal to -2. Using the options above given by the Courant-Weyl inequalities, the second largest eigenvalue will be bounded above by 2, so provided the resulting graph $G+H$ is not cyclotomic again, it will be Salem. Theorem 1.3.6 allows us to think of the family of graphs with $\lambda_n \geq -2$ as either generalised line graphs or the exceptional graphs. Certainly a computer could handle the task of finding which of the exceptional graphs are Salem, as computers are generally very good at dealing with large finite problems. A much more difficult and even lengthier task would be to try to find out for which a and b they are in $\mathcal{P}_{a+b}$ (perhaps by considering every decomposition into graphs G and H). However, the goal here is to classify as many Salem graphs as possible and so knowing this is probably not that interesting (at least not that interesting *to us, at this moment*). With that in mind we will forget about the finite problem of the exceptional graphs for the rest of this chapter and turn our attention to the infinite family of generalised line graphs. We will rely heavily on the structure of generalised line graphs given in Theorem 1.3.2 and often think of generalised line graphs in terms of the partitioning of the edges into GCPs. We will find many interesting infinite families of non-bipartite Salem graphs, however we will unfortunately not be able to say that we have classified all of them.

3.2 The graphs in the set $\mathcal{P}_{0+2}$

For $\mathcal{P}_{0+2}$ we need the graphs G with $\lambda_2 \leq 0$ which, as previously mentioned, are the complete multipartite graphs $K_{n_1,\ldots,n_k} \cup mK_1$ ($\max\{n_i\} \geq 2$, $m \geq 0$), the graphs $K_n \cup mK_1$ ($m \geq 0$), and no others. We can write this more concisely as the graphs $G = K_{n_1,\ldots,n_k} \cup mK_1$, remembering that any extra isolated vertices just add an extra 0 eigenvalue to the spectrum. We know precisely which graphs have $\lambda_2(H) \leq 2$ from Lemma 1.1.2, and they are the cyclotomic graphs in Figure 1.1. As mentioned before, to make bipartite Salem graphs we will only be able to use bipartite graphs G, but to make non-bipartite Salem graphs we will need to consider both bipartite and non-bipartite graphs for G.

3.2.1 Bipartite

Clearly $K_n \cup mK_1$ is non-bipartite for all $n \geq 3$ as it contains a K_3. Furthermore, the graphs $K_{n_1,\ldots,n_k}$ are non-bipartite for all $k \geq 3$ as for each k thereafter we can also induce a K_3. Since $K_2 = K_{1,1}$, for bipartite graphs we can drop the specification that $\max\{n_i\} \geq 2$ and conclude that G is a graph of the form $K_{n_1,n_2} \cup mK_1$.

We can immediately spot which of the cyclotomic graphs in Figure 1.1 can be used in H, because the only ones that are non-bipartite are the odd cycles C_{2n+1}. Therefore we take H to be the union of up to $|V(G)|$ disjoint copies of any of $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{D}_n$, C_{2n} or their subgraphs.

With these G and H, the resulting graphs $G + H$ in $\mathcal{P}_{0+2}$ are then anything that consists of a central graph K_{n_1,n_2} with bipartite cyclotomic graphs attached by sharing a vertex to any of its vertices. A graph from H may even be attached to more than one vertex of G provided $G+H$ remains bipartite. No vertex of G can be attached to more than one graph from H, as this would mean that the two graphs share a vertex in H and would no longer be cyclotomic (unless both are subgraphs of the same cyclotomic graph, but this is redundant as a construction); this is why we limited the number of graphs in H to be less than or equal to $|V(G)|$. This construction yields an infinite family of bipartite graphs in the set $\mathcal{P}_{0+2}$, as both G and H form infinite families of graphs themselves.

Clearly G here is as complete as it can be since we have included every bipartite complete multipartite graph. Similarly, H is as complete as it can be, as we have included every bipartite graph with $\lambda_1 \leq 2$. Furthermore, we have considered all possible ways of attaching them. Therefore this must be the most general possible construction for bipartite graphs with $\lambda_2(G) \leq 0$ and $\lambda_1(H) \leq 2$ in the Courant-Weyl inequality.

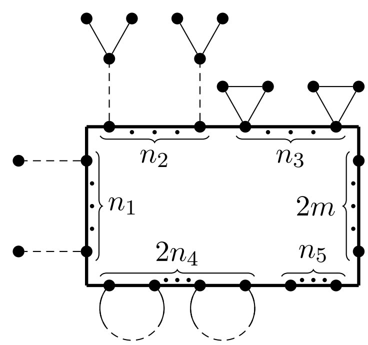
This is a very general description of these graphs, but it is the best we can do as they are a very general family of graphs. This is often the case for bipartite Salem graphs.

Certainly the star graphs, $K_{1,n}$, are bipartite graphs with the property that $\lambda_2 = 0$ and therefore part of this construction. An immediate corollary of this fact is that the bipartite 1-Salem graphs from Theorem 2.2.1 are also in $\mathcal{P}_{0+2}$. The relation here is simple: when we remove a vertex v the induced graph is just missing the vertex and the edges incident to it, say E_v, and this is nothing else but the star $K_{1,|E_v|}$. It is no surprise then that we will also see this for the non-bipartite 1-Salem graphs shortly. Figure 2.2 is clearly also an example of a bipartite graph in $\mathcal{P}_{0+2}$ where $G = K_{1,5} \cup 13K_1$ and $H = C_6 \cup \tilde{D}_5 \cup E_6 \cup K_1$.

3.2.2 Non-bipartite

In Theorem 3.2.1 and Proposition 3.2.3 we will provide two different constructions for non-bipartite Salem generalised line graphs in $\mathcal{P}_{0+2}$. Theorem 3.2.1 is particularly noteworthy, as it produces by far the most general family of non-bipartite Salem graphs so far and demonstrates quite how powerful the Courant-Weyl construction is.

Theorem 3.2.1. *For $0 \le m \le \lfloor n/2 \rfloor$, the graph $L_{GCP} = L_{GCP(n,m)}$ in Figure 3.2 and all of its subgraphs are either Salem or cyclotomic. It consists of a central $GCP\,(n,m)$ and attached to any one of its $n-2m$ vertices of maximal degree there may be a K_3 or a pendent path of any length; pairs of these paths may then be joined to each other (forming a cycle) or individually have a $GCP\,(3,1)$ (or snake's tongue) at the loose end. The only cyclotomic constructions of $L_{GCP(n,m)}$ arise if $(n,m) = (2,0)$ (unless two K_3's are attached), if $(n,m) = (3,1)$, or if $(n,m) = (3,0)$ and nothing is attached.*



$L_{GCP} = L_{GCP(n,m)}$

Figure 3.2: The generalised line graph L_{GCP} from Theorem 3.2.1. The box represents a $GCP\,(n,m)$ where $n = n_1 + n_2 + n_3 + 2n_4 + n_5 + 2m$ and $0 \le m \le \lfloor n/2 \rfloor$. In the GCP the $n-2m$ vertices are of degree $n-1$ and the $2m$ are of degree $n-2$.

Proof. It is easy to see that this graph is Salem using the Courant-Weyl construction. Let the graph in the proposition be called $G+H$ and let G be the aforementioned $GCP\,(n,m)$. We then let H be the remaining edges, which form K_3's, paths and paths with snake's tongues attached (any cycles are just paths attached at both ends). We must also include in both G and H the union of the appropriate number of isolated vertices so they have the correct total number of vertices for the addition.

It is easy to see that H is cyclotomic, as it consists of the disjoint union of $K_3 = C_3$'s and subgraphs of $\tilde{D}_n$, so $\lambda_1(H) \le 2$. Furthermore, we note that for $0 \le m \le \lfloor n/2 \rfloor$

$$GCP\,(n,m) = K_{\underbrace{1,\ldots,1}_{n-2m},\underbrace{2,\ldots,2}_{m}}$$

as each missing independent edge in a GCP is equivalent to a pair of non-adjacent vertices in such a multipartite graph. Since we know that complete multipartite graphs have $\lambda_2 = 0$, we know that our GCP does regardless of the choices of n and m.

Using the Courant-Weyl inequalities as in Equation 3.1, we then have the appropriate bound on the second largest eigenvalue. The graphs in H are clearly generalised line graphs; the K_3's are complete graphs, paths can be partitioned into K_2's and the snake's tongues are $GCP\,(3,1)$'s joined using the only vertex of maximal degree. As we have only attached one path or K_3 to each vertex of maximal degree from the GCP G, it is easy to see that the whole graph $G+H$ is a generalised line graph, ensuring that the least eigenvalue is greater than or equal to -2 (by Theorem 1.3.2). The exception mentioned in the Proposition ensures that the graph is not cyclotomic, so the largest eigenvalue is greater than 2. We have shown that this graph is in $\mathcal{P}_{0+2}$ and is therefore Salem. □

The next corollary strengthens Proposition 2.1.3, as promised earlier.

Corollary 3.2.2. *There are infinitely many non-trivial m-Salem graphs for each $m \geq 1$.*

Proof. Take a K_n and attach to one of its vertices a pendent path of any finite positive length, say P_s, and call this graph G. Clearly G is an example of the graph L_{GCP} from Theorem 3.2.1, so we know it must be Salem. As in the proof of Proposition 2.1.3, we can trap the index of G between strictly $n-1$ and n by considering the graph G', where we have instead attached the path P_{s+1}. Then, for every $s \geq 1$, we have an $(n-2)$-Salem graph. □

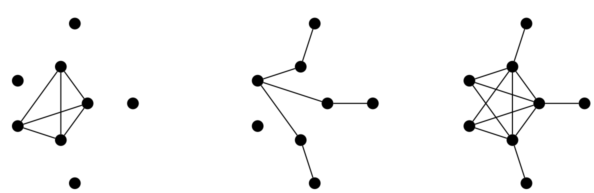
As with the bipartite case, we see that the non-bipartite 1-Salem graphs can also be found using the Courant-Weyl inequalities.

Proposition 3.2.3. *The non-bipartite 1-Salem graphs in Theorem 2.3.2 are in the set $\mathcal{P}_{0+2}$.*

Proof. Each graph in Theorem 2.3.2, $G+H$ say, contains a vertex v such that the induced graph $(G+H)\setminus v$ is cyclotomic. Let the number of edges incident to v be $|E_v|$ then we simply observe that $G+H$ can be constructed from $G = K_{1,|E_v|}$, along with the appropriate number of isolated vertices, and $H = (G+H)\setminus v \cup K_1$. □

A natural question to ask now is: have we considered every possible non-bipartite graph in the set $\mathcal{P}_{0+2}$? On the one hand, L_{GCP} is a very general family, but on the other, there are certainly graphs G and H that have not been looked at. Let us begin by exhaustively splitting the complete multipartite graphs into five groups, firstly by the number of parts k of $G_{n_1,\ldots,n_k}$, then by restricting some of the values of the n_i. Clearly when $k = 1$ we get the complete graphs; for now call this family of graphs X_1. Next let $k = 2$ and let X_2 be the family of complete multipartite graphs where one of the two n_i equals 1, and X_3 be those where both $n_1, n_2 > 1$. We know X_2 to be stars and the graphs in X_3 are the remaining complete bipartite graphs. Finally, let $k \geq 3$ and X_4 be the family of graphs where $n_i \in \{1, 2\}$ for all i and X_5 be the family where $n_i \geq 3$ for some i ($i = 1, \ldots, k$). As we saw in the proof of Theorem 3.2.1, the graphs in X_4 are GCPs.

We know that we have all the possible non-bipartite $\mathcal{P}_{0+2}$ graphs where G is from X_2 as that is precisely what we proved in Section 2.3. The others turn out to be more difficult. Theorem 3.2.1 certainly dealt with graphs from X_1 and X_4 but we did not include every cyclotomic graph in H, just those that are generalised line graphs. Figure 3.3 below shows that we can make generalised line graphs in $\mathcal{P}_{0+2}$ using some of the remaining cyclotomic graphs even though they are not generalised line graphs themselves. The difficulty with the graphs $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ seems to lie in their vertex of degree 3; in the example we have used it forms a vertex of degree 3 in a larger GCP. However, as there are only finitely graphs, it seems reasonable that we might be able to consider all the possible ways of adding GCPs and turning them into generalised line graphs.



$G = K_4 \cup 4K_1$ $\quad$ $H = \tilde{E}_6 \cup K_1$ $\quad$ $G + H$

Figure 3.3: A graph in the set $\mathcal{P}_{0+2}$ where H is not a generalised line graph and not considered by Theorem 3.2.1.

Figures 3.4 and 3.5 show examples of infinite families of generalised line graphs in $\mathcal{P}_{0+2}$ using graphs from the families X_3 and X_5. One possible way of dealing with these graphs is by looking for minimal graphs from X_3 and X_5 that cannot be made into generalised line graphs by adding cyclotomic elements. If this leaves only a finite number of graphs then they could be potentially studied individually. However, we will attempt neither of these suggestions here.

The key observation, however, is that all three of these examples are graphs that can be described by the very general graph L_{GCP} in Theorem 3.2.1. Furthermore, no

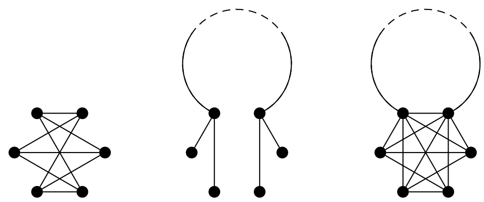


Figure 3.4: An example of a graph in $\mathcal{P}_{0+2}$ where G is from the family X_3. Let $G = K_{3,3} \cup (n-5)\, K_1$ (with $n > 5$), then $G + H$ is in $\mathcal{P}_{0+2}$.

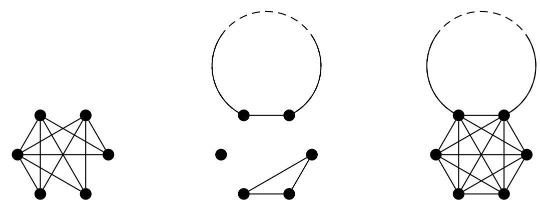


Figure 3.5: An example of a graph in $\mathcal{P}_{0+2}$ where G is from the family X_5. Let $G = K_{1,2,3} \cup (n-2)\, K_1$ (with $n \geq 2$), then $G + H$ is again in $\mathcal{P}_{0+2}$.

examples of generalised line graphs in $\mathcal{P}_{0+2}$ have so far been found that cannot be described by the graphs in Theorem 3.2.1 or Proposition 3.2.3. Perhaps we have found all the generalised line graphs in the set $\mathcal{P}_{0+2}$ but are simply lacking the proof? We will formalise this conjecture in Section 3.5.1.

3.3 The graphs in the set $\mathcal{P}_{1+1}$

In this section we are interested in the graphs G with $\lambda_2(G) \leq 1$ and $H = sK_2 \cup tK_1$ for some $s, t \geq 0$.

3.3.1 Turning disjoint K_2's into paths of any length

The set $\mathcal{P}_{1+1}$ turns out to be far more interesting than we might have initially suspected. The graph H here consists only of those graphs with $\lambda_1 \leq 1$, giving us the forest $H = sK_2 \cup tK_1$. However, below in Proposition 3.3.1 we will show that we can in fact attach paths of any length to G despite the only graphs in H being disconnected K_2's and K_1's. The strange choice of notation in the statement will become clear in the proof.

Proposition 3.3.1. *Let G be a graph with $\lambda_2(G) \leq 1$ and let the graph $G' + H$ be one isomorphic to G but where each vertex has at most one pendent path of any length attached. Cycles may also be made by joining two pendent paths at the loose ends by an extra edge provided the total number of vertices in the pendent paths is even. Then $\lambda_2(G' + H) \leq 2$.*

Furthermore, when $\lambda_1(G' + H) > 2$, if $G' + H$ is bipartite then it is a bipartite Salem graph (in $\mathcal{P}_{1+1}$) and if $\lambda_n(G' + H) \geq -2$ then $G' + H$ is a non-bipartite Salem graph (in $\mathcal{P}_{1+1}$).

Proof. Lemma 1.3.7(i) tells us that the spectrum of a disjoint graph is simply the union of the spectra of the connected components. When G has $\lambda_2(G) \leq 1$ we can add in as many disjoint K_2's and K_1's as we like as they do not push the value of the second largest eigenvalue above 1. Then, for such a G, the disconnected graph $G' = G \cup s'K_2 \cup t'K_1$ has $\lambda_2(G') \leq 1$ for any $s', t' \geq 0$.

Choose $H = sK_2 \cup tK_1$ such that $|V(H)| = |V(G')|$ and that the K_2's in H bridge the gaps between the disconnected parts of G', creating paths of the desired length (s and t can be chosen to suit the choice of s' and t', or vice versa). The resulting graph is what we called $G' + H$ in the statement above. The Courant-Weyl inequalities and Equation (3.1) tell us that

$$\lambda_2\left(G' + H\right) \leq \lambda_2\left(G'\right) + \lambda_1(H) \leq 1 + 1 = 2.$$

To create cycles from the pendent paths we clearly need the total number of vertices to be even (else we will require a P_3 to complete the cycle, but this has too large an index). If the cycle is made from two pendent paths both with even numbers of vertices then the extra edge must be in H, while if it is made from two with odd numbers then the extra edge must be in G'.

To see that these graphs are Salem when $\lambda_1(G' + H) > 2$, we simply observe that in the cases specified they fulfil the conditions in Definition 1.2.1. □

The example below in Figure 3.6 should make this idea clear. The observation that G may be disconnected with as many K_2 (and K_1) components as we like and still have $\lambda_2 = 1$ is key to making this construction much more general than it might have been; without it we would only be considering attaching single vertex pendent paths rather than paths of any length.

As before, we will consider G bipartite and non-bipartite separately.

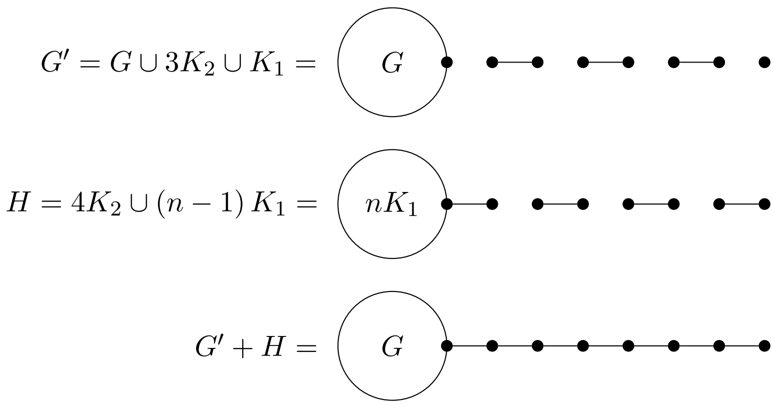


Figure 3.6: An example of how we can attach paths of any length when $\lambda_2 (G) \leq 1$ and $\lambda_1 (H) \leq 1$ (where $|V (G)| = n$). The large circle represents a generic graph G with this property.

3.3.2 Bipartite

Fortunately for us, the bipartite graphs G with $\lambda_2 (G) \leq 1$ have already been characterized by Petrović in 1991. The result is as follows:

Theorem 3.3.2 (see [45], Theorem 3)**.** *A connected bipartite graph G has $\lambda_2 (G) \leq 1$ if and only if G is an induced subgraph of any of the graphs $P_1, \ldots, P_7$ in Figure 3.7.*

By Proposition 3.3.1 we can create seven families of infinite graphs consisting of subgraphs of the graphs $P_1, \ldots, P_7$ above with up to n pendent paths of any length (where n is the size of our subgraph). Two such paths may even be attached at the loose ends by another edge provided the graph remains bipartite. Figure 3.8 shows an example of the construction of one such Salem graph. Again this is a very general construction, but also complete since we have considered every possible graph G and every possible way of adding H to it.

3.3.3 Non-bipartite

The ideal goal of this section would be to give a complete description of all the non-bipartite graphs in the set $\mathcal{P}_{1+1}$, however this turns out to be a tricky problem. Let us first describe the ideal solution to this problem.

Let us start with a family of graphs that are isomorphic to the family with $\lambda_n \geq -2$ but with every possible choice of independent edges removed. If we took our G in Equation 3.1 to be those graphs in this family with $\lambda_2 (G) \leq 1$ then we could use the K_2's in our H to re-insert these independent edges back in and attach pendent paths of

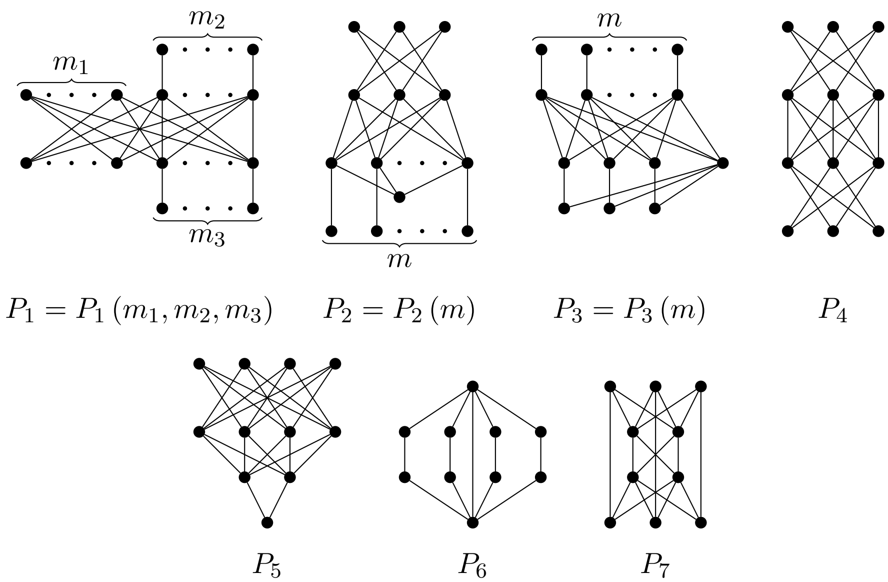


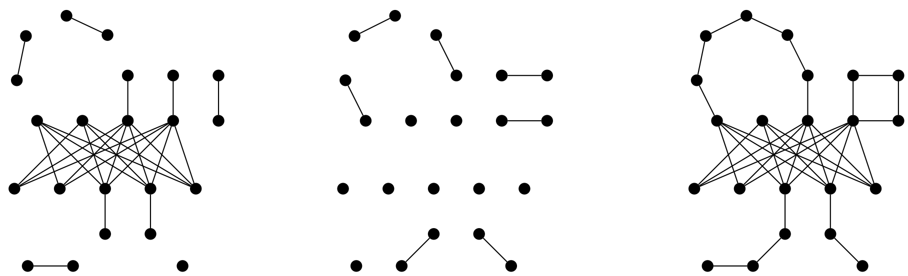
Figure 3.7: The seven graphs from Theorem 3.3.2.

any length to certain vertices. The resulting graph $G+H$ would certainly have $\lambda_n \geq -2$ by construction and $\lambda_2 \leq 2$ by the Courant-Weyl inequalities. Then, provided it is not cyclotomic, $G + H$ will be a Salem graph in $\mathcal{P}_{1+1}$. Moreover, we would have all the non-bipartite graphs in $\mathcal{P}_{1+1}$.

As mentioned before, it seems possible that a computer could find all of the ways of removing independent edges from the finite family of exceptional graphs and check which have $\lambda_2 \leq 1$. Adding these edges back in would give us Salem graphs. However, we already decided that this would not be particularly interesting here, so again we turn our attention to generalised line graphs.

The building blocks of line graphs are cliques, but generalised line graphs are made up of GCPs, which are simply cliques with independent edges removed. Therefore one way of thinking about generalised line graphs is that they are just line graphs with some independent edges removed (albeit in a specific way: recall from Theorem 1.3.2 the extra condition that a vertex in two GCPs must be of maximal degree in both). We would like to know about generalised line graphs with independent edges removed so we could consider these graphs to be "generalised generalised line graphs"; graphs made from "generalised GCPs" subject to similar conditions to Theorem 1.3.2. If we knew these we could then consider which have $\lambda_2 \leq 1$ and create our graphs in $\mathcal{P}_{1+1}$.

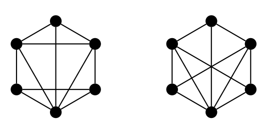
However, these "generalised generalised line graphs" are not so easy to define. The "generalised GCPs" would have vertices of degrees $n-1$, $n-2$ and $n-3$, but we do not have a unique graph for each specific count of vertex degrees like we do for GCPs.



$G' = G \cup 4K_2 \cup K_1$ $H = 7K_2 \cup 8K_1$ $G' + H$

Figure 3.8: An example of a Salem graph $G' + H$ constructed using Proposition 3.3.1 and the graphs in Theorem 3.3.2, where G is a subgraph of $P_1(2, 2, 3)$.

Take the two graphs in Figure 3.9 below; both have three vertices of degree $n-3$, two vertices of degree $n-2$ and one of maximal degree, yet are not isomorphic. On top of that, graph A is itself a line graph which adds further complications to any potential definition. Finally, we would still need to find which of these graphs have $\lambda_2 \leq 1$, and this is generally a difficult task.



A B

Figure 3.9: Two non-isomorphic graphs that both have the same number of vertices of degree $n-3$, $n-2$ and $n-1$ as each other. The graphs were found by consulting the table of connected graphs on six vertices in [20].

Let us instead consider how we can use existing results to create generalised line graphs in $\mathcal{P}_{1+1}$. A result worth mentioning at this point is the following by Cvetković, originally proved using the Courant-Weyl inequalities.

Theorem 3.3.3 (see [15], Theorem 2)**.** *Suppose that G is a graph on n vertices with $\lambda_2(G) \leq 1$. Then either $\lambda_n\left(\overline{G}\right) \geq -2$, or $\overline{G}$ has exactly one eigenvalue less than -2. Conversely, if $\lambda_n\left(\overline{G}\right) \geq -2$ then $\lambda_2(G) \leq 1$.*

Theorem 3.3.3 together with Theorem 1.3.6 tells us that the complements of generalised line graphs and exceptional graphs will have the required bound on λ_2. Furthermore the family of graphs where both the graph and its complement have $\lambda_n \geq -2$ has been classified (see [23], Theorem 7.2.7) so we could use this result to produce some of the desired generalised line graphs (and some exceptional graphs if we so choose). However Theorem 3.3.3 also tells us that there are graphs G with exactly one eigenvalue less than -2 that have $\lambda_2\left(\overline{G}\right) \leq 1$ and we know considerably less about these graphs. A complete classification looks difficult.

An interesting observation from Theorem 3.3.3 is as follows: for G with $\lambda_2(G) \leq 1$ we know that $\overline{G}$ has either no eigenvalues less than -2 or exactly one eigenvalue less than -2. Taking $G' = G \cup s'K_2 \cup t'K_1$ $(t' + s' \geq 1)$, then we still have $\lambda_2(G') \leq 1$ so $\overline{G'}$ also has one or zero eigenvalues less than -2. This graph $\overline{G'}$ is equal to $\overline{G} \triangledown GCP(2t' + s', t')$ (effectively a $GCP(2t' + s', t')$-cone over the graph $\overline{G}$) and, despite this change to $\overline{G}$, we have still not pushed any more eigenvalues below -2.

A result that yields more generalised line graphs than the suggestion above comes from a result of Petrović and Milekić. In 1999 they gave a complete classification of all the generalised line graphs with $\lambda_2 \leq 1$, which can be seen in Theorem 3.3.4 below. The family of graphs where both the graph and its complement are generalised line graphs is contained in this larger family, as would be expected.

Theorem 3.3.4 (see [47], Theorem 3)**.** *A connected generalised line graph G has $\lambda_2 \leq 1$ if and only if G is an induced subgraph of some of the sporadic graphs $F_1, \ldots, F_{10}$ in Figure 3.10 or the infinite family F_{11} in Figure 3.11.*

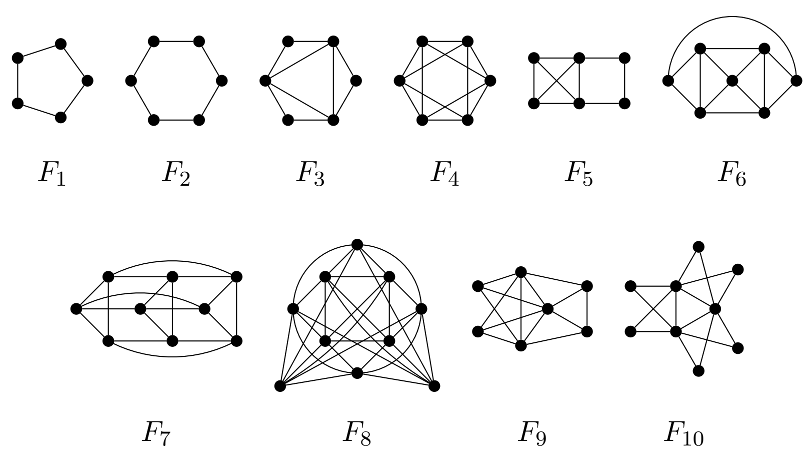


Figure 3.10: The graphs $F_1, \ldots, F_{10}$ from Theorem 3.3.4. Although it is not necessarily easy to see, F_8 comprises five K_4's.

We can certainly produce some generalised line graphs in $\mathcal{P}_{1+1}$ using these graphs. We will begin by looking at the finite graphs $F_1, \ldots, F_{10}$ and return to F_{11} later. We will use the following method:

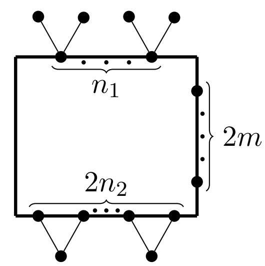


Figure 3.11: The graph F_{11} from Theorem 3.3.4. The box represents a $GCP\,(n, m)$ where $n = n_1 + 2n_2 + 2m$ and $0 \leq m \leq \lfloor n/2 \rfloor$. The $n - 2m$ vertices are of degree $n - 1$ and the $2m$ are of degree $n - 2$ in the GCP.

- Consider all the possible induced subgraphs of $F_1, \ldots, F_{10}$. This can easily be done by hand as we only have a small number of graphs and they are not particularly large.

- To each of these add independent edges where possible such that each subgraph remains a generalised line graph. Knowing that generalised line graphs are constructed from GCPs makes this task very simple too.

- Note which vertices are of maximal degree in their GCP but also only in one GCP. To these we can attach pendent paths of any length by Proposition 3.3.1. We may also join two of these pendent paths together, provided the total number of vertices in the paths is even.

Provided the resulting graph is not cyclotomic, these graphs are then Salem. We let G be our subgraph of F_i (for some $i = 1, \ldots, 10$) along with some disconnected K_2's and K_1's and let H be the remaining independent edges used in the last two steps along with some isolated vertices. This gives us the required bound on $\lambda_2\,(G + H)$, and the fact our graph is a generalised line graph deals with $\lambda_n\,(G + H)$. Working through all the possibilities, we get the twelve infinite families $A_1, \ldots, A_{12}$ in Figures 3.12 and 3.13 and the six sporadic graphs $A_{13}, \ldots, A_{18}$ in Figure 3.14 below.

We now turn our attention to F_{11}. Even though it represents an infinite family, all of its possible subgraphs can be described very easily and are of the form described below in Figure 3.15.

Following a similar method to the above we may add in some extra independent edges provided the graph remains a generalised line graph, although the only interesting change is that we can turn some of the n_2 snake's tongues into pendent K_3's. To the n_1 single vertex pendent paths and n_4 vertices or maximal degree with nothing attached we can add paths of any length, and maybe join two together to make a cycle. This

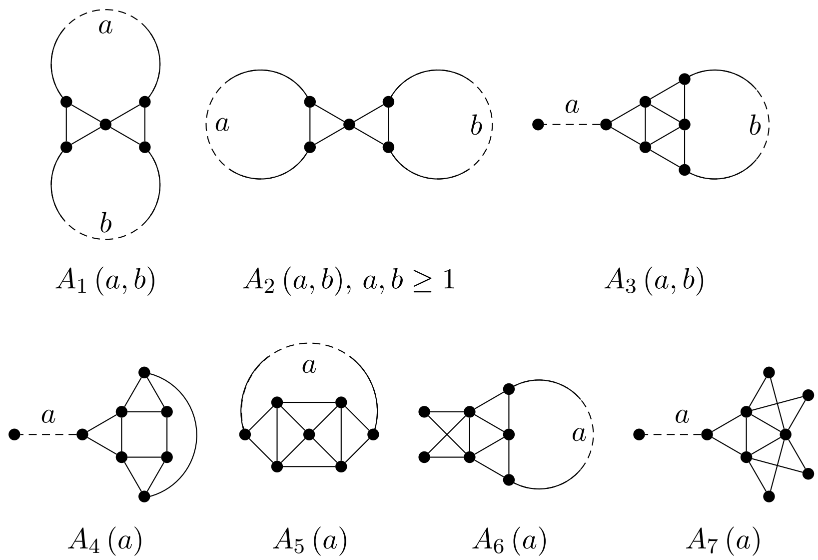
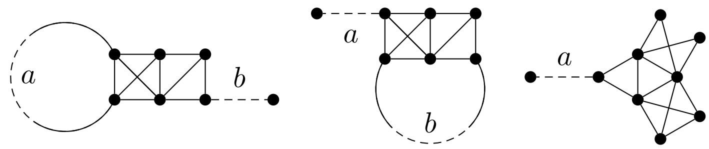


Figure 3.12: Seven of the twelve infinite families of generalised line graphs in the set $\mathcal{P}_{1+1}$. The parameters a and b are the number of vertices in the paths or cycles, and are greater than or equal to 0 unless specified.

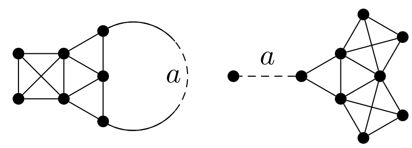
very general family is also Salem, however we note that it is itself a subgraph of the more general family L_{GCP} found in Theorem 3.2.1.

Using the graphs in Theorem 3.3.4 we have managed to find quite a few graphs in $\mathcal{P}_{1+1}$, although this is still far from the full picture. At this point we do not even know all the *line* graphs in $\mathcal{P}_{1+1}$. Whilst generalised line graphs are essentially line graphs with independent edges removed, the third condition to be a generalised line graph in Theorem 1.3.2 tells us that they are not *all* of the line graphs with independent edges removed; there may be graphs that only satisfy the first two conditions that also have $\lambda_2 \leq 1$ which we can use to create line graphs in $\mathcal{P}_{1+1}$.

A natural question to ask is: could we study the family of graphs that only satisfy the first two conditions of Theorem 1.3.2? Or, better still, the "generalised generalised line graphs" mentioned at the start of the section? These would certainly produce even more, if not all, of the generalised line graphs in $\mathcal{P}_{1+1}$. Even if we could, finding which have $\lambda_2 \leq 1$ seems like a particularly difficult question. The proof of Theorem 3.3.4 in the paper by Petrović and Milekić is quite lengthy and requires finding forbidden subgraphs. As the families we are interested in are even more general than theirs, the list of forbidden subgraphs is likely to be even longer and ultimately lead to a much more difficult proof. We concede, then, that a complete classification of generalised line graphs in the set $\mathcal{P}_{1+1}$ is, at the moment, out of reach.

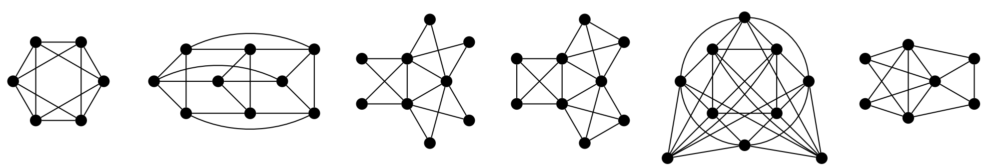


$A_8\,(a,b),\ a \geq 1$ $A_9\,(a,b)$ $A_{10}\,(a)$



$A_{11}\,(a)$ $A_{12}\,(a)$

Figure 3.13: Five of the twelve infinite families of generalised line graphs in the set $\mathcal{P}_{1+1}$. The parameters a and b are the number of vertices in the paths or cycles, and are greater than or equal to 0 unless specified.



$A_{13} = F_4$ $A_{14} = F_7$ $A_{15} = F_{10}$ A_{16} $A_{17} = F_8$ $A_{18} = F_9$

Figure 3.14: Six sporadic generalised line graphs in the set $\mathcal{P}_{1+1}$.

3.4 The graphs in the set $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ for $0 < \epsilon < 1$

In this section we are interested in graphs G with small second largest eigenvalue $\lambda_2\,(G)$ (by which, we mean in the interval $(0,1)$) and H with index bounded by $2 - \lambda_2\,(G)$.

3.4.1 Finding our non-maximal cyclotomic graphs H

The following result by Cao and Yuan allows us to see what the smallest positive second largest eigenvalue can be. Using this we can calculate which non-maximal cyclotomics we can have in H, which in turn tells us in more detail which graphs G we are interested in.

Theorem 3.4.1 (see [10]). *For a graph G we have $0 < \lambda_2\,(G) < 1/3$ if and only if $G = (K_2 \cup K_1)\,\nabla nK_1$.*

As an immediate consequence of this result and of interlacing, they go on to observe

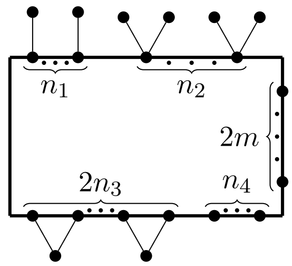


Figure 3.15: The subgraphs of F_{11} from Theorem 3.3.4. Again the box represents a $GCP\,(n,m)$ where $n = n_1 + n_2 + 2n_3 + n_4 + 2m$ and $0 \leq m \leq \lfloor n/2 \rfloor$ and in the GCP the $n - 2m$ vertices are of degree $n - 1$ and the $2m$ are of degree $n - 2$.

that the smallest positive second largest eigenvalue possible is

$$\lambda_2\left(\left(K_2 \cup K_1\right) \triangledown K_1\right) = 0.311$$

(Corollary 2, [10]). With this in mind, we consider the possible sets of non-maximal cyclotomics we can use in this construction. To do this, we list our cyclotomic graphs in increasing order of largest eigenvalue as in Figure 3.16 below.

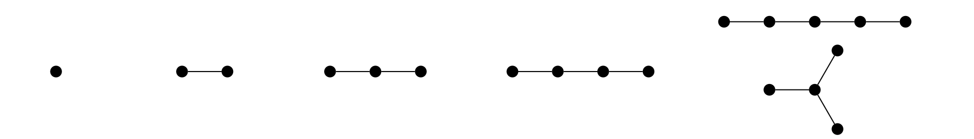


Figure 3.16: The first six cyclotomic graphs in order of smallest index.

If we let H contain a P_5, or $K_{1,3}$, or any other non-maximal cyclotomic with even larger index, then $\lambda_1\,(H)$ would be at least 1.732. Therefore, our graph G would need $0 < \lambda_2\,(G) \leq 2 - 1.732 = 0.268$ in order to get the correct bound on $G + H$. However, the result above shows that no such graph exists.

If $H = qP_4 \cup rP_3 \cup sK_2 \cup tK_1$ for some $q, r, s, t \geq 0$, then we have $\lambda_1\,(H) = 1.618$ so we will want G such that $0 < \lambda_2\,(G) \leq 2 - 1.618 = 0.382$. Similarly, if $H = rP_3 \cup sK_2 \cup tK_1$ then we have $\lambda_1\,(H) = 1.414$ so we will want G such that $0 < \lambda_2\,(G) \leq 2 - 1.414 = 0.586$. We have already considered $H = sK_2 \cup tK_1$ in Section 3.3.

We then conclude that if we are to form Salem graphs with this construction, our set H may only consist of the forest of very short paths (consisting of 2, 3 or 4 vertices each) along with the appropriate number of isolated vertices. Of course these need not be added as paths: in a bipartite graph they can be attached to the graph G by any of their vertices provided the graph remains bipartite. In a generalised line graph we must think of them as GCPs and treat them as paths except for P_3, which may be attached using the degree 2 vertex forming a $GCP\,(3, 1)$ (snake's tongue).

Taking H to be $rP_3 \cup sK_2 \cup tK_1$ (or $qP_4 \cup rP_3 \cup sK_2 \cup tK_1$) we have reduced our search to graphs G with $0 < \lambda_2\,(G) \leq 0.586$ (or 0.382, respectively).

3.4.2 Bipartite

We will now prove the following lemma and show as a corollary that there can be no bipartite graphs with a positive second largest eigenvalue small enough to allow us to add in the non-maximal cyclotomic forests above. We then conclude there exist no bipartite graphs in the set $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ for $\epsilon \in (0, 1)$.

Lemma 3.4.2. *If a connected bipartite graph G contains a C_4 and has $\lambda_2\,(G) < 0.618$ then we must have that $G = K_{m_1,m_2}$ for some $m_1, m_2 \geq 1$ and hence $\lambda_2\,(G) = 0$.*

Proof. Firstly, let us note that $\lambda_2\,(P_4) = 0.618$ and $C_4 = K_{2,2}$ so has $\lambda_2 = 0$. To prove the statement we will use induction on the number of vertices. For use in this, we will quickly define the following graph G_0 in Figure 3.17, which is a C_4 with a single vertex pendent path attached to one of the vertices, and observe that $\lambda_2\,(G_0) = 0.662$.

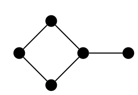


Figure 3.17: The graph G_0 used in the proof of Lemma 3.4.2.

Consider C_4 as the complete bipartite graph $K_{2,2}$ (with bipartitions A and B, say) and think about growing it to a graph with 5 vertices and $0 < \lambda_2 < 0.618$. The new vertex v cannot be adjacent to vertices in both A and B as that would form a triangle and the graph would no longer be bipartite. Therefore we can only attach v to vertices in one bipartition, say A. In $K_{2,2}$ there are only 2 vertices in A; if we attach to one of them we have a graph isomorphic to G_0 and if we attach it to both then we have a graph isomorphic to $K_{3,2}$. The second largest eigenvalue is too large in the former and too small in the latter.

Now consider the graph K_{m_1,m_2} with bipartitions A and B and label the vertices $a_1, \ldots, a_{m_1} \in A$ and $b_1, \ldots, b_{m_2} \in B$. If we attach v to one vertex of A we can induce G_0 on $\{v, a_1, a_2, b_1, b_2\}$. If we attach v to any i vertices of A for $2 \leq i \leq m_1 - 1$ then we can induce a P_4 on $\{v, a_1, b_1, a_{m_1}\}$. Finally if we attach v to all m_1 vertices of A then the graph is isomorphic to K_{m_1+1,m_2}. In each case λ_2 is outside the bounds and we are done. □

Corollary 3.4.3. *There exist no bipartite graphs with* $0 < \lambda_2(G) < 0.618$.

Proof. Let us partition bipartite graphs into those with (even) cycles and those without. The bipartite graphs without cycles are trees and can be partitioned further into those isomorphic to the stars $K_{1,m}$ for some $m \geq 1$, and those not (we can ignore the star with $m = 0$ as this is simply K_1). We know from the previous section that $\lambda_2(K_{1,m}) = 0$ for all m so we turn our attention to trees that are not stars. Consider the family of graphs formed from $K_{1,m}$ ($m \geq 1$) by adding a single vertex pendent path to any one of the m vertices (adding this new vertex to the vertex from the other side of the bipartition simply results in $K_{1,m+1}$). When $m = 1$ we get a graph isomorphic to $K_{1,2}$, a star which we have already dealt with. For $m \geq 2$ we see that our new graph always contains an induced P_4 (recalling that $\lambda_2(P_4) = 0.618$). This is certainly then true of any further trees. Therefore all trees will either be a star with second largest eigenvalue too small, or be a more complicated tree with second largest eigenvalue too large by interlacing with P_4.

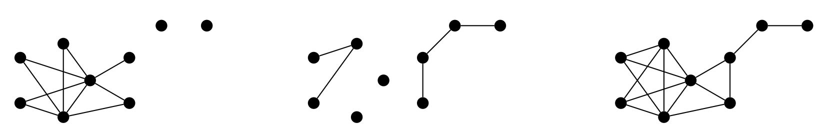
Now consider the bipartite graphs with cycles by the size of their shortest cycle. Clearly any graph that has a cycle of length 6 or longer will contain an induced P_4 so will have $\lambda_2 \geq 0.618$. Then the only graphs remaining are those that contain cycles of maximum length 4 and we simply refer to Lemma 3.4.2 to complete the proof. □

Thus we have proved the following result:

Proposition 3.4.4. *There exist no bipartite graphs in the set* $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ *for* $\epsilon \in (0,1)$.

3.4.3 Non-bipartite

We begin by showing two examples of non-bipartite generalised line graphs in the set $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ in Figures 3.18 and 3.19 below.

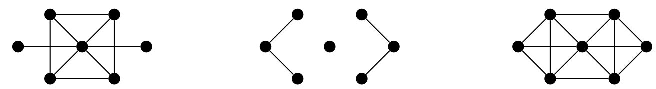


G $\qquad$ H $\qquad$ $G+H$

$\lambda_2(G) = 0.349$ $\qquad$ $\lambda_1(H) = 1.618$ $\qquad$ $\lambda_2(G+H) \leq 1.967$

Figure 3.18: An example of a graph in $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ where $\epsilon = 0.382$.

Having already conceded to not finding all of the generalised line graphs in $\mathcal{P}_{1+1}$, we must admit further defeat in not finding more than a handful of non-bipartite graphs in $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$. The two above in Figures 3.18 and 3.19 are among the very few that



G $\lambda_2(G) = 0.586$ H $\lambda_1(H) = 1.414$ $G+H$ $\lambda_2(G+H) \leq 2$

Figure 3.19: Another example of a graph in $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ but with $\epsilon = 0.586$.

have been found so far but, unlike in the bipartite case, there clearly exist some. In fact, another example can be found using a similar G and H as in Figure 3.18 but where the resulting graph $G+H$ comprises a $GCP(5,1)$ and two K_3's.

Given that we know exactly what our graphs H look like, the natural question to ask is which non-bipartite graphs G have $\lambda_2 \leq 0.382$ or $\lambda_2 \leq 0.586$. It is worth reminding ourselves that at this stage we are not just looking for generalised line graphs with bounded λ_2, but all non-bipartite graphs; the graph G in Figure 3.18 is not itself a generalised line graph, but we can make one by choosing our edges in H carefully.

Certainly there are infinitely many graphs with $\lambda_2 \leq 0.382$ (and indeed with $\lambda_2 \leq 0.586$), as Theorem 3.4.1 contains one such family. In their paper Cao and Yuan ([10]) finished by posing the question of characterising the graphs with $\lambda_2 \leq 0.618$, the golden ratio minus 1. Chapter 7.1 of [23] provides an excellent survey of the results that followed and the progress made on this problem, a few points of which we will repeat here. In 1993 Petrović classified all of the graphs with $\lambda_2 \leq 0.414$ ([46]). Since we are interested in the both the families of graphs with $\lambda_2 \leq 0.382$ and $\lambda_2 \leq 0.586$, knowing this result will certainly help with the former. Cvetković and Simić proved that the number of minimal forbidden subgraphs for the property $\lambda_2 < 0.618$ is finite and went on to classify them. The observation that a graph with $\lambda_2 < 0.618$ cannot contain a $2K_2$ or a P_4 (see Proposition 7.1.4 in [23]) allows us to describe all graphs with this property by using joins and adding isolated vertices. More formally, this is the class $\mathcal{C}$ such that if $G \in \mathcal{C}$ then $G \cup K_1 \in \mathcal{C}$ and $G_1 \nabla G_2 \in \mathcal{C}$ for all $G_1, G_2 \in \mathcal{C}$. We will use this to our advantage to describe our graphs G in a concise way, often grouping joins or unions together when they form graphs with existing names (such as complete graphs or stars).

We know that H is one of the disconnected graphs $rP_3 \cup sK_2 \cup tK_1$ or $qP_4 \cup rP_3 \cup sK_2 \cup tK_1$, which in turn tells us to look for G with λ_2 bounded by 0.586 or 0.382, respectively. Therefore it makes sense to think of these as two separate problems: what are the graphs in $\mathcal{P}_{(0.586)+(1.414)}$ and what are the graphs in $\mathcal{P}_{(0.382)+(1.618)}$? We will make some progress towards the latter and show that there certainly are not infinitely many graphs with this property. The former proves more difficult.

Let us now think about G with $0 < \lambda_2 \leq 0.382$. These graphs are certainly contained in the family of graphs with $\lambda_2 \leq 0.414$ classified by Petrović. We restate that theorem below in a slightly different form. The notation $\nabla_n G$ is used to denote $G\nabla \ldots \nabla G$, the n-fold join of a graph G.

Theorem 3.4.5 (see [46]). *Let G be a connected graph, then $0 < \lambda_2(G) \leq 0.414$ if and only if G is isomorphic to one of the following graphs:*

(i) $R_1 = [\nabla_n (K_1 \cup K_2)] \nabla K_{a_1,\ldots,a_m}$ *for $n \geq 1$ and at least one $a_i \geq 1$ for $i = 1, \ldots, m$;*

(ii) $R_2 = (K_1 \cup K_{a,b}) \nabla K_1$ *for $a, b \geq 1$;*

(iii) $R_3 = (K_1 \cup K_{1,a}) \nabla \overline{K}_b$ *for $a \geq 1$ and $b \geq 2$;*

(iv) $R_4 = (K_1 \cup K_{2,a}) \nabla \overline{K}_2$ *for $a \geq 2$;*

(v) $R_5 = (K_1 \cup K_{2,2}) \nabla \overline{K}_a$ *for $a \geq 3$;*

(vi) $R_6 = (K_1 \cup K_{2,3}) \nabla \overline{K}_a$ *for $a \geq 3$;*

(vii) $R_7 = (K_1 \cup K_2) \nabla K_{a,b}$ *for $a, b \geq 1$;*

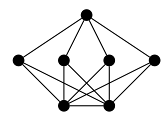
(viii) $R_8 = (K_1 \cup K_{1,2}) \nabla K_{1,a}$ *for $a \geq 1$;*

(ix) $R_9 = (K_1 \cup K_{1,2}) \nabla K_{2,a}$ *for $a \geq 2$;*

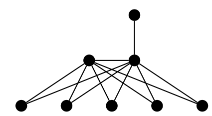
or one of 25 sporadic graphs on up to 14 vertices.

Rather than study which of these graphs have $\lambda_2 \leq 0.382$, with the exception of one infinite family, we will instead find two graphs that are subgraphs of infinitely many of the graphs in Theorem 3.4.5 and not subgraphs for only finitely many. In the exceptional case we will show that only finitely many have $\lambda_2 \leq 0.382$. We will then show that no matter how we add in our edges from $H = qP_4 \cup rP_3 \cup sK_2 \cup tK_1$ to these two graphs, the resulting graph $G + H$ can never be a generalised line graph and therefore is never in $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ for $\epsilon = 0.382$. By the hereditary nature of generalised line graphs, if a subgraph can never become a generalised line graph by adding in certain edges, then nor can any supergraph. We will then have shown that there may only be finitely many graphs in $\mathcal{P}_{(0.382)+(1.618)}$. The two graphs in question are shown below in Figure 3.20.

Proposition 3.4.6. *There are only finitely many graphs in Theorem 3.4.5 with $\lambda_2 \leq 0.382$ for which the graphs $W_1 = (K_1 \cup K_2) \nabla 4K_1$ and $W_2 = (K_1 \cup K_{1,5}) \nabla K_1$ in Figure 3.20 are not induced subgraphs.*



$W_1 = (K_1 \cup K_2) \triangledown 4K_1$
$\lambda_2(W_1) = 0.327$



$W_2 = (K_1 \cup K_{1,5}) \triangledown K_1$
$\lambda_2(W_2) = 0.352$

Figure 3.20: The two graphs W_1 and W_2 from Propositions 3.4.6 and 3.4.7.

Proof. As we are only interested in removing the possibility of infinite families, we can ignore the 25 sporadic graphs on up to 14 vertices mentioned in Theorem 3.4.5. Let us first consider the exceptional case mentioned above and show that only finitely many graphs from it have $\lambda_2 \leq 0.382$.

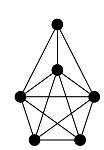
We will partition the family of graphs R_1 as follows: let $R_{1,1}$ be R_1 where $1 \leq a_i \leq 3$ for all $i = 1, \ldots, m$ and $R_{1,2}$ be R_1 where at least one $a_i \geq 4$. The family $R_{1,1}$ are our exceptional graphs mentioned above. The graph

$$(K_1 \cup K_2) \triangledown K_{1,1,1,1} = (K_1 \cup K_2) \triangledown K_4$$

has $\lambda_2 = 0.385$. Clearly only finitely many graphs $R_{1,1}$ do not contain this as a subgraph and, since this graph has $\lambda_2 > 0.382$, we conclude that $R_{1,1}$ can only give us finitely many graphs with suitably bounded λ_2 to worry about.

To finish the proof we simply note that W_1 is not a subgraph for only finitely many of the graphs $R_{1,2}$, R_3, R_5, R_6, R_7, R_8 and R_9 and that W_2 is not a subgraph for only finitely many of the graphs R_2 and R_4. $\square$

For the next proof we will need one of the minimal forbidden subgraphs for the property of being a generalised line graphs that we mentioned in Theorem 1.3.4 which can be seen below in Figure 3.21.



$G^{(31)}$

Figure 3.21: One of the minimal forbidden subgraphs for the property of being a generalized line graph from Theorem 1.3.4. The name is taken from [23].

Proposition 3.4.7. *For $i = 1, 2$, there is no possible graph $H = qP_4 \cup rP_3 \cup sK_2 \cup tK_1$ such that the graph $W_i + H$ is a generalised line graph.*

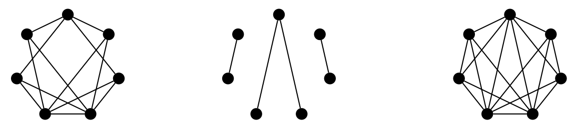
Proof. We prove this result computationally. The hereditary nature of generalised line graphs means that if a subgraph is not a generalised line graph, then nor will any supergraph be. This means that if we are trying to add edges to W_1 and W_2 to make them generalised line graphs, we need not worry about giving them extra isolated vertices and having edges of H connect them. Therefore, we only need to think about the edges missing from W_1 and W_2, which are of course the complements, $\overline{W}_1$ and $\overline{W}_2$. It is easy to see that $\overline{W}_1 = K_4 \cup P_3$ and $\overline{W}_2 = [(K_5 \cup K_1) \triangledown K_1] \cup K_1$, a K_6 with a single vertex pendent path and an isolated vertex.

The next step is to consider all the different ways in which we can embed H into these graphs, and for this we use a computer to grow our graphs. The code is omitted here, but an explanation is given instead to describe the process. For $\overline{W}_1$ we will need to grow all the various ways of embedding H into a K_4 and add either none, one, the other or both of the edges of P_3. For $\overline{W}_2$ we will grow all the ways of embedding H into a K_6 and then add either nothing or the edge from the single vertex pendent path.

A basic description of our growing method is as follows: we firstly define some $(n+1) \times n$ matrices I'_n and I''_n, isomorphic to the identity matrix I_n but with a row of 0's below (in the first case) or above (in the second). We then take the adjacency matrices A of the graphs on n vertices, calculate $I'_n A I'^T_n$ for each one to give a new $(n+1)^{\text{th}}$ row and column of 0's. We then include every possible arrangement of 0's and 1's in this new row and column. To calculate all the possible arrangements of 0's and 1's in the $(n+1)^{\text{th}}$ row and column we simply take the matrices B that contain all the arrangements in the n^{th} row and column twice, calculate $I''_n B I''^T_n$ to give a new first row and column of 0's, and in half the cases place a 1 in the $(1, n+1)$ and $(n+1, 1)$ entries. At each stage we calculate the spectra of our graphs and if there are graphs with $\lambda_1 > 1.618$ we remove them as they will not be subgraphs of H. By $n = 6$ we have finished growing as we will have all the possible ways of embedding H into a K_6. Finally we add in the edges from the previous paragraph (again checking for $\lambda_1 \leq 0.618$) and we have all our possible graphs H.

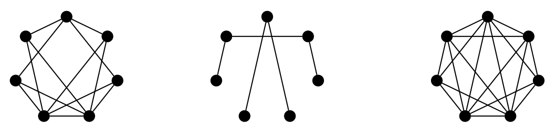
The final step is to add the relevant W_i to each of these graphs. An easy way to check whether any of these graphs are generalised line graphs is to consider the smallest eigenvalue; if $\lambda_n < -2$ then our graph certainly is not a generalised line graph. In all but two non-isomorphic cases we see $\lambda_n < -2$. The remaining two graphs are shown below in Figures 3.22 and 3.23, labeled Y_1 and Y_2, along with the relevant H required to obtain them. Note that both arise from W_1.

However, both of these contain an induced copy of the graph $G^{(31)}$ in Figure 3.21,



$G = W_1$ $H = P_3 \cup 2K_2$ $Y_1 = G + H$

Figure 3.22: The graph Y_1 from the proof of Proposition 3.4.7 along with the component parts G and H. We note that $\lambda_n(Y_1) = -2$.



$G = W_1$ $H = P_4 \cup P_3$ $Y_2 = G + H$

Figure 3.23: The graph Y_2 from the proof of Proposition 3.4.7 along with the component parts G and H. Note that $\lambda_n(Y_1) = -1.825$.

one of the 31 minimal forbidden subgraphs in Theorem 1.3.4, so we see that they cannot be generalised line graphs (meaning that Y_1 and Y_2 must be exceptional graphs). By exhaustively considering every possibility we have proved the statement. □

Firstly, a note on the growing method used above: It is certainly very naive and general. We could have considered isomorphic edges in the $\overline{W}_i$, or arrangements of edges in our new $(n+1)^{\text{th}}$ row and columns that never give a graph with $\lambda_1 < 1.618$, or other methods to reduce the number of calculations needed. However, the method here is more than sufficient for our needs, and relatively simple to implement. Pari/GP was able to do all the necessary calculations in a matter of seconds. Furthermore, the work to show the simpler fact that $W_1 + H$ is never a generalised line graph can be, and was firstly, done by hand.

By ruling out infinitely many of the graphs G with $\lambda_2 \leq 0.382$ as possible candidates for graphs in $\mathcal{P}_{(0.382)+(1.618)}$ and leaving only finitely many we have proved the following theorem:

Theorem 3.4.8. *The number of non-bipartite graphs in the set $\mathcal{P}_{(0.382)+(1.618)}$ is finite.*

Figure 3.18 reminds us that this finite number is certainly greater than zero. In fact, it would certainly be possible to calculate all of them. We could create the finite list of possible candidates by looking at which graphs remain after we have applied Proposition 3.4.6, consider their complements and apply our algorithm in Proposition 3.4.7 to find out when we get generalised line graphs after adding in the edges of H. After this we may even be able to add in further edges (of H) to create short pendent paths or snake's tongues. However, since finite families seem considerably less interesting, we leave this exercise as further work for the future.

Let us not forget that this is only (probably less than) half of the battle; we still need to consider the larger family of graphs G with $\lambda_2 \leq 0.586$ (with, in turn, the smaller family of $H = rP_3 \cup sK_2 \cup tK_1$). Unfortunately there are no descriptions of the family of graphs with $0.414 < \lambda_2 \leq 0.586$. It would be reasonable to suspect that there are quite a lot of graphs in this family; the number of minimal forbidden subgraphs found by Cvetković and Simić for $\lambda_2 \leq 0.618$ is quite large itself so it is reasonable to assume that the number of graphs with bounded λ_2 grows quite fast as the bound increases. Given a family of minimal forbidden subgraphs for $\lambda_2 \leq 0.586$ it might be possible to mimic Petrović's proof, although it would almost certainly involve a lot more work. If we knew more about the structure of these graphs we might be able to take a similar approach and discover that they also only contribute finitely many possible candidates for graphs in $\mathcal{P}_{(0.586)+(1.414)}$. However, this is just conjecture at the moment and, again, we concede to not solving the problem entirely.

3.5 A summary of the families of Salem graphs found using the Courant-Weyl inequalities

In this chapter we have found many infinite families of Salem graphs, both bipartite and non-bipartite, the vast majority of which have not been classified before. In doing so, we have also reclassified the 1-Salem bipartite and generalised line graphs found in Chapter 2. An immediate observation is that classifying Salem graphs using the Courant-Weyl inequalities is a very powerful method indeed. Let us summarise our results.

We have been very successful in our search for bipartite Salem graphs. In Sections 3.2.1 and 3.3.2 we found all of the bipartite graphs in $\mathcal{P}_{0+2}$ and $\mathcal{P}_{1+1}$, respectively. In Corollary 3.4.3 we discovered there that cannot be any bipartite graphs at all in $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ with $\epsilon \in (0,1)$. Furthermore, in Section 3.1.2 we considered every possible way of constructing Salem graphs from the Courant-Weyl inequalities and investigated each one. Therefore we have found every bipartite Salem graph possible using this method.

Our search for non-bipartite Salem graphs gave us many infinite families, some with a very general description, shedding much more light on non-bipartite Salem graphs than has been seen before. However we cannot say that we have a complete description using this method. The family of graphs L_{GCP} in Theorem 3.2.1 is by far the most general family we have found, infinite in infinitely many ways. We also found that all of the 1-Salem graphs from Theorem 2.3.2 are in $\mathcal{P}_{0+2}$. This gives us another 19 infinite families and six sporadic graphs (after noting that G_1, G_2, G_{10}, G_{11}, G_{18} and G_{22} are all subgraphs of L_{GCP}). In Section 3.3.3 we went on to find a further ten infinite families and six more sporadic graphs in Figures 3.12, 3.13 and 3.14 (again after noting that $A_1 = G_5$ and $A_2 = G_7$, two of the 1-Salem graphs, and that the graph in Figure 3.15 is a subgraph of L_{GCP}). Finally we noted in Theorem 3.4.8 that there certainly are not infinitely non-bipartite graphs in $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ when $\epsilon = 0.382$ but conceded to not finding a similar answer for $\epsilon = 0.586$. We did however find at least three sporadic graphs in Figures 3.18, 3.19 and the text that followed. This gives us a grand total of 30 infinite families and 15 sporadic graphs.

3.5.1 Some conjectures

We finish this chapter with some open questions and conjectures.

- Are there generalised line graphs in the set $\mathcal{P}_{0+2}$ that cannot be described by the graphs Theorem 3.2.1 or Theorem 2.3.2? Is there a way to prove that there are not any more graphs, maybe by partitioning generalised line graphs by the number of non-cyclotomic GCPs?

- Can we find a complete classification of the generalised line graphs in $\mathcal{P}_{1+1}$? Are there even any more to be found?

- Are there any infinite families of generalised line graph in $\mathcal{P}_{(0+\epsilon)+(2-\epsilon)}$ for $\epsilon = 0.586$? Can we find all of the sporadic graphs?

- Is it possible to find for which values of a and b the exceptional Salem graphs are in the set $\mathcal{P}_{a+b}$, if at all? Does this provide an interesting insight into those graphs?

Part III

Related work

Chapter 4

An extension of Hoffman and Smith's subdivision and the trivial Salem graphs

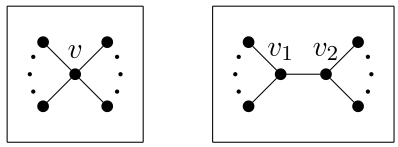
The first result of this chapter is an extension of the subdivision theorem Hoffman and Smith proved in [33]. This new result also appears in [30], a paper by the author on the subject. We then move on to study the families of non-bipartite graphs found in Chapters 2 and 3, observing when they produce trivial Salem graphs. Using our subdivision extension and other results, we will show exactly which graphs, from all but one our families, are trivial.

4.1 Extending Hoffman and Smith's subdivision

In Proposition 2.4 of [33], Hoffman and Smith proved that the index of a graph $G \neq \tilde{D}_n$ strictly decreases each time we subdivide an internal path; that is, for a path with at least one edge where each of the end points have degree greater than 2, we can place a new vertex of degree 2 on that edge (increasing the length of the path) and the value of the largest eigenvalue of the graph will strictly decrease (we mentioned this result in Chapter 1 as Theorem 1.3.11).

We will now extend this result slightly. Instead of requiring that the path has at least one edge, we start with a graph $G \neq K_{1,4}$ with a vertex v of degree at least 4 and split this vertex into two new adjacent vertices, each adjacent to at least two of v's neighbours (so our new graph has $|V(G)| + 1$ vertices and $|E(G)| + 1$ edges). We will show that this new graph also has index strictly less than the original graph. Figure 4.1 shows the relevant vertices and edges of a graph to demonstrate this process, where

G_{K_2} is our subdivided graph (the notation $G_{x,y}$ was used in [33], where (x, y) is an edge of G; the reason for our strange choice of notation will become clear shortly). Informally speaking, Hoffman and Smith proved the result for paths with number of edges from 1 to infinity and in Proposition 4.1.3 we will prove the result for paths of length 0, completing the picture.



G G_{K_2}

Figure 4.1: An illustration of subdividing a vertex v of degree at least 4 from G to create a graph G_{K_2} with an internal path. Only the relevant vertices and edges of the graphs have been drawn. We will prove that $\lambda_1(G_{K_2}) < \lambda_1(G)$ (provided $G \neq K_{1,4}$) in Proposition 4.1.3.

In [50] (also see Theorem 3.2.1 of [23]), Simić showed a similar result about splitting a vertex, but where the two new vertices were not adjacent to each other. The difference between the two results is subtle, and the direct proofs in [23] and [30] use a similar technique, but after proving Proposition 4.1.3 we will show how it can be used to offer a new proof of Simić's theorem.

However, rather than prove our result directly, we shall prove a more general theorem and leave the subdivision extension as a corollary. We will instead take a graph $G \neq K_{1,n^2}$ with a vertex v of degree at least n^2, and expand v into a K_n, where each of these new n vertices are each adjacent to at least n of v's neighbours. We will call our new graph G_{K_n} and soon show that $\lambda_1(G_{K_n}) < \lambda_1(G)$. Figure 4.2 should clarify this idea, again only showing the relevant vertices and edges of G. We will formalise the idea of expanding a vertex below in the statement of Theorem 4.1.2.

The extension of Hoffman and Smith's result follows simply by taking $n = 2$ in Theorem 4.1.2 and the proof of this larger theorem does not take that much more effort than proving the simpler case directly. In fact, the proof follows a similar method to the one used by Hoffman and Smith in their paper. We will refer to the Perron-Frobenius theorem regularly (see Theorem 1.3.10), but first we need to prove a simple lemma.

Lemma 4.1.1. *Let G be a connected graph with a vertex v such that $d(v) \geq n^2$. Then $\lambda_1(G) \geq n$, and $\lambda_1(G) = n$ if and only if $G = K_{1,n^2}$.*

Proof. Consider the vertex v with $d(v) \geq n^2$ along with n^2 of its adjacent vertices. These vertices will form a graph isomorphic to the star K_{1,n^2} or K_{1,n^2} along with some

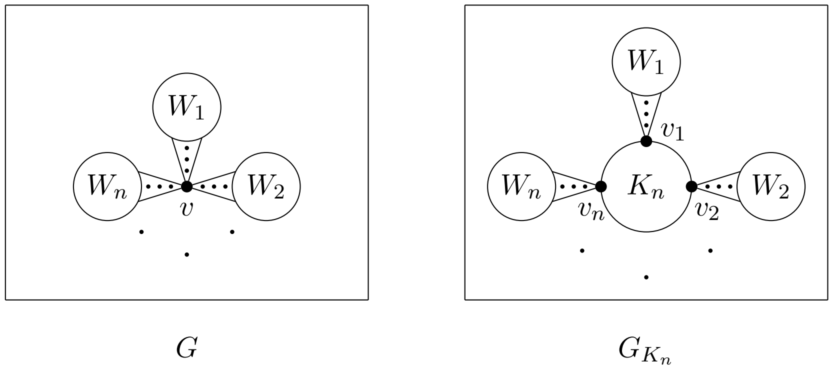


Figure 4.2: An illustration of expanding a vertex v (from G, with $d(v) \geq n^2$) to a K_n to create the graph G_{K_n}, for some $n \geq 2$. Only the relevant vertices and edges of the graphs have been drawn and the W_i are partitions of the neighbours of v, each containing at least n of them. In Theorem 4.1.2 we will show that $\lambda_1(G_{K_n}) < \lambda_1(G)$ provided $G \neq K_{1,n^2}$.

extra edges, $K_{1,n^2} \cup E$ say, where E is the set of extra edges. If the graph G is K_{1,n^2} then the index is n and if K_{1,n^2} is an induced subgraph of G then $\lambda_1(G) > n$ by Perron-Frobenius. If G is $K_{1,n^2} \cup E$ then we note that $A(K_{1,n^2} \cup E) - A(K_{1,n^2})$ is non-negative so by Perron-Frobenius we have that $\lambda_1(K_{1,n^2} \cup E) > \lambda_1(K_{1,n^2}) = n$. Finally, if $K_{1,n^2} \cup E$ is an induced subgraph of G then again we get $\lambda_1(G) > n$ by Perron-Frobenius. $\square$

The proof of the following theorem appears lengthy, but naturally breaks down into four different cases.

Theorem 4.1.2. *Let $G \neq K_{1,n^2}$ be a graph with a vertex v such that $d(v) \geq n^2$. Group the vertices of G adjacent to v into n partitions $W_1, \ldots, W_n$, where each W_i contains at least n vertices, $W_1 \cap \ldots \cap W_n = \emptyset$ and $W_1 \cup \ldots \cup W_n$ is the whole neighbourhood of v. Let the vertices of K_n be labeled $v_1, \ldots, v_n$ and let G_{K_n} be the graph isomorphic to $K_n \cup (G \setminus v)$ along with edges joining the vertices of W_i to v_i (for $i = 1, \ldots, n$). Then $\lambda_1(G_{K_n}) < \lambda_1(G)$.*

Proof. Let $A(G)\boldsymbol{z} = \lambda_1(G)\boldsymbol{z}$ with $\boldsymbol{z} > 0$; that is, $\boldsymbol{z}$ is the eigenvector for the largest eigenvalue λ_1 of G and each coefficient of $\boldsymbol{z}$ is strictly greater than 0. Let $\hat{\boldsymbol{z}}$ be a vector of length $|V(G)| + n - 1 = |V(G_{K_n})|$ where the coordinates corresponding to the vertices of $G_{K_n} \setminus \{v_1, \ldots, v_n\}$ are the same as those in $\boldsymbol{z}$ corresponding to the vertices in $G \setminus v$. Since W_i is a set of vertices, we will let $\boldsymbol{z}_{W_i}$ be a vector of length $|W_i|$ (in fact it will be the vector of coordinates from $\boldsymbol{z}$ that correspond to the vertices in W_i) and let $\sum \boldsymbol{z}_{W_i}$ be the sum of the entries of that vector.

We will choose what the coordinates $\hat{z}_{v_1}, \ldots, \hat{z}_{v_n}$ are and consider the parts of the vector $A(G_{K_n})\,\hat{\boldsymbol{z}}$ that are affected by these. A picture helps and the matrix below shows the multiplication of $A(G_{K_n})$ and $\hat{\boldsymbol{z}}$; the rows and columns of $A(G_{K_n})$ are ordered $v_1, \ldots, v_n$, then the vertices of each W_i, then the other vertices of the graph. The parts that are not filled in are either 0, or not affected by the expansion from G to G_{K_n}. We find

$$A(G_{K_n})\,\hat{\boldsymbol{z}} = \left(\begin{array}{c|c|c|c|c} A(K_n) & \begin{matrix}1 & \cdots & 1\\ & & \\ & & \end{matrix} & \ddots & \begin{matrix} & & \\ & & \\ 1 & \cdots & 1\end{matrix} & \\ \hline \begin{matrix}1 & & \\ \vdots & & \\ 1 & & \end{matrix} & & & & \\ \hline \ddots & & & & \\ \hline \begin{matrix} & & 1\\ & & \vdots\\ & & 1\end{matrix} & & & & \\ \hline & & & & \end{array}\right)\left(\begin{array}{c} \hat{z}_{v_1}\\ \vdots\\ \hat{z}_{v_n}\\ \hline \boldsymbol{z}_{W_1}\\ \hline \vdots\\ \hline \boldsymbol{z}_{W_n}\\ \hline \\ \end{array}\right)$$

and multiplying this out gives, for $i = 1, \ldots, n$, the n equations α_i and n families ($d(v)$ in total) of equations $\boldsymbol{\beta}_i$ below:

$$\begin{aligned} \alpha_i &= \left(\sum_{j=1}^{n} \hat{z}_{v_j}\right) - \hat{z}_{v_i} + \sum \boldsymbol{z}_{W_i} \\ \boldsymbol{\beta}_i &= \lambda_1 \boldsymbol{z}_{W_i} - z_v + \hat{z}_{v_i} \end{aligned}$$

where $\lambda_1 = \lambda_1(G)$.

We then consider 4 cases, as our choices of $\hat{z}_{v_1}, \ldots, \hat{z}_{v_n}$ will vary in each case. It is easy to see that these are exhaustive and non-equivalent, but may require re-labelling of the sets of vertices W_i:

(i) $\sum \boldsymbol{z}_{W_i} \leq z_v$ for all $i = 1, \ldots, n$

(ii) $z_v < \sum \boldsymbol{z}_{W_1}$ and $\sum \boldsymbol{z}_{W_i} \leq z_v$ for $i = 2, \ldots, n$

(iii) there exists an $2 \leq k < n$ such that $z_v < \sum \boldsymbol{z}_{W_i}$ for $i = 1, \ldots, k$ and $\sum \boldsymbol{z}_{W_i} \leq z_v$ for $i = k+1, \ldots, n$

(iv) $z_v < \sum \boldsymbol{z}_{W_i}$ for all $i = 1, \ldots, n$.

In each case we will show that $A(G_{K_n})\,\hat{\boldsymbol{z}} < \lambda_1(G)\,\hat{\boldsymbol{z}}$, which then gives $\lambda_1(G_{K_n}) < \lambda_1(G)$ by Perron-Frobenius.

Case (i). Set $\hat{z}_{v_1}, \ldots, \hat{z}_{v_n} = z_v$. Then for all $i = 1, \ldots, n$ we see that $\boldsymbol{\beta}_i = \lambda_1 \boldsymbol{z}_{W_i}$ and $\alpha_i = (n-1)\,z_v + \sum \boldsymbol{z}_{W_i}$ which is less than or equal to nz_v since $\sum \boldsymbol{z}_{W_i} \le z_v$. However, $\lambda_1 > n$ by Lemma 4.1.1 so we get $\alpha_i < \lambda_1 z_v$ for all i, giving the strict inequality required.

Case (ii). Set $\hat{z}_{v_1} = z_v$ and $\hat{z}_{v_i} = \sum \boldsymbol{z}_{W_i}$ for $i = 2, \ldots, n$. We then see that $\boldsymbol{\beta}_1 = \lambda_1 \boldsymbol{z}_{W_1}$ and $\boldsymbol{\beta}_i \le \lambda_1 \boldsymbol{z}_{W_i}$ since $\sum \boldsymbol{z}_{W_i} \le z_v$ for $i = 2, \ldots, n$. Furthermore, from the definition of $\lambda_1 \boldsymbol{z}$ we get $\alpha_1 = \sum_{j=2}^{n} \hat{z}_{v_j} + \sum \boldsymbol{z}_{W_1} = \lambda_1 z_v = \lambda_1 \hat{z}_{v_1}$.

For $i = 2, \ldots, n$ and each $w \in W_i$ define $\Sigma_w = \lambda_1 z_w - z_v$. Since $\boldsymbol{z} > 0$ we see that $\Sigma_w \ge 0$. Therefore $z_v \le \lambda_1 z_w$ for each $w \in W_i$ and moreover $|W_i|\,z_v \le \lambda_1 \sum \boldsymbol{z}_{W_i}$. Since $|W_i| \ge n$ we deduce the following for $i = 2, \ldots, n$:

$$\begin{aligned}
\alpha_i &= z_v + \left(\sum_{j=2}^{n} \hat{z}_{v_j}\right) - \hat{z}_{v_i} + \sum \boldsymbol{z}_{W_i} \\
&= z_v + \sum_{j=2}^{n} \sum \boldsymbol{z}_{W_j} \\
&\le nz_v \;\le\; |W_i|\,z_v \;\le\; \lambda_1 \sum \boldsymbol{z}_{W_i} \;=\; \lambda_1 \hat{z}_{v_i}.
\end{aligned}$$

To obtain a strict inequality, we consider a few subcases: If $d(w) > 1$ for at least one $w \in W_i$ then we have $\Sigma_w > 0$ so $z_v < \lambda_1 z_w$ giving $|W_i|\,z_v < \lambda_1 \sum \boldsymbol{z}_{W_i}$. If $d(w) = 1$ for all $w \in W_i$ and $|W_i| > n$ then $nz_v < |W_i|\,z_v$. Finally, if $d(w) = 1$ for all $w \in W_i$ and $|W_i| = n$ then the z_w are all the same for all $w \in W_i$, so $|W_i|\,z_w = \sum \boldsymbol{z}_{W_i} \le z_v = \lambda_1 z_w$. However, $\lambda_1 > n$ by Lemma 4.1.1 so $|W_i|\,z_w < z_v$ and then $\alpha_i = z_v + \sum_{j=2}^{n} \sum \boldsymbol{z}_{W_j} < |W_i|\,z_v$ giving our strict inequality.

Case (iii). For $i = 1, \ldots, k$ set $\hat{z}_{v_i} = z_v$ and for $i = k+1, \ldots, n$ set $\hat{z}_{v_i} = \sum \boldsymbol{z}_{W_i}$. Then for $i = 1, \ldots, k$ we get $\boldsymbol{\beta}_i = \lambda_1 \boldsymbol{z}_{W_i}$ and

$$\begin{aligned}
\alpha_i &= \sum_{j=1}^{k} \hat{z}_{v_j} + \sum_{j=k+1}^{n} \hat{z}_{v_j} - \hat{z}_{v_i} + \sum \boldsymbol{z}_{W_i} \\
&= (k-1)\,z_v + \sum_{j=k+1}^{n} \sum \boldsymbol{z}_{W_j} + \sum \boldsymbol{z}_{W_i} \\
&< \sum_{j=1}^{n} \sum \boldsymbol{z}_{W_j} \;=\; \lambda_1 z_v \;=\; \lambda_1 \hat{z}_{v_i}.
\end{aligned}$$

For $i = k+1, \ldots, n$ we get $\boldsymbol{\beta}_i \leq \lambda_1 \boldsymbol{z}_{W_i}$ and

$$\begin{aligned}
\alpha_i &= \sum_{j=1}^{k} \hat{z}_{v_j} + \sum_{j=k+1}^{n} \hat{z}_{v_j} - \hat{z}_{v_i} + \sum \boldsymbol{z}_{W_i} \\
&= kz_v + \sum_{j=k+1}^{n} \sum \boldsymbol{z}_{W_j} \\
&\leq nz_v \leq |W_i| \, z_v \leq \lambda_1 \sum \boldsymbol{z}_{W_i} = \lambda_1 \hat{z}_{v_i}
\end{aligned}$$

where the inequality we require comes from the same reasoning as used in Case (ii).

Case (iv). Now set $\hat{z}_{v_1}, \ldots, \hat{z}_{v_n} = z_v$ again. Then for all $i = 1, \ldots, n$ we see that $\boldsymbol{\beta}_i = \lambda_1 \boldsymbol{z}_{W_i}$ and

$$\begin{aligned}
\alpha_i &= (n-1)\, z_v + \sum \boldsymbol{z}_{W_i} \\
&< \sum_{j=1}^{n} \sum \boldsymbol{z}_{W_j} = \lambda_1 z_v = \lambda_1 \hat{z}_{v_i}
\end{aligned}$$

giving our final strict inequality. □

We can now state our original conjecture as a corollary, using Theorem 1.3.11 and taking $n = 2$ in Theorem 4.1.2. A direct proof is also given in the author's paper [30].

Proposition 4.1.3. *Let $G \neq K_{1,4}$ be a graph with a vertex v such that $d(v) \geq 4$. Then we can expand the vertex v into two adjacent vertices v_1 and v_2 (with $d(v_1) \geq 2$ and $d(v_2) \geq 2$) and continue to subdivide this edge as many times as we like. Each new graph has index strictly less than $\lambda_1(G)$.*

We will make use of this result in our search for trivial Salem graphs shortly in Section 4.2. As a consequence of Proposition 4.1.3 we can provide an alternative proof of the result of Simić in [50] about splitting vertices.

Corollary 4.1.4 (see [50], Theorem 2.4)**.** *Let G be a graph with a vertex v, and let $W_1 \cup W_2$ be a non-trivial bipartition of the vertices adjacent to v. Let the graph G' be formed by taking $G \setminus v$ and including two new non-adjacent vertices v_1 and v_2, where v_i is adjacent to all of the vertices in W_i $(i = 1, 2)$. Then $\lambda_1(G') < \lambda_1(G)$.*

Proof. To prove this we shall consider three cases: $d(v) = 2$, $d(v) = 3$ or $d(v) \geq 4$ (note that the non-trivial bipartition of the vertices adjacent to v excludes the possibility that $d(v) = 1$; in that case the index stays the same).

When $d(v) = 2$, consider whether v is on an internal path or not. If it is, we can subdivide an edge between v and one of its neighbours twice. After doing this we can readily spot a vertex which upon removal induces the graph G'. Subdivision and Perron-Frobenius (Theorem 1.3.11 and Theorem 1.3.10, respectively) then tell us that the index has strictly decreased. If v is not on an internal path it is either on a pendent path, or on a cycle, C_n. In the former case G' consists of two disconnected graphs, both subgraphs of G, so Perron-Frobenius gives us the result. The index of C_n is 2 for all n as it is regular of degree 2 (see Lemma 1.3.7(iii)), and clearly P_n is a subgraph of C_{n+1} so must have a strictly smaller index.

Let $d(v) = 3$ and let the three vertices adjacent to v be x_1, x_2 and x_3. If there exist no walks from the x_i in W_1 to the x_i in W_2 then G' will consist of two disconnected graphs, both subgraphs of G. Again, Perron-Frobenius gives us the result here. If there is a walk between at least one of the x_i in W_1 to at least one of the x_i in W_2 we can follow the same procedure as we did for $d(v) = 2$ with an internal path: subdivide the edge that connects v to the W_i with only one vertex in it and then subdivide the new edge again. Once again, we can find a vertex whose removal induces the graph G', and subdivision and Perron-Frobenius tell us that the index has strictly decreased.

Finally, if $d(v) \geq 4$, then we can compare the adjacency matrix of G' with the same graph found using Proposition 4.1.3. This latter graph is isomorphic to $G' \cup e$, where e is the edge between v_1 and v_2. Applying Perron-Frobenius to these two graphs gives the strict inequality we are after. □

4.2 Finding trivial graphs

Recall from Definition 1.2.1 that a non-bipartite Salem graph G is trivial if $\lambda_1(G) \in \mathbb{Z}$. It is not necessarily easy to say when a Salem graph will be trivial and is an interesting question. Note that it is different to the question of integral graphs (those where $\lambda_i \in \mathbb{Z}$ for $i = 1, \ldots, n$) as we only require the largest eigenvalue to be an integer, however the two problems intersect; any integral Salem graph is trivial, but a trivial Salem graph need not be integral.

In Chapters 2 and 3 we classified a number of families of Salem graphs. The bipartite graphs were largely very general descriptions of families but the non-bipartite graphs were far more explicit. The stronger descriptions of the non-bipartite families make it possible to explore them further and in this section we will study when these non-bipartite graphs produce trivial Salem numbers.

At the end of Chapter 3 we observed that we had in fact found 30 infinite families of non-bipartite Salem graphs: the 1-Salem graphs $G_3, \ldots, G_9, G_{12}, \ldots, G_{17}, G_{19}, \ldots, G_{21}, G_{23}, \ldots, G_{25}$, the graphs $A_3, \ldots, A_{12}$ in the set $\mathcal{P}_{1+1}$ and the graph L_{GCP} from Theorem 3.2.1. We will soon want to make comments about the whole family of non-bipartite

1-Salem graphs, so will include the graphs G_1, G_2, G_{10}, G_{11}, G_{18} and G_{22} again having previously ignored them for being subgraphs of L_{GCP} (giving a total of 36 infinite families). Before describing the method, we will make a few short remarks about reciprocal polynomials.

4.2.1 Reciprocal polynomials

Recall from Section 1.2 that the reciprocal polynomial of a non-bipartite graph G is defined to be $R_G(z) = z^{|V(G)|}\chi_G(z + 1/z)$. The following lemma allows us to calculate the reciprocal polynomial of a non-bipartite graph with a pendent path of length p attached. The beauty of this result is that we can use it to find the indices of a series of graphs without calculating increasingly large adjacency matrices and their characteristic polynomials; we only need the reciprocal polynomials of the graph itself and the graph with a single vertex pendent path attached. After finding the largest root of the reciprocal polynomial, θ, we simply solve $\theta + \frac{1}{\theta} = \lambda_1$ to find the index.

Lemma 4.2.1 (see [41], Lemma 10)**.** *Let G be a graph with a distinguished vertex v. For each $p \geq 0$, let G_p be the graph obtained by attaching one end-vertex of an p-vertex path to the vertex v. Let $R_{G_p}(z)$ be the reciprocal polynomial of G_p, then for $p \geq 2$ we have*

$$R_{G_p}(z) = \frac{z^{2p}-1}{z^2-1}R_{G_1}(z) - \frac{z^{2p}-z^2}{z^2-1}R_{G_0}(z).$$

We also note the following two extensions of Lemma 4.2.1, although they will not be used here. The first gives the reciprocal polynomial of a path of length s attached with a further two vertices in the shape of a snake's tongue, and the second is the reciprocal polynomial for a cycle C_n. Note the subtle differences between Lemma 4.2.1 and Lemma 4.2.2; we now have a $(z^4 - 1)$ term in front and the sign of the 1 and the z^2 in the numerators has changed.

Lemma 4.2.2. *Let G be a graph with a distinguished vertex v. For each $s \geq 0$, let G_s be the graph obtained by attaching one end-vertex of an s-vertex path to the vertex v. Then add a further two vertices in a snake's tongue on the other end of the s-vertex path and call this $G_{\hat{s}}$ (so $G_{\hat{s}}$ has $s+2$ more vertices than G). Let $R_{G_{\hat{s}}}(z)$ be the reciprocal polynomial of $G_{\hat{s}}$, then for $s \geq 2$ we have*

$$R_{G_{\hat{s}}}(z) = \left(z^4 - 1\right)\left(\frac{z^{2s}+1}{z^2-1}R_{G_1}(z) - \frac{z^{2s}+z^2}{z^2-1}R_{G_0}(z)\right).$$

Proof. Let $\chi_{\hat{s}}(x)$ be the characteristic polynomial of $G_{\hat{s}}$. We begin by labelling the two end vertices of the snake's tongue as a and b and then expand the determinant along the row corresponding to a. This gives us two matrices, one of which is $x\chi_{G_{s+1}}(x)$, where G_{s+1} is just the graph G with an $(s+1)$-vertex pendent path attached. The other we then expand twice more: once down the column corresponding to a and then across the row corresponding to b. This matrix is then $-x\chi_{G_{s-1}}(x)$ and we have

$$\chi_{G_{\hat{s}}}(x) = x\chi_{G_{s+1}}(x) - x\chi_{G_{s-1}}(x)$$

and so our reciprocal polynomial is

$$R_{G_{\hat{s}}}(z) = \left(z^2+1\right) R_{G_{s+1}}(z) - \left(z^4+z^2\right) R_{G_{s-1}}(z),$$

taking care that $|V(G)|$ varies depending on the graph in question. We now use Lemma 4.2.1 to deal with the paths of length $s+1$ and $s-1$ attached to the graph and then tidy up to obtain the polynomial stated above. □

Lemma 4.2.3. *The reciprocal polynomial of C_n, the cycle on n vertices, is*

$$R_{C_n}(z) = (z^n - 1)^2.$$

Proof. Expanding the characteristic polynomial of C_n in the usual way gives

$$\chi_{C_n}(x) = x\chi_{P_{n-1}}(x) - 2\chi_{P_{n-2}}(x) - 2$$

where P_n is the graph that is an n-vertex path, as before. Turning this into a reciprocal polynomial and using Lemma 4.2.1 gives the result. □

4.2.2 The method

For our non-bipartite sporadic graphs (either generalised line graphs or exceptional graphs) we simply need to calculate the eigenvalues of the graphs and their subgraphs to see when they are trivial. The infinite families $G_1, \ldots, G_{25}$, $A_3, \ldots, A_{12}$ and L_{GCP} require a slightly more involved approach. For all except L_{GCP} we will find for which path (possibly with a snake's tongue) and cycle lengths they are trivial. For $L_{GCP(n,m)}$ we will show that for each n there are only three possible integral values the index can take (remembering that the n is the number of vertices in the central GCP, not the total number in the graph itself).

The main idea we will use here is one we have used before. In Proposition 2.1.3 and Corollary 3.2.2 we used the idea of bounding the largest eigenvalue strictly between two consecutive integers to show that a graph is not trivial. By finding appropriate graphs we can bound the index of all of our 35 infinite families (not including L_{GCP})

and see exactly which ones produce trivial Salem graphs. Moreover, this will give us a complete list of the trivial non-bipartite 1-Salem graphs. We shall return to L_{GCP} in Section 4.2.5, after looking at the other infinite families.

For each family the method is to find the index of a subgraph that the family always contains to act as a lower bound; Perron-Frobenius (Theorem 1.3.10) tells us that any supergraph must have strictly larger index. This is usually found by looking at the graphs with all pendent paths, cycles and snake's tongues removed.

We then also need to find an upper bound for the index. The method here is to consider an extremal graph, when the index is as large as it can possibly be. Let us call this extremal graph G_E. We know by our extension of Hoffman and Smith's subdivision theorem (Proposition 4.1.3) that as our cycles and paths with snake's tongues get longer, the index will strictly decrease. Therefore, any graph with cycles or paths with snake's tongues attains its largest index when the lengths of these are as short as possible. If we have a pendent path, Perron-Frobenius tells us that the index of a longer path is always strictly larger than that of a shorter one, as the shorter one is simply a subgraph. Therefore, a graph with pendent paths attains its largest index when the paths are infinitely long. Rather than spending too long thinking about what it means to find the index of an infinitely large graph, we shall simply take this to be the limit of the indices as the length of the path tends towards infinity.

If our graph contains both paths and cycles (or paths with snake's tongues) we simply observe that the largest the index can possibly be is when the lengths of the paths are infinitely long and lengths of the cycles and paths with snake's tongues are as short as possible; whilst in this case G_E will not always be a supergraph, we know that each time we increase the length of a cycle or snake's tongue, the index increases too, and to each of these graphs we can add an infinite path to create our supergraph. Essentially, we apply subdivision first, then Perron-Frobenius.

To calculate the largest eigenvalue of a graph with infinitely long paths, we use reciprocal polynomials; we simply apply Lemma 4.2.1 as many times as necessary and let the lengths of our paths tend to infinity in the polynomial. If we let θ be the largest root of $R^*_{G_E}$ then $\lambda_1(G_E) = \theta + \frac{1}{\theta}$ (we use the notation $R^*(z)$ as this polynomial is no longer necessarily reciprocal – the parts that tend to infinity the fastest will dominate and they need not be reciprocal themselves).

4.2.3 An example

We shall work through an example to clarify these ideas. Consider the family of graphs $G_6(\hat{a}, b, c)$ from Theorem 2.3.2 and in Figure 4.3 below. We will prove that each graph has $2 < \lambda_1 < 3$ for all $a, c \geq 0$ and $b \geq 1$.

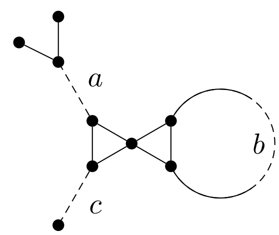


Figure 4.3: The family of graphs $G_6(\hat{a}, b, c)$ from Theorem 2.3.2 where a, b and c are the number of vertices in the paths or cycles, with $a, c \geq 0$ and $b \geq 1$.

A lower bound is easy to spot, as any graph from this family will always contain the bowtie-shaped graph $2K_2 \triangledown K_1$. A quick calculation tells us that this graph has $\lambda_1(2K_2 \triangledown K_1) = 2.562$. In fact, we could have picked K_3 as our lower bound since it is always a proper subgraph of G_6 and $\lambda_1(K_3) = 2$.

For the upper bound, we will let our extremal graph G_E be the graph $G_6(\hat{0}, 1, \infty)$, where we are using ∞ as a symbol to represent a path whose limit we are interested in as the length of the path tends towards infinity (note also that we have defined $b \geq 1$ to avoid multiple edges). The reciprocal polynomial of $G_6(\hat{0}, 1, c)$ for any $c \geq 0$ is

$$R_{G_6(\hat{0},1,c)}(z) = \frac{z^{2c}-1}{z^2-1} R_{G_6(\hat{0},1,1)}(z) - \frac{z^{2c}-1}{z^2-1} R_{G_6(\hat{0},1,0)}(z)$$

by Lemma 4.2.1. When we let c tend to infinity we get the (non-reciprocal) polynomial

$$R^*_{G_E}(z) = R^*_{G_6(\hat{0},1,\infty)}(z) = \frac{1}{z^2-1}\Big(R_{G_6(\hat{0},1,1)}(z) - R_{G_6(\hat{0},1,0)}(z)\Big).$$

The largest root of this polynomial is $\theta = 2.572$ meaning that the largest value the eigenvalue can take is $\lambda_1(G_E) = \theta + \frac{1}{\theta} = 2.961$.

Therefore, since $G_6(\hat{a}, b, c)$ always contains a $2K_2 \triangledown K_1$ we know that it always has index strictly greater than 2, by Perron-Frobenius. Furthermore, we know that the very largest the index can be is $\lambda_1(G_E) = 2.961$, so $\lambda_1(G_6(\hat{a}, b, c)) < 3$ for all possible values of a, b and c. As we have bounded the index strictly between two consecutive integers we see that this graph can never be trivial.

4.2.4 The results

Applying this method to our 35 infinite families (not including L_{GCP}) and 15 sporadic graphs found in Chapters 2 and 3, we find a total of 14 trivial non-bipartite Salem graphs. These are the graphs below in Figure 4.4. It is easy to calculate that $\lambda_1(A_{17}) = 6$, the graphs G_{31}, A_{13}, A_{14} and A_{15} have $\lambda_1 = 4$ and the rest have $\lambda_1 = 3$.

With the exception of these 14 graphs, for all possible parameters, we can bound the indices of G_1, G_2, G_3, $G_6, \ldots, G_{11}$ and G_{15} strictly between 2 and 3, the indices of G_{13}, G_{14}, $G_{16}, \ldots, G_{25}$, $A_3, \ldots, A_{11}$ strictly between 3 and 4, and the indices of A_{12} strictly between 4 and 5. The graphs G_4, G_5 and G_{12} have indices between strictly 2 and 4 but $\lambda_1 = 3$ only occurs twice and both of these are shown in Figure 4.4. Appendix A.3 gives explicit details of these bounds, listing the (potentially infinite) graphs used for the lower and upper bounds.

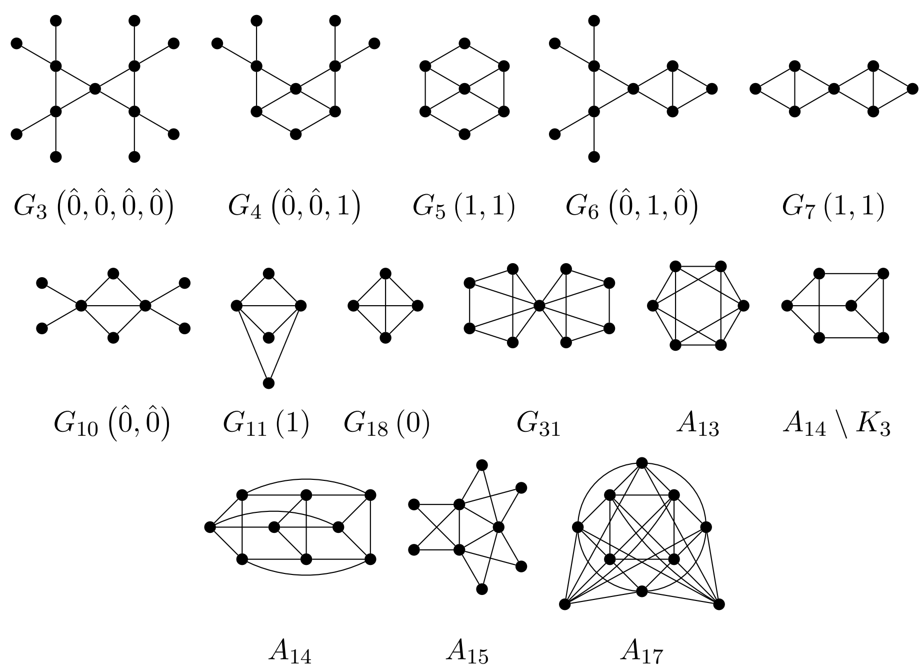


Figure 4.4: The trivial Salem generalised line graphs from the families found in Chapters 2 and 3 (excluding L_{GCP}).

In Chapter 2 we were able to classify the entire family of 1-Salem graphs by partitioning them into bipartite graphs, generalised line graphs and the exceptional graphs. Since the last group is finite (but large) it is certainly possible to calculate which of these are trivial Salem graphs. Jonathan Cooley calculated these for the first version of [31] (see [13]) and they are presented in Figure 4.5. There are seven of them on no more than 10 vertices, two of which (T_4 and T_5) have index 4 while the rest have $\lambda_1 = 3$. These graphs, along with the first nine graphs in Figure 4.4, make up a complete list of the non-bipartite trivial 1-Salem graphs.

It is worth noting that all but five of these trivial non-bipartite Salem graphs are in fact integral, and those five are $G_4(\hat{0}, \hat{0}, 1)$, E_2, E_4, E_5 and E_7. In [12], the authors classified all of the integral graphs with $\lambda_1 \leq 3$, and we note that all of the trivial

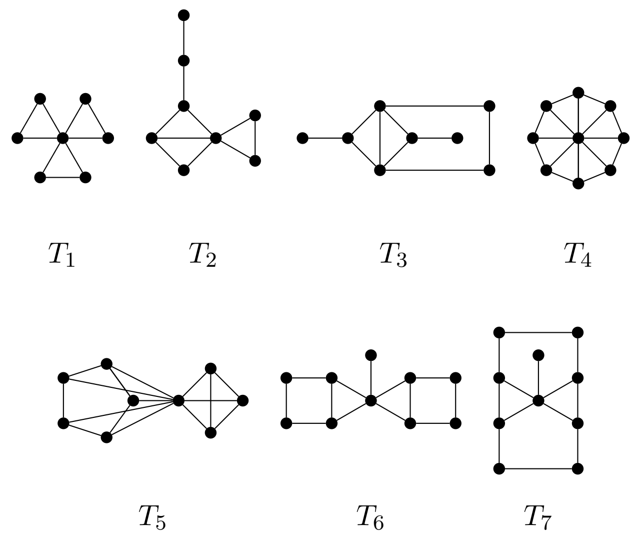


Figure 4.5: The trivial exceptional 1-Salem graphs.

graphs here with $\lambda_1 = 3$ overlap with those in the paper that are also 1-Salem, as would be expected; any integral graph with index 3 which also has the property that on removing one vertex it becomes cyclotomic must be 1-Salem by Theorem 1.3.10. A particularly curious graph is $G_4(\hat{0}, \hat{0}, 1)$ as it is the only non-integral trivial 1-Salem generalised line graph.

4.2.5 The graph L_{GCP}

At the start of this section we put the graph L_{GCP} from Theorem 3.2.1 aside from our calculations. Unlike the other generalised line graph families, L_{GCP} cannot be broken done into a central, fixed-size graph and paths, cycles and paths with snake's tongues of varying length, because it has the family of GCPs at the center; in a sense, it is an infinite family of infinite families. This causes problems, as we cannot merely take each GCP with varying numbers of single vertex pendent paths attached to plug into Lemma 4.2.1 and use the same method. Furthermore, there are just too many possibilities to consider. We shall instead prove the lesser result, that there are only only three possible integers the index of L_{GCP} may take, one of which only occurs in certain cases.

Proposition 4.2.4. *The graph $L_{GCP(n,m)}$ from Theorem 3.2.1 has*

$$n - 2 \leq \lambda_1 \left(L_{GCP}\right) < n + 1,$$

with equality on the left hand side if and only if n is even and $L_{GCP} = GCP\,(n, n/2)$.

Proof. We firstly claim that the smallest possible index we will find for a $GCP\,(n, m)$ will occur when n is even and $m = n/2$: Perron-Frobenius and non-negative matrices tell us that any larger m will have a strictly larger index and, for n even, $GCP\,(n+1, (n-1)/2)$ certainly contains $GCP\,(n, n/2)$ as an induced subgraph (see Theorem 1.3.10). We also know from Lemma 1.3.7(iii) that this graph has $\lambda_1 = n-2$, since $d\,(v) = n-2$ for each $v \in V$. Any supergraph of this graph will have a strictly larger index, proving the lower bound and both sides of our if and only if statement.

For the upper bound we recall from Lemma 1.3.7(ii) that $\lambda_1\,(G) \leq \Delta\,(G)$, the maximum of the vertex degrees. Clearly the largest degree a vertex in L_{GCP} can have is $n+1$, found when a K_3 or a $GCP\,(2, 1)$ is attached to a vertex of maximal degree from the $GCP\,(n, m)$. We can make this a strict inequality by attaching a single vertex pendent path to any vertex v of our graph with $d\,(v) < n+1$, as this graph will be a supergraph but have the same maximum degree (note that we do not even require that this supergraph is still a generalised line graph, as we are only interested in the largest eigenvalue). □

Whilst we can say exactly when $n-2$ occurs as an index of L_{GCP}, it is not so easy for $n-1$ and n. If we take $L_{GCP} = K_n$ (with $n > 2$) then we can have infinitely many graphs where $\lambda_1 = n-1$. Also, an example of L_{GCP} with $\lambda_1 = n$ can be seen in Figure 4.6 below (a non-integral, trivial Salem graph). It is much harder to classify exactly which graphs will have $\lambda_1\,(L_{GCP}) = n-1$ or n.

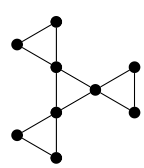


Figure 4.6: An example of a trivial graph L_{GCP} from Theorem 3.2.1 where $\lambda_1 = n$.

Chapter 5

A miscellany

This chapter consists of two results that did not fit in the previous work. The first is a proof of the spectrum of $GCP(n,m)$, which is not a new result, but included for its pleasant and intuitive proof. The second is about the connection between the bipartite complements of line graphs and the line graphs with two positive eigenvalues, which came about from studying the graphs with $\lambda_2 \leq 1$ in Section 3.3. This result was the focus of the paper [29].

5.1 The spectrum of $GCP(n,m)$

Here we present a proof detailing the exact spectrum of $GCP(n,m)$, where $0 < m < n/2$. The result itself is not new, in fact it is considered well-known, but the proof provided arrives at the result in a very pleasing way. We begin with a useful lemma, detailing what happens as we remove vertices from a $GCP(n,m)$.

Lemma 5.1.1. *On removing a vertex from a graph $GCP(n,m)$, where $0 < m < n/2$, we induce either a $GCP(n-1,m)$ or a $GCP(n-1,m-1)$. Furthermore, if $m = 0$ then removing a vertex induces a $GCP(n-1,0)$ and if $m = n/2$ then removing a vertex induces a $GCP(n-1,m-1)$.*

Proof. We prove this by considering the number of vertices of each degree. In a $GCP(n,m)$ there are $n-2m$ vertices of degree $n-1$ and $2m$ vertices of degree $n-2$. We first consider removing a vertex of degree $n-1$, with $0 \leq m < n/2$. This will reduce the degrees of all the other vertices so we are left with a graph on $n-1$ vertices with $n-2m-1$ vertices of degree $n-2$ and $2m$ vertices of degree $n-3$, which is precisely a $GCP(n-1,m)$.

Removing one of the degree $n-2$ vertices with $0 < m \leq n/2$ reduces the degrees of all but one vertex, which will also be of degree $n-2$ as these two correspond to one of the m edges removed. We are then left with a graph on $n-1$ vertices with $n-2m+1$ vertices of degree $n-2$ and $2m-2$ vertices of degree $n-3$, which is a $GCP(n-1,m-1)$. $\square$

We also benefit from knowing the spectra of two well-known GCPs: recall from Theorem 1.3.8 that the complete graph $K_n = GCP(n,0)$ has the spectrum $n-1^{(1)}$ and $-1^{(n-1)}$, and for n even, the cocktail party graph $GCP(n,n/2)$ has the spectrum $n-2^{(1)}$, $0^{(n/2)}$ and $-2^{(n/2-1)}$.

Lemma 5.1.1 tells us we can start with a cocktail party graph, remove vertices until we induce a $GCP(n,m)$ and continue removing vertices until we induce a complete graph; essentially, the GCP sits between the two. The method used to find the spectrum of $GCP(n,m)$ is then to consider how many 0 and -2 eigenvalues it gets from the cocktail party graph above it and how many -1 eigenvalues it gets from the complete graph beneath it. To clarify this idea, we shall look at the specific example of the graph $GCP(6,2)$, which we can induce from the cocktail party graph on 8 vertices and work down to induce a complete graph on 4 vertices. The interlacing below in Figure 5.1 tells us that $GCP(6,2)$ has two 0 eigenvalues, one -1 and one -2.

$GCP(8,4)$:	λ_1	0	0	0	0	-2	-2	-2
$GCP(7,3)$:	λ_1	0	0	0	λ_5	-2	-2	
$GCP(6,2)$:	λ_1	0	0	-1	λ_5	-2		
$GCP(5,1)$:	λ_1	0	-1	-1	λ_5			
$GCP(4,0)$:	λ_1	-1	-1	-1				

Figure 5.1: Interlacing the spectra of five GCPs. The λ_i are the values of the i^{th} eigenvalue for the graph in that line $(i = 1, 5)$.

In this case and, as we shall see, in every case there are only two eigenvalues that are not 0, -1 or -2. Finding these requires some simple facts about the characteristic polynomial of a graph and they are calculated explicitly below in Proposition 5.1.2.

Proposition 5.1.2. *The characteristic polynomial of a graph $G = GCP(n,m)$ with $0 < m < n/2$ is*

$$\chi_G(x) = \left(x^2 - (n-3)x - 2(n-m-1)\right) x^m (x+1)^{n-2m-1} (x+2)^{m-1}$$

and hence has the spectrum

$$\lambda_1^{(1)}, 0^{(m)}, -1^{(n-2m-1)}, \lambda_{n-m-1}^{(1)}, -2^{(m-1)}$$

where

$$\lambda_1 = \frac{1}{2}\left(n-3+\sqrt{(n+1)^2-8m}\right) \text{ and } \lambda_{n-m-1} = \frac{1}{2}\left(n-3-\sqrt{(n+1)^2-8m}\right).$$

Proof. By Lemma 5.1.1 we can remove m vertices from G to induce a $GCP(n-m,0) = K_{n-m}$. We know that $GCP(n-m,0)$ has -1 as an eigenvalue $n-m-1$ times, so by interlacing (Theorem 1.3.9) $GCP(n-m+1,1)$ has -1 as an eigenvalue $(n-m-1)-1$ times. Continuing this we see that G itself has -1 as an eigenvalue $(n-m-1)-m = n-2m-1$ times. We then observe that the characteristic polynomial $\chi_G(x)$ must contain $(x+1)^{n-2m-1}$ as a factor.

Similarly, if we start with a $GCP(2(n-m), n-m)$ then after removing $n-2m$ vertices we can induce the graph G itself. Again we know that the spectrum of $GCP(2(n-m), n-m)$ has 0 with multiplicity $(n-m)$ and -2 with multiplicity $(n-m-1)$. Interlacing tells us that the spectrum of $GCP(2(n-m)-1, n-m-1)$ must have 0 with multiplicity $((n-m)-1)$ and -2 with multiplicity $((n-m-1)-1)$, which means that G has 0 as an eigenvalue $(n-m)-(n-2m) = m$ times and -2 as an eigenvalue $(n-m-1)-(n-2m) = m-1$ times. This tells us that $\chi_G(x)$ contains $x^m(x+2)^{m-1}$ as a factor.

A quick count tells us that all but two of the n eigenvalues are accounted for and these $n-2$ eigenvalues equal either 0, -1 or -2, which means that the characteristic polynomial can be written in the following way

$$\chi_G(x) = \left(\alpha x^2 + \beta x + \gamma\right) x^m (x+1)^{n-2m-1} (x+2)^{m-1}. \tag{5.1}$$

However, recalling from Lemma 1.3.1 that the x^{n-2} term of the characteristic polynomial is equal to the number of edges, we can also write it as

$$\chi_G(x) = x^n + 0x^{n-1} - \left(\frac{a(a-1)}{2} - b\right) x^{n-2} + \ldots, \tag{5.2}$$

noting that the number of edges in a complete graph on a vertices is $a(a-1)/2$. Clearly then $\alpha = 1$. Furthermore we can use the binomial theorem to expand

$$x^m (x+1)^{n-2m-1} (x+2)^{m-1}$$

and compare equations (5.1) and (5.2) to calculate what β and γ are. Working these through we find that $\beta = -(n-3)$ and $\gamma = -2(n-m-1)$ giving the result. □

Two immediate corollaries of this result are the following, the second of which relates it back to the work of the previous chapters.

Corollary 5.1.3. *The graphs $GCP(n,m)$ with $0 \leq m \leq \lfloor n/2 \rfloor$ are integral if and only if $m = 0$ or $m = n/2$ with n even.*

Proof. The graphs $GCP(n,0)$ and, for each even n, $GCP(n,n/2)$ are regular, so their indices are clearly $n-1$ and $n-2$, respectively. Therefore for any other m, Perron-Frobenius then tells us that $n-2 < \lambda_1(GCP(n,m)) < n-1$, so can never be an integer. Similarly, for each odd n the index of $GCP(n,(n-1)/2)$ is strictly greater than $n-2$ so again $\lambda_1(GCP(n,m)) \in (n-2,n-1)$. □

Corollary 5.1.4. *For $n \geq 4$ and $(n,m) \neq (4,2)$, $GCP(n,m)$ is a Salem graph and trivial if and only if $m = 0$ or $m = n/2$ with n even.*

5.2 A connection between the bipartite complements of line graphs and the line graphs with $\lambda_3 \leq 0$

The work in the following section came about whilst studying the graphs with $\lambda_2 \leq 1$ in Section 3.3. Proposition 5.2.4 was observed first, followed by Observation 5.2.3 and then Propositions 5.2.5 and 5.2.6. If Theorems 3.3.2 and 5.2.1 (see below) were not known then this work would have allowed us to classify some, but not all, of the graphs found in Section 3.3.2. We present the results below in a more natural order, as they appeared in [29].

In 1974 Cvetković and Simić showed in [24] the following result.

Theorem 5.2.1 (see [24], Theorem 8)**.** *A graph G is bipartite and the complement of a line graph if and only if G is an induced subgraph of some of the graphs CS_1, $CS_2 = CS_2(n)$ (with $n \geq 0$) and $CS_3 = CS_3(m,n,p)$ (with $p < n \leq m$; $p \geq 0$, $m,n \geq 1$) in Figure 5.2.*

Later, Borovićanin proved the following result.

Theorem 5.2.2 (see [4], Theorem 3)**.** *A connected line graph $L(H)$ has $\lambda_3 \leq 0$ if and only if $L(H)$ is an induced subgraph of some of the graphs B_1, B_2, $B_3 = B_3(n)$ (with $n \geq 0$) and $B_4 = B_4(m,n,p)$ (with $p < n \leq m$; $p \geq 0$, $m,n \geq 1$) in Figure 5.3.*

We make the immediate observation from these:

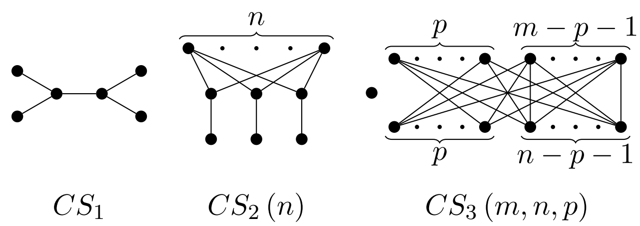


Figure 5.2: The three graphs from Theorem 5.2.1. Note that these are all subgraphs of the graphs in Theorem 3.3.2 and Figure 3.7. This is to be expected by Theorem 3.3.3.

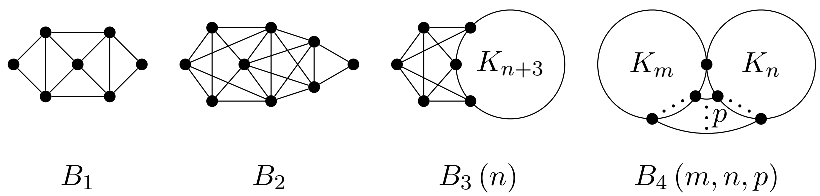


Figure 5.3: The four graphs from Theorem 5.2.2. The large circles are used to denote a complete graph of that size.

Observation 5.2.3. *For the graphs in Figures 5.2 and 5.3, $B_3 = \overline{CS}_2$, $B_4 = \overline{CS}_3$ and $B_2 \supset \overline{CS}_1$.*

Our goal is to explore why these graphs are related and in doing so we offer a new proof of Theorem 5.2.2. In Theorem 3.3.3 we noted the result of Cvetković that if G is the complement of a line graph then $\lambda_2(G) \leq 1$. This was proved using the Courant-Weyl inequalities and, using the same method along with the symmetry of the eigenvalues of a bipartite graph, we show that there can be even more structure in the spectrum of a line graph when it has a bipartite complement.

Proposition 5.2.4. *If a line graph $L(H)$ has a bipartite complement, then we have $\lambda_3(L(H)) \leq 0$.*

Proof. Let $G = \overline{L(H)}$. The spectrum of a bipartite graph is symmetric around 0, meaning that $\lambda_1 = -\lambda_n$, $\lambda_2 = -\lambda_{n-1}$, and so forth. This fact and Theorem 3.3.3 tell us that if G is bipartite then we also have that $\lambda_{n-1}(G) \geq -1$.

Take the second of the Courant-Weyl inequalities in Theorem 3.1.1 and let $A = L(H)$ and $B = G$, so that $A + B = K_n$. Also let $i = 2$ and $j = n-1$ so that $i - j + n = 3$. We know the spectrum of the complete graph from Theorem 1.3.8, and hence $\lambda_i(K_n) = -1$ for $i = 2, \ldots, n$. Bringing these together we get

$$\begin{aligned}\lambda_2(K_n) &\geq \lambda_3(L(H)) + \lambda_{n-1}(G) \\ 0 &\geq \lambda_3(L(H)) + \lambda_{n-1}(G) + 1.\end{aligned}$$

Then for G bipartite we deduce that $\lambda_3(L(H)) \leq 0$, as required. □

We now prove the two directions of Theorem 5.2.2 separately, starting with the reverse.

Proposition 5.2.5. *If a line graph $L(H)$ is an induced subgraph of one of the graphs B_1, B_2, B_3 or B_4, then $\lambda_3(L(H)) \leq 0$.*

Proof. The graphs B_1 and B_2 have only a finite number of vertices so we can easily find their eigenvalues and see that $\lambda_3 \leq 0$. Any subgraphs will also then have $\lambda_3 \leq 0$ by interlacing (Theorem 1.3.9). Since CS_2 and CS_3 are bipartite and the complements of line graphs, Observation 5.2.3 and Proposition 5.2.4 tell us that we must then have $\lambda_3(B_3) \leq 0$ and $\lambda_3(B_4) \leq 0$. □

Proposition 5.2.6. *If a connected line graph $L(H)$ has $\lambda_3 \leq 0$ then it is an induced subgraph of at least one of the graphs B_1, B_2, B_3 and B_4.*

Proof. Let G be the complement of $L(H)$, then G is certainly either bipartite or non-bipartite. If G is bipartite then it is the bipartite complement of a line graph, so by Theorem 5.2.1 it must be an induced subgraph of CS_1, CS_2 or CS_3. Therefore by Observation 5.2.3 $L(H)$ must be an induced subgraph of B_3, B_4 or contained in B_2.

If G is non-bipartite we have a bit more work to do. In this case we know that G contains an odd cycle C_n for some odd n and that $\overline{C}_n$ must be in $L(H)$. For $n \geq 7$, we have the path on 6 vertices P_6 as a subgraph of C_n and $\lambda_2(P_6) > 1$ so by interlacing we also have $\lambda_2(C_n) > 1$. Using the second of the Courant-Weyl inequalities in Theorem 3.1.1 again, with $i = j = 2$, $B = C_n$ and $A + B = K_n$ we get that $\lambda_n\left(\overline{C}_n\right) < -2$. This means that for odd $n \geq 7$ the graph $\overline{C}_n$ cannot be a line graph nor an induced subgraph of a line graph by Theorem 1.3.5. Furthermore, the complement of the cycle C_5 is C_5 itself, which has $\lambda_3 > 0$. The conclusion to all this is that in the complement of a line graph with $\lambda_3 \leq 0$ the only odd cycles we will find will be of length 3.

We now know that G contains a K_3 so that means that $L(H)$ contains $\overline{K}_3 = 3K_1$. To complete the proof we grow line graphs starting with $3K_1$, increasing the number of vertices. To grow line graphs we recall the structure in Theorem 1.3.2 and consider the cliques. At each step we can either

- expand a clique to the next larger one
- expand two non-adjacent cliques and have them share the new vertex
- attach a single vertex pendent path (a K_2) to any vertex currently in only one clique.

Once we have done all these we must then consider adding to each one an edge between any two vertices in only one clique each (in effect, adding in another K_2), except between any two of the original three non-adjacent vertices. At each step we discard any graphs with $\lambda_3 > 0$. We do not add in extra isolated vertices or grow two separate graphs (that is, start growing from one vertex of $3K_1$ and then start growing from another without connecting them) as any resulting connected graphs can be grown using the method above without giving unnecessary extra disconnected graphs along the way.

When the growing has reached 12 vertices we pause and have a look at the graphs we have. There are 37 non-isomorphic line graphs on 12 vertices that contains an induced $3K_1$ and have $\lambda_3 \leq 0$: $B_3\,(4) \cup 2K_1$; $B_3\,(5) \cup K_1$; $B_4\,(m,n,p) \cup 2K_1$ for the triples $(6,5,p)$, $(7,4,p)$, $(8,3,p)$, $(9,2,p)$, $(10,0,0)$ with varying appropriate p; and $B_4\,(m,n,p) \cup K_1$ for the triples $(6,6,p)$, $(7,5,p)$, $(8,4,p)$, $(9,3,p)$, $(10,2,p)$ again with varying p.

We know that ultimately we want a connected line graph. Any attempts to connect these graphs either by adding in edges or growing them further whilst keeping an induced $3K_1$ will result in a graph with $\lambda_3 > 0$ (Figure 3 in [4] has some subgraphs with $\lambda_3 > 0$ that are very easy to spot in this process). Looking at the graphs with 11 vertices or fewer we see the graphs B_1 and B_2, and all their subgraphs (with potentially some extra isolated vertices – if the graph without them contains an induced $3K_1$ then we can safely ignore them) along with some other subgraphs of B_3 and B_4 with one or two isolated vertices but no others. □

This proof highlights the fact that there are only finitely many connected line graphs with $\lambda_3 \leq 0$ with non-bipartite complements, but infinitely many with bipartite ones. By counting the non-isomorphic graphs that appear in the growing process we see that there are in fact only 19 connected line graphs with $\lambda_3 \leq 0$ and non-bipartite complement. Another (easier) way to spot this is to count the number of non-isomorphic non-bipartite induced subgraphs there are of $\overline{B}_1$ and $\overline{B}_2$; there are 24 of these but 5 have disconnected complements. Knowing that there are only finitely many from the original proof of Theorem 5.2.2 in advance helps us know that the growing process used in the proof above will actually terminate.

Appendix A

Appendices

A.1 Numbers as symbols

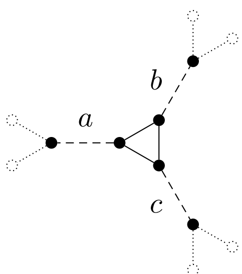
Below is a table of numbers rounded to 3 decimal places and the full number they are used to represent, as mentioned at the start of Chapter 1.

Symbol	Found as		Symbol	Actual number
-1.825	third root of	$x^3 - 3x^2 - 11x - 4$	0.268	$2 - \sqrt{3}$
0.311	second root of	$x^3 - x^2 - 3x + 1$	0.382	$\left(3 - \sqrt{5}\right)/2$
0.327	second root of	$x^3 - x^2 - 12x + 4$	0.414	$\sqrt{2} - 1$
0.349	second root of	$x^4 - 10x^2 - 8x + 4$	0.586	$2 - \sqrt{2}$
0.352	second root of	$x^4 - 12x^2 - 10x + 5$	0.618	$\left(-1 + \sqrt{5}\right)/2$
0.385	second root of	$x^3 - 4x^2 - 9x + 4$	1.414	$\sqrt{2}$
0.662	second root of	$x^4 - 5x^2 + 2$	1.618	$\left(1 + \sqrt{5}\right)/2$
1.117	second root of	$x^4 - 2x^3 - 5x^2 + 4x + 3$	1.732	$\sqrt{3}$
1.176	largest root of	$x^{10} + x^9 - x^7 - x^6 - x^5$	2.562	$\left(1 + \sqrt{17}\right)/2$
		$-x^4 - x^3 + x + 1$		
1.967	solution of	$0.349 + 1.618$		
2.572	largest root of	$x^6 - 3x^5 + 2x^4 - 3x^3$		
		$+2x^2 - x + 1$		
2.961	solution of	$2.572 + 1/2.572$		

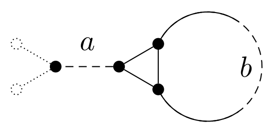
Table A.1: The 3 decimal place symbols used throughout, and the full number they represent.

A.2 The complete list of 1-Salem generalised line graphs from Theorem 2.3.2

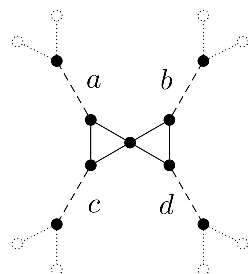
Below in Figures A.1, A.2 and A.3 are the the 25 infinite families of 1-Salem generalised line graphs and the six sporadic 1-Salem generalised line graphs from Theorem 2.3.2. Recall that a dashed edge indicates a path of arbitrarily many vertices, and a dotted edge and vertex indicates that the edge and vertex may or may not be present.



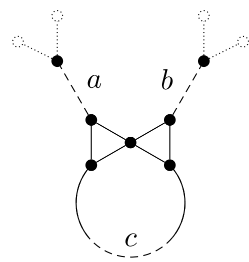
$G_1\,(a,b,c),\ a \geq 1$ or $a = \hat{0}$



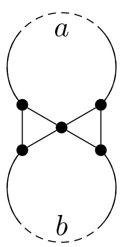
$G_2\,(a,b),\ b \geq 1$



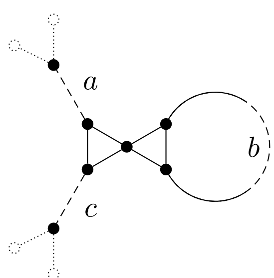
$G_3\,(a,b,c,d)$



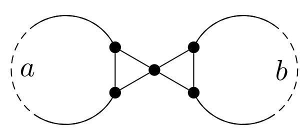
$G_4\,(a,b,c)$



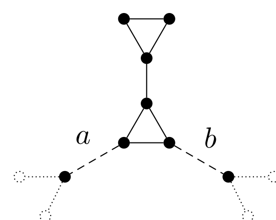
$G_5\,(a,b)$



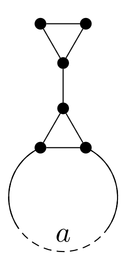
$G_6\,(a,b,c),\ b \geq 1$



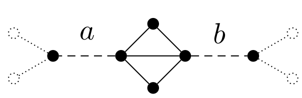
$G_7\,(a,b),\ a,b \geq 1$



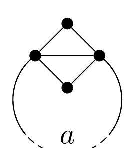
$G_8\,(a,b)$



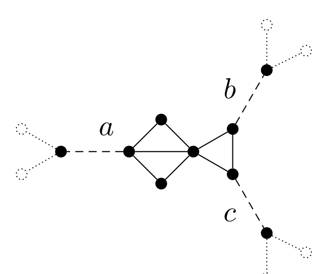
$G_9\,(a),\ a \geq 1$



$G_{10}\,(a,b)$



$G_{11}\,(a),\ a \geq 1$



$G_{12}\,(a,b,c)$

Figure A.1: Twelve of the 25 infinite families of 1-Salem generalised line graphs. The parameters a, b, c, d are the numbers of vertices in the paths and cycles and are greater than or equal to 0 unless specified.

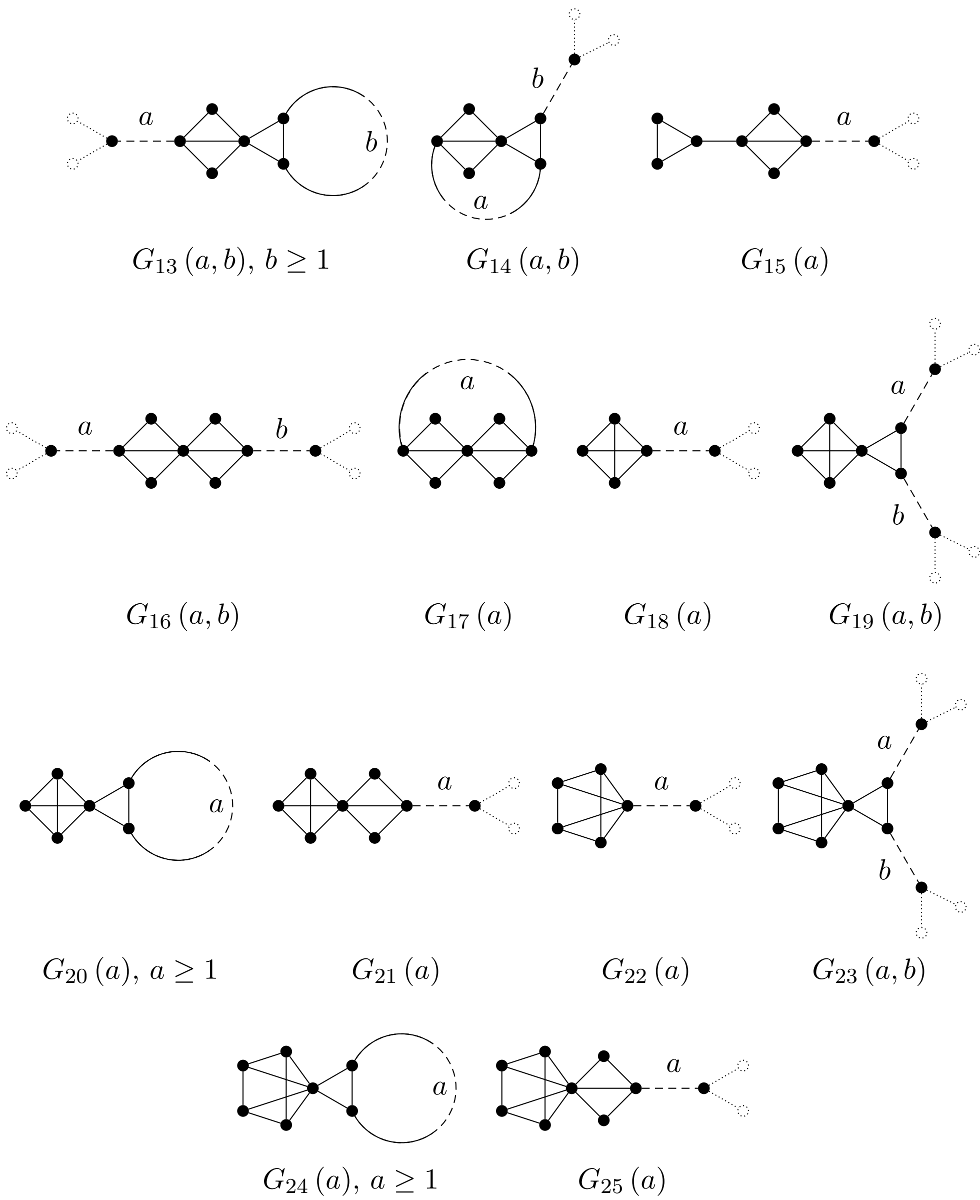
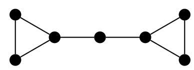
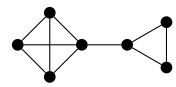


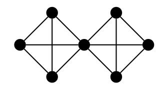
Figure A.2: Thirteen of the 25 infinite families of 1-Salem generalised line graphs. The parameters a, b are the numbers of vertices in the paths and cycles and are greater than or equal to 0 unless specified.



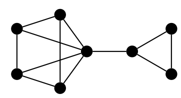
G_{26}



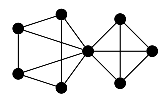
G_{27}



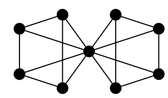
G_{28}



G_{29}



G_{30}



G_{31}

Figure A.3: The six sporadic 1-Salem generalised line graphs.

A.3 Index bounds for our Salem generalized line graphs

Below are the bounds for the largest eigenvalues of the infinite families found in Chapters 2 and 3 using the method outlined in Section 4.2. The lower bounds are subgraphs of the graphs in question, so have a smaller index by Perron-Frobenius. The upper bounds show the maximum value the index can take; when we say, for example, $\lambda_1\big(G_1(\infty,\infty,\infty)\big)$ we mean the limit of the eigenvalues for $G_1(a,b,c)$ as the path-lengths a, b and c all tend towards infinity. Tables A.2, A.3, A.4 and A.5 show the bounds for the 1-Salem graphs $G_1,\dots,G_{25}$. In Figure A.4 we define a few subgraphs of the graphs $A_3,\dots,A_{12}$ needed for our calculations, and in Table A.6 we bound the indices of ten infinite families in the set $\mathcal{P}_{1+1}$ found in Section 3.3.3.

Graph	Lower bound graph	Upper bound graph
$G_1(a,b,c)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_1(\infty,\infty,\infty)\big) = \quad 2.5$
$G_1(\hat{a},b,c)$		$\lambda_1\big(G_1(\hat{0},\infty,\infty)\big) < \quad 2.6$
$G_1(\hat{a},\hat{b},c)$		$\lambda_1\big(G_1(\hat{0},\hat{0},\infty)\big) < \quad 2.7$
$G_1(\hat{a},\hat{b},\hat{c})$		$\lambda_1\big(G_1(\hat{0},\hat{0},\hat{0})\big) < \quad 2.8$
$G_2(a,b)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_2(\infty,1)\big) < \quad 2.7$
$G_2(\hat{a},b)$		$\lambda_1\big(G_2(\hat{0},1)\big) < \quad 2.8$
$G_3(a,b,c,d)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_3(\infty,\infty,\infty,\infty)\big) < \quad 2.9$
$G_3(\hat{a},b,c,d)$		$\lambda_1\big(G_3(\hat{0},\infty,\infty,\infty)\big) < \quad 2.9$
$G_3(\hat{a},\hat{b},c,d)$		$\lambda_1\big(G_3(\hat{0},\hat{0},\infty,\infty)\big) < \quad 3$
$G_3(\hat{a},b,\hat{c},d)$		$\lambda_1\big(G_3(\hat{0},\infty,\hat{0},\infty)\big) < \quad 3$
$G_3(\hat{a},\hat{b},\hat{c},d)$		$\lambda_1\big(G_3(\hat{0},\hat{0},\hat{0},\infty)\big) < \quad 3$
$G_3(\hat{a},\hat{b},\hat{c},\hat{d})$		$\lambda_1\big(G_3(\hat{0},\hat{0},\hat{0},\hat{0})\big) = \quad 3$

Table A.2: The index bounds for the graphs G_1, G_2, G_3.

Graph	Lower bound graph	Upper bound graph
$G_4(a,b,c)$		
$a=0,\ b=0,\ c=0$	$\lambda_1\big(G_4(0,0,0)\big) \approx 2.94$	
$a\geq 1,\ b=0,\ c=0$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_4(\infty,0,0)\big) < \quad 3$
$a\geq 1,\ b\geq 1,\ c=0$	$3 \quad < \lambda_1\big(G_4(1,1,0)\big)$	$\lambda_1\big(G_4(\infty,\infty,0)\big) < \quad 3.1$
$a\geq 1,\ b\geq 1,\ c\geq 1$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_4(\infty,\infty,1)\big) < \quad 3$
$G_4(\hat{a},b,c)$		
$a=0,\ b=0,\ c=0$	$\lambda_1\big(G_4(\hat{0},0,0)\big) \approx 3.03$	
$a\geq 1,\ b=0,\ c=0$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_4(\hat{1},0,0)\big) < \quad 3$
$a=0,\ b\geq 1,\ c=0$	$3 \quad < \lambda_1\big(G_4(\hat{0},0,0)\big)$	$\lambda_1\big(G_4(\hat{0},\infty,0)\big) < \quad 3.1$
$a\geq 1,\ b\geq 1,\ c=0$	$3 \quad < \lambda_1\big(G_4(1,1,0)\big)$	$\lambda_1\big(G_4(\hat{1},\infty,0)\big) < \quad 3.1$
$a\geq 1,\ b\geq 1,\ c\geq 1$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_4(\hat{0},\infty,1)\big) < \quad 3$
$G_4(\hat{a},\hat{b},c)$		
$a\geq 0,\ b\geq 0,\ c=0$	$3 \quad < \lambda_1\big(G_4(1,1,0)\big)$	$\lambda_1\big(G_4(\hat{0},\hat{0},0)\big) < \quad 3.2$
$a=0,\ b=0,\ c=1$	$\lambda_1\big(G_4(\hat{0},\hat{0},1)\big) = 3$	
$a\geq 1,\ b\geq 0,\ c\geq 1$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_4(\hat{1},\hat{0},1)\big) < \quad 3$
$G_5(a,b)$		
$a=0,\ b=0$	$\lambda_1\big(G_5(0,0)\big) \approx 3.24$	
$a=0,\ b=1$	$\lambda_1\big(G_5(0,1)\big) \approx 3.11$	
$a=0,\ b=2$	$\lambda_1\big(G_5(0,2)\big) \approx 3.07$	
$a=0,\ b\geq 3$	$3 \quad < \lambda_1\big(G_4(1,1,0)\big)$	$\lambda_1\big(G_5(0,0)\big) < \quad 3.3$
$a\geq 1,\ b\geq 1$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_5(1,1)\big) = \quad 3$
$G_6(a,b,c)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_6(\infty,1,\infty)\big) < \quad 3$
$G_6(\hat{a},b,c)$		$\lambda_1\big(G_6(\hat{0},1,\infty)\big) < \quad 3$
$G_6(\hat{a},b,\hat{c})$		$\lambda_1\big(G_6(\hat{0},1,\hat{0})\big) = \quad 3$
$G_7(a,b)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_7(1,1)\big) = \quad 3$
$G_8(a,b)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_8(\infty,\infty)\big) < \quad 2.6$
$G_8(a,b)$		$\lambda_1\big(G_8(\hat{0},\infty)\big) < \quad 2.7$
$G_8(a,b)$		$\lambda_1\big(G_8(\hat{0},\hat{0})\big) < \quad 2.8$
$G_9(a)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_9(1)\big) < \quad 2.8$
$G_{10}(a,b)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_{10}(\infty,\infty)\big) < \quad 2.9$
$G_{10}(\hat{a},b)$		$\lambda_1\big(G_{10}(\hat{0},\infty)\big) < \quad 3$
$G_{10}(\hat{a},\hat{b})$		$\lambda_1\big(G_{10}(\hat{0},\hat{0})\big) = \quad 3$
$G_{11}(a)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_{11}(1)\big) = \quad 3$

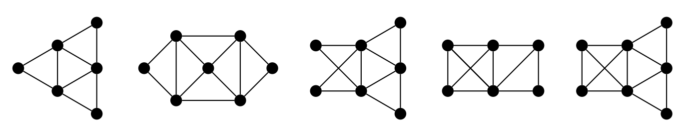
Table A.3: The index bounds for the graphs $G_4, \ldots, G_{11}$.

Graph	Lower bound graph	Upper bound graph
$G_{12}(a,b,c)$		
$a=0,\ b=0,\ c\geq 0$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_{12}(0,0,\infty)\big) < \quad 3$
$a=0,\ b\geq 1,\ c\geq 1$	$3 \quad < \lambda_1\big(G_{12}(0,1,1)\big)$	$\lambda_1\big(G_{12}(0,\infty,\infty)\big) < \quad 3.1$
$a\geq 1,\ b\geq 1,\ c\geq 1$	$3 \quad < \lambda_1\big(G_{12}(1,0,0)\big)$	$\lambda_1\big(G_{12}(\infty,\infty,\infty)\big) < \quad 3.1$
$G_{12}(\hat{a},b,c)$	$3 \quad < \lambda_1\big(G_{12}(1,0,0)\big)$	$\lambda_1\big(G_{12}(\hat{0},\infty,\infty)\big) < \quad 3.2$
$G_{12}(a,\hat{b},c)$		
$a=0,\ b=0,\ c=0$	$\lambda_1\big(G_{12}(0,\hat{0},0)\big) \approx 3.02$	
$a=0,\ b\geq 1,\ c=0$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_{12}(0,\hat{1},0)\big) < \quad 3$
$a=0,\ b\geq 0,\ c\geq 1$	$3 \quad < \lambda_1\big(G_{12}(0,1,1)\big)$	$\lambda_1\big(G_{12}(0,\hat{0},\infty)\big) < \quad 3.1$
$a\geq 1,\ b\geq 0,\ c\geq 0$	$3 \quad < \lambda_1\big(G_{12}(1,1,0)\big)$	$\lambda_1\big(G_{12}(\infty,\hat{0},\infty)\big) < \quad 3.2$
$G_{12}(\hat{a},\hat{b},c)$	$3 \quad < \lambda_1\big(G_{12}(1,0,0)\big)$	$\lambda_1\big(G_{12}(\hat{0},\hat{0},\infty)\big) < \quad 3.2$
$G_{12}(a,\hat{b},\hat{c})$	$3 \quad < \lambda_1\big(G_{12}(0,1,1)\big)$	$\lambda_1\big(G_{12}(\infty,\hat{0},\hat{0})\big) < \quad 3.2$
$G_{12}(\hat{a},\hat{b},\hat{c})$	$3 \quad < \lambda_1\big(G_{12}(1,0,0)\big)$	$\lambda_1\big(G_{12}(\hat{0},\hat{0},\hat{0})\big) < \quad 3.3$
$G_{13}(a,b)$		
$a=0,\ b=1$	$\lambda_1\big(G_{13}(0,1)\big) \approx 3.10$	
$a=0,\ b=2$	$\lambda_1\big(G_{13}(0,2)\big) \approx 3.06$	
$a=0,\ b\geq 2$	$3 \quad < \lambda_1\big(G_{12}(0,1,1)\big)$	$\lambda_1\big(G_{13}(0,1)\big) < \quad 3.2$
$a=\geq 1,\ b\geq 1$	$3 \quad < \lambda_1\big(G_{12}(1,0,0)\big)$	$\lambda_1\big(G_{13}(\infty,1)\big) < \quad 3.2$
$G_{13}(\hat{a},b)$	$3 \quad < \lambda_1\big(G_{12}(1,0,0)\big)$	$\lambda_1\big(G_{13}(\hat{0},1)\big) < \quad 3.3$
$G_{14}(a,b)$		
$a=0,\ b=0$	$\lambda_1\big(G_{14}(0,0)\big) \approx 3.28$	
$a=1,\ b=0$	$\lambda_1\big(G_{14}(1,0)\big) \approx 3.14$	
$a=2,\ b=0$	$\lambda_1\big(G_{14}(2,0)\big) \approx 3.09$	
$a\geq 3,\ b=0$	$3 \quad < \lambda_1\big(G_{12}(1,0,1)\big)$	$\lambda_1\big(G_{14}(0,0)\big) < \quad 3.3$
$a=0,\ b\geq 1$	$3.2 \quad < \lambda_1\big(G_{14}(0,0)\big)$	$\lambda_1\big(G_{14}(0,\infty)\big) < \quad 3.4$
$a=1,\ b\geq 1$	$3.1 \quad < \lambda_1\big(G_{14}(1,0)\big)$	$\lambda_1\big(G_{14}(1,\infty)\big) < \quad 3.2$
$a\geq 2,\ b\geq 1$	$3 \quad < \lambda_1\big(G_{12}(0,1,1)\big)$	$\lambda_1\big(G_{14}(2,\infty)\big) < \quad 3.1$
$G_{14}(a,\hat{b})$		
$a=0,\ b\geq 0$	$3.2 \quad < \lambda_1\big(G_{14}(0,0)\big)$	$\lambda_1\big(G_{14}(0,\hat{0})\big) < \quad 3.4$
$a=1,\ b\geq 0$	$3.1 \quad < \lambda_1\big(G_{14}(0,1)\big)$	$\lambda_1\big(G_{14}(1,\hat{0})\big) < \quad 3.3$
$a\geq 2,\ b\geq 0$	$3 \quad < \lambda_1\big(G_{12}(0,1,1)\big)$	$\lambda_1\big(G_{14}(2,\hat{0})\big) < \quad 3.2$
$G_{15}(a)$	$2 \quad = \lambda_1(K_3)$	$\lambda_1\big(G_{15}(\infty)\big) < \quad 2.9$
$G_{15}(\hat{a})$		$\lambda_1\big(G_{15}(\hat{0})\big) < \quad 3$

Table A.4: The index bounds for the graphs $G_{12},\ldots,G_{15}$.

Graph	Lower bound graph	Upper bound graph
$G_{16}(a,b)$	$3.2 \quad < \lambda_1\big(G_{16}(0,0)\big)$	$\lambda_1\big(G_{16}(\infty,\infty)\big) < \quad 3.4$
$G_{16}(\hat{a},b)$		$\lambda_1\big(G_{16}(\hat{0},\infty)\big) < \quad 3.4$
$G_{16}(\hat{a},\hat{b})$		$\lambda_1\big(G_{16}(\hat{0},\hat{0})\big) < \quad 3.5$
$G_{17}(a)$		
$a = 0$	$\lambda_1\big(G_{17}(0)\big) \approx 3.56$	
$a \geq 1$	$3.2 \quad < \lambda_1\big(G_{16}(0,0)\big)$	$\lambda_1\big(G_{17}(1)\big) < \quad 3.5$
$G_{18}(a)$	$3 \quad = \lambda_1(K_4)$	$\lambda_1\big(G_{18}(\infty)\big) < \quad 3.1$
$G_{18}(\hat{a})$		$\lambda_1\big(G_{18}(\hat{0})\big) < \quad 3.2$
$G_{19}(a,b)$	$3 \quad = \lambda_1(K_4)$	$\lambda_1\big(G_{19}(\infty,\infty)\big) < \quad 3.4$
$G_{19}(\hat{a},b)$		$\lambda_1\big(G_{19}(\hat{0},\infty)\big) < \quad 3.4$
$G_{19}(\hat{a},\hat{b})$		$\lambda_1\big(G_{19}(\hat{0},\hat{0})\big) < \quad 3.4$
$G_{20}(a)$	$3 \quad = \lambda_1(K_4)$	$\lambda_1\big(G_{20}(1)\big) < \quad 3.4$
$G_{21}(a)$	$3 \quad = \lambda_1(K_4)$	$\lambda_1\big(G_{21}(\infty)\big) < \quad 3.6$
$G_{21}(\hat{a})$		$\lambda_1\big(G_{21}(\hat{0})\big) < \quad 3.6$
$G_{22}(a)$	$3 \quad < \lambda_1\big(GCP(5,2)\big)$	$\lambda_1\big(G_{22}(\infty)\big) < \quad 3.4$
$G_{22}(\hat{a})$		$\lambda_1\big(G_{22}(\hat{0})\big) < \quad 3.5$
$G_{23}(a,b)$	$3 \quad < \lambda_1\big(GCP(5,2)\big)$	$\lambda_1\big(G_{23}(\infty,\infty)\big) < \quad 3.6$
$G_{23}(\hat{a},b)$		$\lambda_1\big(G_{23}(\hat{0},\infty)\big) < \quad 3.6$
$G_{23}(\hat{a},\hat{b})$		$\lambda_1\big(G_{23}(\hat{0},\hat{0})\big) < \quad 3.6$
$G_{24}(a)$	$3 \quad < \lambda_1\big(GCP(5,2)\big)$	$\lambda_1\big(G_{24}(1)\big) < \quad 3.6$
$G_{25}(a)$	$3 \quad < \lambda_1\big(GCP(5,2)\big)$	$\lambda_1\big(G_{25}(\infty)\big) < \quad 3.8$
$G_{25}(\hat{a})$		$\lambda_1\big(G_{25}(\hat{0})\big) < \quad 3.8$

Table A.5: The index bounds for the graphs $G_{16}, \ldots, G_{25}$.



A_3^* A_5^* A_6^* A_8^* A_{11}^*

Figure A.4: Five subgraphs of the graphs in Figures 3.12 and 3.13 required for our calculations.

Graph	Lower bound graph	Upper bound graph
$A_3(a,b)$		
$a \geq 0,\ b = 0$	$3.4 \quad < \lambda_1\big(A_3(0,0)\big)$	$\lambda_1\big(A_3(\infty,0)\big) < \quad 3.5$
$a \geq 0,\ b \geq 1$	$3.2 \quad < \lambda_1\big(A_3^*\big)$	$\lambda_1\big(A_3(\infty,1)\big) < \quad 3.4$
$A_4(a)$	$3.2 \quad < \lambda_1\big(A_4(0)\big)$	$\lambda_1\big(A_4(\infty)\big) < \quad 3.3$
$A_5(a)$		
$a = 0$	$\lambda_1\big(A_5(0)\big) \approx 3.78$	
$a \geq 1$	$3.6 \quad < \lambda_1\big(A_5^*\big)$	$\lambda_1\big(A_5(1)\big) < \quad 3.8$
$A_6(a)$		
$a = 0$	$\lambda_1\big(A_6(0)\big) \approx 3.71$	
$a \geq 1$	$3.5 \quad < \lambda_1\big(A_6^*\big)$	$\lambda_1\big(A_6(1)\big) < \quad 3.7$
$A_7(a)$	$3.7 \quad < \lambda_1\big(A_7(0)\big)$	$\lambda_1\big(A_7(\infty)\big) < \quad 3.8$
$A_8(a,b)$	$3.5 \quad < \lambda_1\big(A_8^*\big)$	$\lambda_1\big(A_8(1,\infty)\big) < \quad 3.8$
$A_9(a,b)$		
$a \geq 0,\ b = 0$	$3.7 \quad < \lambda_1\big(A_9(0,0)\big)$	$\lambda_1\big(9_3(\infty,0)\big) < \quad 3.9$
$a \geq 0,\ b \geq 1$	$3.5 \quad < \lambda_1\big(A_8^*\big)$	$\lambda_1\big(A_9(\infty,1)\big) < \quad 3.8$
$A_{10}(a)$	$3.9 \quad < \lambda_1\big(A_{10}(0)\big)$	$\lambda_1\big(A_{10}(\infty)\big) < \quad 4$
$A_{11}(a)$		
$a = 0$	$\lambda_1\big(A_{11}(0)\big) \approx 3.88$	
$a \geq 1$	$3.6 \quad < \lambda_1\big(A_{11}^*\big)$	$\lambda_1\big(A_{11}(1)\big) < \quad 3.9$
$A_{12}(a)$	$4 \quad < \lambda_1\big(A_{12}(0)\big)$	$\lambda_1\big(A_{12}(\infty)\big) < \quad 4.1$

Table A.6: The index bounds for the graphs $A_3, \dots, A_{12}$.

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