COHOMOLOGICAL DIMENSION OF MACKEY FUNCTORS FOR INFINITE GROUPS

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ABSTRACT. We consider the cohomology of Mackey functors for infinite groups and define the Mackey-cohomological dimension $cd_{\mathfrak{M}}G$ of a group G. We will relate this dimension to other cohomological dimensions such as the Bredon-cohomological dimension $cd_{\mathfrak{F}}G$ and the relative cohomological dimension $\mathfrak{F}$-cdG. In particular we show that for virtually torsion free groups the Mackey-cohomological dimension is equal to both $\mathfrak{F}$-cdG and the virtual cohomological dimension.

1. Introduction

Mackey-functors for finite groups have been around for a long time thanks to having common properties of natural functors for finite groups such as group cohomology, the Burnside ring, the representation ring, algebraic K-theory or topological K-theory for classifying spaces to name a few. Motivation for this work on Mackey functors for finite groups was representation theory, see [18, 19, 20] as well as equivariant cohomology theory [5, 6]. The study of Mackey functors for infinite groups is a fairly recent phenomenon, see for example [11]. In connection with the Baum-Connes conjecture, Bredon homology with coefficients in Mackey functors, especially with coefficients in the representation ring, seem to be of importance [14].

Let G be a group and denote by $\mathfrak{O}_{\mathfrak{F}}G$ the orbit category, which has as objects cosets G/K, where $K \in \mathfrak{F}$ the family of finite subgroups of G and where morphisms are G-maps $G/L \to G/K$ for $G/L, G/K \in \mathfrak{O}_{\mathfrak{F}}G$. The most common definition of a Mackey-functor is a pair of functors

$$(M^*, M_*) : \mathfrak{O}_{\mathfrak{F}}G \to \mathcal{A}b,$$

where M^* is contravariant, M_* is covariant and which coincide on objects. Furthermore they satisfy a certain pull-back condition, which we will describe later. A different but equivalent definition turns out to be better suited for our purposes. We shall introduce this in Section 3. As we are dealing with an abelian category, things can be done pointwise and we therefore have the notion of exact sequences. Also, the category of Mackey functors has enough projectives and hence there is the notion of cohomology of Mackey-functors and of cohomological dimension $cd_{\mathfrak{M}_{\mathfrak{F}}}G$.

Our motivation comes from classifying spaces for proper actions and their algebraic mirror, Bredon cohomology. Bredon functors are slightly less complicated gadgets. A Bredon functor, or Bredon module, is a contravariant functor $T : \mathfrak{O}_{\mathfrak{F}}G \to \mathcal{A}b$ and there is a natural way to define cohomology and the cohomological dimension $cd_{\mathfrak{F}}G$ of a group G. This is the projective dimension in the Bredon category of the constant functor $\underline{\mathbb{Z}}$. A classifying space for proper actions, denoted

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$\underline{E}G$ is a G-CW-complex X satisfying the following: the fixed point complex X^K is contractible if K is a finite subgroup of G and empty otherwise. Constructions by Milnor [13] and Segal [17] imply that these always exist but these constructions give us very big models. We denote by $gd_{\mathfrak{F}}G$ the minimal dimension of a model for an $\underline{E}G$. By taking fixed points, the augmented cellular chain complex of an $\underline{E}G$ gives us a projective resolution of Bredon-functors

$$C_*(X^{(-)}) \twoheadrightarrow \underline{\mathbb{Z}}$$

and hence $cd_{\mathfrak{F}}G \leq gd_{\mathfrak{F}}G$. Work of Dunwoody [3] for dimension one and Lück [8] for higher dimensions implies, that unless $cd_{\mathfrak{F}}G = 2$, both $cd_{\mathfrak{F}}G = gd_{\mathfrak{F}}G$. Furthermore, there are examples where $cd_{\mathfrak{F}}G = 2$ but $gd_{\mathfrak{F}}G = 3$ [1]. In Section 3 we shall compare the Bredon cohomology with the cohomology of Mackey functors and will show (Corollary 3.9) that for every group G

$$cd_{\mathfrak{M}_{\mathfrak{F}}}G \leq cd_{\mathfrak{F}}G.$$

Another quantity of interest is the relative cohomological dimension $\mathfrak{F}$-cdG. This is defined to be the length of the shortest relative projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. A relative projective resolution $P_* \twoheadrightarrow \mathbb{Z}$ is an exact sequence of $\mathbb{Z}G$-modules, which splits when restricted to each finite subgroup of G and where the P_i are direct summands of direct sums of modules induced up from finite subgroups. In particular, permutation modules $\mathbb{Z}[G/K]$ with K finite are relative projective. It can be shown that $cd_{\mathbb{Q}}G \leq \mathfrak{F}$-$cdG$. For detail on relative cohomology see [15]. We shall show (Theorem 4.3) that always $\mathfrak{F}$-$cdG \leq cd_{\mathfrak{M}_{\mathfrak{F}}}G$ and we therefore have the following chain of inequalities:

$$cd_{\mathbb{Q}}G \leq \mathfrak{F}\text{-}cdG \leq cd_{\mathfrak{M}_{\mathfrak{F}}}G \leq cd_{\mathfrak{F}}G.$$

The main motivation for studying Mackey functors came from looking at the behaviour of $\mathfrak{F}$-cdG and $cd_{\mathfrak{F}}G$ for virtually torsion free groups. A group G is said to be virtually torsion-free if it has a torsion-free subgroup H of finite index. The virtual cohomological dimension $vcdG$ is defined to be equal to the cohomological dimension of H over $\mathbb{Z}$. By Serre's Theorem, see [2], this is well defined. Serre's Theorem also implies that whenever $vcdG = n$ is finite then there is a model for $\underline{E}G$ of dimension $|G : H|n$. Furthermore, $vcdG \leq \mathfrak{F}$-cdG [1]. A question of interest, which has become known as Brown's conjecture, is whether we can always find a model for $\underline{E}G$ of dimension equal to $vcdG$. In [7] examples were exhibited, where this is not so. In particular for these examples and positive integers m $3m = vcdG = \mathfrak{F}$-$cdG < cd_{\mathfrak{F}}G = gd_{\mathfrak{F}}G = 4m$. As Mackey functors seem to have a more "symmetric" structure and seem to behave more naturally under induction from finite index subgroups, see Theorem 3.3, one would expect that things are slightly more straightforward, which is indeed the case.

Theorem 5.1 *Let G be a virtually torsion-free group. Then*

$$vcdG = \mathfrak{F}\text{-}cdG = cd_{\mathfrak{M}_{\mathfrak{F}}}G.$$

2. Induction and restriction between categories of functors.

Let $\mathfrak{C}$ be a category and R be a commutative ring with 1. Denote by $\mathrm{Mod}_{R\mathfrak{C}}$ the category with objects contravariant functors

$$\mathfrak{C} \to R\text{-mod}$$

and morphisms natural transformations between functors. As mentioned in the introduction one can consider things pointwise and therefore we have coproducts, colimits and the notion of exactness. We say a sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Mod}_{R\mathfrak{C}}$ is exact if after evaluating at every object in $\mathfrak{C}$, we obtain an exact sequence of R-modules. Given two objects $A, B \in \mathrm{Ob}\mathfrak{C}$, let $R_{\mathrm{mor}_{\mathfrak{C}}(A,B)}$ be the free R-module generated by the maps in $\mathfrak{C}$ from A to B. For any $M \in \mathrm{Mod}_{R\mathfrak{C}}$ there are Yoneda Lemmas

$$\mathrm{Hom}_{\mathfrak{C}}(R_{\mathrm{mor}_{\mathfrak{D}}(-,A)}, M) = M(A)$$
$$M \otimes_{\mathfrak{C}} R_{\mathrm{mor}_{\mathfrak{D}}(A,-)} = M(A)$$

Assume we have another category $\mathfrak{D}$ and a functor

$$F : \mathfrak{C} \to \mathfrak{D}.$$

Then, associated to F there are functors which we call restriction, induction and coinduction given by

$$\begin{aligned} \mathrm{res}_F : \mathrm{Mod}_{R\mathfrak{D}} &\to \mathrm{Mod}_{R\mathfrak{C}} \\ M &\mapsto MF \\ \mathrm{ind}_F : \mathrm{Mod}_{R\mathfrak{C}} &\to \mathrm{Mod}_{R\mathfrak{D}} \\ T &\mapsto R_{\mathrm{mor}_{\mathfrak{D}}(-,F*)} \otimes_{\mathfrak{C}} T(*) \\ \mathrm{coind}_F : \mathrm{Mod}_{R\mathfrak{C}} &\to \mathrm{Mod}_{R\mathfrak{D}} \\ T &\mapsto \mathrm{Hom}_{\mathfrak{C}}(R_{\mathrm{mor}_{\mathfrak{D}}(F*,-)}, T(*)). \end{aligned}$$

By the Yoneda Lemmas, the restriction functor can also be defined as

$$\mathrm{res}_F M(*) = \mathrm{Hom}_{\mathfrak{D}}(R_{\mathrm{mor}_{\mathfrak{D}}(-,F*)}, M(-)) = M(-) \otimes_{\mathfrak{D}} R_{\mathrm{mor}_{\mathfrak{D}}(F*,-)}.$$

For any bifunctor $N : \mathfrak{C} \times \mathfrak{D} \to R$-mod and $T \in \mathrm{Mod}_{\mathfrak{C}}$, $M \in \mathrm{Mod}_{\mathfrak{D}}$ there is a natural adjunction

$$\mathrm{Hom}_{\mathfrak{D}}(T \otimes_{\mathfrak{C}} N, M) \cong \mathrm{Hom}_{\mathfrak{C}}(T, \mathrm{Hom}_{\mathfrak{D}}(N, M))$$

which yields

$$\mathrm{Hom}_{\mathfrak{D}}(\mathrm{ind}_F T, M) \cong \mathrm{Hom}_{\mathfrak{C}}(T, \mathrm{res}_F M)$$
$$\mathrm{Hom}_{\mathfrak{D}}(M, \mathrm{coind}_F T) \cong \mathrm{Hom}_{\mathfrak{C}}(\mathrm{res}_F M, T).$$

Lemma 2.1. *Assume we have categories $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{L}$ and functors*

$$\mathfrak{C} \xrightarrow{F} \mathfrak{D} \xrightarrow{H} \mathfrak{L}.$$

Then $\mathit{ind}_H \mathit{ind}_F = \mathit{ind}_{HF}$ and $\mathit{coind}_H \mathit{coind}_F = \mathit{coind}_{HF}$.

3. The Mackey category.

To define Mackey functors, we follow the third approach used in [19]. Let $\mathfrak{M}_{\mathfrak{F}}G$ be the category with objects G/K with $K \in \mathfrak{F}$ and morphisms from G/S to G/K the diagrams

$$G/S \leftarrow G/L \rightarrow G/K$$

of G-maps up to the following equivalence relationship: two such diagrams $G/S \leftarrow G/L \rightarrow G/K$ and $G/S \leftarrow G/L_1 \rightarrow G/K$ are equivalent if $L_1 = L^x$ for some $x \in G$ such that the following diagram commutes

$$\begin{array}{ccccc} & & G/L & & \\ & \swarrow & \big\downarrow x & \searrow & \\ G/S & & & & G/K \\ & \nwarrow & & \nearrow & \\ & & G/L_1 & & \end{array}$$

Composition is given by pull-back. We define now the category of Mackey functors as

$$\mathrm{Mack}_{\mathfrak{F}}G = \{\text{functors } \mathfrak{M}_{\mathfrak{F}}G \to R\text{-mod}\}.$$

We will write $M(K)$ instead of $M(G/K)$.

As already mentioned in the introduction, the most common definition of Mackey functors is as a pair of functors, (M^*, M_*) which coincide on objects and with M^* contravariant and M_* covariant. This pair of functors now satisfies a pull-back condition as follows: Let $L, S, K \in \mathfrak{F}$ and

$$\begin{array}{ccc} \coprod_{x \in L^g \backslash S / K^{g'}} G/H^g \cup K^{g'x^{-1}} & \longrightarrow & G/K \\ \big\downarrow & & \big\downarrow g' \\ G/L & \xrightarrow{\quad g \quad} & G/S \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_{x \in L^g \backslash S / K^{g'}} M(G/H^g \cup K^{g'x^{-1}}) & \longleftarrow & M(G/K) \\ \big\downarrow & & \big\downarrow M_*(g') \\ M(G/L) & \xleftarrow{\quad M^*(g) \quad} & M(G/S). \end{array}$$

The relationship between both definitions can be seen if we consider the map corresponding to

$$G/S \stackrel{g}{\leftarrow} G/L \stackrel{1}{\rightarrow} G/L$$

as the contravariant structure and the map associated to

$$G/S \stackrel{1}{\leftarrow} G/S \stackrel{h}{\rightarrow} G/L$$

as the covariant. In particular, if M is a Mackey functor, then for any $S \in \mathfrak{F}$, $M(S)$ is a bi-$\mathrm{N}_G(S)/S$-module and the equivalence of the diagrams

$$G/S \stackrel{g}{\leftarrow} G/S \stackrel{1}{\rightarrow} G/S$$

$$G/S \stackrel{1}{\leftarrow} G/S \stackrel{g^{-1}}{\rightarrow} G/S$$

implies that for any $x \in \mathrm{N}_G(S)$,

$$xm = mx^{-1}.$$

Given an RG-module V the fixed and cofixed points functors which we denote V^- and V_- respectively are examples of Mackey functors.

Let Ω ba a subset of $\mathfrak{F}$ and M a Mackey functor. The subfuctor of M generated by $M(K)$, $K \in \Omega$ is defined as the intersection of all the subfunctors N of M with $N(K) = M(K)$ for all $K \in \Omega$. If this subfunctor equals M, then we say that M is generated by those values. Note that for any $K \in \mathfrak{F}$, $g \in G$, the morphism associated to $G/K \overset{1}{\leftarrow} G/K \overset{g}{\rightarrow} G/K$ maps $M(K)$ isomorphically onto $M(K^g)$ so in such a set Ω it suffices to consider one representative of each G-conjugacy class of subgroups.

Let $S, K \leq G$ and

$$A^G(S,K) = \text{free } R\text{-module generated by the morphisms from } S \text{ to } K \text{ in } \mathfrak{M}_{\mathfrak{F}}G$$

Then, by [19], if $K \in \mathfrak{F}$, $A^G(-,K)$ is a projective Mackey functor and

$$A^G(S,K) = \bigoplus_{g \in S\backslash G/K} \; \bigoplus_{\substack{L \leq S^g \cap K \\ \text{up to } S^g \cap K\text{-conjugacy}}} R_{L,g}.$$

Note that there is an obvious isomorphism between the R-modules $A^G(S,K)$ and $A^G(K,S)$. Moreover

$$A^G(-,K) \cong A^G(K,-)$$

as Mackey functors.

Later on we need to be able to describe induction and coinduction of Mackey functors from a subgroup H of G. Hence, consider the inclusion functor

$$i : \mathfrak{F} \cap H \to \mathfrak{F},$$

where $\mathfrak{F} \cap H$ is the family of all finite subgroups of H. Denote by res_H^G, ind_H^G and coind_H^G the restriction, induction and coinduction functors as in Section 2. First, we consider restriction of projective Mackey functors.

Proposition 3.1. *For $H \leq G$ and $K \in \mathfrak{F}$*

$$res_H^G A^G(-,K) = \bigoplus_{x \in H\backslash G/K} A^H(-, K^{x^{-1}} \cap H).$$

<u>Proof:</u> For any $S \in \mathfrak{F}$ with $S \leq H$, we have

$$S\backslash G/K = \coprod_{x \in H\backslash G/K} S\backslash H/K^{x^{-1}} \cap H = \coprod_{x \in H\backslash G/K} \; \coprod_{h \in S\backslash H/K^{x^{-1}} \cap H} hx.$$

Taking into account the previous description of $A^G(S,K)$ this implies that

$$A^G(S,K) = \bigoplus_{x \in H\backslash G/K} A^H(S, K^{x^{-1}} \cap H).$$

Finally, one easily checks that this is in fact an isomorphism of Mackey functors. □

Using the above formula we may also describe explicitely ind_H^G and coind_H^G.

Lemma 3.2. *Let T be any Mackey functor over $\mathfrak{F}\cap H$, then*

$$\mathit{coind}\,{}^G_H T(S) = \prod_{x\in H\backslash G/S} T(S^{x^{-1}}\cap H)$$

$$\mathit{ind}^G_H T(S) = \bigoplus_{x\in H\backslash G/S} T(S^{x^{-1}}\cap H)$$

Proof:

$$\begin{aligned}\text{coind}\,{}^G_H T(S) &= \text{Hom}_{\mathfrak{M}_{\mathfrak{F}\cap H}}(A^G(S,*),T(*))) = \\ \text{Hom}_{\mathfrak{M}_{\mathfrak{F}\cap H}}(&\bigoplus_{x\in H\backslash G/S} A^H(S^{x^{-1}}\cap H,*),T(*))) = \\ \prod_{x\in H\backslash G/S} &\text{Hom}_{\mathfrak{M}_{\mathfrak{F}\cap H}}(A^H(S^{x^{-1}}\cap H,*),T(*)) = \\ &\prod_{x\in H\backslash G/S} T(S^{x^{-1}}\cap H)\ \text{(by the Yoneda Lemma).}\end{aligned}$$

The proof for induction is analogous. □

Theorem 3.3. *For any Mackey functor T over $\mathfrak{F}\cap H$, if the inclusion of the coproduct into the product is an isomorphism between $ind^G_H T(K)$ and $coind^G_H T(K)$ for each $K\in\mathfrak{F}$, then*

$$\mathit{ind}^G_H T \cong \mathit{coind}^G_H T,$$

that is, they are isomorphic as Mackey functors for $\mathfrak{F}$. In particular, this happens if the index $|G:H|$ is finite.

Proof: One easily checks that for $S,K\in\mathfrak{F}\cap H$

$$A^G(K,S)\to A^H(K,S)$$

$$K\overset{1}{\leftarrow} L\overset{g}{\to} S\mapsto\begin{cases}K\overset{1}{\leftarrow} L\overset{g}{\to} S \text{ if } g\in H\\ 0 \text{ otherwise.}\end{cases}$$

yields a morphism of Mackey bifunctors for $\mathfrak{F}\cap H$ (that is, it behaves well with respect to morphisms in both variables). From this, using Yoneda, we deduce the existence of a Mackey functor morphism

$$\begin{aligned}T(-) = &\text{Hom}_{\mathfrak{M}_{\mathfrak{F}\cap H}}(A^H(*,-),T(*))\to\\ &\text{Hom}_{\mathfrak{M}_{\mathfrak{F}\cap H}}(A^G(*,-),T(*)) = (\text{res}^G_H\text{coind}^G_H T)(-).\end{aligned}$$

Using the adjoint isomorphism

$$\text{Hom}_{\mathfrak{M}_{\mathfrak{F}\cap H}}(T,\text{res}^G_H\text{coind}^G_H T)\cong\text{Hom}_{\mathfrak{M}_{\mathfrak{F}}}(\text{ind}^G_H T,\text{coind}^G_H T)$$

we obtain a morphism of Mackey functors

$$\text{ind}^G_H T\to\text{coind}^G_H T.$$

After evaluating at objects this is precisely the inclusion of the coproduct into the product. Therefore, if this inclusion is an isomorphism on objects, it must in fact be an ismorphism of Mackey functors. □

Definition 3.4. *Let $B^G = A^G(G,-)$ given by*

$$B^G(K) = \bigoplus_{\substack{L\le K\\ \text{up to } K\text{-conjugacy}}} R_L$$

for $K \in \mathfrak{F}$.
Next, we define

$$H^n_{\mathfrak{M}_\mathfrak{F}}(G,M) := Ext^n_{\mathfrak{M}_\mathfrak{F}}(B^G,M)$$
$$\text{and } cd_{\mathfrak{M}_\mathfrak{F}}G := max\{n : H^n_{\mathfrak{M}_\mathfrak{F}}(G,M) \neq 0 \text{ for } M \in Mack_\mathfrak{F}G\}$$

One more consequence of 3.1 is that res^G_H takes projectives to projectives. Moreover, note that $\mathrm{res}^G_H B^G = B^H$ and that res^G_H is an exact functor. Therefore there is a Shapiro Lemma, that is

$$\mathrm{H}^n_{\mathfrak{M}_\mathfrak{F}}(G, \mathrm{coind}^G_H N) = \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap H}}(H,N).$$

We give now another description of $A^G(-,K)$.

Proposition 3.5.

$$A^G(S,K) = \bigoplus_{\substack{L\le S\\ \text{up to } S\text{-conjugacy}}} P^G_K(L)/\sim_{\mathrm{N}_S(L)}$$

with $g, g_1 : G/L \to G/K$ $g \sim g_1$ if $xg_1K = gK$ for some $x \in \mathrm{N}_S(L)$.

Proof: By [19], any morphism in the Mackey category from G/S to G/K is equivalent to one of the form

$$G/S \overset{1}{\leftarrow} G/L \overset{g}{\rightarrow} G/K.$$

Moreover, we may represent such a diagram by a pair (L,g) with $g \in P^G_K(L)$ and (L,g) and (L,g_1) are equivalent if and only if for some $x \in S$, $L_1 = L^x$ and $xg_1K = gK$. Clearly in that case L and L_1 are S-conjugated. And conversely, if $L_1 = L^x$ for some $x \in S$ then (L,g) and $(L_1, x^{-1}g)$ are equivalent.

This means that we only have to choose pairs with one subgroup for each S-conjugacy class of subgroups of S and pairs for different classes are not equivalent. Finally note that (L,g) and (L,g_1) are equivalent if and only if for some $x \in S$ with $L^x = L$ (thus $x \in \mathrm{N}_S(L)$), $xg_1K = gK$. □

Note that if we see each $P^G_S(L)$ as a left module for the group $\mathrm{N}_G(L)/L$ (using the contravariant structure of P^G_S), then the above formula yields

Proposition 3.6. *As covariant Bredon functors we have*

$$A^G(S,-) = \bigoplus_{\substack{L\le S\\ \text{up to } S\text{-conjugacy}}} R \otimes_{\mathrm{N}_S(L)/L} P^G_-(L).$$

Proof: 3.5 may be rewritten as

$$A^G(S,K) = \bigoplus_{\substack{L\le S\\ \text{up to } S\text{-conjugacy}}} R \otimes_{\mathrm{N}_S(L)/L} P^G_K(L).$$

Moreover, the covariant structure on $A^G(S,-)$ is given by

$$G/S \leftarrow G/L \rightarrow G/K \rightarrow G/K_1$$

which is precisely the covariant structure on the right hand side. □

Let $\mathfrak{O}_{\mathfrak{F}}G$ be the orbit category. The functor $\mathfrak{O}_{\mathfrak{F}}G \to \mathfrak{M}_{\mathfrak{F}}G$ given by the identity on objects and taking a G-map $G/S \xrightarrow{g} G/K$ to

$$G/S \xleftarrow{g} G/K \xrightarrow{1} G/K$$

provides, by Section 2, restriction and induction maps

$$\begin{aligned} \text{res} : \text{Mack}_{\mathfrak{F}}G &\to \text{Mor}_{\mathfrak{F}}G \\ M &\mapsto M \end{aligned}$$

$$\begin{aligned} \text{ind} : \text{Mor}_{\mathfrak{F}}G &\to \text{Mack}_{\mathfrak{F}}G \\ T &\mapsto T(*) \otimes_{\mathfrak{F}} A^G(-,*) \end{aligned}$$

and an adjoint isomorphism

$$\text{Hom}_{\mathfrak{F}}(T, \text{res}M) = \text{Hom}_{\mathfrak{M}_{\mathfrak{F}}}(\text{ind}T, M).$$

As res is clearly exact, the adjoint isomorphism implies that induction takes projectives to projectives. But this is also a consequence of the next result.

Theorem 3.7. *For any Bredon contra-module T and $S \in \mathfrak{F}$*

$$(\text{ind}T)(S) = \bigoplus_{\substack{L \leq S \\ \text{up to } S\text{-conjugacy}}} R \otimes_{\text{N}_S(L)/L} T(L).$$

Moreover for $K \in \mathfrak{F}$

$$\text{ind}\underline{R} = B^G \text{ and}$$
$$\text{ind}P^G_K = A^G(-,K)$$

<u>Proof:</u> By 3.6 we have

$$\begin{aligned}(\text{ind}T)(S) = A^G(S,-) \otimes_{\mathfrak{F}} T = (&\bigoplus_{\substack{L \leq S \\ \text{up to } S\text{-conjugacy}}} R \otimes_{\text{N}_S(L)/L} P^G_-(G/L)) \otimes_{\mathfrak{F}} T = \\ &\bigoplus_{\substack{L \leq S \\ \text{up to } S\text{-conjugacy}}} (R \otimes_{\text{N}_S(L)/L} P^G_-(G/L)) \otimes_{\mathfrak{F}} T = \\ &\bigoplus_{\substack{L \leq S \\ \text{up to } S\text{-conjugacy}}} R \otimes_{\text{N}_S(L)/L} (P^G_-(G/L) \otimes_{\mathfrak{F}} T) = \\ &\bigoplus_{\substack{L \leq S \\ \text{up to } S\text{-conjugacy}}} R \otimes_{\text{N}_S(L)/L} T(G/L).\end{aligned}$$

This implies that on the one hand ind$\underline{R}$ and B^G and on the other indP^G_K and $A^G(-,K)$ coincide in objects and one easily checks that they are in fact isomorphic as Mackey functors. □

Ind is not exact in general. However, we prove in the next result that it is exact when applied to projective resolutions.

Theorem 3.8. *Let*

$$P_* \twoheadrightarrow \underline{R}$$

be a projective resolution of Bredon contramodules. Then

$$\text{ind}P_* \twoheadrightarrow B^G$$

is a projective resolution of Mackey modules. Therefore for any Mackey functor

$$H^n_{\mathfrak{M}_{\mathfrak{F}}}(G, M) = H^n_{\mathfrak{F}}(G, M)$$

Proof: For each $L \in \mathfrak{F}$,

$$P_*(L) \twoheadrightarrow \underline{R}(L) = R$$

is an exact sequence of $\mathrm{N}_G(L)/L$-modules. Moreover, by 3.2 of [15], this sequence is split when restricted to the finite subgroups of $\mathrm{N}_G(L)/L$. In particular this is split when restricted to $\mathrm{N}_S(L)/L = (\mathrm{N}_G(L) \cap S)/L$ for each $S \in \mathfrak{F}$. This implies that

$$(\mathrm{ind}P_*)(S) = \bigoplus_{\substack{L \leq S \\ \text{up to } S\text{-conjugacy}}} R \otimes_{\mathrm{N}_S(L)/L} P_i(L) \twoheadrightarrow (\mathrm{ind}\underline{R})(K) = B^G(K)$$

is an exact sequence. Now, by 3.7 each $\mathrm{ind}P_i$ is a projective Mackey functor and therefore

$$\mathrm{ind}P_* \twoheadrightarrow \mathrm{ind}\underline{R} = B^G$$

is a projective resolution. The result follows from the adjoint isomorphism. □

Corollary 3.9.

$$cd_{\mathfrak{M}_{\mathfrak{F}}} G \leq cd_{\mathfrak{F}} G.$$

4. Modules for WK.

In this section we consider some results which are proven in [19] for finite groups to check that they remain true in the general case.

Let τ be the functor between the Mackey categories for G associated to the familes $\{1\}$ and $\mathfrak{F}$ given by

$$\tau : \{1\} \to \mathfrak{F}$$
$$1 \mapsto 1.$$

As we may identify $\mathrm{Mack}_{\{1\}}(G)$ and RG-mod, the coinduction and restriction functors associated to τ as in Section 2 are given by

$$\mathrm{coind}_\tau : RG\text{-mod} \to \mathrm{Mack}_{\mathfrak{F}}(G)$$
$$V \mapsto \mathrm{Hom}_{RG}(A^G(1, -), V)$$

and

$$\mathrm{res}_\tau : \mathrm{Mack}_{\mathfrak{F}}(G) \to RG\text{-mod}$$
$$M \mapsto M(1).$$

We have

Lemma 4.1. *For $S \in \mathfrak{F}$*

$$A^G(1, S) = R \uparrow_S^G \quad \textit{(induced module).}$$

Proof: By 3.6 there is an isomorphism of right RG-modules

$$A^G(1, S) = A^G(S, 1) = R \otimes_S P_1^G(1) = R \otimes_S RG = R \uparrow_S^G .$$

□

From this we recover one of the adjunctions of [19]:

Lemma 4.2. [19] *There are isomorphisms of Mackey functors*

$$coind_\tau V = V^-$$
$$ind_\tau V = V_-$$

and for any Mackey functor M we have adjunctions

$$\mathrm{Hom}_{RG}(M(1),V) \cong \mathrm{Hom}_{\mathfrak{M}_{\mathfrak{F}}}(M,V^-)$$
$$\mathrm{Hom}_{RG}(V,M(1)) \cong \mathrm{Hom}_{\mathfrak{M}_{\mathfrak{F}}}(V_-,M).$$

Proof: Using the previous result we have

$$\mathrm{coind}_\tau V(S) = \mathrm{Hom}_{RG}(R\uparrow_S^G, V) = \mathrm{Hom}_{RS}(R, V\downarrow_S^G) = V^S$$

and it is easy to check that this yields in fact an isomorphism of Mackey functors. The proof for ind_τ is analogous.

The adjunctions are a consequence of the adjunctions of Section 2. □

Theorem 4.3.

$$\mathfrak{F}\text{-}cdG \leq cd_{\mathfrak{M}_{\mathfrak{F}}} G$$

Proof: For any Bredon contramodule T, 3.7 implies that $(\mathrm{ind}T)(1) = T(1)$ and this is in fact an isomorphism of left RG-modules. Therefore by the previous result there is an adjunction

$$\mathrm{Hom}_{RG}(T(1),V) \cong \mathrm{Hom}_{\mathfrak{M}_{\mathfrak{F}}}(\mathrm{ind}T,V^-) \cong \mathrm{Hom}_{\mathfrak{F}}(T,V^-)$$

It suffices now to use the fact that by [15], evaluating a Bredon projective resolution of $\underline{R}$ at 1 we obtain a relative $\mathfrak{F}$-projective resolution of R. □

Assume for a moment that we have $K \in \mathfrak{F}$, $K \triangleleft G$ and N any Mackey functor for G/K with respect to the family $\bar{\mathfrak{F}} = \{LK/K : L \in \mathfrak{F}\}$. From N we can get a Mackey functor for G defined as

$$\mathrm{inf}_{G/K}^G N(S) = \begin{cases} N(S/K) \text{ if } K \leq S \\ 0 \text{ in other case.} \end{cases}$$

Conversely, a Mackey functor for G, say M, yields Mackey functors for G/K defined in the following way. For each $L, J \in \mathfrak{F}$ with $J \leq L$ we put R_J^L resp. I_J^L for the morphisms

$$R_J^L : M(L) \to M(J)$$
$$I_J^L : M(J) \to M(L)$$

asociated to the maps $G/L \leftarrow G/J \to G/J$ resp. $G/J \leftarrow G/J \to G/L$ in the Mackey category. With the same notation as in [19] we put

$$SM(L) = \bigcap_{\substack{J\leq L\\ K\not\leq J}} \mathrm{Ker} R_J^L$$
$$TM(L) = M(L)/\Sigma_{\substack{J\leq L\\ K\not\leq J}} \mathrm{Im} I_J^L.$$

These functors are respectively left and right adjoint to inflation. This is proven in [18] for finite groups and the proof works equally well for arbitrary groups.

$$\mathrm{Hom}_{\mathfrak{M}_{\bar{\mathfrak{F}}}}(N, SM) \cong \mathrm{Hom}_{\mathfrak{M}_{\mathfrak{F}}}(\mathrm{inf}_{G/K}^K N, M)$$
$$\mathrm{Hom}_{\mathfrak{M}_{\bar{\mathfrak{F}}}}(TM, N) \cong \mathrm{Hom}_{\mathfrak{M}_{\mathfrak{F}}}(\mathrm{M}, \mathrm{inf}_{G/K}^K N).$$

We turn now to the general case and fix $K \in \mathfrak{F}$. Let $WK = \mathrm{N}_G(K)/K$ be the Weyl group of K and denote

$$S_K M = S(\mathrm{res}^G_{\mathrm{N}_G(K)} M)(K) = \bigcap_{J \lneq L} \mathrm{Ker} R^K_J$$

for any Mackey functor M. We may compose the following maps

$$RWK\text{-mod} \overset{\mathrm{coind}_\tau}{\to} \mathrm{Mack}_{\mathfrak{F}\cap \mathrm{N}_G(K)} WK \overset{\mathrm{inf}^{\mathrm{N}_G(K)}_{WK}}{\to}$$

$$\mathrm{Mack}_{\mathfrak{F}\cap \mathrm{N}_G(K)} \mathrm{N}_G(K) \overset{\mathrm{ind}^G_{\mathrm{N}_G(K)}}{\to} \mathrm{Mack}_{\mathfrak{M}_\mathfrak{F}} G$$

and we get an adjunction for any RWK-module U as in [19]

$$\mathrm{Hom}_{RWK}(U, S_K M) \cong \mathrm{Hom}_{\mathfrak{M}_\mathfrak{F}}(\mathrm{ind}^G_{\mathrm{N}_G(K)} \mathrm{inf}^{\mathrm{N}_G(K)}_{WK} U_-, M).$$

In particular if we take $U = S_K M$ we get a morphism of Mackey functors such that the evaluation in K is the inclusion $S_K M \hookrightarrow M(K)$.

Proposition 4.4. *If $K \lhd G$ and we have $K \leq L \leq G$, then for any Mackey module M for L/K we have*

$$\mathit{inf}^G_{G/K} \mathit{ind}^{G/K}_{L/K} M = \mathit{ind}^G_L \mathit{inf}^L_{L/K} M.$$

Proof: Let $S \in \mathfrak{F}$. Then if $K \leq S$,

$$\mathrm{inf}^G_{G/K} \mathrm{ind}^{G/K}_{L/K} M(S) = \mathrm{ind}^{G/K}_{L/K} M(S/K) = \bigoplus_{x \in L/K \backslash G/K/S/K} M(S^{x^{-1}} \cap L/K) =$$

$$\bigoplus_{x \in L \backslash G/S} \mathrm{inf}^L_{L/K} M(S^{x^{-1}} \cap L) = \mathrm{ind}^G_L \mathrm{inf}^L_{L/K} M(S).$$

If $K \not\leq S$ one easily checks that both functors annihilate on S. Hence they coincide in objects. It is also easy to check that this yields an isomorphisms of Mackey functors between them. □

5. Virtually torsion free groups.

Let G be a virtually torsion free group. Then there is a subgroup $H \lhd G$ with $|G : H| < \infty$. The virtual cohomological dimension of G is

$$\mathrm{vcd} G = \mathrm{cd} H.$$

Theorem 5.1. *Let G be a virtually torsion free group. Then*

$$\mathit{cd}_{\mathfrak{M}_\mathfrak{F}} G = \mathit{vcd} G.$$

Proof: Let H be a subgroup $H \lhd G$ with $|G : H| < \infty$.

If $\mathrm{cd} H = \infty$ then obviously

$$\mathrm{cd}_{\mathfrak{M}_\mathfrak{F}} G = \mathrm{cd} H = \infty.$$

Therefore we may assume $\mathrm{cd} H < \infty$ and then by Serre's Theorem, see [2], $\mathrm{cd}_\mathfrak{F} G < \infty$ thus also $\mathrm{cd}_{\mathfrak{M}_\mathfrak{F}} G < \infty$. Note also that there is a bound on the orders of the finite subgroups of G.

Put $n = \mathrm{cd}_{\mathfrak{M}_{\mathfrak{F}}} G$ and let M be a Mackey functor such that

$$\mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}}}(G, M) \neq 0.$$

For a certain set Ω of subgroups of G M is generated by $M(K)$, $K \in \Omega$. Note that we may assume that Ω contains at most one representativeof each conjugacy class of finite subgroups. The strategy of the proof is to find a group Q and a Mackey functor T such that $\mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F} \cap Q}}(Q, T) \neq 0$. and such that Q is the extension of a finite by a torsion free group. Theorem 3.8 allows us to consider Bredon-cohomology. Applying spectral sequences as in [12] will then reduce the problem to considering a torsion-free subgroup of H. We proceed in six steps: Step 1 to Step 4 are needed to construct Q and T, while we show in Step 5 that indeed $\mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F} \cap Q}}(Q, T) \neq 0$. The last step then consists of the reduction to a suitable torsion-free subgroup of G, which will finish the proof.

Step 1: For each $K \in \Omega$ let M_K be the subfunctor of M generated by $M(K)$. The homomorphisms $M_K \hookrightarrow M$ yield an epimorphism

$$\bigoplus_{K \in \Omega} M_K \twoheadrightarrow M.$$

We get

$$0 \neq \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}}}(G, \bigoplus_{K \in \Omega} M_K).$$

Step 2: We shall now show that we may assume the following: For any $K \in \Omega$ and

$S \in \mathfrak{F}$ with $S < K$, $M_K(S) = 0$.

We argue by induction on

$$\max\{|K| : K \in \Omega \text{ does not satisfy this claim}\}.$$

Note that as the order of the finite subgroups of G is bounded, the previous maximum is finite.

Now, if $K \in \Omega$ satisfies the claim, put $S = K$. Otherwise choose an $S < K$ with $M_K(S) \neq 0$. In both cases let N_K be the subfunctor of M_K generated by $M_K(S)$. We have a s.e.s.

$$\bigoplus_{K \in \Omega} N_K \rightarrowtail \bigoplus_{K \in \Omega} M_K \twoheadrightarrow \bigoplus_{K \in \Omega} M_K/N_K$$

which yields a l.e.s. in cohomology

$$\ldots \to \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}}}(G, \bigoplus_{K \in \Omega} N_K) \to \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}}}(G, \bigoplus_{K \in \Omega} M_K) \to \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}}}(G, \bigoplus_{K \in \Omega} M_K/N_K) \to \ldots$$

Thus we have either

$$\mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}}}(G, \bigoplus_{K \in \Omega} M_K/N_K) \neq 0$$

or

$$\mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}}}(G, \bigoplus_{K \in \Omega} N_K) \neq 0.$$

In the first case, as N_K is generated by $N_K(S)$, the claim follows by induction.

In the second case note that M_K/N_K is generated by $M_K/N_K(K)$. Moreover, $M_K/N_K(S) = 0$. As the number of subgroups of any $K \in \mathfrak{F}$ is bounded after a finite number of steps this process yields a Mackey functor with no trivial cohomology which is a direct sum of functors generated by their value at a single subgroup and satisfying the conditions in the claim.

Step 3: We show that there exists a subgroup L with $H \leq L \leq G$ such that $\mathrm{H}^n_{\mathfrak{M}_\mathfrak{F}}(G, \bigoplus_{K\in\Omega_L} M_K) \neq 0$, where $\Omega_L = \{K \in \Omega \,|\, HK = L\}$.

To see this, note that Ω is the disjoint union

$$\Omega = \bigcup_{H \leq L \leq G} \{\Omega_L\}$$

with $\Omega_L = \{K \in \Omega : HK = L\}$. As the number of possible subgroups L is finite we get

$$0 \neq \mathrm{H}^n_{\mathfrak{M}_\mathfrak{F}}(G, \bigoplus_{K\in\Omega} M_K) = \bigoplus_{H\leq L\leq G} \mathrm{H}^n_{\mathfrak{M}_\mathfrak{F}}(G, \bigoplus_{K\in\Omega_L} M_K)$$

so that for some L, the corresponding cohomology is not trivial and the claim follows. From now on we keep L fixed and denote $\Omega = \Omega_L$.

Step 4: We now fix a subgroup $K \in \Omega$ and take a closer look at the situation for M_K. Our objective is to find a suitable Mackey functor mapping epimorphically onto M_K. In the process we shall define the group Q and the functor T, which will later enable us to make the necessary reductions. For simplicity we put $M = M_K$. We have

$$S_K M = \bigcap_{S<K} \mathrm{Ker} R^K_S = M(K).$$

Let $U = S_K M$ and consider the adjunction

$$\mathrm{Hom}_{RWK}(U, S_K M) = \mathrm{Hom}_{\mathfrak{M}_\mathfrak{F}}(\mathrm{ind}^G_{\mathrm{N}_G(K)} \mathrm{inf}^{\mathrm{N}_G(K)}_{WK}(U_-), M).$$

When we evaluate in K the map ϕ corresponding to the identity on the right hand side we get

$$\begin{array}{rcll} \mathrm{ind}^G_{\mathrm{N}_G(K)} \mathrm{inf}^{\mathrm{N}_G(K)}_{WK}(U_-)(K) = & A^G(K,*) \otimes \mathrm{inf}^{\mathrm{N}_G(K)}_{WK}(U_-)(*) & \stackrel{\phi_K}{\to} & M(K) \\ & \cup & & \| \\ & A^G(K,K) \otimes U_K = A^G(K,K) \otimes U & \twoheadrightarrow & U \end{array}$$

so ϕ_K is surjective. Consider now the subgroups L and $Q = L \cap \mathrm{N}_G(K) = K\mathrm{N}_H(K)$ and denote $V = U \downarrow^{WK}_{Q/K}$. As the index $|\mathrm{N}_G(K) : Q|$ is finite we have an epimorphism of RWK-modules

$$V \uparrow^{WK}_{Q/K} \twoheadrightarrow U$$

which induces an epimorphism of Mackey functors for WK (since the functor taking V to V_- is right exact)

$$(V \uparrow^{WK}_{Q/K})_- \twoheadrightarrow U_-.$$

As the functor inf is also right exact we get an epimorphism of Mackey functors for $\mathrm{N}_G(K)$

$$\mathrm{inf}^{\mathrm{N}_G(K)}_{WK}(V \uparrow^{WK}_{Q/K})_- \twoheadrightarrow \mathrm{inf}^{\mathrm{N}_G(K)}_{WK} U_-.$$

From this and inducing to G we have

$$\mathrm{ind}_{\mathrm{N}_G(K)}^G \mathrm{inf}_{WK}^{\mathrm{N}_G(K)}(V\uparrow_{Q/K}^{WK})_- \to \mathrm{ind}_{\mathrm{N}_G(K)}^G \mathrm{inf}_{WK}^{\mathrm{N}_G(K)} U_- \stackrel{\phi}{\to} M.$$

When we evaluate this composition on K we get epimorphisms and therefore, as M is generated by $M(K)$, it is in fact an epimorphism of Mackey functors

Next, we describe the functor on the left hand side of the previous expression in a different way. Using 2.1 (recall that by 4.2, $V_- = \mathrm{ind}_\tau V$) and 4.4 we get

$$\mathrm{inf}_{WK}^{\mathrm{N}_G(K)}(V\uparrow_{Q/K}^{WK}) = \mathrm{inf}_{WK}^{\mathrm{N}_G(K)} \mathrm{ind}_{Q/K}^{WK}(V_-) = \mathrm{ind}_Q^{\mathrm{N}_G(K)} \mathrm{inf}_{Q/K}^Q(V_-).$$

Therefore again by 2.1

$$\mathrm{ind}_{\mathrm{N}_G(K)}^G \mathrm{ind}_Q^{\mathrm{N}_G(K)} \mathrm{inf}_{Q/K}^Q(V_-) = \mathrm{ind}_L^G \mathrm{ind}_Q^L \mathrm{inf}_{Q/K}^Q(V_-).$$

Put

$$T = \mathrm{inf}_{Q/K}^Q(V_-) \text{ and}$$
$$F_K = \mathrm{ind}_Q^L T.$$

The previous formula implies we have an epimorphism

$$\mathrm{ind}_L^G F_K \twoheadrightarrow M.$$

We may describe quite explicitly the functor F_K as follows. Taking into account that Q/K is torsion free, it follows that T is a Mackey functor in $\mathfrak{F} \cap Q$ given by

$$T(S) = \begin{cases} 0 \text{ for } K \neq S \\ V \text{ for } K = S. \end{cases}$$

We compute now the value

$$F_K(S) = \mathrm{ind}_Q^L T(S) = \bigoplus_{x \in S\backslash L/Q} T(S^{x^{-1}} \cap Q).$$

That module is trivial unless $K = S^{x^{-1}} \cap Q \leq S^{x^{-1}}$. In that case, and as $S^{x^{-1}} \leq L$, we have $K \cong L/H = HS^{x^{-1}}/H \cong S^{x^{-1}}$ thus $K^x = S$. But then, if that is the case for $x_1 \neq x_2$, we deduce $x_1 x_2^{-1} \in L \cap \mathrm{N}_G(K) = Q$ thus $Sx_1Q = Sx_2Q$. Therefore

$$F_K(S) = \begin{cases} 0 \text{ for } K \text{ not conjugated to } S \\ V \text{ for } K \text{ conjugated to } S. \end{cases}$$

The same computation for coinduction yields

$$F_K(S) = \mathrm{ind}_Q^L T(S) = \mathrm{coind}_Q^L T(S)$$

for any $S \in \mathfrak{F} \cap L$. So by 3.3, both Mackey functors are isomorphic.

Step 5: Now, we go back to the general case and show that $\mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap Q}}(Q, T) \neq 0$ for certain T. One more consequence of the previous formula is that we have an isomorphism of Mackey functors

$$\bigoplus_{K \in \Omega} F_K \cong \prod_{K \in \Omega} F_K.$$

To see this, consider the inclusion of the direct sum into the product and note that it is an isomorphism on objects, thus it is in fact an isomorphism. Now as the index $|G:L|$ is finite we have by 3.3

$$\begin{aligned}\bigoplus_{K\in\Omega}\operatorname{ind}_L^G F_K = \operatorname{ind}_L^G \bigoplus_{K\in\Omega} F_K &= \\ = \operatorname{coind}_L^G \bigoplus_{K\in\Omega} F_K &= \\ \operatorname{coind}_L^G \prod_{K\in\Omega} F_K&.\end{aligned}$$

Moreover we have for any $K \in \Omega$ an epimorphism $\operatorname{ind}_L^G F_K \twoheadrightarrow M_K$ and this yields an epimorphism

$$\operatorname{coind}_L^G \prod_{K\in\Omega} F_K = \bigoplus_{K\in\Omega}\operatorname{ind}_L^G F_K \twoheadrightarrow \bigoplus_{K\in\Omega} M_K.$$

This implies that

$$0 \neq \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}}}(G, \operatorname{coind}_L^G \prod_{K\in\Omega} F_K) = \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap L}}(L, \prod_{K\in\Omega} F_K) = \prod_{K\in\Omega} \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap L}}(L, F_K).$$

Therefore, for some $K \in \Omega$, $0 \neq \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap L}}(L, F_K)$. Now, with the same notation as in Step 4 we have

$$\begin{aligned}0 \neq \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap L}}(L, F_K) = \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap L}}(L, \operatorname{ind}_Q^L T) &= \\ = \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap L}}(L, \operatorname{coind}_Q^L T) &= \\ = \mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap Q}}(Q, T)&.\end{aligned}$$

<u>*Step 6*</u>: We have reduced the problem to the group Q and the functor T, which

now enables to finish the proof. As

$$\mathrm{H}^n_{\mathfrak{M}_{\mathfrak{F}\cap Q}}(Q, T) = \mathrm{H}^n_{\mathfrak{F}\cap Q}(Q, T)$$

we may use the spectral sequence of [12] for K, Q and Q/K and we get

$$\mathrm{H}^p(Q/K, \mathrm{H}^q_{\mathfrak{F}\cap K}(K, T)) \Rightarrow \mathrm{H}^{p+q}_{\mathfrak{F}\cap Q}(Q, T).$$

As K is finite,

$$\mathrm{H}^q_{\mathfrak{F}\cap K}(K, T) = \begin{cases} 0 \text{ for } q > 0 \\ T(K) = V \text{ for } q = 0\end{cases}$$

thus

$$\mathrm{H}^n(Q/K, V) = \mathrm{H}^n_{\mathfrak{F}\cap Q}(Q, T) \neq 0$$

and we have (note that $Q/K \cong \mathrm{N}_H(K)$)

$$n \leq \mathrm{cd}\mathrm{N}_H(K) \leq \mathrm{cd}H \leq n$$

which yields the result. □

6. Open Questions

There are still a number of open problems relating to the cohomological dimension for Mackey-functors over $\mathfrak{M}_{\mathfrak{F}}G$ and for some of them partial answers are known.

Since the main motivation for this work was the relationship between the cohomology of Mackey-functors and Bredon cohomology, the following two questions spring to mind.

Question 6.1. *For which classes of groups G do we have equality*

$$cd_{\mathfrak{M}_{\mathfrak{F}}}G = cd_{\mathfrak{F}}G?$$

Question 6.2. *Let $cd_{\mathfrak{M}_{\mathfrak{F}}}G$ be finite. Can we conclude that also $cd_{\mathfrak{F}}G$ is finite?*

Question 6.1 is obviously true for polycyclic-by-finite groups and countable virtually torsion-free nilpotent groups as here $vcdG = cd_{\mathfrak{F}}G$ [10, 16]. Also, Question 6.2 is trivial for elementary amenable groups of finite Hirsch length since their Bredon-cohomological dimension is finite [4] . It is also conceivable that Question 6.1 has a positive answer at least when they are virtually torsion-free. Here the problem can be reduced to checking whether for soluble groups of type VFP $cd_{\mathfrak{F}}G = hd_{\mathfrak{F}}G$. There are not many naturally occuring examples for which it is known whether $cd_{\mathbb{Q}}G = cd_{\mathfrak{F}}G$. Dunwoody's result [3] implies that this is so for groups acting on trees with finite stabilizers, where $cd_{\mathbb{Q}} = 1 = cd_{\mathfrak{F}}G$. But even for hyperbolic groups or mapping class groups , which both admit cocompact models for $\underline{E}G$, see [10], this is not known to hold. An answer to Question 6.1 would at least give a partial solution.

Since in Theorem 5.1 we have only dealt with a very special kind of group extension, namely with torsion-free -by finite groups, it is natural to ask whether this can be extended to more general settings.

Question 6.3. *Let $N \hookrightarrow G \twoheadrightarrow Q$ be a group extension with $cd_{\mathfrak{M}_{\mathfrak{F}}}N$ and $cd_{\mathfrak{M}_{\mathfrak{F}}}Q$ finite. What can be said about $cd_{\mathfrak{M}_{\mathfrak{F}}}G$?*

By Theorem 3.8 it is possible to obtain a similar spectral sequence to the one in [12] and in particular a Hochschild-Serre spectral sequence [12, Theorem5.1]. These however require a bound on the orders of the finite subgroups of Q. In view of Theorem 5.1 one would expect that for Q a finite group one gets that $cd_{\mathfrak{M}_{\mathfrak{F}}}N = cd_{\mathfrak{M}_{\mathfrak{F}}}G$. A slightly more optimistic conjecture to make would be that in general $cd_{\mathfrak{M}_{\mathfrak{F}}}G \leq cd_{\mathfrak{M}_{\mathfrak{F}}}N + cd_{\mathfrak{M}_{\mathfrak{F}}}Q$.

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