

ON GROUPS ACTING ON CONTRACTIBLE SPACES WITH STABILIZERS OF PRIME POWER ORDER

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ABSTRACT. Let \mathfrak{F} denote the class of finite groups, and let \mathfrak{P} denote the subclass consisting of groups of prime power order. We study group actions on topological spaces in which either (1) all stabilizers lie in \mathfrak{P} or (2) all stabilizers lie in \mathfrak{F} . We compare the classifying spaces for actions with stabilizers in \mathfrak{F} and \mathfrak{P} , the Kropholler hierarchies built on \mathfrak{F} and \mathfrak{P} , and group cohomology relative to \mathfrak{F} and to \mathfrak{P} . In terms of standard notations, we show that $\mathfrak{F} \subset \mathbf{H}_1\mathfrak{P} \subset \mathbf{H}_1\mathfrak{F}$, with all inclusions proper; that $\mathbf{H}\mathfrak{F} = \mathbf{H}\mathfrak{P}$; that $\mathfrak{F}H^*(G; -) = \mathfrak{P}H^*(G; -)$; and that $E_{\mathfrak{P}}G$ is finite-dimensional if and only if $E_{\mathfrak{F}}G$ is finite-dimensional and every finite subgroup of G is in \mathfrak{P} .

1. INTRODUCTION

Let \mathcal{F} denote a class of groups, by which we mean a collection of groups which is closed under isomorphism and taking subgroups. A G -CW-complex X is said to be a model for $E_{\mathcal{F}}G$, the classifying space for actions of G with stabilizers in \mathcal{F} , if for each $H \leq G$, one has that the fixed point set X^H is contractible for $H \in \mathcal{F}$ and is empty for $H \notin \mathcal{F}$. The most common classes considered are the class of trivial groups and the class \mathfrak{F} consisting of all finite groups. In these cases $E_{\mathcal{F}}G$ is often denoted EG and $\underline{E}G$ respectively. Note that EG is the total space of the universal principal G -bundle, or equivalently the universal covering space of an Eilenberg-Mac Lane space for G . The space $\underline{E}G$ is called the classifying space for proper actions of G . Recently there has been much interest in finiteness conditions for the spaces $E_{\mathcal{F}}G$, especially for $\underline{E}G$. Milnor and Segal's constructions of EG both generalize easily to construct models for any $E_{\mathcal{F}}G$, and one can show that any two models for $E_{\mathcal{F}}G$ are naturally equivariantly homotopy equivalent.

For some purposes the structure of the fixed point sets for subgroups in \mathcal{F} is irrelevant. For example, a group is in Kropholler's class $\mathbf{H}_1\mathcal{F}$ if there is any finite-dimensional contractible G -CW-complex X with

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all stabilizers in \mathcal{F} . The class $\mathbf{H}_1\mathcal{F}$ is the first stage of a hierarchy whose union is Kropholler's class $\mathbf{H}\mathcal{F}$ of hierarchically decomposable groups [10]. (These definitions were first considered for the class \mathfrak{F} of all finite groups, but work for any class \mathcal{F} .)

A priori, the class $\mathbf{H}_1\mathcal{F}$ may contain groups G that do not admit a finite-dimensional model for $E_{\mathcal{F}}G$, and we shall give such examples in the case when $\mathcal{F} = \mathfrak{P}$, the class of groups of prime power order. By contrast, in the case when $\mathcal{F} = \mathfrak{F}$, no group G is known to lie in $\mathbf{H}_1\mathfrak{F}$ without also admitting a finite-dimensional model for $\underline{E}G$. A construction due to Serre shows that every group G in $\mathbf{H}_1\mathfrak{F}$ that is virtually torsion-free has a finite-dimensional $\underline{E}G$ [4], and the authors have given examples of G for which the minimal dimension of a contractible G -CW-complex is lower than the minimal dimension of a model for $\underline{E}G$ [14]. These examples G also have the property that they admit a contractible G -CW-complex with finitely many orbits of cells, but that they do not admit any model for $\underline{E}G$ with finitely many orbits of cells.

Throughout this paper, \mathfrak{F} will denote the class of finite groups, and \mathfrak{P} will denote the class of finite groups of prime power order. We compare the classifying space for G -actions with stabilizers in \mathfrak{P} with the more well-known $\underline{E}G$, and we compare the Kropholler hierarchies built on \mathfrak{F} and \mathfrak{P} . We show that a finite group G that is not of prime power order cannot admit a finite-dimensional $E_{\mathfrak{P}}G$, but that every finite group is in $\mathbf{H}_1\mathfrak{P}$. We also construct a group that is in $\mathbf{H}_1\mathfrak{F}$ but not in $\mathbf{H}_1\mathfrak{P}$, and we show that $\mathbf{H}\mathfrak{P} = \mathbf{H}\mathfrak{F}$.

In the final section we shall contrast this with cohomology relative to all finite subgroups. The relative cohomological dimension can be viewed as a generalisation of the virtual cohomological dimension, since for virtually torsion free groups these are equal, see [17]. By a result of Bouc [2, 12] it follows that groups belonging to $\mathbf{H}_1\mathfrak{F}$ have finite relative cohomological dimension, but the converse is not known. In contrast to our results concerning classifying spaces, we show that cohomology relative to subgroups in \mathfrak{F} is naturally isomorphic to cohomology relative to subgroups in \mathfrak{P} .

2. CLASSIFYING SPACES FOR ACTIONS WITH STABILIZERS IN \mathfrak{P}

Theorem 2.1. *Let G be a finite group. Then G has a finite dimensional model for $E_{\mathfrak{P}}G$ if and only if G has prime power order.*

Proof. If G has prime power order, then a single point may be taken as a model for $E_{\mathfrak{P}}G$. Now let G be an arbitrary finite group, let p be a prime dividing the order of G , and assume that there is a p -subgroup

$P < G$, such that $N_G(P)$ is not a p -group. Then the Weyl-group $WP = N_G(P)/P$ contains a subgroup H of order prime to p . Assume G has a finite dimensional model for $E_{\mathfrak{p}}G$, X say. Then the augmented cellular chain complex of the P -fixed point set, X^P , is a finite length resolution of \mathbb{Z} by free H -modules. This gives a contradiction, since \mathbb{Z} has infinite projective dimension as an H -module for any non-trivial finite group H .

Therefore we may suppose that for each subgroup $P \leq G$ which lies in \mathfrak{P} , the normalizer $N_G(P)$ is also in \mathfrak{P} . It follows from the Frobenius normal p -complement theorem [7, 5.26] that G has a normal p -complement for each prime p . Hence G is nilpotent and equal to the direct product of its Sylow subgroups. If P is a non-trivial Sylow subgroup of G , it follows that $G = N_G(P)$ is in \mathfrak{P} . \square

Remark 2.2. The above proof was suggested to the authors by Yoav Segev. It is considerably shorter than our original proof, which did not quote the Frobenius normal p -complement theorem.

Corollary 2.3. *For a group G , the following are equivalent.*

- (i) G admits a finite-dimensional $E_{\mathfrak{p}}G$;
- (ii) Every finite subgroup of G is in \mathfrak{P} and G admits a finite-dimensional $\underline{E}G$. \square

Remark 2.4. We conclude the section with a remark on the type of $E_{\mathfrak{p}}G$. It can be proved analogously to Lück's proof for $\underline{E}G$ [15] that a group G admits a finite type model for $E_{\mathfrak{p}}G$ if and only if G has finitely many conjugacy classes of groups of prime power order and the Weyl-groups $N_G(P)/P$ for all subgroups P of prime power order are finitely presented and of type FP_{∞} . Hence any group admitting a finite type $\underline{E}G$ also admits a finite type $E_{\mathfrak{p}}G$. Recall that a finite extension of a group admitting a finite model for $\underline{E}G$ always has finitely many conjugacy classes of subgroups of prime power order [4, IX.13.2]. Hence the groups exhibited in [14, Example 7.4] are groups admitting a finite type $E_{\mathfrak{p}}G$ which do not admit a finite type $\underline{E}G$.

This behaviour is in stark contrast to that of $E_{\mathcal{V}\mathcal{C}}G$, the classifying space with virtually cyclic isotropy. Any group admitting a finite dimensional model for $E_{\mathcal{V}\mathcal{C}}G$ admits a finite dimensional model for $\underline{E}G$, see [16] and the converse also holds for a large class of groups including all polycyclic-by-finite and all hyperbolic groups [8, 16]. Furthermore, any group admitting a finite type model for $E_{\mathcal{V}\mathcal{C}}G$ also admits a finite type model for $\underline{E}G$ [9], but it is conjectured [8] that any group admitting a finite model for $E_{\mathcal{V}\mathcal{C}}G$ has to be virtually cyclic. This has been

shown for a class of groups containing all hyperbolic groups [8] and for elementary amenable groups [9].

3. THE HIERARCHIES $\mathbf{H}\mathfrak{F}$ AND $\mathbf{H}\mathfrak{B}$

Proposition 3.1. *Let X be a finite dimensional contractible G -CW-complex such that all stabilizers are finite. If there is a bound on the orders of the stabilizers then there exists a finite dimensional contractible G -CW-complex Y and an equivariant map $f : Y \rightarrow X$ such that $Y^H = \emptyset$ if H is not a p -group.*

Proof. Using the equivariant form of the simplicial approximation theorem, we may assume that X is a simplicial G -CW-complex. To simplify notation the phrase ‘ G -space’ shall mean ‘simplicial G -CW-complex’ and ‘ G -map’ will mean ‘ G -equivariant simplicial map’ throughout the rest of the proof. The space Y will be a G -space in this sense and the map $f : Y \rightarrow X$ will be a G -map in this sense. The G -space Y is constructed in two stages. Firstly, for each finite $K \leq G$ we build a finite-dimensional contractible K -space Y_K with the property that all simplex stabilizers in Y_K lie in \mathfrak{B} .

Suppose for now that each such K -space Y_K has been constructed. Using the G -equivariant form of the construction used in [11, Section 8] the space Y is constructed as follows. Let I be an indexing set for the G -orbits of vertices in X . For each $i \in I$, let v_i be a representative of the corresponding orbit, and let K_i be the stabilizer of v_i . Let X^0 denote the 0-skeleton of X . Define a G -space Y^0 by

$$Y^0 = \coprod_{i \in I} G \times_{K_i} Y_{K_i},$$

and define a G -map $f : Y^0 \rightarrow X^0$ by $f(g, y) = g.v_i$ for all $i \in I$, for all $g \in G$ and for all $y \in Y_{K_i}$. For each vertex w of X , let $Y(w) = f^{-1}(w) \subset Y^0$. Each $Y(w)$ is a contractible subspace of Y^0 , and the stabilizer of w acts on $Y(w)$.

Now for $\sigma = (w_0, \dots, w_n)$ an n -simplex of X , define a space $Y(\sigma)$ as the join

$$Y(\sigma) = Y(w_0) * Y(w_1) * \dots * Y(w_n).$$

Each vertex of $Y(\sigma)$ is already a vertex of one of the $Y(w_i)$, and so the map $f : Y^0 \rightarrow X^0$ defines a unique simplicial map $f : Y(\sigma) \rightarrow \sigma$. By construction, whenever τ is a face of σ , the space $Y(\tau)$ is identified with a subspace of $Y(\sigma)$. This allows us to define Y and $f : Y \rightarrow X$ as the colimit over the simplices σ of X of the subspaces $Y(\sigma)$, and to define $f : Y \rightarrow X$, which is a G -map of G -spaces. Since each $Y(\sigma)$ is

contractible, it follows that f is a homotopy equivalence, and hence Y is also contractible (see [11, Corollary 8.6]).

It remains to build the K -space Y_K for each finite group $K \leq G$. In the case when $K \in \mathfrak{P}$ we may take a single point to be Y_K , and so we may suppose that $K \notin \mathfrak{P}$. Fix such a subgroup K , and suppose that we are able to construct a finite-dimensional contractible K -space Z_K in which each stabilizer is a proper subgroup of K . We may assume by induction that for each $L < K$ we have already constructed the L -space Y_L . The K -space Y_K can now be constructed from Z_K and the spaces Y_L using a process similar to the construction of Y from X and the spaces Y_K . It remains to construct the K -space Z_K .

An explicit construction of an K -space Z_K with the required properties is given in [13]. We therefore provide only a sketch of the argument. We may assume that K is not in \mathfrak{P} . Let S be the unit sphere in the reduced regular complex representation of K , so that S is a topological space with K -action such that the stabilizer of every point of S is a proper subgroup of K . Since K is not in \mathfrak{P} , there are K -orbits in S of coprime lengths. Using this property, it can be shown that the sphere S admits an K -equivariant self-map $g : S \rightarrow S$ of degree zero. The K -space Z_K is defined to be the infinite mapping telescope (suitably triangulated) of the map g . \square

Corollary 3.2. *If G is in $\mathbf{H}_1\mathfrak{F}$ and there is a bound on the orders of the finite subgroups of G , then G is in $\mathbf{H}_1\mathfrak{P}$.* \square

Remark 3.3. In Proposition 3.1, the bound on the orders of the stabilizers of X is used only to give a bound on the dimensions of the spaces Y_K . In Theorem 3.8 we shall show that $\mathbf{H}_1\mathfrak{F} \neq \mathbf{H}_1\mathfrak{P}$.

Remark 3.4. The construction in Proposition 3.1 does not preserve cocompactness, because for most finite groups K , the space Y_K used in the construction cannot be chosen to be finite. A result similar to Proposition 3.1 but preserving cocompactness can be obtained by replacing \mathfrak{P} by a larger class \mathfrak{D} of groups. Here \mathfrak{D} is defined to be the class of \mathfrak{P} -by-cyclic-by- \mathfrak{P} -groups. A theorem of Oliver [20] implies that any finite group K that is not in \mathfrak{D} admits a *finite* contractible K -CW-complex Z'_K in which all stabilizers are proper subgroups of K . Applying the same argument as in the proof of Proposition 3.1, one can show that given any contractible G -CW-complex X with all stabilizers in \mathfrak{F} , there is a contractible G -CW-complex Y' with all stabilizers in \mathfrak{D} and a proper equivariant map $f' : Y' \rightarrow X$. (By proper, we mean that the inverse image of any compact subset of X is compact.)

For X a G -CW-complex with stabilizers in \mathfrak{F} , and p a prime, let $X_{\text{sing}(p)}$ denote the subcomplex consisting of points whose stabilizer has order divisible by p . For G a group and p a prime, let $S_p(G)$ denote the poset of non-trivial finite p -subgroups of G .

Proposition 3.5. *Suppose that X is a finite-dimensional contractible G -CW-complex with all stabilizers in \mathfrak{P} . For each prime p , the mod- p homology of $X_{\text{sing}(p)}$ is isomorphic to the mod- p homology of the (realization of the) poset $S_p(G)$.*

Proof. Fix a prime p , and let S denote the realization of the poset $S_p(G)$. For P a non-trivial p -subgroup of G , let X^P denote the points fixed by P , and let $S_{\geq P}$ denote the realization of the subposet of $S_p(G)$ consisting of all p -subgroups that contain P . By the P. A. Smith theorem [3], each X^P is mod- p acyclic. Each $S_{\geq P}$ is contractible since it is equal to a cone with apex P . Let P and Q be p -subgroups of G , and let $R = \langle P, Q \rangle$, the subgroup of G generated by P and Q . If R is a p -group then $X^P \cap X^Q = X^R$, and otherwise $X^P \cap X^Q$ is empty. Similarly, $S_{\geq P} \cap S_{\geq Q} = S_{\geq R}$ if R is a p -group and $S_{\geq P} \cap S_{\geq Q}$ is empty if R is not a p -group.

Since each X^P is mod- p acyclic, the mod- p homology $H_*(X_{\text{sing}(p)})$ is isomorphic to the mod- p homology of the nerve of the covering $X_{\text{sing}(p)} = \bigcup_P X^P$. Similarly, the mod- p homology $H_*(S)$ is isomorphic to the mod- p homology of the nerve of the covering $S = \bigcup_P S_{\geq P}$. By the remarks in the first paragraph, these two nerves are isomorphic. \square

Proposition 3.6. *Let k be a finite field, and let G be the group of k -points of a reductive algebraic group over k , whose commutator subgroup has k -rank n . (For example, $G = SL_{n+1}(k)$, or $GL_{n+1}(k)$.) Any finite-dimensional contractible G -CW-complex with stabilizers in \mathfrak{P} has dimension at least n .*

Proof. The hypotheses on G imply that G acts on a spherical building Δ of dimension $n - 1$ [1, 5, Appendix on algebraic groups]. Any such building is homotopy equivalent to a wedge of $(n - 1)$ -spheres. Quillen has shown that Δ is homotopy equivalent to the realization of $S_p(G)$, where p is the characteristic of the field k [21, Proposition 2.1 and Theorem 3.1]. It follows that $S_p(G)$ is homotopy equivalent to a wedge of $(n - 1)$ -spheres, and in particular the mod- p homology group $H_{n-1}(S_p(G))$ is non-zero.

Now suppose that X is a finite-dimensional contractible G -CW-complex with stabilizers in \mathfrak{P} . Using Proposition 3.5, one sees that

the mod- p homology group $H_{n-1}(X_{\text{sing}(p)})$ is non-zero. It follows that X must have dimension at least n . \square

Remark 3.7. In [22] it is shown that if G is a finite simple group of Lie type, of Lie rank n , then any contractible G -CW-complex of dimension strictly less than n contains a point fixed by G . (Theorem 1 of [22] contains the additional hypothesis that the G -CW-complex should be finite, but this is not used in the proof.) A similar argument to that used in [22, Theorem 2] was used in [13] to show that when $G = SL_{n+1}(\mathbb{F}_p)$, every contractible G -CW-complex without a global fixed point has dimension at least n . Note that Proposition 3.6 applies in greater generality than these results. For example, the Conner-Floyd construction [6] shows that whenever the multiplicative group of k does not have prime-power order, there is, for any $n \geq 1$, a 4-dimensional contractible $Gl_n(k)$ -CW-complex without a global fixed point.

Theorem 3.8. *There are the following strict containments and equalities between classes of groups:*

- (i) $\mathfrak{F} \subsetneq \mathbf{H}_1\mathfrak{P}$;
- (ii) $\mathbf{H}_1\mathfrak{P} \subsetneq \mathbf{H}_1\mathfrak{F}$;
- (iii) $\mathbf{H}\mathfrak{F} = \mathbf{H}\mathfrak{P}$.

Proof. Corollary 3.2 shows that $\mathfrak{F} \subseteq \mathbf{H}_1\mathfrak{P}$. The free product of two cyclic groups of prime order is in $\mathbf{H}_1\mathfrak{P}$ and is not finite. The claim that $\mathbf{H}\mathfrak{F} = \mathbf{H}\mathfrak{P}$ follows from the inequalities $\mathfrak{P} \subseteq \mathfrak{F} \subseteq \mathbf{H}_1\mathfrak{P}$, and the claim $\mathbf{H}_1\mathfrak{P} \subseteq \mathbf{H}_1\mathfrak{F}$ follows from $\mathfrak{P} \subseteq \mathfrak{F}$.

It remains to exhibit a group G that is in $\mathbf{H}_1\mathfrak{F}$ but not in $\mathbf{H}_1\mathfrak{P}$. Let $G = SL_\infty(\mathbb{F}_p)$, the direct limit of the groups $G_n = SL_n(\mathbb{F}_p)$, where G_n is included in G_{n+1} as the ‘top corner’. As a countable locally-finite group, G acts with finite stabilizers on a tree. (Explicitly, the vertex set V and edge set E are both equal as G -sets to the disjoint union of the sets of cosets $G/G_1 \cup G/G_2 \cup \dots$, with the edge gG_i joining the vertex gG_i to the vertex gG_{i+1} .) It follows that $G \in \mathbf{H}_1\mathfrak{F}$. By Proposition 3.6, G cannot be in $\mathbf{H}_1\mathfrak{P}$. \square

Remark 3.9. Let G be a group in $\mathbf{H}\mathfrak{F}$ that is also of type FP_∞ . By a result of Kropholler [10], there is a bound on the orders of finite subgroups of G , and Kropholler-Mislin show that G is in $\mathbf{H}_1\mathfrak{F}$ [11]. Corollary 3.2 shows that G is in $\mathbf{H}_1\mathfrak{P}$.

4. COHOMOLOGY RELATIVE TO A CLASS OF GROUPS

Let Δ denote a G -set, and let $\mathbb{Z}\Delta$ denote the corresponding G -module. For $\delta \in \Delta$, we write G_δ for the stabilizer of δ . A short

exact sequence $A \twoheadrightarrow B \twoheadrightarrow C$ of G -modules is said to be Δ -split if and only if it splits as a sequence of G_δ -modules for each $\delta \in \Delta$. Equivalently, the sequence is Δ -split if and only if the following sequence of $\mathbb{Z}G$ -modules splits: $A \otimes \mathbb{Z}\Delta \twoheadrightarrow B \otimes \mathbb{Z}\Delta \twoheadrightarrow C \otimes \mathbb{Z}\Delta$ [18].

We say a G module is Δ -projective if it is a direct summand of a G -module of the form $N \otimes \mathbb{Z}\Delta$, where N is an arbitrary G -module. Δ -projectives satisfy analogue properties to ordinary projectives. Furthermore, for each δ , and each G_δ -module M , the induced module $\text{Ind}_{G_\delta}^G M$ is Δ -projective. Given two G -sets Δ_1 and Δ_2 and a G -map $\Delta_1 \rightarrow \Delta_2$ then Δ_1 -projectives are Δ_2 -projective and Δ_2 -split sequences are Δ_1 -split. For more detail the reader is referred to [18].

Now suppose that \mathcal{F} is a class of groups closed under taking subgroups. We consider G -sets Δ satisfying the following condition, for all $H \leq G$:

$$(*) \quad \Delta^H \neq \emptyset \iff H \in \mathcal{F}.$$

There are G -maps between any two G -sets satisfying condition $(*)$, and so we may define an \mathcal{F} -projective module to be a Δ -split module for any such Δ . Similarly, an \mathcal{F} -split exact sequence of G -modules is defined to be a Δ -split sequence. If Δ satisfies $(*)$ and M is any G -module, the module $M \otimes \mathbb{Z}\Delta$ is \mathcal{F} -projective and admits an \mathcal{F} -split surjection to M . This leads to a construction of homology relative to \mathcal{F} . An \mathcal{F} -projective resolution of a module M is an \mathcal{F} -split exact sequence

$$\cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where all P_i are \mathcal{F} -projective. Group cohomology relative to \mathcal{F} , denoted $\mathcal{F}H^*(G; N)$ can now be defined as the cohomology of the cochain complex $\text{Hom}_G(P_*, N)$, where P_* is an \mathcal{F} -projective resolution of \mathbb{Z} .

We say that a module M is of type $\mathcal{F}\text{FP}_n$ if M admits an \mathcal{F} -projective resolution in which P_i is finitely generated for $0 \leq i \leq n$. It has been shown that modules of type $\mathcal{F}\text{FP}_n$ are of type FP_n [18]. We will say that a group G is of type $\mathcal{F}\text{FP}_n$ if the trivial G -module \mathbb{Z} is of type $\mathcal{F}\text{FP}_n$.

We now specialize to the cases when $\mathcal{F} = \mathfrak{F}$ and $\mathcal{F} = \mathfrak{P}$.

Theorem 4.1. *The following properties hold.*

- (i) *A short exact sequence of G -modules is \mathfrak{F} -split if and only if it is \mathfrak{P} -split.*
- (ii) *A G module is \mathfrak{F} -projective if and only if it is \mathfrak{P} -projective.*
- (iii) $\mathfrak{F}H^*(G, -) \cong \mathfrak{P}H^*(G, -)$

Proof: (i) It is obvious that any \mathfrak{F} -split sequence is \mathfrak{P} -split, and the converse follows from a standard averaging argument. Let H be an

arbitrary finite subgroup of G . Then $|H| = \prod_{i=1, \dots, n} p_i^{a_i}$ where p_i are distinct primes and $0 < a_i \in \mathbb{Z}$. For each i , let n_i be the index $n_i = [H : P_i]$. Now consider a \mathfrak{P} -split surjection $A \xrightarrow{\pi} B$. Let σ_i be a P_i -splitting of π , and define a map s_i by summing σ_i over the cosets of P_i :

$$s_i(b) = \sum_{t \in H/P_i} t\sigma_i(t^{-1}b).$$

For each P_i we obtain a map $s_i : B \rightarrow A$, such that $\pi \circ s_i = n_i \times id_B$. There exist $m_i \in \mathbb{Z}$ so that $\sum_i m_i n_i = 1$, and the map $s = \sum_i m_i s_i$ is the required H -splitting.

(ii) It is obvious that a \mathfrak{P} -projective module is \mathfrak{F} -projective. Now let P be \mathfrak{F} -projective. We may take a \mathfrak{P} -split surjection $M \twoheadrightarrow P$ with M a \mathfrak{P} -projective. By (i) this surjection is \mathfrak{F} -split, and hence split. Thus P is a direct summand of a \mathfrak{P} -projective and so is \mathfrak{P} -projective.

(iii) now follows directly from (i) and (ii). \square

Proposition 4.2. *A group G is of type $\mathfrak{F}FP_0$ if and only if G has only finitely many conjugacy classes of subgroups of prime power order.*

Proof: Suppose that G has only finitely many conjugacy classes of subgroups in \mathfrak{P} . Let I be a set of representatives for the conjugacy classes of \mathfrak{P} -subgroups and set

$$\Delta_0 = \bigsqcup_{P \in I} G/P.$$

This G -set satisfies condition $(*)$ for \mathfrak{P} and therefore the surjection $\mathbb{Z}\Delta_0 \twoheadrightarrow \mathbb{Z}$ is \mathfrak{F} -split and also $\mathbb{Z}\Delta_0$ is finitely generated.

To prove the converse we consider an arbitrary \mathfrak{F} -split surjection $P_0 \twoheadrightarrow \mathbb{Z}$ with P_0 a finitely generated \mathfrak{F} -projective. As in [18, 6.1] we can show that P_0 is a direct summand of a module $\bigoplus_{\delta \in \Delta_f} \text{Ind}_{G_\delta}^G P_\delta$, where Δ_f is a finite G -set, the G_δ are finite groups and P_δ are finitely generated G_δ -modules. Therefore we might assume from now on that P_0 is of the above form. Since there is a G -map $\Delta_f \rightarrow \Delta$, where Δ satisfies condition $(*)$ the \mathfrak{F} -split surjection $P_0 \xrightarrow{\varepsilon} \mathbb{Z}$ is also Δ_f -split [18]. Consider now the following commutative diagram:

$$\begin{array}{ccc} P_0 & \xrightarrow{\varepsilon} & \mathbb{Z} \\ \alpha \uparrow & \beta \downarrow & \parallel \\ \mathbb{Z}\Delta_f & \xrightarrow{\varepsilon_f} & \mathbb{Z} \end{array}$$

That we can find such an α follows from the fact that ε is Δ_f -split, and β exists since P_0 is Δ_f -projective being a direct sum of induced modules, induced from G_δ , ($\delta \in \Delta_f$) to G .

As a next step we'll show that ε_f is \mathfrak{F} -split. Take an arbitrary finite subgroup H of G and show that ε_f splits when restricted to H . Since ε is split by s , say, when restricted to H we can define the required splitting by $\beta \circ s$.

Now let P be an arbitrary p -subgroup of G . Since the module $\mathbb{Z}[G/P]$ is \mathfrak{F} -projective, there exists a G -map φ , such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}\Delta_f & \xrightarrow{\varepsilon_f} & \mathbb{Z} \\ \varphi \uparrow & & \parallel \\ \mathbb{Z}[G/P] & \longrightarrow & \mathbb{Z} \end{array}$$

The image $\varphi(P)$ of the identity coset P is a point of $\mathbb{Z}\Delta$ fixed by the action of P . If H is any group and $\mathbb{Z}\Omega$ is any permutation module, then the H -fixed points are generated by the orbit sums $H.\omega$. Hence P must stabilize some point of Δ_f , since otherwise we would have that p divides $\varepsilon_f\varphi(P) = \varepsilon\alpha(P) = 1$, a contradiction. It follows that P is a subgroup of G_δ for some $\delta \in \Delta_f$. \square

Note that being of type $\mathfrak{F}FP_0$ does not imply that there are finitely many conjugacy classes of finite subgroups. In fact, the authors have examples with infinitely many conjugacy classes of finite subgroups, see [14]. Nevertheless this gives rise to the following conjecture:

Conjecture 4.3. *A group G is of type $\mathfrak{F}FP_\infty$ if and only if G is of type FP_∞ and has finitely many conjugacy classes of p -subgroups.*

It is shown in [18] that any G of type $\mathfrak{F}FP_\infty$ is of type FP_∞ , which together with Proposition 4.2 proves one implication in the above conjecture.

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