# Continuous-time trading and the emergence of probability

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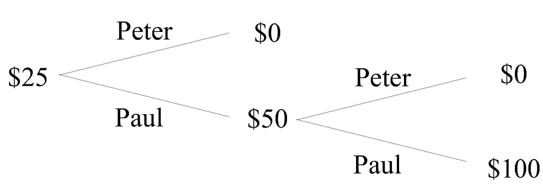


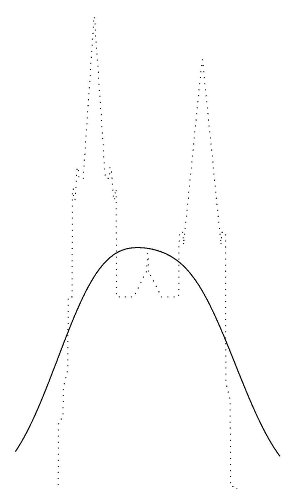

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## Abstract

This paper establishes a non-stochastic analogue of the celebrated result by Dubins and Schwarz about reduction of continuous martingales to Brownian motion via time change. We consider an idealized financial security with continuous price path, without making any stochastic assumptions. It is shown that typical price paths possess quadratic variation, where "typical" is understood in the following game-theoretic sense: there exists a trading strategy that earns infinite capital without risking more than one monetary unit if the process of quadratic variation does not exist. Replacing time by the quadratic variation process, we show that the price path becomes Brownian motion. This is essentially the same conclusion as in the Dubins–Schwarz result, except that the probabilities (constituting the Wiener measure) emerge instead of being postulated. We also give an elegant statement, inspired by Peter McCullagh's unpublished work, of this result in terms of game-theoretic probability theory.

# Contents

# 1 Introduction

This paper is a contribution to the game-theoretic approach to probability. This approach was explored (by, e.g., von Mises, Wald, and Ville) as a possible basis for probability theory at the same time as the now standard measure-theoretic approach (Kolmogorov), but then became dormant. The current revival of interest in it started with A. P. Dawid's prequential principle ([10], Section 5.1, [11], Section 3), and recent work on game-theoretic probability includes monographs [37, 40] and papers [26, 22, 25, 27].

Treatment of continuous-time processes in game-theoretic probability often involves non-standard analysis (see, e.g., [37], Chapters 11–14). Recent paper [41] suggested avoiding non-standard analysis and introduced the key technique of "high-frequency limit order strategies", also used in this paper and its predecessors, [44] and [45].

An advantage of game-theoretic probability is that one does not have to start with a full-fledged probability measure from the outset to arrive at interesting conclusions, even in the case of continuous time. For example, [44] shows that continuous price paths satisfy many standard properties of Brownian motion (such as the absence of isolated zeroes) and [45] (developing [48] and [41]) shows that the variation index of a non-constant continuous price path is 2, as in the case of Brownian motion. The standard qualification "with probability one" is replaced with "unless a specific trading strategy increases the capital it risks manyfold" (the formal definitions, assuming zero interest rate, will be given in Section 2). This paper makes the next step, showing that the Wiener measure emerges in a natural way in the continuous trading protocol. Its main result contains all main results of [44, 45], together with several refinements, as special cases.

Other results about the emergence of the Wiener measure in game-theoretic probability can be found in [43] and [46]. However, the protocols of those papers are much more restrictive, involving an externally given quadratic variation (a game-theoretic analogue of predictable quadratic variation, generally chosen by a player called Forecaster). In this paper the Wiener measure emerges in a situation with surprisingly little *a priori* structure, involving only two players: the market and a trader.

The reader will notice that not only our main result but also many of our definitions resemble those in Dubins and Schwarz's paper [14], which can be regarded as the measure-theoretic counterpart of this paper. The main difference of this paper is that we do not assume a given probability measure from outset. A less important difference is that our main result will not assume that the price path is unbounded and nowhere constant (among other things, this generalization is important to include the main results of [44, 45] as special cases). A result similar to that of Dubins and Schwarz was almost simultaneously proved by Dambis [8]; however, Dambis, unlike Dubins and Schwarz, dealt with predictable quadratic variation, and his result can be regarded as the measure-theoretic counterpart of [43] and [46].

Another related result is the well-known observation (see, e.g., [4], Proposition 2.22) that in the binomial model of a financial market every contingent claim can be replicated by a self-financing portfolio whose initial price is the expected value (suitably discounted if the interest rate is not zero) of the claim with respect to the risk-neutral probability measure. This insight is, essentially, extended in this paper to the case of an incomplete market (the price for completeness in the binomial model is the artificial assumption that at each step the price can only go up or down by specified factors) and continuous time (continuous-time mathematical finance always starts from an underlying probability measure). More generally, our results are related to the First and, especially, the Second Fundamental Theorems of Asset Pricing; this will be discussed in Section 12.

The main part of the paper starts with the description of our continuous-time trading protocol and the definition of game-theoretic versions of the notion of probability (outer and inner content) in Section 2. In Section 3 we state our main result (Theorem 1), which becomes especially intuitive if we restrict our attention to the case of the initial price equal to 0 and price paths that do not converge to a finite value and are nowhere constant: the game-theoretic probability of any event that is invariant with respect to time changes then exists and coincides with its Wiener measure (Corollary 1). This simple statement was made possible by Peter McCullagh's unpublished work on Fisher's fiducial probability: McCullagh's idea was that fiducial probability is only defined on the $\sigma$-algebra of events invariant with respect to a certain group of transformations. Section 4 presents several applications (connected with [44, 45]) demonstrating the power of Theorem 1. The fact that typical price paths possess quadratic variation is proved in Section 8. It is, however, used earlier, in Section 5, where it allows us to state a constructive version of Theorem 1. The constructive version, Theorem 2, says that replacing time by the quadratic variation process turns the price path into Brownian motion. The easy part of Theorem 1 is proved in Section 6. Sections 7 and 9 prepare the ground for the proof of Theorem 2 (in Section 10) and the non-trivial part of Theorem 1 (in Section 11).

The words such as "positive", "negative", "before", "after", "increasing", and "decreasing" will be understood in the wide sense of $\geq$ or $\leq$, as appropriate; when necessary, we will add the qualifier "strictly".

As usual, $C(E)$ is the space of all continuous functions on a topological space $E$ equipped with the sup norm. We usually omit the parentheses around $E$ in expressions such as $C[0,T] := C([0,T])$.

## 2 Outer content

We consider a game between two players, Reality (a financial market) and Sceptic (a trader), over the time interval $[0,\infty)$. First Sceptic chooses his trading strategy and then Reality chooses a continuous function $\omega : [0,\infty) \to \mathbb{R}$ (the price path of a security).

Let $\Omega$ be the set of all continuous functions $\omega : [0,\infty) \to \mathbb{R}$. For each $t \in [0,\infty)$, $\mathcal{F}_t$ is defined to be the smallest $\sigma$-algebra that makes all functions

$\omega \mapsto \omega(s)$, $s \in [0,t]$, measurable. A *process* $\mathfrak{S}$ is a family of functions $\mathfrak{S}_t : \Omega \to \mathbb{R}$, $t \in [0,\infty)$, each $\mathfrak{S}_t$ being $\mathcal{F}_t$-measurable; its *sample paths* are the functions $t \mapsto \mathfrak{S}_t(\omega)$. An *event* is an element of the $\sigma$-algebra $\mathcal{F}_\infty := \vee_t \mathcal{F}_t$ (also denoted by $\mathcal{F}$). Stopping times $\tau : \Omega \to [0,\infty]$ w.r.t. to the filtration $(\mathcal{F}_t)$ and the corresponding $\sigma$-algebras $\mathcal{F}_\tau$ are defined as usual; $\omega(\tau(\omega))$ and $\mathfrak{S}_{\tau(\omega)}(\omega)$ will be simplified to $\omega(\tau)$ and $\mathfrak{S}_\tau(\omega)$, respectively (occasionally, the argument $\omega$ will be omitted in other cases as well).

The class of allowed strategies for Sceptic is defined in two steps. A *simple trading strategy* $G$ consists of an increasing sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$ and, for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable function $h_n$. It is required that, for each $\omega \in \Omega$, $\lim_{n\to\infty} \tau_n(\omega) = \infty$. To such $G$ and an *initial capital* $c \in \mathbb{R}$ corresponds the *simple capital process*

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega)\bigl(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\bigr), \quad t \in [0,\infty) \tag{1}$$

(with the zero terms in the sum ignored, which makes the sum finite for each $t$); the value $h_n(\omega)$ will be called Sceptic's *bet* (or *stake*) at time $\tau_n$, and $\mathcal{K}_t^{G,c}(\omega)$ will be referred to as Sceptic's capital at time $t$.

A *positive capital process* is any process $\mathfrak{S}$ that can be represented in the form

$$\mathfrak{S}_t(\omega) := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n,c_n}(\omega), \tag{2}$$

where the simple capital processes $\mathcal{K}_t^{G_n,c_n}(\omega)$ are required to be positive, for all $t$ and $\omega$, and the positive series $\sum_{n=1}^{\infty} c_n$ is required to converge. The sum (2) is always positive but allowed to take value $\infty$. Since $\mathcal{K}_0^{G_n,c_n}(\omega) = c_n$ does not depend on $\omega$, $\mathfrak{S}_0(\omega)$ also does not depend on $\omega$ and will sometimes be abbreviated to $\mathfrak{S}_0$.

**Remark 1.** The financial interpretation of a positive capital process (2) is that it represents the total capital of a trader who splits his initial capital into a countable number of accounts and on each account runs a simple trading strategy making sure that this account never goes into debit.

The *outer content* of a set $E \subseteq \Omega$ is defined as

$$\overline{\mathbb{P}}(E) := \inf\bigl\{\mathfrak{S}_0 \bigm| \forall\omega \in \Omega : \liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge \mathbf{1}_E(\omega)\bigr\}, \tag{3}$$

where $\mathfrak{S}$ ranges over the positive capital processes and $\mathbf{1}_E$ stands for the indicator function of $E$. It is easy to see that the $\liminf_{t\to\infty}$ in (3) can be replaced by $\sup_t$ (and, therefore, by $\limsup_{t\to\infty}$): we can always stop (i.e., set all bets to 0) when $\mathfrak{S}$ reaches the level 1 (or a level arbitrarily close to 1).

We say that a set $E \subseteq \Omega$ is *null* if $\overline{\mathbb{P}}(E) = 0$. If $E$ is null, there is a positive capital process $\mathfrak{S}$ such that $\mathfrak{S}_0 = 1$ and $\lim_{t\to\infty} \mathfrak{S}_t(\omega) = \infty$ for all $\omega \in E$ (it suffices to sum over $\epsilon = 1/2, 1/4, \ldots$ positive capital processes $\mathfrak{S}^\epsilon$ satisfying $\mathfrak{S}_0^\epsilon = \epsilon$ and $\lim_{t\to\infty} \mathfrak{S}_t^\epsilon \ge \mathbf{1}_E$). A property of $\omega \in \Omega$ will be said to hold *for* *typical* $\omega$ if the set of $\omega$ where it fails is null. Correspondingly, a set $E \subseteq \Omega$ is *full* if $\overline{\mathbb{P}}(E^c) = 0$, where $E^c := \Omega \setminus E$ stands for the complement of $E$.

We can also define *inner content*:

$$\underline{\mathbb{P}}(E) := 1 - \overline{\mathbb{P}}(E^c).$$

This notion of inner content will not be useful in this paper (but its simple modification will be).

**Remark 2.** Another natural setting is where $\Omega$ is defined as the set of all continuous functions $\omega : [0, T] \to \mathbb{R}$ for a given constant $T$ (the time horizon). In this case the definition of outer content simplifies: instead of $\liminf_{t\to\infty} \mathfrak{S}_t(\omega)$ we will have simply $\mathfrak{S}_T(\omega)$ in (3).

**Remark 3.** Alternative names (used in, e.g., [37]) for outer and inner content are upper and lower probability in the case of sets and upper and lower expectation in the case of functions (the latter case will be considered in Section 7). Our terminology essentially follows [21] and [38], but we drop "probability" in outer/inner probability content. We also avoid expressions such as "for almost all" and "almost surely". Hopefully, this terminology will remind the reader that we do not start from a probability measure on $\Omega$.

## 2.1 Connection with the standard notion of a self-financing trading strategy

Readers accustomed to the standard definition of a self-financed trading strategy specifying explicitly the cash position (as in [39], Section VII.1a) might find it helpful to have the connection between our notion of a simple trading strategy and the standard definition spelled out in detail. The main difference of the standard definition (apart from not being "simple", i.e., not trading at discrete times) is that it specifies not only the process of trading but also the initial capital. In the standard definition, we have $d + 1$ assets (a bank account and $d$ securities) with prices $X_t^0, \dots, X_t^d$ at time $t$ (we are using the notation of [39]). In this paper, $d = 1$, it is assumed that $X_t^0 = 1$ for all $t$ (i.e., the interest rate is zero) and the notation for $X_t^1$ is $\omega(t)$; since $X_t^0$ does not carry any information, it is not mentioned explicitly.

Suppose we are given an initial capital $c$ and a simple trading strategy $G$, as described above. The corresponding standard trading strategy is defined as a pair of predictable processes $(\pi_t^0, \pi_t^1)$; intuitively, $\pi_t^0$ (resp. $\pi_t^1$) is the number of units of $X_t^0$ (resp. $X_t^1$) in the trader's portfolio. We will now describe how the pair $(G, c)$ determines $(\pi_t^0, \pi_t^1)$; first we define $\pi_t^1$ and then explain how $\pi_t^0$ is determined by the condition that the trading strategy is self-financing. The process $\pi_t^1$ is piecewise constant and is defined by

$$\pi_t^1 = \begin{cases} 0 & \text{if } t \le \tau_1 \\ h_1 & \text{if } \tau_1 < t \le \tau_2 \\ h_2 & \text{if } \tau_2 < t \le \tau_3 \\ \dots; \end{cases}$$

in particular, $\pi_0^1 = 0$. Being làdcàg (left-continuous with limits on the right), this process is predictable. The *gain process* of the standard trading strategy $(\pi_t^0, \pi_t^1)$ is

$$Y_t^\pi := \int_0^t \pi_s^0 dX_s^0 + \int_0^t \pi_s^1 dX_s^1 = \int_0^t \pi_s^1 dX_s^1 = \mathcal{K}_t^{G,0},$$

in the notation of (1), and its *value process* is

$$X_t^\pi := \pi_t^0 X_t^0 + \pi_t^1 X_t^1 = \pi_t^0 + \pi_t^1 X_t^1.$$

Since the initial capital is $c$, we have to define $\pi_0^0 := c$. In order to be self-financing, the trading strategy $(\pi_t^0, \pi_t^1)$ must satisfy $X_t^\pi = X_0^\pi + Y_t^\pi$, i.e.,

$$\pi_t^0 + \pi_t^1 X_t^1 = c + \mathcal{K}_t^{G,0} = \mathcal{K}_t^{G,c}.$$

Therefore, defining

$$\pi_t^0 := \mathcal{K}_t^{G,c} - \pi_t^1 X_t^1$$

(which agrees with $\pi_0^0 := c$) makes the strategy $(\pi_t^0, \pi_t^1)$ self-financing.

It remains to check that the process $\pi_t^0$ is làdcàg: for each $t \in (0, \infty)$,

$$\begin{aligned}\pi_t^0 - \pi_{t-}^0 &= (\mathcal{K}_t^{G,c} - \mathcal{K}_{t-}^{G,c}) - \pi_t^1 (X_t^1 - X_{t-}^1) \\ &= h_n(\omega)(\omega(t) - \omega(t-)) - \pi_t^1 (X_t^1 - X_{t-}^1) = 0,\end{aligned}$$

where $n$ is defined from the condition $t \in (\tau_n, \tau_{n+1}]$.

# 3 Main result: abstract version

A *time change* is defined to be a continuous increasing (not necessarily strictly increasing) function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$. Equipped with the binary operation of composition, $(f \circ g)(t) := f(g(t))$, $t \in [0, \infty)$, the time changes form a (non-commutative) monoid, with the identity time change $t \mapsto t$ as the unit. The *action* of a time change $f$ on $\omega \in \Omega$ is defined to be the composition $\omega^f := \omega \circ f \in \Omega$, $(\omega \circ f)(t) := \omega(f(t))$. The *trail* of $\omega \in \Omega$ is the set of all $\psi \in \Omega$ such that $\psi^f = \omega$ for some time change $f$. (These notions are often defined for groups rather than monoids: see, e.g., [32]; in this case the trail is called the orbit. In their "time-free" considerations Dubins and Schwarz [14, 35, 36] make simplifying assumptions that make the monoid of time changes a group; we will make similar assumptions in Corollary 1.) A subset $E$ of $\Omega$ is *time-superinvariant* if together with any $\omega \in \Omega$ it contains the whole trail of $\omega$; in other words, if for each $\omega \in \Omega$ and each time change $f$ it is true that

$$\omega^f \in E \Longrightarrow \omega \in E. \tag{4}$$

The *time-superinvariant class* $\mathcal{K}$ is defined to be the family of those events (elements of $\mathcal{F}$) that are time-superinvariant.

**Remark 4.** The time-superinvariant class $\mathcal{K}$ is closed under countable unions and intersections; in particular, it is a monotone class. However, it is not closed under complementation, and so is not a $\sigma$-algebra (unlike McCullagh's invariant $\sigma$-algebras). An example of a time-superinvariant event $E$ such that $E^c$ is not time-superinvariant is the set of all increasing (not necessarily strictly increasing) $\omega \in \Omega$ satisfying $\lim_{t\to\infty} \omega(t) = \infty$: implication (4) is violated for $\omega$ the identity function (i.e., $\omega(t) = t$ for all $t$), $f = 0$, and $E^c$ in place of $E$.

Let $c \in \mathbb{R}$. The probability measure $\mathcal{W}_c$ on $\Omega$ is defined by the conditions that $\omega(0) = c$ with probability one and, for all $0 \le s < t$, $\omega(t) - \omega(s)$ is independent of $\mathcal{F}_s$ and has the Gaussian distribution with mean 0 and variance $t - s$. (In other words, $\mathcal{W}_c$ is the distribution of Brownian motion started at $c$.)

**Theorem 1.** *Let $c \in \mathbb{R}$. Each event $E \in \mathcal{K}$ such that $\omega(0) = c$ for all $\omega \in E$ satisfies*

$$\overline{\mathbb{P}}(E) = \mathcal{W}_c(E). \tag{5}$$

The main part of (5) is the inequality $\le$, whose proof will occupy us in Sections 7–11. The easy part $\ge$ will be established in Section 6.

**Remark 5.** By the Dubins–Schwarz result [14], we can replace the $\mathcal{W}_c$ in the statement of Theorem 1 by any probability measure $P$ on $(\Omega, \mathcal{F})$ such that the process $X_t(\omega) := \omega(t)$ is a martingale w.r.t. to $P$ and the filtration $(\mathcal{F}_t)$, is unbounded $P$-a.s., is nowhere constant $P$-a.s., and satisfies $X_0 = c$ $P$-a.s.

Because of its generality, some aspects of Theorem 1 may appear counterintuitive. (For example, the conditions we impose on $E$ imply that $E$ contains all $\omega \in \Omega$ satisfying $\omega(0) = c$ whenever $E$ contains constant $c$.) In the rest of this section we will specialize Theorem 1 to the more intuitive case of divergent and nowhere constant price paths.

Formally, we say that $\omega \in \Omega$ is *nowhere constant* if there is no interval $(t_1, t_2)$, where $0 \le t_1 < t_2$, such that $\omega$ is constant on $(t_1, t_2)$, we say that $\omega$ is *divergent* if there is no $c \in \mathbb{R}$ such that $\lim_{t\to\infty} \omega(t) = c$, and we let DS $\subseteq \Omega$ stand for the set of all $\omega \in \Omega$ that are divergent and nowhere constant. Intuitively, the condition that the price path $\omega$ should be nowhere constant means that trading never stops completely, and the condition that $\omega$ should be divergent will be satisfied if $\omega$'s volatility does not eventually die away (cf. Remark 7 in Section 5 below). The conditions of being divergent and nowhere constant in the definition of DS are similar to, but weaker than, Dubins and Schwarz's [14] conditions of being unbounded and nowhere constant.

All unbounded and strictly increasing time changes $f : [0, \infty) \to [0, \infty)$ form a group, which will be denoted $\mathcal{G}$. Let us say that an event $E$ is *time-invariant* if it contains the whole orbit $\{\omega^f \mid f \in \mathcal{G}\}$ of each of its elements $\omega \in E$. It is clear that DS is time-invariant. Unlike $\mathcal{K}$, the time-invariant events form a $\sigma$-algebra: $E^c$ is time-invariant whenever $E$ is (cf. Remark 4). It is not difficult to see that for subsets of DS there is no difference between time-invariance and time-superinvariance:

**Lemma 1.** *An event $E \subseteq \mathrm{DS}$ is time-superinvariant if and only if it is time-invariant.*

*Proof.* If $E$ (not necessarily $E \subseteq \mathrm{DS}$) is time-superinvariant, $\omega \in \Omega$, and $f \in \mathcal{G}$, we have $\psi := \omega^f \in E$ as $\psi^{f^{-1}} = \omega$. Therefore, time-superinvariance always implies time-invariance.

It is clear that, for all $\psi \in \Omega$ and time changes $f$, $\psi^f \notin \mathrm{DS}$ unless $f \in \mathcal{G}$. Let $E \subseteq \mathrm{DS}$ be time-invariant, $\omega \in E$, $f$ be a time change, and $\psi^f = \omega$. Since $\psi^f \in \mathrm{DS}$, we have $f \in \mathcal{G}$, and so $\psi = \omega^{f^{-1}} \in E$. Therefore, time-invariance implies time-superinvariance for subsets of DS. □

**Lemma 2.** *An event $E \subseteq \mathrm{DS}$ is time-superinvariant if and only if $\mathrm{DS} \setminus E$ is time-superinvariant.*

*Proof.* This follows immediately from Lemma 1. □

For time-invariant events in DS, (5) can be strengthened to assert the coincidence of the outer and inner content of $E$ with $\mathcal{W}_c(E)$. However, the notions of outer and inner content have to be modified slightly.

For any $B \subseteq \Omega$, a restricted version of outer content can be defined by

$$\overline{\mathbb{P}}(E; B) := \inf\left\{\mathfrak{S}_0 \;\middle|\; \forall \omega \in B : \liminf_{t\to\infty} \mathfrak{S}_t(\omega) \geq \mathbf{1}_E(\omega)\right\} = \overline{\mathbb{P}}(E \cap B),$$

with $\mathfrak{S}$ again ranging over the positive capital processes. Intuitively, this is the definition obtained when $\Omega$ is replaced by $B$: we are told in advance that $\omega \in B$. The corresponding restricted version of inner content is

$$\underline{\mathbb{P}}(E; B) := 1 - \overline{\mathbb{P}}(E^c; B) = \underline{\mathbb{P}}(E \cup B^c).$$

We will use these definitions only in the case where $\overline{\mathbb{P}}(B) = 1$. Lemma 7 below shows that in this case $\underline{\mathbb{P}}(E; B) \leq \overline{\mathbb{P}}(E; B)$.

We will say that $\overline{\mathbb{P}}(E; B)$ and $\underline{\mathbb{P}}(E; B)$ are *restricted to* $B$. It should be clear by now that these notions are not related to conditional probability $\mathbb{P}(E \mid B)$. Their analogues in measure-theoretic probability are the function $E \mapsto \mathbb{P}(E \cap B)$, in the case of outer content, and the function $E \mapsto \mathbb{P}(E \cup B^c)$, in the case of inner content (assuming $B$ is measurable). Both functions coincide with $\mathbb{P}$ when $\mathbb{P}(B) = 1$.

We will also use the "restricted" versions of the notions "null", "for typical", and "full". For example, $E$ being $B$*-null* means $\overline{\mathbb{P}}(E; B) = 0$.

Theorem 1 immediately implies the following statement about the emergence of the Wiener measure in our trading protocol (another such statement, more general and constructive but also more complicated, will be given in Theorem 2(b)).

**Corollary 1.** *Let $c \in \mathbb{R}$. Each event $E \in \mathcal{K}$ satisfies*

$$\overline{\mathbb{P}}(E; \omega(0) = c, \mathrm{DS}) = \underline{\mathbb{P}}(E; \omega(0) = c, \mathrm{DS}) = \mathcal{W}_c(E) \tag{6}$$

*(in this context, $\omega(0) = c$ stands for the event $\{\omega \in \Omega \mid \omega(0) = c\}$ and a comma stands for the intersection).*

*Proof.* Events $E \cap \mathrm{DS} \cap \{\omega \mid \omega(0) = c\}$ and $E^c \cap \mathrm{DS} \cap \{\omega \mid \omega(0) = c\}$ belong to $\mathcal{K}$: for the first of them, this immediately follows from $\mathrm{DS} \in \mathcal{K}$ and $\mathcal{K}$ being closed under intersections (cf. Remark 4), and for the second, it suffices to notice that $E^c \cap \mathrm{DS} = \mathrm{DS} \setminus (E \cap \mathrm{DS}) \in \mathcal{K}$ (cf. Lemma 2). Applying (5) to these two events and making use of the inequality $\underline{\mathbb{P}} \le \overline{\mathbb{P}}$ (cf. Lemma 7 and Equation (15) below), we obtain:

$$\begin{aligned}\mathcal{W}_c(E) = 1 - \mathcal{W}_c(E^c) &= 1 - \overline{\mathbb{P}}(E^c) \le 1 - \overline{\mathbb{P}}(E^c; \omega(0) = c, \mathrm{DS}) \\ &= \underline{\mathbb{P}}(E; \omega(0) = c, \mathrm{DS}) \le \overline{\mathbb{P}}(E; \omega(0) = c, \mathrm{DS}) \le \overline{\mathbb{P}}(E) = \mathcal{W}_c(E). \quad \square\end{aligned}$$

We can express the equality (6) by saying that the game-theoretic probability of $E$ exists and is equal to $\mathcal{W}_c(E)$ when we restrict our attention to $\omega$ in DS satisfying $\omega(0) = c$.

# 4 Applications

The main goal of this section is to demonstrate the power of Theorem 1; in particular, we will see that it implies the main results of [44] and [45]. (We will deduce these and other results as corollaries of Theorem 1 and the corresponding results for measure-theoretic Brownian motion; it is, however, still important to have direct game-theoretic proofs such as those given in [44, 45].) One corollary (Corollary 4) of Theorem 1 solves an open problem posed in [45], and two other corollaries (Corollaries 5 and 6) give much more precise results. At the end of the section we will draw the reader's attention to several events such that: Theorem 1 together with very simple game-theoretic arguments show that they are full; the fact that they are full does not follow from Theorem 1 alone.

## 4.1 Points of increase

Let us say that $t \in (0, \infty)$ is a *point of increase* for $\omega \in \Omega$ if there exists $\delta > 0$ such that $\omega(t_1) \le \omega(t) \le \omega(t_2)$ for all $t_1 \in ((t - \delta)^+, t)$ and $t_2 \in (t, t + \delta)$. Points of decrease are defined in the same way except that $\omega(t_1) \le \omega(t) \le \omega(t_2)$ is replaced by $\omega(t_1) \ge \omega(t) \ge \omega(t_2)$. We say that $\omega$ is *locally constant to the right of* $t \in [0, \infty)$ if there exists $\delta > 0$ such that $\omega$ is constant over the interval $[t, t + \delta]$.

A slightly weaker form of the following corollary was proved directly (by adapting Burdzy's [7] proof) in [44].

**Corollary 2.** *Typical $\omega$ have no points $t$ of increase or decrease such that $\omega$ is not locally constant to the right of $t$.*

This result (without the clause about local constancy) was established by Dvoretzky, Erdős, and Kakutani [17] for Brownian motion, and Dubins and Schwarz [14] noticed that their reduction of continuous martingales to Brownian motion shows that it continues to hold for all almost surely unbounded continuous martingales that are almost surely nowhere constant. We will apply Dubins and Schwarz's observation in the game-theoretic framework.

*Proof of Corollary 2.* Let us first consider only the $\omega \in \Omega$ satisfying $\omega(0) = 0$. Theorem 1 and the Dvoretzky–Erdős–Kakutani result show that typical $\omega$ have no points $t$ of increase or decrease such that $\omega$ is not locally constant to the right of $t$ and $\omega$ is not locally constant to the left of $t$ (with the obvious definition of local constancy to the left of $t$). A simple game-theoretic argument (as in [44], Theorem 1) shows that the event that $\omega$ is locally constant to the left but not locally constant to the right of a point of increase or decrease is null.

Let us now get rid of the restriction $\omega(0) = 0$. Fix a positive capital process $\mathfrak{S}$ satisfying $\mathfrak{S}_0 < \epsilon$ and reaching 1 on $\omega$ with $\omega(0) = 0$ that have at least one point $t$ of increase or decrease such that $\omega$ is not locally constant to the right of $t$. Applying $\mathfrak{S}$ to $\omega - \omega(0)$ gives another positive capital process, which will achieve the same goal but without the restriction $\omega(0) = 0$. □

It is easy to see that the qualification about local constancy to the right of $t$ in Corollary 2 is essential.

**Proposition 1.** *The outer content of the following event is one: there is a point $t$ of increase such that $\omega$ is locally constant to the right of $t$.*

*Proof.* This proof uses Lemma 6 stated and proved in Section 7 below. Consider the continuous martingale which is Brownian motion that starts at 0 and is stopped as soon as it reaches 1. □

## 4.2 Variation index

For each interval $[u, v] \subseteq [0, \infty)$ and each $p \in (0, \infty)$, the *p-variation* of $\omega \in \Omega$ over $[u, v]$ is defined as

$$\mathrm{v}_p^{[u,v]}(\omega) := \sup_{\kappa} \sum_{i=1}^{n_\kappa} |\omega(t_i) - \omega(t_{i-1})|^p ,$$

where $\kappa$ ranges over all partitions $u = t_0 \leq t_1 \leq \cdots \leq t_{n_\kappa} = v$ of the interval $[u, v]$. It is obvious that there exists a unique number $\mathrm{vi}^{[u,v]}(\omega) \in [0, \infty]$, called the *variation index* of $\omega$ over $[u, v]$, such that $\mathrm{v}_p^{[u,v]}(\omega)$ is finite when $p > \mathrm{vi}^{[u,v]}(\omega)$ and infinite when $p < \mathrm{vi}^{[u,v]}(\omega)$; notice that $\mathrm{vi}^{[u,v]}(\omega) \notin (0, 1)$.

The following result was obtained in [45] (by adapting Bruneau's [6] proof); in measure-theoretic probability it was established by Lepingle ([28], Theorem 1 and Proposition 3) for continuous semimartingales and Lévy [29] for Brownian motion.

**Corollary 3.** *For typical $\omega \in \Omega$, the following is true. For any interval $[u, v] \subseteq [0, \infty)$ such that $u < v$, either $\mathrm{vi}^{[u,v]}(\omega) = 2$ or $\omega$ is constant over $[u, v]$.*

(The interval $[u, v]$ was assumed fixed in [45], but this assumption is easy to get rid of.)

*Proof.* Without loss of generality we restrict our attention to the $\omega$ satisfying $\omega(0) = 0$ (see the proof of Corollary 2). Consider the set of $\omega \in \Omega$ such that, for some interval $[u,v] \subseteq [0,\infty)$, neither $\mathrm{vi}^{[u,v]}(\omega)=2$ nor $\omega$ is constant over $[u,v]$. This set is time-superinvariant, and so in conjunction with Theorem 1 Lévy's result implies that it is null. □

Corollary 3 says that, for typical $\omega$,

$$\mathrm{v}_p(\omega) \begin{cases} <\infty & \text{if } p>2 \\ =\infty & \text{if } p<2 \text{ and } \omega \text{ is not constant.} \end{cases}$$

However, it does not say anything about the situation for $p=2$. The following result completes the picture (solving the problem posed in [45], Section 5).

**Corollary 4.** *For typical $\omega \in \Omega$, the following is true. For any interval $[u,v] \subseteq [0,\infty)$ such that $u<v$, either $\mathrm{v}_2^{[u,v]}(\omega)=\infty$ or $\omega$ is constant over $[u,v]$.*

*Proof.* Lévy [29] proves for Brownian motion that $\mathrm{v}_2^{[u,v]}(\omega)=\infty$ almost surely (for fixed $[u,v]$, which implies the statement for all $[u,v]$). Consider the set of $\omega \in \Omega$ such that, for some interval $[u,v] \subseteq [0,\infty)$, neither $\mathrm{v}_2^{[u,v]}(\omega)=\infty$ nor $\omega$ is constant over $[u,v]$. This set is time-superinvariant, and so in conjunction with Theorem 1 Lévy's result implies that it is null. □

### 4.3 More precise results

Theorem 1 allows us to deduce much stronger results than Corollaries 3 and 4 from known results about Brownian motion.

Define $\ln^* u := 1 \vee |\ln u|$ and let $\psi : [0,\infty) \to [0,\infty)$ be Taylor's [42] function

$$\psi(u) := \frac{u^2}{2 \ln^* \ln^* u}$$

(with $\psi(0)=0$). For $\omega \in \Omega$, $T \in [0,\infty)$, and $\phi : [0,\infty) \to [0,\infty)$, set

$$\mathrm{v}_{\phi,T}(\omega) := \sup_{\kappa} \sum_{i=1}^{n_\kappa} \phi\left(|\omega(t_i)-\omega(t_{i-1})|\right),$$

where $\kappa$ ranges over all partitions $0=t_0 \le t_1 \le \cdots \le t_{n_\kappa}=T$ of $[0,T]$. In the previous subsection we considered the case $\phi(u) := u^p$; another interesting case is $\phi := \psi$. See [5] for a much more explicit expression for $\mathrm{v}_{\psi,T}(\omega)$.

**Corollary 5.** *For typical $\omega$,*

$$\forall T \in [0,\infty) : \mathrm{v}_{\psi,T}(\omega) < \infty.$$

*Suppose $\phi : [0,\infty) \to [0,\infty)$ is such that $\psi(u) = o(\phi(u))$ as $u \to 0$. For typical $\omega$,*

$$\forall T \in [0,\infty) : \omega \text{ is constant on } [0,T] \text{ or } \mathrm{v}_{\phi,T}(\omega)=\infty.$$

Corollary 5 refines Corollaries 3 and 4; it will be further strengthened by Corollary 6.

The quantity $\mathrm{v}_{\psi,T}(\omega)$ is not nearly as fundamental as the following quantity introduced by Taylor [42]: for $\omega \in \Omega$ and $T \in [0,\infty)$, set

$$\mathrm{w}_T(\omega) := \lim_{\delta\to 0} \sup_{\kappa\in K_\delta[0,T]} \sum_{i=1}^{n_\kappa} \psi\left(|\omega(t_i) - \omega(t_{i-1})|\right), \tag{7}$$

where $K_\delta[0,T]$ is the set of all partitions $0 = t_0 \le \cdots \le t_{n_\kappa} = T$ of $[0,T]$ whose mesh is less than $\delta$: $\max_i(t_i - t_{i-1}) < \delta$. Notice that the expression after the $\lim_{\delta\to0}$ in (7) is increasing in $\delta$; therefore, $\mathrm{w}_T(\omega) \le \mathrm{v}_{\psi,T}(\omega)$.

The following corollary contains Corollaries 3–5 as special cases. It is similar to Corollary 5 but is stated in terms of w.

**Corollary 6.** *For typical* $\omega$*,*

$$\forall T \in [0,\infty) : \omega \textit{ is constant on } [0,T] \textit{ or } \mathrm{w}_T(\omega) \in (0,\infty). \tag{8}$$

*Proof.* First let us check that under the Wiener measure (8) holds for almost all $\omega$. It is sufficient to prove that $\mathrm{w}_T = T$ for all $T \in [0,\infty)$ a.s. Furthermore, it is sufficient to consider only rational $T \in [0,\infty)$. Therefore, it is sufficient to consider a fixed rational $T \in [0,\infty)$. And for a fixed $T$, $\mathrm{w}_T = T$ a.s. follows from Taylor's result ([42], Theorem 1).

As usual, let us restrict our attention to the case $\omega(0) = 0$. In view of Theorem 1 it suffices to check that the complement of the event (8) is time-superinvariant. It is sufficient to check (4), where $E$ is the complement of (8). In other words, it is sufficient to check that $\omega^f = \omega \circ f$ satisfies (8) whenever $\omega$ satisfies (8). This follows from Lemma 3 below, which says that $\mathrm{w}_T(\omega \circ f) = \mathrm{w}_{f(T)}(\omega)$. ☐

**Lemma 3.** *Let* $T \in [0,\infty)$*,* $\omega \in \Omega$*, and* $f$ *be a time change. Then* $\mathrm{w}_T(\omega \circ f) = \mathrm{w}_{f(T)}(\omega)$*.*

*Proof.* Fix $T \in [0,\infty)$, $\omega \in \Omega$, a time change $f$, and $c \in [0,\infty]$. Our goal is to prove

$$\lim_{\delta\to0} \sup_{\kappa\in K_\delta[0,f(T)]} \sum_{i=1}^{n_\kappa} \psi\left(|\omega(t_i) - \omega(t_{i-1})|\right) = c \\ \Longrightarrow \lim_{\delta\to0} \sup_{\kappa\in K_\delta[0,T]} \sum_{i=1}^{n_\kappa} \psi\left(|\omega(f(t_i)) - \omega(f(t_{i-1}))|\right) = c, \tag{9}$$

in the notation of (7). Suppose the antecedent in (9) holds. Notice that the two $\lim_{\delta\to0}$ in (9) can be replaced by $\inf_{\delta>0}$.

To prove that the limit on the right-hand side of (9) is $\le c$, take any $\epsilon > 0$. We will assume $c < \infty$ (the case $c = \infty$ is trivial). Let $\delta > 0$ be so small that

$$\sup_{\kappa\in K_\delta[0,f(T)]} \sum_{i=1}^{n_\kappa} \psi\left(|\omega(t_i) - \omega(t_{i-1})|\right) < c + \epsilon.$$

Let $\delta' > 0$ be so small that $|t - t'| < \delta' \implies |f(t) - f(t')| < \delta$. Since $f(\kappa) \in K_\delta[0, f(T)]$ whenever $\kappa \in K_{\delta'}[0, T]$,

$$\sup_{\kappa \in K_{\delta'}[0,T]} \sum_{i=1}^{n_\kappa} \psi\left(|\omega(f(t_i)) - \omega(f(t_{i-1}))|\right) < c + \epsilon.$$

To prove that the limit on the right-hand side of (9) is $\geq c$, take any $\epsilon > 0$ and $\delta' > 0$. We will assume $c < \infty$ (the case $c = \infty$ can be considered analogously). Place a finite number $N$ of points including 0 and $T$ onto the interval $[0, T]$ so that the distance between any pair of adjacent points is less than $\delta'$; this set of points will be denoted $\kappa_0$. Let $\delta > 0$ be so small that $\psi(u) < \epsilon/N$ whenever $0 < u < \delta$. Choose a partition $\kappa = \{t_0, \ldots, t_n\} \in K_\delta[0, f(T)]$ satisfying

$$\sum_{i=1}^{n} \psi\left(|\omega(t_i) - \omega(t_{i-1})|\right) > c - \epsilon.$$

Let $\kappa' = \{t'_0, \ldots, t'_n\}$ be a partition of the interval $[0, T]$ satisfying $f(\kappa') = \kappa$. This partition will satisfy

$$\sum_{i=1}^{n} \psi\left(\left|\omega(f(t'_i)) - \omega(f(t'_{i-1}))\right|\right) > c - \epsilon,$$

and the union $\kappa'' = \{t''_0, \ldots, t''_{N+n}\}$ (with its elements listed in the increasing order) of $\kappa_0$ and $\kappa'$ will satisfy

$$\sum_{i=1}^{N+n} \psi\left(\left|\omega(f(t''_i)) - \omega(f(t''_{i-1}))\right|\right) > c - 2\epsilon.$$

Since $\kappa'' \in K_{\delta'}[0, T]$ and $\epsilon$ and $\delta'$ can be taken arbitrarily small, this completes the proof. $\square$

The value $\mathrm{w}_T(\omega)$ defined by (7) can be interpreted as the quadratic variation of the price path $\omega$ over the time interval $[0, T]$. Another non-stochastic definition of quadratic variation will serve us in Section 5 as the basis for the proof of Theorem 1.

## 4.4 Limitations of Theorem 1

We said earlier that Theorem 1 implies the main result of [44] (see Corollary 2). This is true in the sense that the extra game-theoretic argument used in the proof of Corollary 2 was very simple. But this simple argument was essential: in this subsection we will see that Theorem 1 *per se* does not imply the full statement of Corollary 2.

Let $c \in \mathbb{R}$ and $E \subseteq \Omega$ be such that $\omega(0) = c$ for all $\omega \in E$. Suppose the set $E$ is null. We can say that the equality $\overline{\mathbb{P}}(E) = 0$ can be deduced from Theorem 1 and the properties of Brownian motion if (and only if) $\mathcal{W}_c(\overline{E}) = 0$, where $\overline{E}$

is the smallest time-superinvariant set containing $E$ (it is clear that such a set exists and is unique). It would be nice if all equalities $\overline{\mathbb{P}}(E) = 0$, for all null sets $E$ satisfying $\forall \omega \in E : \omega(0) = c$, could be deduced from Theorem 1 and the properties of Brownian motion. We will see later (Proposition 2) that this is not true even for some fundamental null events $E$; an example of such an event will now be given.

Let us say that a closed interval $[t_1, t_2] \subseteq [0, \infty)$ is an *interval of local maximum* for $\omega \in \Omega$ if (a) $\omega$ is constant on $[t_1, t_2]$ but not constant on any larger interval containing $[t_1, t_2]$, and (b) there exists $\delta > 0$ such that $\omega(s) \le \omega(t)$ for all $s \in ((t_1 - \delta)^+, t_1) \cup (t_2, t_2 + \delta)$ and all $t \in [t_1, t_2]$. In the case where $t_1 = t_2$ we will say "point" instead of "interval". It is shown in [44] (Corollary 3) that, for typical $\omega$, all intervals of local maximum are points; this also follows from Corollary 2, and is very easy to check directly. Let $E$ be the null event that $\omega(0) = c$ and not all intervals of local maximum of $\omega$ are points. Proposition 2 says that $\overline{\mathbb{P}}(E) = 0$ cannot be deduced from Theorem 1 and the properties of Brownian motion. This implies that Corollary 2 also cannot be deduced from Theorem 1 and the properties of Brownian motion, despite the fact that the deduction is possible with the help of a very easy game-theoretic argument.

Before stating and proving Proposition 2, we will introduce formally the operator $E \mapsto \overline{E}$ and show that it is a bona fide closure operator. For each $E \subseteq \Omega$, $\overline{E}$ is defined to be the union of the trails of all points in $E$. It can be checked that $E \mapsto \overline{E}$ satisfies the standard properties of closure operators: $\overline{\emptyset} = \emptyset$ and $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$ are obvious, and $\overline{\overline{E}} = \overline{E}$ and $E \subseteq \overline{E}$ follow from the fact that the time changes constitute a monoid. Therefore ([18], Theorem 1.1.3 and Proposition 1.2.7), $E \mapsto \overline{E}$ is the operator of closure in some topology on $\Omega$, which will be called the *time-superinvariant topology*. A set $E \subseteq \Omega$ is closed in this topology if and only if it contains the trail of any of its elements.

**Proposition 2.** *Let $c \in \mathbb{R}$ and $E$ be the set of all $\omega \in \Omega$ such that $\omega(0) = c$ and $\omega$ has an interval of local maximum that is not a point. Then $\overline{\mathbb{P}}(E) = 0$ but*

$$\overline{\mathbb{P}}\left(\overline{E}\right) = \overline{\mathbb{P}}\left(\overline{E}; \omega(0) = c\right) = \underline{\mathbb{P}}\left(\overline{E}; \omega(0) = c\right) = \mathcal{W}_c\left(\overline{E}\right) = 1.$$

*Proof.* Let us see that almost every trajectory $\omega$ of Brownian motion starting at $c$ is an element of $\overline{E}$ (the rest follows from Theorem 1 and Lemmas 5 and 7). For a given $\omega$, let $\tau = \tau(\omega) \in [0, 1]$ be the smallest element of $\arg\max_{t \in [0,1]} \omega(t)$. Suppose that $\tau \in (0, 1)$ (by the local law of the iterated logarithm, this is true with probability one) and that the local maximum of $\omega$ at $\tau$ is strict (this also happens with probability one). Applying the time change

$$f(t) := \begin{cases} t & \text{if } t < \tau \\ \tau & \text{if } \tau \le t \le \tau + 1 \\ t - 1 & \text{if } t > \tau + 1, \end{cases}$$

we obtain an element of $E$. □

Proposition 2 shows that Theorem 1 does not make all other game-theoretic arguments redundant. What is interesting is that already very simple arguments suffice to deduce all results in [44, 45].

**Remark 6.** All results discussed in this section are about sets of outer content zero or inner content one, and one might suspect that the class $\mathcal{K}$ is so small that $\mathcal{W}_c(E) \in \{0,1\}$ for all $c \in \mathbb{R}$ and all $E \in \mathcal{K}$ such that $\omega(0) = c$ when $\omega \in \mathcal{K}$; this would have been another limitation of Theorem 1. However, it is easy to check that for each $p \in [0,1]$ and each $c \in \mathbb{R}$ there exists $E \in \mathcal{K}$ satisfying $\omega(0) = c$ for all $\omega \in E$ and satisfying $\mathcal{W}_c(E) = p$. Indeed, without loss of generality we can take $c := p$, and we can then define $E$ to be the event that $\omega(0) = p$, $\omega$ reaches levels 0 and 1, and $\omega$ reaches level 1 before reaching level 0.

# 5 Main result: constructive version

For each $n \in \{0,1,\ldots\}$, let $\mathbb{D}_n := \{k2^{-n} \mid k \in \mathbb{Z}\}$ and define a sequence of stopping times $T_k^n$, $k = -1,0,1,2,\ldots$, inductively by $T_{-1}^n := 0$,

$$T_0^n(\omega) := \inf\{t \geq 0 \mid \omega(t) \in \mathbb{D}_n\},$$
$$T_k^n(\omega) := \inf\left\{t \geq T_{k-1}^n \mid \omega(t) \in \mathbb{D}_n \;\&\; \omega(t) \neq \omega(T_{k-1}^n)\right\}, \quad k = 1,2,\ldots$$

(as usual, $\inf\emptyset := \infty$). For each $t \in [0,\infty)$ and $\omega \in \Omega$, define

$$A_t^n(\omega) := \sum_{k=0}^{\infty} \left(\omega(T_k^n \wedge t) - \omega(T_{k-1}^n \wedge t)\right)^2, \quad n = 0,1,2,\ldots, \tag{10}$$

and set

$$\overline{A}_t(\omega) := \limsup_{n\to\infty} A_t^n(\omega), \quad \underline{A}_t(\omega) := \liminf_{n\to\infty} A_t^n(\omega).$$

We will see later (Theorem 2(a)) that the event $(\forall t \in [0,\infty) : \overline{A}_t = \underline{A}_t)$ is full and that for typical $\omega$ the functions $\overline{A}(\omega) : t \in [0,\infty) \mapsto \overline{A}_t(\omega)$ and $\underline{A}(\omega) : t \in [0,\infty) \mapsto \underline{A}_t(\omega)$ are elements of $\Omega$ (in particular, they are finite). But in general we can only say that $\overline{A}(\omega)$ and $\underline{A}(\omega)$ are positive increasing functions (not necessarily strictly increasing) that can even take value $\infty$. For each $s \in [0,\infty)$, define the stopping time

$$\tau_s := \inf\left\{t \geq 0 \mid \overline{A}|_{[0,t)} = \underline{A}|_{[0,t)} \in C[0,t) \;\&\; \sup_{u<t} \overline{A}_u = \sup_{u<t} \underline{A}_u \geq s\right\}. \tag{11}$$

(We will see in Section 8, Lemma 11, that this is indeed a stopping time.) It will be convenient to use the following convention: an event stated in terms of $A_\infty$, such as $A_\infty = \infty$, happens if and only if $\overline{A} = \underline{A}$ and $A_\infty := \overline{A}_\infty = \underline{A}_\infty$ satisfies the given condition.

Let $P$ be a function defined on the power set of $\Omega$ and taking values in $[0,1]$ (such as $\overline{\mathbb{P}}$ or $\underline{\mathbb{P}}$), and let $f : \Omega \to \Psi$ be a mapping from $\Omega$ to another set $\Psi$. The *pushforward* $Pf^{-1}$ of $P$ by $f$ is the function on the power set of $\Psi$ defined by

$$Pf^{-1}(E) := P(f^{-1}(E)), \quad E \subseteq \Psi.$$

An especially important mapping for this paper is the *normalizing time change* $\mathrm{ntc} : \Omega \to \mathbb{R}^{[0,\infty)}$ defined as follows: for each $\omega \in \Omega$, $\mathrm{ntc}(\omega)$ is the time-changed price path $s \mapsto \omega(\tau_s)$, $s \in [0,\infty)$ (with $\omega(\infty)$ set to, e.g., 0). For each $c \in \mathbb{R}$, let

$$\overline{Q}_c := \overline{\mathbb{P}}(\,\cdot\,; \omega(0) = c, A_\infty = \infty)\,\mathrm{ntc}^{-1} \tag{12}$$

$$\underline{Q}_c := \underline{\mathbb{P}}(\,\cdot\,; \omega(0) = c, A_\infty = \infty)\,\mathrm{ntc}^{-1} \tag{13}$$

(as before, the commas stand for conjunction in this context) be the pushforwards of the restricted outer and inner content

$$E \subseteq \Omega \mapsto \overline{\mathbb{P}}(E; \omega(0) = c, A_\infty = \infty)$$
$$E \subseteq \Omega \mapsto \underline{\mathbb{P}}(E; \omega(0) = c, A_\infty = \infty),$$

respectively, by normalizing time change ntc.

As mentioned earlier, we use restricted outer and inner content $\overline{\mathbb{P}}(E; B)$ and $\underline{\mathbb{P}}(E; B)$ only when $\overline{\mathbb{P}}(B) = 1$. In Section 7, (16), we will see that indeed $\overline{\mathbb{P}}(\omega(0) = c, A_\infty = \infty) = 1$.

The next theorem shows that the pushforwards of $\overline{\mathbb{P}}$ and $\underline{\mathbb{P}}$ we have just defined are closely connected with the Wiener measure. Remember that, for each $c \in \mathbb{R}$, $\mathcal{W}_c$ is the probability measure on $(\Omega, \mathcal{F})$ which is the pushforward of the Wiener measure $\mathcal{W}_0$ by the mapping $\omega \in \Omega \mapsto \omega + c$ (i.e., $\mathcal{W}_c$ is the distribution of Brownian motion over time period $[0,\infty)$ started from $c$).

**Theorem 2.** *(a) For typical $\omega$, the function*

$$A(\omega) : t \in [0,\infty) \mapsto A_t(\omega) := \overline{A}_t(\omega) = \underline{A}_t(\omega)$$

*exists, is an increasing element of $\Omega$ with $A_0(\omega) = 0$, and has the same intervals of constancy as $\omega$. (b) For all $c \in \mathbb{R}$, the restriction of both $\overline{Q}_c$ and $\underline{Q}_c$ to $\mathcal{F}$ coincides with the measure $\mathcal{W}_c$ on $\Omega$ (in particular, $\underline{Q}_c(\Omega) = 1$).*

**Remark 7.** The value $A_t(\omega)$ can be interpreted as the total volatility of the price path $\omega$ over the time period $[0,t]$. Theorem 2(b) implies that typical $\omega$ satisfying $A_\infty(\omega) = \infty$ are unbounded (in particular, divergent). If $A_\infty(\omega) < \infty$, the total volatility $A_{t+1}(\omega) - A_t(\omega)$ of $\omega$ over $[t, t+1]$ tends to 0 as $t \to \infty$, and so the volatility of $\omega$ can be said to die away.

**Remark 8.** Theorem 2 will continue to hold if the restriction "$; \omega(0) = c, A_\infty = \infty$)" in the definitions (12) and (13) is replaced by "$; \omega(0) = c, \omega$ is unbounded)" (in analogy with [14]).

**Remark 9.** Theorem 2 depends on the arbitrary choice $(\mathbb{D}_n)$ of the sequence of grids to define the quadratic variation process $A_t$. To make this less arbitrary, we could consider all grids whose mesh tends to zero fast enough and which are definable in the standard language of set theory (similarly to Wald's [49] suggested requirement for von Mises's collectives). Dudley's [15] result suggests that the rate of convergence $o(1/\log n)$ of the mesh to zero is sufficient, and de la Vega's [12] result suggests that this rate is slowest possible.

**Remark 10.** In this paper we construct quadratic variation $A$ and define the stopping times $\tau$ in terms of $A$. Dubins and Schwarz [14] construct $\tau$ directly (in a very similar way to our construction of $A$). An advantage of our construction (the game-theoretic counterpart of that in [23]) is that the function $A(\omega)$ is continuous for typical $\omega$, whereas the event that the function $s \mapsto \tau_s(\omega)$ is continuous has inner content zero (Dubins and Schwarz's extra assumptions make this function continuous for almost all $\omega$).

**Remark 11.** Theorem 1 implies that the two notions of quadratic variation that we have discussed so far, $\mathrm{w}_t(\omega)$ defined by (7) and $A_t(\omega)$, coincide for all $t$ for typical $\omega$: remember that, in the case of Brownian motion, $\forall t \in [0,\infty) : \mathrm{w}_t = A_t = t$ a.s., and that the complement of the event $\forall t \in [0,\infty) : \mathrm{w}_t = A_t$ belongs to $\mathcal{K}$ (cf. Lemma 3).

The rest of the paper is mainly devoted to the proof of Theorems 2 and 1. The general scheme of the proof will mainly follow the proof of Theorem 2 in [46] (although the steps are often implemented differently).

# 6 Proof of the inequality $\geq$ in Theorem 1

When applied to $\mathcal{W}_c$, the following lemma asserts the inequality $\geq$ in (5).

**Lemma 4.** *Let $P$ be a probability measure on $(\Omega, \mathcal{F})$ such that the process $X_t(\omega) := \omega(t)$ is a martingale w.r. to $P$ and the filtration $(\mathcal{F}_t)$. Then $P(E) \leq \overline{\mathbb{P}}(E)$ for any event $E$.*

*Proof.* First notice that the equality in (3) will continue to hold when $\geq$ is replaced by $>$. Fix an event $E$ and $\epsilon > 0$. Find a positive capital process $\mathfrak{S}$ of the form (2) such that $\mathfrak{S}_0 < \overline{\mathbb{P}}(E) + \epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) > 1$ for all $\omega \in E$. It can be checked using the optional sampling theorem (it is here that the boundedness of Sceptic's bets is used) that each addend in (1) is a martingale, and so each partial sum in (1) is a martingale and (1) itself is a local martingale. Since each addend in (2) is a positive local martingale, it is a supermartingale. (We use the definition of supermartingale that does not require integrability and right continuity, as in, e.g., [33].) We can see that each partial sum in (2) is a positive continuous supermartingale. Using Fatou's lemma and the maximal inequality for positive supermartingales, we now obtain

$$\begin{aligned} P(E) \leq P\left(\liminf_{t\to\infty} \mathfrak{S}_t > 1\right) &\leq \liminf_{t\to\infty} P\left(\mathfrak{S}_t > 1\right) \\ &\leq \liminf_{t\to\infty} P\left(\sum_{n=1}^{N_t} \mathcal{K}_t^{G_n,c_n} > 1\right) + \epsilon \leq \liminf_{t\to\infty} \sum_{n=1}^{N_t} c_n + \epsilon \\ &\leq \mathfrak{S}_0 + \epsilon \leq \overline{\mathbb{P}}(E) + 2\epsilon, \quad (14) \end{aligned}$$

where $N_t$ is chosen large enough for each $t$ (which can be assumed to take only integer values). Since $\epsilon$ can be arbitrarily small, this implies the statement of the lemma. □

# 7 Coherence and outer content for functionals

The following trivial result says that our trading game is *coherent*, in the sense that $\overline{\mathbb{P}}(\Omega) = 1$ (i.e., no positive capital process increases its value between time 0 and $\infty$ by more than a strictly positive constant for all $\omega \in \Omega$).

**Lemma 5.** $\overline{\mathbb{P}}(\Omega) = 1$. *Moreover, for each* $c \in \mathbb{R}$, $\overline{\mathbb{P}}(\omega(0) = c) = 1$.

*Proof.* No positive capital process can strictly increase its value on a constant $\omega \in \Omega$. $\square$

Lemma 5, however, does not even guarantee that the set of non-constant elements of $\Omega$ has outer content one. The theory of measure-theoretic probability provides us with a plethora of non-trivial events of outer content one.

**Lemma 6.** *Let $E$ be an event that almost surely contains the sample path of a continuous martingale with time interval* $[0, \infty)$*. Then* $\overline{\mathbb{P}}(E) = 1$.

*Proof.* This is a special case of Lemma 4. $\square$

In particular, applying Lemma 6 to Brownian motion started at $c \in \mathbb{R}$ gives

$$\overline{\mathbb{P}}(\omega(0) = c, \omega \in \mathrm{DS}) = 1 \tag{15}$$

and

$$\overline{\mathbb{P}}(\omega(0) = c, A_\infty = \infty) = 1 \tag{16}$$

(by Lévy's result about quadratic variation of Brownian motion, [29], Section 4.1). Both (15) and (16) have been used above.

**Lemma 7.** *Let* $\overline{\mathbb{P}}(B) = 1$*. For every set* $E \subseteq \Omega$, $\underline{\mathbb{P}}(E; B) \le \overline{\mathbb{P}}(E; B)$.

*Proof.* Suppose $\underline{\mathbb{P}}(E; B) > \overline{\mathbb{P}}(E; B)$ for some $E$; by the definition of $\underline{\mathbb{P}}$, this would mean that $\overline{\mathbb{P}}(E; B) + \overline{\mathbb{P}}(E^c; B) < 1$. Since $\overline{\mathbb{P}}(\cdot; B)$ is finitely subadditive (this is formally stated in Lemma 8 below), this would imply $\overline{\mathbb{P}}(\Omega; B) < 1$, which is equivalent to $\overline{\mathbb{P}}(B) < 1$ and, therefore, contradicts our assumption. $\square$

The *outer content* of a positive functional $F : \Omega \to [0, \infty]$ *restricted to* a set $B \subseteq \Omega$ with $\overline{\mathbb{P}}(B) = 1$ is defined by

$$\overline{\mathbb{E}}(F; B) := \inf\bigl\{\mathfrak{S}_0 \bigm| \forall \omega \in B : \liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge F(\omega)\bigr\},$$

where $\mathfrak{S}$ ranges over the positive capital processes. Restricted outer content for functionals generalizes restricted outer content for sets: $\overline{\mathbb{P}}(E; B) = \overline{\mathbb{E}}(\mathbf{1}_E; B)$ for all $E \subseteq \Omega$.

It is clear that restricted outer content for functionals and, therefore, restricted outer content for sets are countably (in particular, finitely) subadditive:

**Lemma 8.** *For any $B \subseteq \Omega$ and any sequence of positive functionals $F_1, F_2, \dots$ on $\Omega$,*

$$\overline{\mathbb{E}}\left(\sum_{n=1}^{\infty} F_n; B\right) \le \sum_{n=1}^{\infty} \overline{\mathbb{E}}(F_n; B).$$

*In particular, for any sequence of subsets $E_1, E_2, \dots$ of $\Omega$,*

$$\overline{\mathbb{P}}\left(\bigcup_{n=1}^{\infty} E_n; B\right) \le \sum_{n=1}^{\infty} \overline{\mathbb{P}}(E_n; B).$$

*In particular, a countable union of $B$-null sets is $B$-null.*

# 8 Quadratic variation

In this paper, the set $\Omega$ is always equipped with the metric

$$\rho(\omega_1, \omega_2) := \sum_{d=1}^{\infty} 2^{-d} \sup_{t \in [0, 2^d]} (|\omega_1(t) - \omega_2(t)| \wedge 1) \tag{17}$$

(and the corresponding topology and Borel $\sigma$-algebra, the latter coinciding with $\mathcal{F}$). This makes it a complete and separable metric space. The main goal of this section is to prove that the sequence of continuous functions $t \in [0, \infty) \mapsto A_t^n(\omega)$ is convergent in $\Omega$ for typical $\omega$; this is done in Lemma 10. This will establish the existence of $A(\omega) \in \Omega$ for typical $\omega$, which is part of Theorem 2(a). It is obvious that, when it exists, $A(\omega)$ is increasing and $A_0(\omega) = 0$. The last part of Theorem 2(a), asserting that the intervals of constancy of $\omega$ and $A(\omega)$ coincide for typical $\omega$, will be proved in the next section (Lemma 15).

**Lemma 9.** *For each $T > 0$, for typical $\omega$, $t \in [0, T] \mapsto A_t^n$ is a Cauchy sequence of functions in $C[0, T]$.*

*Proof.* Fix a $T > 0$ and fix temporarily an $n \in \{1, 2, \dots\}$. Let $\kappa \in \{0, 1\}$ be such that $T_0^{n-1} = T_\kappa^n$ and, for each $k = 1, 2, \dots$, let

$$\xi_k := \begin{cases} 1 & \text{if } \omega(T^n_{\kappa+2k}) = \omega(T^n_{\kappa+2k-2}) \\ -1 & \text{otherwise} \end{cases}$$

(this is only defined when $T^n_{\kappa+2k} < \infty$). If $\omega$ were generated by Brownian motion, $\xi_k$ would be a random variable taking value $j$, $j \in \{1, -1\}$, with probability $1/2$; in particular, the expected value of $\xi_k$ would be 0. As the standard backward induction procedure shows, this remains true in our current framework in the following game-theoretic sense: there exists a simple trading strategy that, when started with initial capital 0 at time $T^n_{\kappa+2k-2}$, ends with $\xi_k$ at time $T^n_{\kappa+2k}$, provided both times are finite; moreover, the corresponding simple capital process is always between $-1$ and 1. (Namely, at time $T^n_{\kappa+2k-1}$ bet $-2^n$ if

$\omega(T^n_{\kappa+2k-1}) > \omega(T^n_{\kappa+2k-2})$ and bet $2^n$ otherwise.) Notice that the increment of the process $A^n_t - A^{n-1}_t$ over the time interval $[T^n_{\kappa+2k-2}, T^n_{\kappa+2k}]$ is

$$\eta_k := \begin{cases} 2(2^{-n})^2 = 2^{-2n+1} & \text{if } \xi_k = 1 \\ 2(2^{-n})^2 - (2^{-n+1})^2 = -2^{-2n+1} & \text{if } \xi_k = -1, \end{cases}$$

i.e., $\eta_k = 2^{-2n+1}\xi_k$.

Let us say that a positive process $\mathfrak{S}$ is a *positive supercapital process* if there exists a positive capital process $\mathfrak{T}$ such that, for all $0 \le t_1 < t_2 < \infty$, $\mathfrak{S}(t_2) - \mathfrak{S}(t_1) \le \mathfrak{T}(t_2) - \mathfrak{T}(t_1)$. The game-theoretic version of Hoeffding's inequality (see Theorem 3 in Appendix below) shows that for any constant $\lambda \in \mathbb{R}$ there exists a positive supercapital process $\mathfrak{S}$ with $\mathfrak{S}_0 = 1$ such that, for all $K = 0, 1, 2, \ldots$,

$$\mathfrak{S}_{T^n_{\kappa+2K}} = \prod_{k=1}^{K} \exp\left(\lambda\eta_k - 2^{-4n+1}\lambda^2\right).$$

Equation (41) in Appendix shows that $\mathfrak{S}$ can be chosen positive. Fix temporarily $\alpha > 0$. It is easy to see that, since the sum of these positive supercapital processes over $n = 1, 2, \ldots$ with weights $2^{-n}$ will also be a positive supercapital process, none of these processes will ever exceed $2^n 2/\alpha$ except for a set of $\omega$ of outer content at most $\alpha/2$. The inequality

$$\prod_{k=1}^{K} \exp\left(\lambda\eta_k - 2^{-4n+1}\lambda^2\right) \le 2^n \frac{2}{\alpha} \le e^n \frac{2}{\alpha}$$

can be equivalently rewritten as

$$\lambda \sum_{k=1}^{K} \eta_k \le K\lambda^2 2^{-4n+1} + n + \ln\frac{2}{\alpha}. \tag{18}$$

Plugging in the identities

$$K = \frac{A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}}{2^{-2n+1}},$$

$$\sum_{k=1}^{K} \eta_k = \left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \left(A^{n-1}_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_\kappa}\right),$$

and taking $\lambda := 2^n$, we can transform (18) to

$$\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \left(A^{n-1}_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_\kappa}\right) \le 2^{-n}\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) + \frac{n + \ln\frac{2}{\alpha}}{2^n}, \tag{19}$$

which implies

$$A^n_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_{\kappa+2K}} \le 2^{-n} A^n_{T^n_{\kappa+2K}} + 2^{-2n+1} + \frac{n + \ln\frac{2}{\alpha}}{2^n}. \tag{20}$$

This is true for any $K = 0, 1, 2, \ldots$; choosing the largest $K$ such that $T^n_{\kappa+2K} \le t$, we obtain

$$A^n_t - A^{n-1}_t \le 2^{-n} A^n_t + 2^{-2n+2} + \frac{n + \ln \frac{2}{\alpha}}{2^n}, \tag{21}$$

for any $t \in [0, \infty)$ (the simple case $t < T^n_\kappa$ has to be considered separately). Proceeding in the same way but taking $\lambda := -2^n$, we obtain

$$\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \left(A^{n-1}_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_\kappa}\right) \ge -2^{-n} \left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \frac{n + \ln \frac{2}{\alpha}}{2^n}$$

instead of (19) and

$$A^n_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_{\kappa+2K}} \ge -2^{-n} A^n_{T^n_{\kappa+2K}} - 2^{-2n+1} - \frac{n + \ln \frac{2}{\alpha}}{2^n}$$

instead of (20), which gives

$$A^n_t - A^{n-1}_t \ge -2^{-n} A^n_t - 2^{-2n+2} - \frac{n + \ln \frac{2}{\alpha}}{2^n} \tag{22}$$

instead of (21). We know that that (21) and (22) hold for all $t \in [0, \infty)$ and all $n = 1, 2, \ldots$ except for a set of $\omega$ of outer content at most $\alpha$.

Now we have all ingredients to complete the proof. Suppose there exists $\alpha > 0$ such that (21) and (22) hold for all $n = 1, 2, \ldots$ (this is true for typical $\omega$). First let us show that the sequence $A^n_T$, $n = 1, 2, \ldots$, is bounded. Define a new sequence $B^n$, $n = 0, 1, 2, \ldots$, as follows: $B^0 := A^0_T$ and $B^n$, $n = 1, 2, \ldots$, are defined inductively by

$$B^n := \frac{1}{1 - 2^{-n}} \left(B^{n-1} + 2^{-2n+2} + \frac{n + \ln \frac{2}{\alpha}}{2^n}\right) \tag{23}$$

(notice that this is equivalent to (21) with $B^n$ in place of $A^n_t$ and $=$ in place of $\le$). As $A^n_T \le B^n$ for all $n$, it suffices to prove that $B^n$ is bounded. If it is not, $B^N \ge 1$ for some $N$. By (23), $B^n \ge 1$ for all $n \ge N$. Therefore, again by (23),

$$B^n \le B^{n-1} \frac{1}{1 - 2^{-n}} \left(1 + 2^{-2n+2} + \frac{n + \ln \frac{2}{\alpha}}{2^n}\right), \quad n > N,$$

and the boundedness of the sequence $B^n$ follows from $B^N < \infty$ and

$$\prod_{n=N+1}^{\infty} \frac{1}{1 - 2^{-n}} \left(1 + 2^{-2n+2} + \frac{n + \ln \frac{2}{\alpha}}{2^n}\right) < \infty.$$

Now it is obvious that the sequence $A^n_t$ is Cauchy in $C[0, T]$: (21) and (22) imply

$$\left|A^n_t - A^{n-1}_t\right| \le 2^{-n} A^n_T + 2^{-2n+2} + \frac{n + \ln \frac{2}{\alpha}}{2^n} = O(n/2^n). \qquad \square$$

Lemma 9 implies that, for typical $\omega$, the sequence $t \in [0, \infty) \mapsto A_t^n$ is Cauchy in $\Omega$. Therefore, we have the following implication.

**Lemma 10.** *The event that the sequence of functions $t \in [0, \infty) \mapsto A_t^n$ converges in $\Omega$ is full.*

We can see that the first term in the conjunction in (11) holds for typical $\omega$; let us check that $\tau_s$ itself is a stopping time.

**Lemma 11.** *For each $s \geq 0$, the function $\tau_s$ defined by (11) is a stopping time.*

*Proof.* It suffices to notice that the set $\{\tau_s \leq t\}$ can be written as

$$\Big\{ \underline{A}_t \geq s \;\&\; (\forall q \in (0,t) \cap \mathbb{Q} : \underline{A}_q < s \implies \overline{A}_q = \underline{A}_q) \\ \& \; (\forall q_1, q_2 \in (0,s) \cap \mathbb{Q} \; \exists q \in (0,t) \cap \mathbb{Q} : \overline{A}_q = \underline{A}_q \in (q_1, q_2)) \Big\}. \quad \square$$

# 9 Tightness

In this section we will do some groundwork for the proof of part (b) of Theorem 2 and will also finish the proof of part (a). We start from the results that show (see the next section) that $\underline{Q}_c$ is tight in the topology given by (17).

**Lemma 12.** *For each $\alpha > 0$ and $S \in \{1, 2, 4 \ldots\}$,*

$$\underline{\mathbb{P}}\big(\forall \delta \in (0,1) \; \forall s_1, s_2 \in [0, S] : (0 \leq s_2 - s_1 \leq \delta \;\&\; \tau_{s_2} < \infty) \\ \implies |\omega(\tau_{s_2}) - \omega(\tau_{s_1})| \leq 230\, \alpha^{-1/2} S^{1/4} \delta^{1/8}\big) \geq 1 - \alpha. \quad (24)$$

*Proof.* Let $S = 2^d$, where $d \in \{0, 1, 2, \ldots\}$. For each $m = 1, 2, \ldots$, divide the interval $[0, S]$ into $2^{d+m}$ equal subintervals of length $2^{-m}$. Fix, for a moment, such an $m$, and set $\beta = \beta_m := (2^{1/4} - 1) 2^{-m/4} \alpha$ (where $2^{1/4} - 1$ is the normalizing constant ensuring that the $\beta_m$ sum to $\alpha$) and

$$t_i := \tau_{i 2^{-m}}, \quad \omega_i := \omega(t_i), \qquad i = 0, 1, \ldots, 2^{d+m} \quad (25)$$

(we will be careful to use $\omega_i$ only when $t_i < \infty$).

We will first replace the quadratic variation process $A$ (in terms of which the stopping times $\tau_s$ are defined) by a version of $A^l$ for a large enough $l$. If $\tau$ is any stopping time (we will be interested in $\tau = t_i$ for various $i$), set, in the notation of (10),

$$A_t^{n,\tau}(\omega) := \sum_{k=0}^{\infty} \big(\omega(\tau \vee T_k^n \wedge t) - \omega(\tau \vee T_{k-1}^n \wedge t)\big)^2, \quad t \geq \tau, \quad n = 1, 2, \ldots$$

(we omit parentheses in expressions of the form $x \vee y \wedge z$ since $(x \vee y) \wedge z = x \vee (y \wedge z)$, provided $x \leq z$). The intuition is that $A_t^{n,\tau}(\omega)$ is the version of $A_t^n(\omega)$ that starts at time $\tau$ rather than 0.

For $i = 0, 1, \ldots, 2^{d+m} - 1$, let $\mathfrak{E}_i$ be the event that $t_i < \infty$ implies that (22), with $\alpha$ replaced by $\gamma > 0$ and $A^n_t$ replaced by $A^{n,t_i}_t$, holds for all $n = 1, 2, \ldots$ and $t \in [t_i, \infty)$. Applying a trading strategy similar to that used in the proof of Lemma 9 but starting at time $t_i$ rather than 0, we can see that the inner content of $\mathfrak{E}_i$ is at least $1 - \gamma$. The inequality

$$A^{n,t_i}_t - A^{n-1,t_i}_t \geq -2^{-n} A^{n,t_i}_t - 2^{-2n+2} - \frac{n + \ln \frac{2}{\gamma}}{2^n}$$

holds for all $t \in [t_i, t_{i+1}]$ and all $n$ on the event $\{t_i < \infty\} \cap \mathfrak{E}_i$. For the value $t := t_{i+1}$ this inequality implies

$$A^{n,t_i}_{t_{i+1}} \geq \frac{1}{1 + 2^{-n}} \left( A^{n-1,t_i}_{t_{i+1}} - 2^{-2n+2} - \frac{n + \ln \frac{2}{\gamma}}{2^n} \right)$$

(including the case $t_{i+1} = \infty$). Applying the last inequality to $n = l+1, l+2, \ldots$ (where $l$ will be chosen later), we obtain that

$$A^{\infty,t_i}_{t_{i+1}} \geq \left( \prod_{n=l+1}^{\infty} \frac{1}{1 + 2^{-n}} \right) A^{l,t_i}_{t_{i+1}} - \sum_{n=l+1}^{\infty} \left( 2^{-2n+2} + \frac{n + \ln \frac{2}{\gamma}}{2^n} \right) \qquad (26)$$

holds on the whole of $\{t_i < \infty\} \cap \mathfrak{E}_i$ except perhaps a null set. The qualification "except a null set" allows us not only to assume that $A^{\infty,t_i}_{t_{i+1}}$ exists in (26) but also to assume that $A^{\infty,t_i}_{t_{i+1}} = A_{t_{i+1}} - A_{t_i} = 2^{-m}$. Let $\gamma := \frac{1}{3} 2^{-d-m} \beta$ and choose $l = l(m)$ so large that (26) implies $A^{l,t_i}_{t_{i+1}} \leq 2^{-m+1/2}$ (this can be done as both the product and the sum in (26) are convergent, and so the product can be made arbitrarily close to 1 and the sum can be made arbitrarily close to 0). Doing this for all $i = 0, 1, \ldots, 2^{d+m} - 1$ will ensure that the inner content of

$$t_i < \infty \Longrightarrow A^{l,t_i}_{t_{i+1}} \leq 2^{-m+1/2}, \quad i = 0, 1, \ldots, 2^{d+m} - 1, \qquad (27)$$

is at least $1 - \beta/3$.

An important observation for what follows is that the process defined as $(\omega(t) - \omega(t_i))^2 - A^{l,t_i}_t$ for $t \geq t_i$ and as 0 for $t < t_i$ is a simple capital process (corresponding to betting $2(\omega(T^l_k) - \omega(t_i))$ at each time $T^l_k > t_i$). Now we can see that

$$\sum_{i=1,\ldots,2^{d+m} : t_i < \infty} (\omega_i - \omega_{i-1})^2 \leq 2^{1/2} \frac{3}{\beta} S \qquad (28)$$

will hold on the event (27), except for a set of $\omega$ of outer content at most $\beta/3$: indeed, there is a positive simple capital process taking value at least $2^{1/2} S + \sum_{i=1}^{j} (\omega_i - \omega_{i-1})^2 - j 2^{-m+1/2}$ on the conjunction of events (27) and $t_j < \infty$ at time $t_j$, $j = 0, 1, \ldots, 2^{d+m}$, and this simple capital process will make at least $2^{1/2} \frac{3}{\beta} S$ at time $\tau_S$ (in the sense of $\liminf$ if $\tau_S = \infty$) out of initial capital $2^{1/2} S$ if (27) happens but (28) fails to happen.

For each $\omega \in \Omega$, define

$$J(\omega) := \left\{ i = 1, \dots, 2^{d+m} : t_i < \infty \;\&\; |\omega_i - \omega_{i-1}| \ge \epsilon \right\},$$

where $\epsilon = \epsilon_m$ will be chosen later. It is clear that $|J(\omega)| \le 2^{1/2} 3S/\beta\epsilon^2$ on the set (28). Consider the simple trading strategy whose capital increases by $(\omega(t_i) - \omega(\tau))^2 - A^{l,\tau}_{t_i}$ between each time $\tau \in [t_{i-1}, t_i] \cap [0, \infty)$ when $|\omega(\tau) - \omega_{i-1}| = \epsilon$ for the first time during $[t_{i-1}, t_i] \cap [0, \infty)$ (this is guaranteed to happen when $i \in J(\omega)$) and the corresponding time $t_i$, $i = 1, \dots, 2^{d+m}$, and which is not active (i.e., sets the bet to 0) otherwise. (Such a strategy exists, as explained in the previous paragraph.) This strategy will make at least $\epsilon^2$ out of $(2^{1/2} 3S/\beta\epsilon^2) 2^{-m+1/2}$ provided all three of the events (27), (28), and

$$\exists i \in \{1, \dots, 2^{d+m}\} : t_i < \infty \;\&\; |\omega_i - \omega_{i-1}| \ge 2\epsilon$$

happen. (And we can make the corresponding simple capital process positive by being active for at most $2^{1/2} 3S/\beta\epsilon^2$ values of $i$ and setting the bet to 0 as soon as (27) becomes violated.) This corresponds to making at least 1 out of $(2^{1/2} 3S/\beta\epsilon^4) 2^{-m+1/2}$. Solving the equation $(2^{1/2} 3S/\beta\epsilon^4) 2^{-m+1/2} = \beta/3$ gives $\epsilon = (2^{1/2} 3^2 S 2^{-m+1/2}/\beta^2)^{1/4}$. Therefore,

$$\max_{i=1,\dots,2^{d+m}: t_i < \infty} |\omega_i - \omega_{i-1}| \le 2\epsilon = 2(2 \times 3^2 S 2^{-m}/\beta^2)^{1/4}$$
$$= 2^{5/4} 3^{1/2} \left(2^{1/4} - 1\right)^{-1/2} \alpha^{-1/2} S^{1/4} 2^{-m/8} \qquad (29)$$

except for a set of $\omega$ of outer content $\beta$. By the countable subadditivity of outer content (Lemma 8), (29) holds for all $m = 1, 2, \dots$ except for a set of $\omega$ of outer content at most $\sum_m \beta_m = \alpha$.

We will now allow $m$ to vary and so will write $t^m_i$ instead of $t_i$ defined by (25). Fix an $\omega \in \Omega$ satisfying $A(\omega) \in \Omega$ and (29) for $m = 1, 2, \dots$. Intervals of the form $[t^m_{i-1}(\omega), t^m_i(\omega)] \subseteq [0, \infty)$, for $m \in \{1, 2, \dots\}$ and $i \in \{1, 2, 3, \dots, 2^{d+m}\}$, will be called *predyadic* (*of order* $m$). Given an interval $[s_1, s_2] \subseteq [0, S]$ of length at most $\delta \in (0, 1)$ and with $\tau_{s_2} < \infty$, we can cover $(\tau_{s_1}(\omega), \tau_{s_2}(\omega))$ (without covering any points in the complement of $[\tau_{s_1}(\omega), \tau_{s_2}(\omega)]$) by adjacent predyadic intervals with disjoint interiors such that, for some $m \in \{1, 2, \dots\}$: there are between one and two predyadic intervals of order $m$; for $i = m+1, m+2, \dots$, there are at most two predyadic intervals of order $i$ (start from finding the point in $[s_1, s_2]$ of the form $j2^{-k}$ with integer $j$ and $k$ and the smallest possible $k$, and cover $(\tau_{s_1}(\omega), \tau_{j2^{-k}}]$ and $[\tau_{j2^{-k}}, \tau_{s_2}(\omega))$ by predyadic intervals in the greedy manner). Combining (29) and $2^{-m} \le \delta$, we obtain

$$|\omega(\tau_{s_2}) - \omega(\tau_{s_1})| \le 2^{9/4} 3^{1/2} \left(2^{1/4} - 1\right)^{-1/2} \alpha^{-1/2} S^{1/4}$$
$$\times \left(2^{-m/8} + 2^{-(m+1)/8} + 2^{-(m+2)/8} + \cdots\right)$$
$$= 2^{9/4} 3^{1/2} \left(2^{1/4} - 1\right)^{-1/2} \left(1 - 2^{-1/8}\right)^{-1} \alpha^{-1/2} S^{1/4} 2^{-m/8}$$

$$\leq 2^{9/4}3^{1/2}\left(2^{1/4}-1\right)^{-1/2}\left(1-2^{-1/8}\right)^{-1}\alpha^{-1/2}S^{1/4}\delta^{1/8},$$

which is stronger than (24). □

Now we can prove the following elaboration of Lemma 12, which will be used in the next two sections.

**Lemma 13.** *For each $\alpha > 0$,*

$$\underline{\mathbb{P}}\big(\forall S \in \{1,2,4,\ldots\}\ \forall \delta \in (0,1)\ \forall s_1, s_2 \in [0,S] : \\ (0 \leq s_2 - s_1 \leq \delta\ \&\ \tau_{s_2} < \infty) \\ \Longrightarrow |\omega(\tau_{s_2}) - \omega(\tau_{s_1})| \leq 430\,\alpha^{-1/2}S^{1/2}\delta^{1/8}\big) \geq 1-\alpha. \quad (30)$$

*Proof.* Replacing $\alpha$ in (24) by $\alpha_S := (1-2^{-1/2})S^{-1/2}\alpha$ for $S = 1,2,4,\ldots$ (where $1-2^{-1/2}$ is the normalizing constant ensuring that the $\alpha_S$ sum to $\alpha$ over $S$), we obtain

$$\underline{\mathbb{P}}\big(\forall \delta \in (0,1)\ \forall s_1, s_2 \in [0,S] : (0 \leq s_2 - s_1 \leq \delta\ \&\ \tau_{s_2} < \infty) \\ \Longrightarrow |\omega(\tau_{s_2}) - \omega(\tau_{s_1})| \leq 230\,(1-2^{-1/2})^{-1/2}\alpha^{-1/2}S^{1/2}\delta^{1/8}\big) \\ \geq 1-(1-2^{-1/2})S^{-1/2}\alpha.$$

The countable subadditivity of outer content now gives

$$\underline{\mathbb{P}}\big(\forall S \in \{1,2,4,\ldots\}\ \forall \delta \in (0,1)\ \forall s_1, s_2 \in [0,S] : \\ (0 \leq s_2 - s_1 \leq \delta\ \&\ \tau_{s_2} < \infty) \Longrightarrow \\ |\omega(\tau_{s_2}) - \omega(\tau_{s_1})| \leq 230\,(1-2^{-1/2})^{-1/2}\alpha^{-1/2}S^{1/2}\delta^{1/8}\big) \geq 1-\alpha,$$

which is stronger than (30). □

The following lemma develops inequality (28) and will be useful in the proof of Theorem 2.

**Lemma 14.** *For each $\alpha > 0$,*

$$\underline{\mathbb{P}}\left(\forall S \in \{1,2,4,\ldots\}\ \forall m \in \{1,2,\ldots\} : \\ \sum_{i=1,\ldots,S2^m : t_i < \infty}\left(\omega(t_i)-\omega(t_{i-1})\right)^2 \leq 64\,\alpha^{-1}S^2 2^{m/16}\right) \geq 1-\alpha, \quad (31)$$

*in the notation of (25).*

*Proof.* Replacing $\beta/3$ in (28) with $2^{-1}(2^{1/16}-1)S^{-1}2^{-m/16}\alpha$, where $S$ ranges over $\{1,2,4,\ldots\}$ and $m$ over $\{1,2,\ldots\}$, we obtain

$$\underline{\mathbb{P}}\Biggl(\sum_{i=1,\dots,S2^m:t_i<\infty}\Bigl(\omega(t_i)-\omega(t_{i-1})\Bigr)^2 \le 2^{3/2}(2^{1/16}-1)^{-1}\alpha^{-1}S^2 2^{m/16}\Biggr) \ge 1-2^{-1}(2^{1/16}-1)S^{-1}2^{-m/16}\alpha.$$

By the countable subadditivity of outer content this implies

$$\underline{\mathbb{P}}\Biggl(\forall S\in\{1,2,4,\dots\}\ \forall m\in\{1,2,\dots\}: \sum_{i=1,\dots,S2^m:t_i<\infty}\Bigl(\omega(t_i)-\omega(t_{i-1})\Bigr)^2 \le 2^{3/2}(2^{1/16}-1)^{-1}\alpha^{-1}S^2 2^{m/16}\Biggr) \ge 1-\alpha,$$

which is stronger than (31). □

The following lemma completes the proof of Theorem 2(a).

**Lemma 15.** *For typical $\omega$, $A(\omega)$ has the same intervals of constancy as $\omega$.*

*Proof.* The definition of $A$ immediately implies that $A(\omega)$ is always constant on every interval of constancy of $\omega$ (provided $A(\omega)$ exists). Therefore, we are only required to prove that typical $\omega$ are constant on every interval of constancy of $A(\omega)$.

The proof can be extracted from the proof of Lemma 12. It suffices to prove that, for any $\alpha>0$, $S\in\{1,2,4,\dots\}$, $c>0$, and interval $[a,b]$ with rational end-points $a$ and $b$ such that $a<b$, the outer content of the following event is at most $\alpha$: $\omega$ changes by at least $c$ over $[a,b]$, $A$ is constant over $[a,b]$, and $[a,b]\subseteq[0,\tau_S]$. Fix such $\alpha$, $S$, $c$, and $[a,b]$, and let $E$ stand for the event described in the previous sentence. Choose $m\in\{1,2,\dots\}$ such that $2^{-m+1/2}/c^2\le\alpha/2$ and choose the corresponding $l=l(m)$, as in the proof of Lemma 12. The positive simple capital process $2^{-m+1/2}+(\omega(t)-\omega(a))^2-A^{l,a}_t$, started at time $a$ and stopped when $t$ reaches $b\wedge\tau_S$, when $A^{l,a}_t$ reaches $2^{-m+1/2}$, or when $|\omega(t)-\omega(a)|$ reaches $c$, whatever happens first, makes $c^2$ out of $2^{-m+1/2}$ on the conjunction of (27) and the event $E$. Therefore, the outer content of the conjunction is at most $\alpha/2$, and the outer content of $E$ is at most $\alpha$. □

In view of Lemma 15 we can strengthen (30) to

$$\underline{\mathbb{P}}\bigl(\forall S\in\{1,2,4,\dots\}\forall\delta\in(0,1)\ \forall t_1,t_2\in[0,\infty): \bigl(|A_{t_2}-A_{t_1}|\le\delta\ \&\ A_{t_1}\in[0,S]\ \&\ A_{t_2}\in[0,S]\bigr)\Longrightarrow |\omega(t_2)-\omega(t_1)|\le 430\,\alpha^{-1/2}S^{1/2}\delta^{1/8}\bigr)\ge 1-\alpha.$$

## 10 Proof of Theorem 2(b)

Let $c \in \mathbb{R}$ be a fixed constant. Results of the previous section imply the tightness of $\underline{Q}_c$:

**Lemma 16.** *For each $\alpha > 0$ there exists a compact set $\mathfrak{K} \subseteq \Omega$ such that $\underline{Q}_c(\mathfrak{K}) \geq 1 - \alpha$.*

In particular, Lemma 16 asserts that $\underline{Q}_c(\Omega) = 1$.

More precise results can be stated in terms of the *modulus of continuity* of a function $\psi \in \mathbb{R}^{[0,\infty)}$ on an interval $[0, S] \subseteq [0, \infty)$:

$$\mathrm{m}_\delta^S(\psi) := \sup_{s_1,s_2 \in [0,S]:|s_1-s_2|\leq\delta} |\psi(s_1) - \psi(s_2)|, \quad \delta > 0;$$

it is clear that $\lim_{\delta\to 0} \mathrm{m}_\delta^S(\psi) = 0$ if and only if $\psi$ is continuous on $[0, S]$.

**Lemma 17.** *For each $\alpha > 0$,*

$$\underline{Q}_c\left(\forall S \in \{1, 2, 4, \dots\}\ \forall\delta \in (0, 1) : \mathrm{m}_\delta^S \leq 430\, \alpha^{-1/2} S^{1/2} \delta^{1/8}\right) \geq 1 - \alpha.$$

Lemma 17 immediately follows from Lemma 13, and Lemma 16 immediately follows from Lemma 17 and the Arzelà–Ascoli theorem (as stated in [24], Theorem 2.4.9).

We start the proof proper from a series of reductions:

(a) It suffices to prove that, for any $E \in \mathcal{F}$, $\overline{Q}_c(E) \leq \mathcal{W}_c(E)$. Indeed, this will imply

$$\begin{aligned}
\underline{Q}_c(E) &= \underline{\mathbb{P}}(\mathrm{ntc}^{-1}(E); \omega(0) = c, A_\infty = \infty) \\
&= 1 - \overline{\mathbb{P}}\left(\mathrm{ntc}^{-1}(E^c) \cup \left(\mathrm{ntc}^{-1}(\Omega)\right)^c; \omega(0) = c, A_\infty = \infty\right) \\
&= 1 - \overline{\mathbb{P}}(\mathrm{ntc}^{-1}(E^c); \omega(0) = c, A_\infty = \infty) \qquad (32) \\
&\geq 1 - \mathcal{W}_c(E^c) = \mathcal{W}_c(E)
\end{aligned}$$

and so, by Lemma 7 and (16),

$$\overline{Q}_c(E) = \underline{Q}_c(E) = \mathcal{W}_c(E)$$

for all $E \in \mathcal{F}$. The equality in line (32) follows from $\underline{\mathbb{P}}(\mathrm{ntc}^{-1}(\Omega); \omega(0) = c, A_\infty = \infty) = 1$, which in turn follows from (and is in fact equivalent to) $\underline{Q}_c(\Omega) = 1$.

(b) Furthermore, it suffices to prove that, for any bounded positive $\mathcal{F}$-measurable functional $F : \Omega \to [0, \infty)$,

$$\overline{\mathbb{E}}(F \circ \mathrm{ntc}; \omega(0) = c, A_\infty = \infty) \leq \int_\Omega F(\psi)\mathcal{W}_c(d\psi) \qquad (33)$$

(with $\circ$ standing for composition of two functions and the important convention that $(F \circ \mathrm{ntc})(\omega) := 0$ when $\omega \notin \mathrm{ntc}^{-1}(\Omega)$). Indeed, this will imply

$$\begin{aligned}\overline{Q}_c(E) &= \overline{\mathbb{P}}(\mathrm{ntc}^{-1}(E); \omega(0) = c, A_\infty = \infty) \\ &= \overline{\mathbb{E}}(\mathbf{1}_E \circ \mathrm{ntc}; \omega(0) = c, A_\infty = \infty) \le \int_\Omega \mathbf{1}_E(\psi) \mathcal{W}_c(d\psi) = \mathcal{W}_c(E)\end{aligned}$$

for all $E \in \mathcal{F}$. To establish (33) we only need to establish $\overline{\mathbb{E}}(F \circ \mathrm{ntc}; \omega(0) = c, A_\infty = \infty) < \int F d\mathcal{W}_c + \epsilon$ for each positive constant $\epsilon$.

(c) We can assume that $F$ in (33) is lower semicontinuous on $\Omega$. Indeed, if it is not, by the Vitali–Carathéodory theorem (see, e.g., [34], Theorem 2.24) for any compact $\mathfrak{K} \subseteq \Omega$ (assumed non-empty) there exists a lower semicontinuous function $G$ on $\mathfrak{K}$ such that $G \ge F$ on $\mathfrak{K}$ and $\int_{\mathfrak{K}} G d\mathcal{W}_c \le \int_{\mathfrak{K}} F d\mathcal{W}_c + \epsilon$. Without loss of generality we assume $\sup G \le \sup F$, and we extend $G$ to all of $\Omega$ by setting $G := \sup F$ outside $\mathfrak{K}$. Choosing $\mathfrak{K}$ with large enough $\mathcal{W}_c(\mathfrak{K})$ (which can be done since the probability measure $\mathcal{W}_c$ is tight: see, e.g., [3], Theorem 1.4), we will have $G \ge F$ and $\int G d\mathcal{W}_c \le \int F d\mathcal{W}_c + 2\epsilon$. Achieving $\mathfrak{S}_0 \le \int G d\mathcal{W}_c + \epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (G \circ \mathrm{ntc})(\omega)$, where $\mathfrak{S}$ is a positive capital process, will automatically achieve $\mathfrak{S}_0 \le \int F d\mathcal{W}_c + 3\epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (F \circ \mathrm{ntc})(\omega)$.

(d) We can further assume that $F$ is continuous on $\Omega$. Indeed, since each lower semicontinuous function on a metric space is a limit of an increasing sequence of continuous functions (see, e.g., [18], Problem 1.7.15(c)), given a lower semicontinuous positive function $F$ on $\Omega$ we can find a series of positive continuous functions $G^n$ on $\Omega$, $n = 1, 2, \ldots$, such that $\sum_{n=1}^\infty G^n = F$. The sum $\mathfrak{S}$ of positive capital processes $\mathfrak{S}^1, \mathfrak{S}^2, \ldots$ achieving $\mathfrak{S}_0^n \le \int G^n d\mathcal{W}_c + 2^{-n}\epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t^n(\omega) \ge (G^n \circ \mathrm{ntc})(\omega)$, $n = 1, 2, \ldots$, will achieve $\mathfrak{S}_0 \le \int F d\mathcal{W}_c + \epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (F \circ \mathrm{ntc})(\omega)$.

(e) We can further assume that $F$ depends on $\psi \in \Omega$ only via $\psi|_{[0,S]}$ for some $S \in (0, \infty)$. Indeed, let us fix $\epsilon > 0$ and prove $\overline{\mathbb{E}}(F \circ \mathrm{ntc}; \omega(0) = c, A_\infty = \infty) \le \int F d\mathcal{W}_c + C\epsilon$ for some positive constant $C$ assuming $\overline{\mathbb{E}}(G \circ \mathrm{ntc}; \omega(0) = c, A_\infty = \infty) \le \int G d\mathcal{W}_c$ for all continuous positive $G$ that depend on $\psi \in \Omega$ only via $\psi|_{[0,S]}$ for some $S \in (0, \infty)$. Choose a compact set $\mathfrak{K} \subseteq \Omega$ with $\mathcal{W}_c(\mathfrak{K}) > 1 - \epsilon$ and $\underline{Q}_c(\mathfrak{K}) > 1 - \epsilon$ (cf. Lemma 16). Set $F^S(\psi) := F(\psi^S)$, where $\psi^S$ is defined by $\psi^S(s) := \psi(s \wedge S)$ and $S$ is sufficiently large in the following sense. Since $F$ is uniformly continuous on $\mathfrak{K}$ and the metric is defined by (17), $F$ and $F^S$ can be made arbitrarily close in $C(\mathfrak{K})$; in particular, let $\|F - F^S\|_{C(\mathfrak{K})} < \epsilon$. Choose positive capital processes $\mathfrak{S}^0$ and $\mathfrak{S}^1$ such that

$$\begin{aligned}\mathfrak{S}_0^0 &\le \int F^S d\mathcal{W}_c + \epsilon, & \liminf_{t\to\infty} \mathfrak{S}_t^0(\omega) &\ge (F^S \circ \mathrm{ntc})(\omega), \\ \mathfrak{S}_0^1 &\le \epsilon, & \liminf_{t\to\infty} \mathfrak{S}_t^1(\omega) &\ge (\mathbf{1}_{\mathfrak{K}^c} \circ \mathrm{ntc})(\omega),\end{aligned}$$

for all $\omega \in \Omega$ satisfying $\omega(0) = c$ and $A_\infty(\omega) = \infty$. The sum $\mathfrak{S} := \mathfrak{S}^0 + (\sup F)\mathfrak{S}^1 + \epsilon$ will satisfy

$$\begin{aligned}\mathfrak{S}_0 &\le \int F^S d\mathcal{W}_c + (\sup F + 2)\epsilon \le \int_{\mathfrak{K}} F^S d\mathcal{W}_c + (2\sup F + 2)\epsilon \\ &\le \int_{\mathfrak{K}} F d\mathcal{W}_c + (2\sup F + 3)\epsilon \le \int F d\mathcal{W}_c + (2\sup F + 3)\epsilon\end{aligned}$$

and

$$\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (F^S \circ \mathrm{ntc})(\omega) + (\sup F)(\mathbf{1}_{\mathfrak{K}^c} \circ \mathrm{ntc})(\omega) + \epsilon \ge (F \circ \mathrm{ntc})(\omega),$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$. We assume $S \in \{1, 2, 4, \ldots\}$, without loss of generality.

(f) We can further assume that $F(\psi)$ depends on $\psi \in \Omega$ only via the values $\psi(iS/N)$, $i = 1, \ldots, N$ (remember that we are interested in the case $\psi(0) = c$), for some $N \in \{1, 2, \ldots\}$. Indeed, let us fix $\epsilon > 0$ and prove $\overline{\mathbb{E}}(F \circ \mathrm{ntc}; \omega(0) = c, A_\infty = \infty) \le \int F d\mathcal{W}_c + C\epsilon$ for some positive constant $C$ assuming $\overline{\mathbb{E}}(G \circ \mathrm{ntc}; \omega(0) = c, A_\infty = \infty) \le \int G d\mathcal{W}_c$ for all continuous positive $G$ that depend on $\psi \in \Omega$ only via $\psi(iS/N)$, $i = 1, \ldots, N$, for some $N$. Let $\mathfrak{K} \subseteq \Omega$ be the compact set in $\Omega$ defined as $\mathfrak{K} := \{\psi \in \Omega \mid \psi(0) = c \;\&\; \forall \delta > 0 : m_\delta^S(\psi) \le f(\delta)\}$ for some $f : (0, \infty) \to (0, \infty)$ satisfying $\lim_{\delta\to 0} f(\delta) = 0$ (cf. the Arzelà–Ascoli theorem) and chosen in such a way that $\mathcal{W}_c(\mathfrak{K}) > 1 - \epsilon$ and $\underline{Q}_c(\mathfrak{K}) > 1 - \epsilon$. Let $g$ be the modulus of continuity of $F$ on $\mathfrak{K}$, $g(\delta) := \sup_{\psi_1,\psi_2\in\mathfrak{K}:\rho(\psi_1,\psi_2)\le\delta} |F(\psi_1) - F(\psi_2)|$; we know that $\lim_{\delta\to 0} g(\delta) = 0$. Set $F_N(\psi) := F(\psi_N)$, where $\psi_N$ is the piecewise linear function whose graph is obtained by joining the points $(iS/N, \psi(iS/N))$, $i = 0, 1, \ldots, N$, and $(\infty, \psi(S))$, and $N$ is so large that $g(f(S/N)) \le \epsilon$. Since

$$\psi \in \mathfrak{K} \implies \|\psi - \psi_N\|_{C[0,S]} \le f(S/N) \implies \rho(\psi, \psi_N) \le f(S/N)$$

(we assume, without loss of generality, that the graph of $\psi$ is horizontal over $[S, \infty)$), we have $\|F - F_N\|_{C(\mathfrak{K})} \le \epsilon$. Choose positive capital processes $\mathfrak{S}^0$ and $\mathfrak{S}^1$ such that

$$\mathfrak{S}_0^0 \le \int F_N d\mathcal{W}_c + \epsilon, \qquad \liminf_{t\to\infty} \mathfrak{S}_t^0(\omega) \ge (F_N \circ \mathrm{ntc})(\omega),$$
$$\mathfrak{S}_0^1 \le \epsilon, \qquad \liminf_{t\to\infty} \mathfrak{S}_t^1(\omega) \ge (\mathbf{1}_{\mathfrak{K}^c} \circ \mathrm{ntc})(\omega),$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$. The sum $\mathfrak{S} := \mathfrak{S}^0 + (\sup F)\mathfrak{S}^1 + \epsilon$ will satisfy

$$\begin{aligned}\mathfrak{S}_0 &\le \int F_N d\mathcal{W}_c + (\sup F + 2)\epsilon \le \int_{\mathfrak{K}} F_N d\mathcal{W}_c + (2\sup F + 2)\epsilon \\ &\le \int_{\mathfrak{K}} F d\mathcal{W}_c + (2\sup F + 3)\epsilon \le \int F d\mathcal{W}_c + (2\sup|F| + 3)\epsilon\end{aligned}$$

and

$$\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (F_N \circ \mathrm{ntc})(\omega) + (\sup F)(\mathbf{1}_{\mathfrak{K}^c} \circ \mathrm{ntc})(\omega) + \epsilon \ge (F \circ \mathrm{ntc})(\omega),$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$.

(g) We can further assume that

$$F(\psi) = U\left(\psi(S/N), \psi(2S/N), \dots, \psi(S)\right) \tag{34}$$

where the function $U : \mathbb{R}^N \to [0, \infty)$ is not only continuous but also has compact support. (We will sometimes say that $U$ is the *generator* of $F$.) Indeed, let us fix $\epsilon > 0$ and prove $\overline{\mathbb{E}}(F \circ \mathrm{ntc}; \omega(0) = c, A_\infty = \infty) \le \int F d\mathcal{W}_c + C\epsilon$ for some positive constant $C$ assuming $\overline{\mathbb{E}}(G \circ \mathrm{ntc}; \omega(0) = c, A_\infty = \infty) \le \int G d\mathcal{W}_c$ for all $G$ whose generator has compact support. Let $B_R$ be the open ball of radius $R$ and centred at the origin in the space $\mathbb{R}^N$ with the $\ell_\infty$ norm. We can rewrite (34) as $F(\psi) = U(\sigma(\psi))$ where $\sigma : \Omega \to \mathbb{R}^N$ reduces each $\psi \in \Omega$ to $\sigma(\psi) := (\psi(S/N), \psi(2S/N), \dots, \psi(S))$. Choose $R > 0$ so large that $\mathcal{W}_c(\sigma^{-1}(B_R)) > 1 - \epsilon$ and $\underline{Q}_c(\sigma^{-1}(B_R)) > 1 - \epsilon$ (the existence of such $R$ follows from the Arzelà–Ascoli theorem and Lemma 16). Alongside $F$, whose generator is denoted $U$, we will also consider $F^*$ with generator

$$U^*(z) := \begin{cases} U(z) & \text{if } z \in \overline{B_R} \\ 0 & \text{if } z \in B_{2R}^c \end{cases}$$

(where $\overline{B_R}$ is the closure of $B_R$ in $\mathbb{R}^N$); in the remaining region $B_{2R} \setminus \overline{B_R}$, $U^*$ is defined arbitrarily (but making sure that $U^*$ is continuous and takes values in $[\inf U, \sup U]$; this can be done by the Tietze–Urysohn theorem, [18], Theorem 2.1.8). Choose positive capital processes $\mathfrak{S}^0$ and $\mathfrak{S}^1$ such that

$$\begin{aligned} \mathfrak{S}^0_0 &\le \int F^* d\mathcal{W}_c + \epsilon, & \liminf_{t\to\infty} \mathfrak{S}^0_t(\omega) &\ge (F^* \circ \mathrm{ntc})(\omega), \\ \mathfrak{S}^1_0 &\le \epsilon, & \liminf_{t\to\infty} \mathfrak{S}^1_t(\omega) &\ge (\mathbf{1}_{(\sigma^{-1}(B_R))^c} \circ \mathrm{ntc})(\omega), \end{aligned}$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$. The sum $\mathfrak{S} := \mathfrak{S}^0 + (\sup F)\mathfrak{S}^1$ will satisfy

$$\begin{aligned} \mathfrak{S}_0 &\le \int F^* d\mathcal{W}_c + (\sup|F| + 1)\epsilon \le \int_{\sigma^{-1}(B_R)} F^* d\mathcal{W}_c + (2\sup|F| + 1)\epsilon \\ &= \int_{\sigma^{-1}(B_R)} F d\mathcal{W}_c + (2\sup|F| + 1)\epsilon \le \int F d\mathcal{W}_c + (2\sup|F| + 1)\epsilon \end{aligned}$$

and

$$\begin{aligned} \liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (F^* \circ \mathrm{ntc})(\omega) + (\sup F)(\mathbf{1}_{(\sigma^{-1}(B_R))^c} \circ \mathrm{ntc})(\omega) \\ \ge (F \circ \mathrm{ntc})(\omega), \end{aligned}$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$.

(h) Since every continuous $U : \mathbb{R}^N \to [0,\infty)$ with compact support can be arbitrarily well approximated in $C(\mathbb{R}^N)$ by an infinitely differentiable (positive) function with compact support (see, e.g., [1], Theorem 2.29(d)), we can further assume that the generator $U$ of $F$ is an infinitely differentiable function with compact support.

(i) By Lemma 16, it suffices to prove that, given $\epsilon > 0$ and a compact set $\mathfrak{K}$ in $\Omega$, some positive capital process $\mathfrak{S}$ with $\mathfrak{S}_0 \le \int F d\mathcal{W}_c + \epsilon$ achieves $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (F \circ \mathrm{ntc})(\omega)$ for all $\omega \in \mathrm{ntc}^{-1}(\mathfrak{K})$ such that $\omega(0) = c$ and $A_\infty(\omega) = \infty$. Indeed, we can choose $\mathfrak{K}$ with $\underline{Q}_c(\mathfrak{K})$ so close to 1 that the sum of $\mathfrak{S}$ and a positive capital process eventually attaining $\sup F$ on $(\mathrm{ntc}^{-1}(\mathfrak{K}))^c$ will give a positive capital process starting from at most $\int F d\mathcal{W}_c + 2\epsilon$ and attaining $(F \circ \mathrm{ntc})(\omega)$ in the limit, provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$.

From now on we fix a compact $\mathfrak{K} \subseteq \Omega$, assuming, without loss of generality, that the statements inside the outer parentheses in (30) and (31) are satisfied for some $\alpha > 0$ when $\mathrm{ntc}(\omega) \in \mathfrak{K}$.

In the rest of the proof we will be using, often following [37], Section 6.2, the standard method going back to Lindeberg [30]. For $i = N-1$, define a function $\overline{U}_i : \mathbb{R} \times [0,\infty) \times \mathbb{R}^i \to \mathbb{R}$ by

$$\overline{U}_i(x, D; x_1, \ldots, x_i) := \int_{-\infty}^{\infty} U_{i+1}(x_1, \ldots, x_i, x+z)\mathcal{N}_{0,D}(dz), \tag{35}$$

where $U_N$ stands for $U$ and $\mathcal{N}_{0,D}$ is the Gaussian probability measure on $\mathbb{R}$ with mean 0 and variance $D \ge 0$. Next define, for $i = N-1$,

$$U_i(x_1, \ldots, x_i) := \overline{U}_i(x_i, S/N; x_1, \ldots, x_i). \tag{36}$$

Finally, we can alternately use (35) and (36) for $i = N-2, \ldots, 1, 0$ to define inductively other $\overline{U}_i$ and $U_i$ (with (36) interpreted as $U_0 := \overline{U}_0(c, S/N)$ when $i = 0$). Notice that $U_0 = \int F d\mathcal{W}_c$.

Informally, the functions (35) and (36) constitute Sceptic's goal: assuming $\mathrm{ntc}(\omega) \in \mathfrak{K}$, $\omega(0) = c$, and $A_\infty(\omega) = \infty$, he will keep his capital at time $\tau_{iS/N}$, $i = 0, 1, \ldots, N$, close to $U_i(\omega(\tau_{S/N}), \omega(\tau_{2S/N}), \ldots, \omega(\tau_{iS/N}))$ and his capital at any other time $t \in [0, \tau_S]$ close to $\overline{U}_i(\omega(t), D; \omega(\tau_{S/N}), \omega(\tau_{2S/N}), \ldots, \omega(\tau_{iS/N}))$ where $i := \lfloor N A_t / S \rfloor$ and $D := (i+1)S/N - A_t$. This will ensure that his capital at time $\tau_S$ is close to or exceeds $(F \circ \mathrm{ntc})(\omega)$ when his initial capital is $U_0 = \int F d\mathcal{W}_c$, $\omega(0) = c$, and $A_\infty(\omega) = \infty$.

The proof is based on the fact that each function $\overline{U}_i(x, D; x_1, \ldots, x_i)$ satisfies the heat equation in the variables $x$ and $D$:

$$\frac{\partial \overline{U}_i}{\partial D}(x, D; x_1, \ldots, x_i) = \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial x^2}(x, D; x_1, \ldots, x_i) \tag{37}$$

for all $x \in \mathbb{R}$, all $D > 0$, and all $x_1, \ldots, x_i \in \mathbb{R}$. This can be checked by direct differentiation.

Sceptic will only bet at the times of the form $\tau_{kS/LN}$, where $L \in \{1,2,\dots\}$ is a constant that will later be chosen large and $k$ is integer. For $i = 0,\dots,N$ and $j = 0,\dots,L$ let us set

$$t_{i,j} := \tau_{iS/N+jS/LN}, \quad X_{i,j} := \omega(t_{i,j}), \quad D_{i,j} := S/N - jS/LN.$$

For any array $Y_{i,j}$, we set $dY_{i,j} := Y_{i,j+1} - Y_{i,j}$.

Using Taylor's formula and omitting the arguments $\omega(\tau_{S/N}),\dots,\omega(\tau_{iS/N})$, we obtain, for $i = 0,\dots,N-1$ and $j = 0,\dots,L-1$,

$$\begin{aligned} d\overline{U}_i(X_{i,j},D_{i,j}) &= \frac{\partial \overline{U}_i}{\partial x}(X_{i,j},D_{i,j})dX_{i,j} + \frac{\partial \overline{U}_i}{\partial D}(X_{i,j},D_{i,j})dD_{i,j} \\ &\quad + \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial x^2}(X'_{i,j},D'_{i,j})(dX_{i,j})^2 + \frac{\partial^2 \overline{U}_i}{\partial x \partial D}(X'_{i,j},D'_{i,j})dX_{i,j}dD_{i,j} \\ &\quad + \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial D^2}(X'_{i,j},D'_{i,j})(dD_{i,j})^2, \end{aligned} \tag{38}$$

where $(X'_{i,j},D'_{i,j})$ is a point strictly between $(X_{i,j},D_{i,j})$ and $(X_{i,j+1},D_{i,j+1})$. Applying Taylor's formula to $\partial^2\overline{U}_i/\partial x^2$, we find

$$\begin{aligned} \frac{\partial^2 \overline{U}_i}{\partial x^2}(X'_{i,j},D'_{i,j}) &= \frac{\partial^2 \overline{U}_i}{\partial x^2}(X_{i,j},D_{i,j}) \\ &\quad + \frac{\partial^3 \overline{U}_i}{\partial x^3}(X''_{i,j},D''_{i,j})\Delta X_{i,j} + \frac{\partial^3 \overline{U}_i}{\partial D \partial x^2}(X''_{i,j},D''_{i,j})\Delta D_{i,j}, \end{aligned}$$

where $(X''_{i,j},D''_{i,j})$ is a point strictly between $(X_{i,j},D_{i,j})$ and $(X'_{i,j},D'_{i,j})$, and $\Delta X_{i,j}$ and $\Delta D_{i,j}$ satisfy $|\Delta X_{i,j}| \le |dX_{i,j}|$, $|\Delta D_{i,j}| \le |dD_{i,j}|$. Plugging this equation and the heat equation (37) into (38), we obtain

$$\begin{aligned} d\overline{U}_i(X_{i,j},D_{i,j}) &= \frac{\partial \overline{U}_i}{\partial x}(X_{i,j},D_{i,j})dX_{i,j} + \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial x^2}(X_{i,j},D_{i,j})\left((dX_{i,j})^2 + dD_{i,j}\right) \\ &\quad + \frac{1}{2}\frac{\partial^3 \overline{U}_i}{\partial x^3}(X''_{i,j},D''_{i,j})\Delta X_{i,j}(dX_{i,j})^2 + \frac{1}{2}\frac{\partial^3 \overline{U}_i}{\partial D \partial x^2}(X''_{i,j},D''_{i,j})\Delta D_{i,j}(dX_{i,j})^2 \\ &\quad + \frac{\partial^2 \overline{U}}{\partial x \partial D}(X'_{i,j},D'_{i,j})dX_{i,j}dD_{i,j} + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial D^2}(X'_{i,j},D'_{i,j})(dD_{i,j})^2. \end{aligned} \tag{39}$$

To show that Sceptic can achieve his goal, we will describe a simple trading strategy that results in increase of his capital of approximately (39) during the time interval $[t_{i,j}, t_{i,j+1}]$ (we will make sure that the cumulative error of our approximation is small with high probability, which will imply the statement of the theorem). We will see that there is a trading strategy resulting in the capital increase equal to the first addend on the right-hand side of (39), that there is another trading strategy resulting in the capital increase approximately equal to the second addend, and that the last four addends are negligible. The sum of the two trading strategies will achieve our goal.

The trading strategy whose capital increase over $[t_{i,j}, t_{i,j+1}]$ is the first addend is obvious: it bets $\partial\overline{U}_i/\partial x$ at time $t_{i,j}$. The bet is bounded as average of

$\partial U_{i+1}/\partial x_{i+1}$ and so, eventually, average of $\partial U/\partial x$ ($x$ being the last argument of $U$).

The second addend involves the expression $(dX_{i,j})^2 + dD_{i,j} = (\omega_{i,j+1} - \omega_{i,j})^2 - S/LN$. To analyze it, we will need the following lemma.

**Lemma 18.** *For all $\delta > 0$ and $\beta > 0$, there exists a positive integer $l$ such that*

$$t_{i,j+1} < \infty \Longrightarrow \left| \frac{A^{l,t_{i,j}}_{t_{i,j+1}}}{S/LN} - 1 \right| < \delta$$

*holds for all $i = 0, \ldots, N-1$ and $j = 0, \ldots, L-1$ except for a set of $\omega$ of outer content at most $\beta$.*

Lemma 18 can be proved similarly to (27). (The inequality in (27) is one-sided, so it was sufficient to use only (22); for Lemma 18 both (22) and (21) should be used.)

We know that $(\omega(t) - \omega(t_{i,j}))^2 - A^{l,t_{i,j}}_t$ is a simple capital process (see the proof of Lemma 12). Therefore, there is indeed a simple trading strategy resulting in capital increase approximately equal to the second addend on the right-hand side of (39), with the cumulative approximation error that can be made arbitrarily small on a set of $\omega$ of inner content arbitrarily close to 1. (Analogously to the analysis of the first addend, $\partial^2\overline{U}_i/\partial x^2$ is bounded as average of $\partial^2 U_{i+1}/\partial x^2_{i+1}$ and, eventually, average of $\partial^2 U/\partial x^2$.)

Let us show that the last four terms on the right-hand side of (39) are negligible when $L$ is sufficiently large (assuming $S$, $N$, and $U$ fixed). All the partial derivatives involved in those terms are bounded: the heat equation implies

$$\begin{aligned}
\frac{\partial^3\overline{U}_i}{\partial D \partial x^2} &= \frac{\partial^3\overline{U}_i}{\partial x^2 \partial D} = \frac{1}{2}\frac{\partial^4\overline{U}_i}{\partial x^4}, \\
\frac{\partial^2\overline{U}_i}{\partial x \partial D} &= \frac{1}{2}\frac{\partial^3\overline{U}_i}{\partial x^3}, \\
\frac{\partial^2\overline{U}_i}{\partial D^2} &= \frac{1}{2}\frac{\partial^3\overline{U}_i}{\partial D \partial x^2} = \frac{1}{4}\frac{\partial^4\overline{U}_i}{\partial x^4},
\end{aligned}$$

and $\partial^3\overline{U}_i/\partial x^3$ and $\partial^4\overline{U}_i/\partial x^4$, being averages of $\partial^3 U_{i+1}/\partial x^3_{i+1}$ and $\partial^4 U_{i+1}/\partial x^4_{i+1}$, and eventually averages of $\partial^3 U/\partial x^3$ and $\partial^4 U/\partial x^4$, are bounded. We can assume that

$$|dX_{i,j}| \le C_1 L^{-1/8}, \quad \sum_{i=0}^{N-1}\sum_{j=0}^{L-1}(dX_{i,j})^2 \le C_2 L^{1/16}$$

(cf. (30) and (31), respectively) for $\mathrm{ntc}(\omega) \in \mathfrak{K}$ and some constants $C_1$ and $C_2$ (remember that $S$, $N$, $U$, and, of course, $\alpha$ are fixed; without loss of generality we can assume that $N$ and $L$ are powers of 2). This makes the cumulative contribution of the four terms have at most the order of magnitude $O(L^{-1/16})$; therefore, Sceptic can achieve his goal for $\mathrm{ntc}(\omega) \in \mathfrak{K}$ by making $L$ sufficiently large.

To ensure that his capital is always positive, Sceptic stops playing as soon as his capital hits 0. Increasing his initial capital by a small amount we can make sure that this will never happen when $\mathrm{ntc}(\omega) \in \mathfrak{K}$ (for $L$ sufficiently large).

## 11 Proof of the inequality $\le$ in Theorem 1

Let $a := \mathcal{W}_c(E)$; our goal is to show that $\overline{\mathbb{P}}(E) \le a$. Define $E'$ to be the set of all $\omega \in E$ for which $\forall t \in [0,\infty) : \overline{A}_t(\omega) = \underline{A}_t(\omega) = t$. Notice that $\mathcal{W}_c(E') = a$. It is clear that $\tau_s(\omega) = s$ for all $\omega \in E'$, and so $\mathrm{ntc}(\omega) = \omega$ for all $\omega \in E'$. By Theorem 2(b), $\overline{\mathbb{P}}(E') \le a$. Therefore, for any $\epsilon > 0$ there exists a positive capital process $\mathfrak{S}$ such that $\mathfrak{S}_0 \le a + \epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t \ge 1$ on $E'$. Moreover, the proof of Theorem 2 shows that $\mathfrak{S}$ can be chosen *time-invariant*, in the sense that $\mathfrak{S}_{f(t)}(\omega) = \mathfrak{S}_t(\omega \circ f)$ for all time changes $f$ and all $t \in [0,\infty)$. This property will be assumed to be satisfied until the end of this section. In conjunction with the time-superinvariance of $E$ and Theorem 2(a), it implies, for typical $\omega \in E$ satisfying $A_\infty(\omega) = \infty$,

$$\liminf_{t\to\infty} \mathfrak{S}_t(\omega) = \liminf_{t\to\infty} \mathfrak{S}_t(\psi^f) = \liminf_{t\to\infty} \mathfrak{S}_{f(t)}(\psi) \ge 1, \tag{40}$$

where $\psi$ is any element of $E'$ that satisfies $\psi^f = \omega$ for some time change $f$. It is easy to modify $\mathfrak{S}$ so that (40) becomes true for all, rather than for typical, $\omega \in E$ satisfying $A_\infty(\omega) = \infty$.

Let us now consider $\omega \in E$ such that $A_\infty(\omega) = \infty$ is not satisfied. Without loss of generality we assume that $A(\omega)$ exists and is an element of $\Omega$ with the same intervals of constancy as $\omega$. Set $b := A_\infty(\omega) < \infty$. Suppose $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \le 1-\delta$ for some $\delta > 0$; to complete the proof, it suffices to arrive at a contradiction. The definition of quadratic variation shows that the function $\mathrm{ntc}(\omega)|_{[0,b)}$ can be continued to the closed interval $[0,b]$ so that it becomes an element $g$ of $C[0,b]$. It is easy to see that all $\Omega$-extensions (i.e., extensions that are elements of $\Omega$) $\psi$ of $g$ are elements of $E$. Since $\liminf_{t\to b-} \mathfrak{S}_t(\psi) \le 1-\delta$ (remember that $\mathfrak{S}$ is time-invariant) and the function $t \mapsto \mathfrak{S}_t$ is lower semicontinuous (see (2)), $\mathfrak{S}_b(\psi) \le 1-\delta$, for each $\Omega$-extension $\psi$ of $g$. Let us continue $g$, which is now fixed, by measure-theoretic Brownian motion starting from $g(b)$, so that the extension is an element of $E'$ with probability one. Then $\mathfrak{S}_t(\xi)$, $t \ge b$, where $\xi$ is $g$ extended by the trajectory of Brownian motion starting from $g(b)$, is a measure-theoretic stochastic process which is the limit of an increasing sequence of positive continuous supermartingales over the time interval $[b,\infty)$. Let us represent $\mathfrak{S}$ in the form (2) and use the argument in the proof of Lemma 4. For a sufficiently small $\epsilon > 0$, we have the following analogue of (14):

$$\begin{aligned} &P\left(\liminf_{t\to\infty} \mathfrak{S}_t > 1-\delta/2\right) \le \liminf_{t\to\infty} P\left(\mathfrak{S}_t > 1-\delta/2\right) \\ &\le \liminf_{t\to\infty} P\left(\sum_{n=1}^{N_t} \mathcal{K}_t^{G_n,c_n} > 1-\delta/2\right) + \epsilon \le \liminf_{t\to\infty} \frac{1}{1-\delta/2} \sum_{n=1}^{N_t} \mathcal{K}_b^{G_n,c_n} + \epsilon \end{aligned}$$

$$\leq \frac{\mathfrak{S}_b}{1-\delta/2} + \epsilon \leq \frac{1-\delta}{1-\delta/2} + \epsilon < 1,$$

$P$ referring to the underlying probability measure of the Brownian motion. We can see that $\liminf_{t\to\infty} \mathfrak{S}_t \leq 1-\delta/2 < 1$ holds with strictly positive probability, and so $\liminf_{t\to\infty} \mathfrak{S}_t(\psi) < 1$ holds for some extension $\psi \in E'$ of $g$, which contradicts the choice of $\mathfrak{S}$.

# 12 Other connections with literature

Two areas of the theory of stochastic processes and mathematical finance are especially closely connected with the definitions and results of this paper: stochastic integration and the Second Fundamental Theorem of Asset Pricing.

## 12.1 Stochastic integration

The natural financial interpretation of the stochastic integral is that $\int_0^t \pi_s dX_s$ is the trader's profit from holding $\pi_s$ units of a financial security with price path $X$ at time $s$ (see, e.g., [39], Remark III.5.5a.2). It is widely believed that $\int_0^t \pi_s dX_s$ cannot in general be defined pathwise; since our picture does not involve a probability measure on $\Omega$, we restricted ourselves to countable combinations (see (2)) of integrals of simple integrands (see (1)). This definition served our purposes well, but in this subsection we will discuss other possible definitions, always assuming that $X_s$ is a continuous function of $s$.

The pathwise definition of $\int_0^t \pi_s dX_s$ is straightforward when the total variation (i.e., 1-variation in the terminology of Subsection 4.2) of $X_s$ over $[0,t]$ is finite; it can be defined as, e.g., the Lebesgue–Stiltjes integral. It has been known for a long time that the Riemann–Stiltjes definition also works in the case $1/\operatorname{vi}(\pi) + 1/\operatorname{vi}(X) > 1$ (Youngs' theory; see, e.g., [16], Section 2.2). Unfortunately, in the most interesting case $\operatorname{vi}(\pi) = \operatorname{vi}(X) = 2$ this condition is not satisfied.

Another pathwise definition of stochastic integral is due to Föllmer [19]. Föllmer considers a sequence of partitions of the interval $[0,\infty)$ and assumes that the quadratic variation of $X$ exists, in a suitable sense, along this sequence. Our definition of quadratic variation given in Section 5 resembles Föllmer's definition; in particular, our Theorem 2(a) implies that Föllmer's quadratic variation exists for typical $\omega$ along the sequence of partitions $T^n$ (as defined at the beginning of Section 5). In the statement of his theorem ([19], p. 144), Föllmer defines the pathwise integral $\int_0^t f(X_s)dX_s$ for a $C^1$ function $f$ assuming that the quadratic variation of $X$ exists and proves Itô's formula for his integral. In particular, Föllmer's pathwise integral $\int_0^t f(\omega(s))d\omega(s)$ along $T^n$ exists for typical $\omega$ and satisfies Itô's formula. There are two obstacles to using Föllmer's definition in this paper: in order to prove the existence of the quadratic variation we already need our simple notion of integration (which defines the notion of "typical" in Theorem 2(a)); the class of integrals $\int_0^t f(\omega(s))d\omega(s)$ with $f \in C^1$ is too restrictive for our purposes, and using it would complicate the proofs.

An interesting development of Youngs' theory is Lyons's [31] theory of rough paths. In Lyons's theory, we can deal directly only with the rough paths $X$ satisfying $\mathrm{vi}(X) < 2$ (by means of Youngs' theory). In order to treat rough paths satisfying $\mathrm{vi}(X) \in [n, n+1)$, where $n = 2, 3, \ldots$, we need to postulate the values of the iterated integrals $X^i_{s,t} := \int_{s<u_1<\cdots<u_i<t} dX_{u_1} \cdots dX_{u_i}$ for $i = 2, \ldots, n$ (satisfying so-called Chen's consistency condition). According to Corollary 3, only the case $n = 2$ is relevant for our idealized market, and in this case Lyons's theory is much simpler than in general (but to establish Corollary 3 we already used our simple integral). Even in the case $n = 2$ there are different natural choices of $X^2_{s,t}$ (e.g., those leading to Itô-type and to Stratonovich-type integrals); and in the case $n > 2$ the choice would inevitably become even more *ad hoc*.

Another obstacle to using Lyons's theory in this paper is that the smoothness restrictions that it imposes are too strong for our purposes. In principle, we could use the integral $\int_0^t G d\omega$ to define the capital brought by a strategy $G$ for trading in $\omega$ by time $t$. However, similarly to Föllmer's, Lyons's theory requires that $G$ should take a position of the form $f(\omega(t))$ at time $t$, where $f$ is a differentiable function whose derivative $f'$ is a Lipschitz function ([9], Theorems 3.2 and 3.6). This restriction would again complicate the proofs.

## 12.2 Fundamental Theorems of Asset Pricing

The First and Second Fundamental Theorems of Asset Pricing (FTAPs, for brevity) are families of mathematical statements; e.g., we have different statements for one-period, multi-period, discrete-time, and continuous-time markets. A very special case of the Second FTAP, the one covering binomial models, was already discussed briefly in Section 1. In the informal comparisons of our results and the FTAPs in this subsection we only consider the case of one security whose price path $X_t$ is assumed to be continuous.

The First FTAP says that a stochastic model for the security price path $X_t$ admits no arbitrage (or a suitable modification of this condition, such as no free lunch with vanishing risk) if and only if there is an equivalent martingale measure (or a suitable modification thereof, such as an equivalent sigma-martingale measure). The Second FTAP says that the market is complete if and only if there is only one equivalent martingale measure (as, e.g., in the case of the classical Black–Scholes model). The completeness of the market means that each contingent claim has a unique fair price defined in terms of hedging.

Theorem 1 is closely connected with the Second FTAP, namely with its part saying that each contingent claim has a unique fair price provided there is a unique equivalent martingale measure. Theorem 1 and Corollary 1 essentially say that each contingent claim of the form $\mathbf{1}_E$, where $E \in \mathcal{K}$ and $\omega(0) = c$ for all $\omega \in E$, has a fair price and its fair price is equal to the Wiener measure $\mathcal{W}_c(E)$. The scarcity of contingent claims that have a fair price is not surprising as our market is heavily incomplete. According to Remark 5, we can replace the Wiener measure by many other measures. The proofs of both the Second FTAP and Theorem 1 construct fair prices of contingent claims using hedging arguments;

the key technical tool used in Section 10 was the reduction of pricing contingent claims of the form $\mathbf{1}_E$ to pricing smooth contingent claims. Extending this paper's results to a wider class of contingent claims is an interesting direction of further research.

Despite the resemblance, the part of the Second FTAP discussed in the previous paragraph and Theorem 1 are not comparable: the latter leads to a much weaker conclusion (we only get fair prices for a narrow class of contingent claims), but its conditions are also much weaker. The conditions of both Second and First FTAP include a given probability measure on the sample space (our stochastic model of the market). The hedging arguments used in the proof of the Second FTAP depend very much on the postulated probability measure, which allows one to use Itô's notion of stochastic integral. No such condition is needed in the case of our results.

The First FTAP is less closely connected with Theorem 1; it also weakens the two statements whose equivalence is asserted by the Second FTAP, but it weakens them in a direction very different from Theorem 1. However, there are still important, albeit less direct, connections. The main financial notion used in the First FTAP is the no-arbitrage condition. There are two places where the arbitrage-type notions enter the picture in this paper.

First, we used the notion of coherence in Section 7. The most standard notion of arbitrage is that no trading strategy can start from zero capital and end up with positive capital that is strictly positive with a strictly positive probability. Our condition of coherence is very similar but weaker; and of course, it does not involve probabilities. We show that this condition is satisfied automatically in our framework.

The second place where we need arbitrage-type notions is in the interpretation of results such as Corollaries 2–6. For example, Corollary 3 implies that $\mathrm{vi}^{[0,1]}(\omega) \in \{0, 2\}$ for typical $\omega$. Remembering our definitions, this means that either $\mathrm{vi}^{[0,1]}(\omega) \in \{0, 2\}$ or a predefined trading strategy makes infinite capital (at time 1) starting from one monetary unit and never risking going into debt. If we do not believe that making infinite capital risking only one monetary unit is possible for a predefined trading strategy (i.e., that the market is "efficient", in a very weak sense), we should expect $\mathrm{vi}^{[0,1]}(\omega) \in \{0, 2\}$. This looks like an arbitrage-type argument, but there are two important differences:

- Our condition of market efficiency is only needed for the interpretation of our results; the results themselves do not depend on it. The standard no-arbitrage conditions are conditions in mathematical theorems (such as various versions of the First FTAP).

- The usual no-arbitrage conditions are conditions on our stochastic models of the market. On the contrary, our condition of market efficiency describes what we expect to happen, or not to happen, on the actual price path.

It should be noted that our condition of market efficiency (a predefined trading strategy is not expected to make infinite capital risking only one monetary unit) is much closer to Delbaen and Schachermayer's [13] version of the no-arbitrage condition, which is known as NFLVR (no free lunch with vanishing risk), than to the classical no-arbitrage condition. The classical no-arbitrage condition only considers trading strategies that start from 0 and never go into debt, whereas the NFLVR condition allows trading strategies that start from 0 and are permitted to go into slight debt. Our condition of market efficiency allows risking one monetary unit, but this can be rescaled so that the trading strategies considered start from zero and are only allowed to go into debt limited by an arbitrarily small $\epsilon > 0$.

**Remark 12.** Mathematical statements of the First FTAP sometimes involve the condition that $X_t$ should be a semimartingale: see, e.g., Delbaen and Schachermayer's version in [13], Theorem 1.1. However, this condition is not a big restriction: in the same paper, Delbaen and Schachermayer show that the NFLVR condition already implies that $X_t$ is a semimartingale (under some conditions, such as $X_t$ being locally bounded; see [13], Theorem 7.2). A direct proof of the last result, using financial arguments and not depending on the Bichteler–Dellacherie theorem, is given in the recent paper [2].

# Appendix: Hoeffding's process

In this appendix we will check that Hoeffding's original proof of his inequality ([20], Theorem 2) remains valid in the game-theoretic framework. This observation is fairly obvious, but all details will be spelled out for convenience of reference. This appendix is concerned with the case of discrete time, and it will be convenient to redefine some notions (such as "process").

Perhaps the most useful product of Hoeffding's method is a positive supermartingale starting from 1 and attaining large values when the sum of bounded martingale differences is large. Hoeffding's inequality can be obtained by applying the maximal inequality to this supermartingale (see, e.g., [47], Section A.7). However, we do not need Hoeffding's inequality in this paper, and instead of Hoeffding's positive supermartingale we will have a positive "supercapital process", to be defined below.

This is a version of the basic forecasting protocol from [37]:

GAME OF FORECASTING BOUNDED VARIABLES

**Players:** Sceptic, Forecaster, Reality

**Protocol:**

Sceptic announces $\mathcal{K}_0 \in \mathbb{R}$.
FOR $n = 1, 2, \dots$:
  Forecaster announces interval $[a_n, b_n] \subseteq \mathbb{R}$ and number $\mu_n \in (a_n, b_n)$.
  Sceptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in [a_n, b_n]$.
  Sceptic announces $\mathcal{K}_n \leq \mathcal{K}_{n-1} + M_n(x_n - \mu_n)$.

On each round $n$ of the game Forecaster outputs an interval $[a_n, b_n]$ which, in his opinion, will cover the actual observation $x_n$ to be chosen by Reality, and also outputs his expectation $\mu_n$ for $x_n$. The forecasts are being tested by Sceptic, who is allowed to gamble against them. The expectation $\mu_n$ is interpreted as the price of a ticket which pays $x_n$ after Reality's move becomes known; Sceptic is allowed to buy any number $M_n$, positive or negative (perhaps zero), of such tickets. When $x_n$ falls outside $[a_n, b_n]$, Sceptic becomes infinitely rich; without loss of generality we include the requirement $x_n \in [a_n, b_n]$ in the protocol; furthermore, we will always assume that $\mu_n \in (a_n, b_n)$. Sceptic is allowed to choose his initial capital $\mathcal{K}_0$ and is allowed to throw away part of his money at the end of each round.

It is important that the game of forecasting bounded variables is a perfect-information game: each player can see the other players' moves before making his or her (Forecaster and Sceptic are male and Reality is female) own move; there is no randomness in the protocol.

A *process* is a real-valued function defined on all finite sequences

$$(a_1, b_1, \mu_1, x_1, \ldots, a_N, b_N, \mu_N, x_N), \quad N = 0, 1, \ldots,$$

of Forecaster's and Reality's moves in the game of forecasting bounded variables. If we fix a strategy for Sceptic, Sceptic's capital $\mathcal{K}_N$, $N = 0, 1, \ldots$, become a function of Forecaster's and Reality's previous moves; in other words, Sceptic's capital becomes a process. The processes that can be obtained this way are called *supercapital processes*.

The following theorem is essentially inequality (4.16) in [20].

**Theorem 3.** *For any $h \in \mathbb{R}$, the process*

$$\prod_{n=1}^{N} \exp\left(h(x_n - \mu_n) - \frac{h^2}{8}(b_n - a_n)^2\right)$$

*is a supercapital process.*

*Proof.* Assume, without loss of generality, that Forecaster is additionally required to always set $\mu_n := 0$. (Adding the same number to $a_n$, $b_n$, and $\mu_n$ on each round will not change anything for Sceptic.) Now we have $a_n < 0 < b_n$.

It suffices to prove that on round $n$ Sceptic can make a capital of $\mathcal{K}$ into a capital of at least

$$\mathcal{K} \exp\left(hx_n - \frac{h^2}{8}(b_n - a_n)^2\right);$$

in other words, that he can obtain a payoff of at least

$$\exp\left(hx_n - \frac{h^2}{8}(b_n - a_n)^2\right) - 1$$

using the available tickets (paying $x_n$ and costing 0). This will follow from the inequality

$$\exp\left(hx_n - \frac{h^2}{8}(b_n - a_n)^2\right) - 1 \le x_n \frac{e^{hb_n} - e^{ha_n}}{b_n - a_n} \exp\left(-\frac{h^2}{8}(b_n - a_n)^2\right), \quad (41)$$

which can be rewritten as

$$\exp(hx_n) \le \exp\left(\frac{h^2}{8}(b_n - a_n)^2\right) + x_n \frac{e^{hb_n} - e^{ha_n}}{b_n - a_n}. \tag{42}$$

Our goal is to prove (42). By the convexity of the function exp, it suffices to prove

$$\frac{x_n - a_n}{b_n - a_n} e^{hb_n} + \frac{b_n - x_n}{b_n - a_n} e^{ha_n} \le \exp\left(\frac{h^2}{8}(b_n - a_n)^2\right) + x_n \frac{e^{hb_n} - e^{ha_n}}{b_n - a_n},$$

i.e.,

$$\frac{b_n e^{ha_n} - a_n e^{hb_n}}{b_n - a_n} \le \exp\left(\frac{h^2}{8}(b_n - a_n)^2\right), \tag{43}$$

i.e.,

$$\ln\left(b_n e^{ha_n} - a_n e^{hb_n}\right) \le \frac{h^2}{8}(b_n - a_n)^2 + \ln(b_n - a_n). \tag{44}$$

(Notice that the numerator of the left-hand side of (43) is strictly positive, and so the logarithm on the left-hand side of (44) is well defined.) The derivative of the left-hand side of (44) in $h$ is

$$\frac{a_n b_n e^{ha_n} - a_n b_n e^{hb_n}}{b_n e^{ha_n} - a_n e^{hb_n}}$$

and the second derivative, after cancellations and regrouping, is

$$(b_n - a_n)^2 \frac{\left(b_n e^{ha_n}\right)\left(-a_n e^{hb_n}\right)}{\left(b_n e^{ha_n} - a_n e^{hb_n}\right)^2}.$$

The last ratio is of the form $u(1-u)$ where $0 < u < 1$. Hence it does not exceed $1/4$, and the second derivative itself does not exceed $(b_n - a_n)^2/4$. Inequality (44) now follows from the second-order Taylor expansion of the left-hand side around $h = 0$. □


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