

Strong confidence intervals for autoregression

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February 1, 2008

Abstract

In this short preliminary note I apply the methodology of game-theoretic probability to calculating non-asymptotic confidence intervals for the coefficient of a simple first order scalar autoregressive model. The most distinctive feature of the proposed procedure is that with high probability it produces confidence intervals that always cover the true parameter value when applied sequentially.

1 Introduction

Game-theoretic probability (see, e.g., [6], with the basic idea going back to Ville [7]) provides a means of testing probabilistic models. In this note the game-theoretic methodology is extended to statistical models; it will be demonstrated on the first-order scalar autoregressive model

$$y_t = \alpha y_{t-1} + \epsilon_t, \quad t = 1, 2, \dots, \quad (1)$$

without the intercept term, with constant y_0 , and with independent $N(0, 1)$ innovations ϵ_t .

We will be interested in procedures for computing, for each $t = 1, 2, \dots$, a confidence interval $[l_t, u_t]$ for α given y_0, \dots, y_t . Let us fix a confidence level $1 - \delta$, and let α be the true parameter value. The usual procedures are “batch”, in that they only guarantee that $\alpha \in [l_t, u_t]$ with high probability for a fixed t . It is usually true that, when they are applied sequentially, the intersection $\bigcap_{t=1}^{\infty} [l_t, u_t]$ is empty with probability one. Our goal is to guarantee that

$$\alpha \in \bigcap_{t=1}^{\infty} [l_t, u_t] \quad (2)$$

with probability at least $1 - \delta$.

Analogously to the usual classification of the limit theorems of probability theory into “strong” (involving the conjunction over all t) and “weak” (applicable to individual t), let us call such confidence intervals *strong*. In particular, confidence intervals satisfying (2) with probability at least $1 - \delta$ will be called

strong $(1 - \delta)$ -confidence intervals. Accordingly, confidence intervals produced by the standard procedures will be referred to as *weak*; weak $(1 - \delta)$ -confidence intervals satisfy $\alpha \in [l_t, u_t]$ with probability at least $1 - \delta$ for each individual t . (This probability is sometimes required to be precisely $1 - \delta$, but we will only consider the “conservative” definitions.)

To achieve the goal (2), for each possible value of the parameter α we construct random variables S_t^α , $t = 0, 1, \dots$, that form a nonnegative martingale under the probability measure \mathbb{P}_α corresponding to the probabilistic model (1) with the given α . It will also be true that $S_0^\alpha = 1$; such sequences (nonnegative martingales starting from 1) will be called *martingale tests*. We can then set

$$[l_t, u_t] = \{\alpha : S_t^\alpha < 1/\delta\}$$

(assuming that the set on the right-hand side is an interval, which it will be in our case). The special case

$$\mathbb{P}_\alpha \left\{ \sup_{t=0,1,\dots} S_t^\alpha \geq 1/\delta \right\} \leq \delta$$

(due to Ville; see, e.g., [7], p. 100, or [6], (2.12)) of Doob’s inequality shows that (2) will indeed be true with probability at least $1 - \delta$.

2 Derivation of strong confidence intervals

If the true probability density of y_t (conditional on the past) is

$$\frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(y_t - \alpha^{\text{true}} y_{t-1})^2}{2} \right)$$

and we want to reject the hypothesis

$$\frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(y_t - \alpha y_{t-1})^2}{2} \right),$$

the best, in many respects¹, martingale test is the likelihood ratio sequence with the relative increments

$$\begin{aligned} & \frac{\frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(y_t - \alpha^{\text{true}} y_{t-1})^2}{2} \right)}{\frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(y_t - \alpha y_{t-1})^2}{2} \right)} \\ &= \exp \left(\frac{(\alpha^2 - (\alpha^{\text{true}})^2) y_{t-1}^2 + 2(\alpha^{\text{true}} - \alpha) y_{t-1} y_t}{2} \right). \end{aligned}$$

¹Cf., e.g., the nonnegativity of the Kullback–Leibler divergence, Neyman–Pearson lemma, and the optimality property of the probability ratio test in sequential analysis.

The product over $t = 1, \dots, T$ is the martingale test itself:

$$S_T^{\alpha, \alpha^{\text{true}}} = \exp\left(\frac{(\alpha^2 - (\alpha^{\text{true}})^2)\Gamma_0 + 2(\alpha^{\text{true}} - \alpha)\Gamma_1}{2}\right), \quad (3)$$

where

$$\Gamma_0 = \sum_{t=1}^T y_{t-1}^2$$

and

$$\Gamma_1 = \sum_{t=1}^T y_{t-1}y_t.$$

To get rid of the parameter α^{true} , let us integrate (3) over the probability distribution $N(\alpha, a^2)$ on the α^{true} s:

$$\begin{aligned} S_T^\alpha &= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} \exp\left(\frac{(\alpha^2 - (\alpha^{\text{true}})^2)\Gamma_0 + 2(\alpha^{\text{true}} - \alpha)\Gamma_1}{2}\right) \\ &\quad \times \exp\left(-\frac{(\alpha - \alpha^{\text{true}})^2}{2a^2}\right) d\alpha^{\text{true}} \\ &= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{1}{2a^2} + \frac{\Gamma_0}{2}\right)x^2 + (\Gamma_1 - \alpha\Gamma_0)x\right) dx \end{aligned}$$

(where I made the substitution $x = \alpha^{\text{true}} - \alpha$). Now the formula

$$\int_{-\infty}^{\infty} \exp(-Ax^2 + Bx) dx = \sqrt{\frac{\pi}{A}} \exp\left(\frac{B^2}{4A}\right)$$

gives

$$S_T^\alpha = \frac{1}{\sqrt{a^2\Gamma_0 + 1}} \exp\left(\frac{a^2(\Gamma_1 - \alpha\Gamma_0)^2}{2(a^2\Gamma_0 + 1)}\right). \quad (4)$$

To find the confidence intervals corresponding to (4), fix a confidence level $1 - \delta$. The $(1 - \delta)$ -confidence interval corresponding to (4) is defined as the set of α s satisfying

$$\frac{1}{\sqrt{a^2\Gamma_0 + 1}} \exp\left(\frac{a^2(\Gamma_1 - \alpha\Gamma_0)^2}{2(a^2\Gamma_0 + 1)}\right) \leq \frac{1}{\delta}.$$

Solving this in α gives the confidence interval

$$\left|\alpha - \frac{\Gamma_1}{\Gamma_0}\right| \leq \sqrt{\frac{a^2\Gamma_0 + 1}{a^2\Gamma_0^2} \ln \frac{a^2\Gamma_0 + 1}{\delta^2}}. \quad (5)$$

Notice that, in the stationary case $|\alpha| < 1$, where Γ_0 has the order of magnitude T , the size of the confidence interval (5) is $O(\sqrt{\ln T/T})$ as $T \rightarrow \infty$. This is worse than the usual iterated-logarithm behaviour ($O(\sqrt{\ln \ln T/T})$) but agrees with [4], Theorem 2.5 (although the latter result is just an upper bound). One

can speculate that, in the stationary case, the $O(\sqrt{\ln \ln T/T})$ behaviour will be recovered if the $N(\alpha, a^2)$ is replaced by a probability distribution that is more concentrated around α , as in Ville's [7] proof of the law of the iterated logarithm (see also [6], Chapter 5).

Most of the terms in the confidence interval (5) are familiar from the literature (which, however, mainly covers the case of weak confidence intervals). The centre $\frac{\Gamma_1}{\Gamma_0}$ of the interval is just the least-squares estimate of α from the given sample. The statistic

$$\tau_T = \left(\frac{\Gamma_1}{\Gamma_0} - \alpha \right) \sqrt{\Gamma_0} \approx \frac{\frac{\Gamma_1}{\Gamma_0} - \alpha}{\sqrt{\frac{a^2 \Gamma_0 + 1}{a^2 \Gamma_0^2}}} \quad (6)$$

(for a fixed sample size T) has been studied extensively. In describing the known results I will follow [2]. Mann and Wald [3] showed that τ_T is $N(0, 1)$ asymptotically when $|\alpha| < 1$. Anderson [1] extended this to the case $|\alpha| > 1$. White [8] and Rao [5] showed that, in the case $|\alpha| = 1$, τ_T converges in distribution to

$$\frac{1}{2} \frac{W^2(1) - 1}{\sqrt{\int_0^1 W^2(s) ds}} \quad (7)$$

where W is a standard Brownian motion.

Suppose, for concreteness, that (6) is asymptotically $N(0, 1)$. The central asymptotic weak confidence interval for α based on the statistic given after the “ \approx ” in (6) will be different from (5) in that

$$\sqrt{\ln \frac{a^2 \Gamma_0 + 1}{\delta^2}} = \sqrt{2 \ln \frac{1}{\delta} + \ln(a^2 \Gamma_0 + 1)} \quad (8)$$

will be replaced by the upper $\delta/2$ -quantile of $N(0, 1)$, essentially by

$$\sqrt{2 \ln \frac{2}{\delta}}$$

for a small δ . This is close to the first addend on the right-hand side of (8), and so the second addend represents the price that we are paying for our confidence intervals being strong.

3 Empirical results

To test the test martingales (4) empirically, I generated y_0, \dots, y_{1000} from the model (1) with $y_0 = 0$ and $\alpha = 0.8, 1$. The case $\alpha = 0.8$ illustrates the stationary behaviour ($|\alpha| < 1$), and the “unit-root” case $\alpha = 1$ is intermediate between the stationary and “explosive” ($|\alpha| > 1$) behaviour. Tables 1 and 2 give the approximate weak central 99%-confidence intervals based on the above approximations for τ_T (normal for $\alpha = 0.8$ and (7) for $\alpha = 1$) and the strong 99%-confidence intervals computed from (5).

Type of the interval	Confidence interval	Its width
Weak (approximate)	[0.736, 0.837]	0.101
Strong	[0.716, 0.857]	0.141

Table 1: Weak and strong 99%-confidence intervals obtained for $T = 1000$ and $\alpha = 0.8$ (stationary case). The value of the constant a is 0.1.

Type of the interval	Confidence interval	Its width
Weak (approximate)	[0.982, 1.003]	0.022
Strong	[0.977, 1.010]	0.033

Table 2: The analogue of Table 1 for $\alpha = 1$ (unit root case).

The intuition behind the value of a in (5) is that it should be of the same order of magnitude as the expected width of the confidence interval (since a represents the order of magnitude of the distance to the bulk of α^{true} that we are competing with). It is taken as 0.1 in the tables, but the results will not be drastically different if $a = 1$, which is intuitively more “neutral”, is chosen: e.g., the width 0.141 in Table 1 would go up to 0.162, and the width 0.033 in Table 2 would go up to 0.037.

Figures 1 and 2 give the final values S_T^α for the same data set and the same value of a , $a = 0.1$.

4 Directions of further research

These are some possible areas in which the methods of martingale testing could be applied:

Online testing of statistical models. When the strong confidence interval $[l_t, u_t]$ becomes empty, the statistical model can be rejected. Of course, efficient testing of statistical models will require different martingale tests: it will not be sufficient to consider, as in this note, different values of parameters as alternatives.

Prediction. In the simplest case, the prediction interval at step t might be computed as the union of the prediction intervals corresponding to all $\alpha \in [l_t, u_t]$.

Alternative assumptions about innovations. For example, the assumption that ϵ_t have zero medians (conditional on the past) might lead to feasible statistical procedures.

Acknowledgments

I am grateful to Bent Nielsen, Clive Bowsher, David Hendry, and Jennifer Castle for useful discussions. This work was partially supported by EPSRC (grant

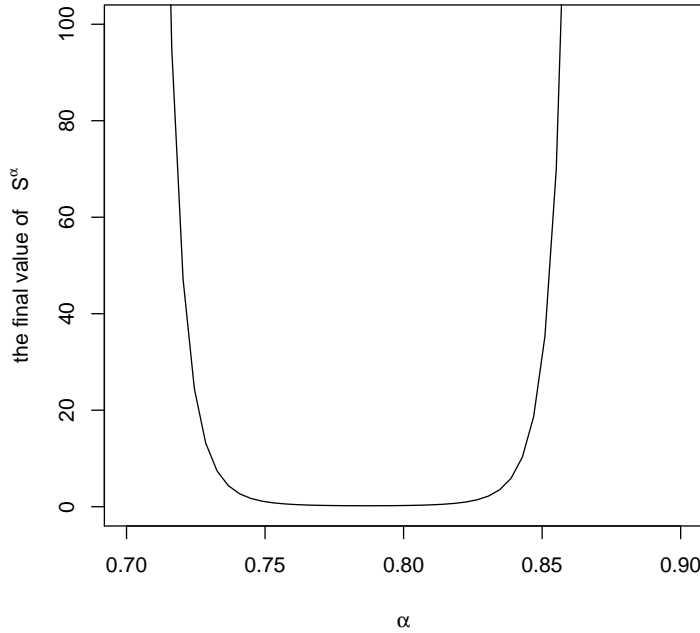


Figure 1: The capital S_{1000}^α achieved for various values of α when the true coefficient α is 0.8.

EP/F002998/1), MRC (grant G0301107), and the Cyprus Research Promotion Foundation.

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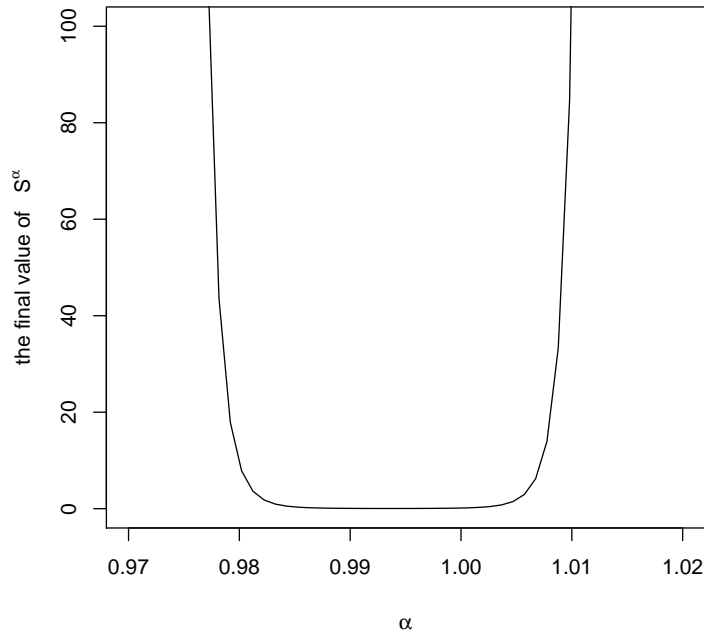


Figure 2: The analogue of Figure 1 for $\alpha = 1$.

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