Three attractive osculating walkers and a polymer collapse transition.

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Abstract

Consider n interacting lock-step walkers in one dimension which start at the points $\{0, 2, 4, \ldots, 2(n-1)\}$ and at each tick of a clock move unit distance to the left or right with the constraint that if two walkers land on the same site the directions of their next steps must be in the opposite direction so that crossing is avoided. When two walkers visit and then leave the same site an osculation is said to take place. The space-time paths of these walkers may be taken to represent the configurations of n fully directed polymer chains of length t embedded on a directed square lattice. If a weight λ is associated with each of the i osculations the partition function is $Z_t^{(n)}(\lambda) = \sum_{i=0}^{\lfloor \frac{(n-1)t}{2} \rfloor} z_{t,i}^{(n)} \lambda^i$ where $z_{t,i}^{(n)}$ is the number of $t-$step configurations having i osculations. When $\lambda = 0$ the partition function reduces to the number of vicious walker configurations for which an explicit formula is known. The asymptotics of such configurations was discussed by Fisher in his Boltzmann medal lecture. Also for $n = 2$ the partition function for arbitrary λ is easily obtained by Fisher's necklace method. For $n > 2$ and $\lambda \neq 0$ the only exact result so far is that of Guttmann and Vöge who obtained the generating function $G^{(n)}(\lambda, u) \equiv \sum_{t=0}^{\infty} Z_t^{(n)}(\lambda) u^t$ for $\lambda = 1$ and $n = 3$. The main result of this paper is to extend their result to arbitrary λ. By fitting computer generated data it is conjectured that $Z_t^{(3)}(\lambda)$ satisfies a third order inhomogeneous difference equation with constant coefficients which was used to obtain

$$G^{(3)}(\lambda, u) = \frac{(\lambda-3)(\lambda+2) - \lambda\left(12 - 5\lambda + \lambda^2\right)u - 2\lambda^3 u^2 + 2(\lambda-4)\left(\lambda^2 u^2 - 1\right)c(2u)}{(\lambda - 2 - \lambda^2 u)\,(\lambda - 1 - 4\lambda u - 4\lambda^2 u^2)}$$

where $c(u) = \frac{1-\sqrt{1-4u}}{2u}$, the generating function for Catalan numbers.

The nature of the collapse transition which occurs at $\lambda = 4$ is discussed and extensions to higher values of n are considered. It is argued that the position of the collapse transition is independent of n.

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1 Introduction and Summary.

We consider n parallel walks, or paths, on the directed square lattice. Each walker makes one step forward at every time step and a walker at the lattice site (x, y) may move along a bond to either of the points $(x + 1, y \pm 1)$ but only one walker may traverse a given bond and the paths may not cross. The walkers start at the points $(0, 0), (0, 2), \dots (0, 2(n - 1))$ and may terminate anywhere subject to the non-crossing condition. A site where two walks intersect is said to be an osculation and the combinatorial problem is to enumerate the number $z_{t,i}^{(n)}$ of $t-$step path configurations which have i osculations. A distinction will be made between configurations where two walkers intersect on the final step and those which do not. Intersection on the last step is not counted as an osculation. Figure 1 shows three such paths with 21 steps and 9 osculations.

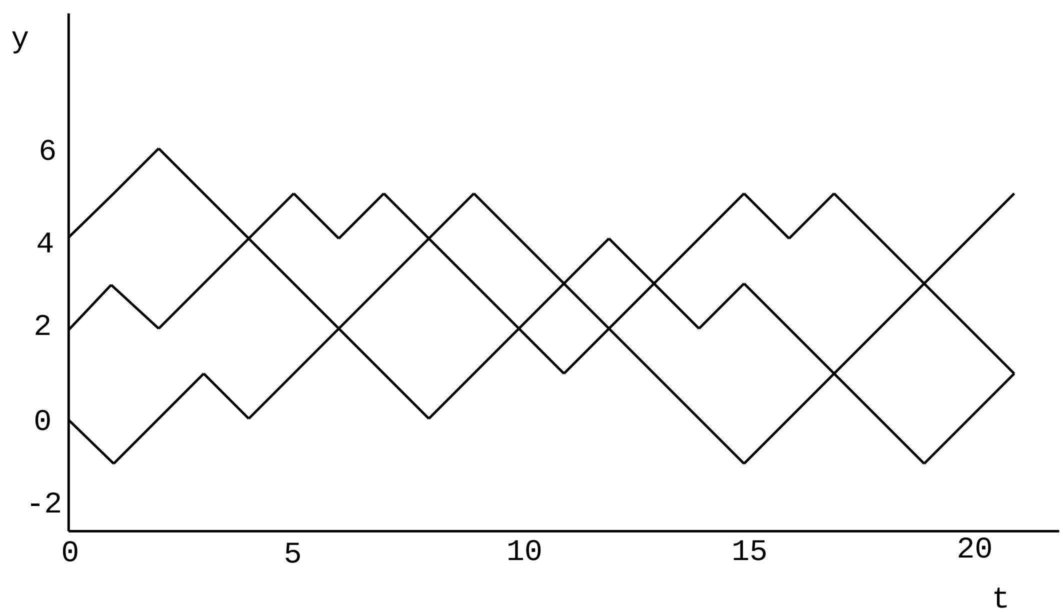

Figure 1: *A configuration of three osculating walkers. Each walker has made 21 steps and there are 9 osculations.*

The physical problem is to investigate the statistical properties of directed polymer networks. The walks are then the individual polymer chains and the osculations are interactions which modify the energy of the network. The generating function

$$Z_t^{(n)}(\lambda) \equiv \sum_{i=0}^{\lfloor \frac{(n-1)t}{2} \rfloor} z_{t,i}^{(n)} \lambda^i \epsilon^f \quad (1)$$

is the partition function of the polymer network the derivative of which determines the expected number of interactions. The Boltzmann factor λ^i makes the chains mutually repulsive or attractive depending on whether $\lambda < 1$ or $\lambda > 1$. The repulsive case is less interesting since it is qualitatively similar to the non-intersecting walk or "vicious walker" problem $\lambda = 0$ which is soluble for any number of walkers [1, 2]. This similarity persists into the attractive region as far as the collapse transition point λ_c at which the chains begin to stick together. It will be argued that $\lambda_c = 4$ for any number of chains.

The factor ϵ^f, where f is the number of intersections on the final step, is included so that different network topologies may be distinguished. Subtracting the $\epsilon = 0$ part gives the partition function for networks having at least one final intersection. For two or three chains $f = 0$ or 1 since only one walk can traverse a given bond. In the case of two chains the coefficient of ϵ is the partition function for "watermelon" configurations [1] and for three chains it is the partition function for networks in which the endpoints of just two of the chains are joined together. Such networks have been called "ceratic" [3].

The two variable generating function

$$G^{(n)}(\lambda, u) \equiv \sum_{t=0}^{\infty} Z_t^{(n)}(\lambda) u^t = \sum_{t=0}^{\infty} \sum_{i=0}^{\lfloor \frac{(n-1)t}{2} \rfloor} z_{t,i}^{(n)} \epsilon^f \lambda^i u^t \quad (2)$$

is the grand partition function of the polymer network and $G_\epsilon^{(n)}(\lambda, u)$ will denote the coefficient of ϵ.

Brak [4] has conjectured a general formula for the generating function of n osculating walks for general λ but with fixed endpoints. This extends the determinantal vicious walker formula of Gessel and Viennot [5]. The formula involves multiple summations and a further summation over endpoints to obtain the formula for three walkers given here is far from straightforward.

Recently Guttmann and Vöge [6] found an explicit formula for $G^{(3)}(1, u)$ in the case $\epsilon = 1$, that is the generating function for the total number of three walk osculating star configurations. The work here generalises their result to arbitrary λ and ϵ. In [6] the method of differential approximants reviewed in [7] was used to determine a recurrence relation, having polynomial coefficients, satisfied by the sequence of partition functions for increasing t. Although the recurrence relation was not proven, the values of t used made it inconceivable that it would ever fail. For general λ the same method leads to recurrence relation of order 5 with coefficients linear in t from which the generating function may be deduced by solving a first order differential equation. However we present a different approach which gives a much neater recurrence relation with constant coefficients and an inhomgeneous term which is the product of an exponential and a linear combination of Catalan numbers. For example, when $\epsilon = 1$

$$2(\lambda-1)(\lambda-2)Z_t^{(3)} - 2\lambda(\lambda^2+3\lambda-8)Z_{t-1}^{(3)} + 16\lambda^2 Z_{t-2}^{(3)} + 8\lambda^4 Z_{t-3}^{(3)} = 2^t R_t$$

where $R_t = (\lambda-4)(\lambda^2 C_{t-2} - 4C_t)$ and C_t is a Catalan number. The form is motivated by the corresponding relation for two walkers which we are able to derive from the known generating function [8]. It is hoped that a similar approach may work for more than three walkers if the appropriate form of the right-hand side can be found. Also the relative simplicity of the relation gives rise to the hope that a proof may be possible.

For $t \to \infty$ we find that

$$Z_t^{(n)}(\lambda) \sim [\mu^{(n)}(\lambda)]^t t^g \tag{3}$$

where the growth factor $\mu^{(n)}(\lambda)$ varies continuously with λ but is independent of ϵ. The exponent g is piecewise constant and for $\lambda \leq \lambda_c$ depends on the topology of the network. The exponent for networks with at least one endpoint intersection will be denoted by g_ϵ.

$G^{(n)}(\lambda, u)$, as a function of u, will have at least one singular point on the positive real axis. The closest such point will be at $u_c^{(n)}(\lambda) = 1/\mu^{(n)}(\lambda)$ and as $u \to u_c^{(n)}(\lambda)$ from below we find that, for $\lambda \neq \lambda_c$,

$$G^{(n)}(\lambda, u) \sim |u - u_c^{(n)}(\lambda)|^{-\gamma} \tag{4}$$

where $\gamma = g + 1$. For $\lambda = \lambda_c$ there are confluent singularities.

Our results for the growth factor and exponents are summarised in table 1. The collapse point is found to be same for two and three walks, that is

	Two walks			Three walks		
	μ	g	g_ϵ	μ	g	g_ϵ
$\lambda < 4$	4	$-\frac{1}{2}$	$-\frac{3}{2}$	8	$-\frac{3}{2}$	$-\frac{5}{2}$
$\lambda = 4$	4	$0, -\frac{1}{2}$	$-\frac{1}{2}$	8	$0, -\frac{1}{2}$	$-\frac{1}{2}$
$\lambda > 4$	$\frac{\lambda}{\sqrt{\lambda-1}}$	0	0	$\frac{\lambda^2}{\lambda-2}$	0	0

Table 1: Growth factors and exponents.

$\lambda_c = 4$. This may be understood by considering the case when two walkers arrive at the same site. In the case of osculating walkers there is only one way for them to leave but if the walks were allowed to share the same bond and to cross there would be 4 ways, so placing weight 4 on an osculating vertex is equivalent to allowing the walks to move independently. This additional freedom is responsible for the change of exponent. When $\epsilon = 1$ the final steps are also independent and $Z_t^{(n)}(4) = 2^{nt}$ which is supported by our findings for two and three walks. The value of g is therefore zero.

We shall see that for $\lambda > \lambda_c$, $G^{(n)}(\lambda, u)$ has several singular points on the real axis the positions of which depend on λ. The collapse transition is marked by a subset of these coming together at $u = 1/2^n$ as $\lambda \to \lambda_c$. In general the result of this will be the occurrence of confluent singularities and the critical point will have more than one exponent. The value $\epsilon = 1$ is special in this respect since the confluent singularities at $\lambda = \lambda_c$ cancel out. For $\epsilon \neq 1$ the confluent singularity for two and three walks has an exponent which is independent of ϵ and is equal to g_ϵ. The value $g_\epsilon = -\frac{1}{2}$ is that for two walks which are independent except that they must intersect after $t-$steps [2, 1]. It is the same for both two and three walk configurations because in the latter case only two of the walks have a common endpoint.

For $\lambda < 4$ the growth factor $\mu^{(n)}(\lambda) = 2^n$ and the exponent g is the same as for vicious walker star configurations, that is [1],

$$g = g_S = -\frac{n(n-1)}{4} \tag{5}$$

These results are expected to be true for any number of walks. For two walks $g_\epsilon = -\frac{3}{2}$ is the exponent for staircase polygons [9]. For three walks the exponent $g_\epsilon = -\frac{5}{2}$ for the ceratic network is in agreement with the exponent formula for general directed polymer networks [10].

For $\lambda > 4$ the walks tend to stick together and for both two and three walks the exponent is the same as that for a single walk. In this region the growth factor varies with λ and as $\lambda \to \infty$, $\mu^{(n)}(\lambda) \sim \lambda^{\frac{n-1}{2}}$ which is the expected form when the osculating walks are completely bound together in a single rod-like configuration. In this configuration there is one osculation every two steps in the case of two walks but one on every step for three walks which explains the power of λ. Again these results are expected to be valid for any number of walks.

2 Two walks.

The two walker problem may be solved exactly in various ways [1, 6, 8] and we present results for this case as a guide to the solution of the three walk problem. It is also of interest to compare the formulae for the two problems since they have some common features.

First consider $t-$step configurations with no osculations and denote the generating function by $G_0^{(2)}(u)$.

- Configurations in which the walks terminate on the same site are equinumerous with staircase polygons of length $t+1$ the number of which is known [9] to be the Catalan number

$$C_t = \frac{1}{t+1}\binom{2t}{t} \tag{6}$$

having generating function

$$c^*(u) = \sum_{t=1}^{\infty} C_t u^t = \frac{1-2u-\sqrt{1-4u}}{2u} \tag{7}$$

which satisfies the equation

$$uc^{*2} - (1-2u)c^* + u = 0. \tag{8}$$

- Configurations in which the walks terminate on different sites biject to pairs of $(t+1)-$step walks which start together and never meet again. The generating function for such walks was shown in [2] (equation (14)) to be $(1-4u)^{-\frac{1}{2}}$ but includes the zero length configuration which must be subtracted and configurations with $t>1$ were counted twice because the walks were considered distinguishable.

To keep account of the two types of configuration we give weight ϵ to configurations in which the walks end on the same site thus, using (7)

$$G_0^{(2)}(u) = \epsilon\, c^*(u) + \frac{1}{2u}\left(\frac{1}{\sqrt{1-4u}} - 1\right) = \epsilon\, c^*(u) + \frac{1-c^*(u)}{1-4u}. \tag{9}$$

Following Fisher [1] we find the generating function $G^{(2)}(\lambda, u)$ using the bubble chaining technique. Any configuration with i osculations may be obtained by concatenating i staircase polygons, moving the first pair of steps to the end and appending a configuration with no osculations. The generating function for configurations with i osculations is therefore

$$G_i^{(2)}(u) = (uc^*(u))^i G_0^{(2)}(u) \tag{10}$$

and hence

$$G^{(2)}(\lambda, u) = \sum_{i=0}^{\infty} G_i^{(2)}(u)\lambda^i = \frac{G_0^{(2)}(u)}{1-\lambda uc^*(u)}. \tag{11}$$

Now [11]

$$c^*(u)^y = \sum_{t=y}^{\infty} B_{2t-1,2y-1} u^t \tag{12}$$

where $B_{-1,-1} = 1$; for $j \neq 1$, $B_{-1,j} = B_{j,-1} = 0$ and for $j,k \geq 0$, $B_{j,k}$ is the Ballot number

$$B_{j,k} = \frac{(k+1)j!}{(\frac{1}{2}(j+k)+1)!\,(\frac{1}{2}(j-k))!} \tag{13}$$

Expanding (10) gives the number of configurations with i osculations as

$$z^{(2)}_{t,i} = \epsilon B_{2t-2i-1,2i+1} + \sum_{\ell=0}^{t-2i} 4^{\ell}(B_{2t-2i-2\ell-1,2i-1} - B_{2t-2i-2\ell-1,2i+1}) \tag{14}$$

for $t \geq 2i$ and zero otherwise.

To obtain a recurrence relation for $Z^{(2)}_t(\lambda)$ we note that, using (8)

$$\frac{1}{1-\lambda uc^*(u)} = \frac{\lambda - 1 - 2\lambda u - \lambda uc^*(u)}{\lambda - 1 - 2\lambda u - \lambda^2 u^2} \tag{15}$$

and combining this with (11) and (9) and using (8) gives

$$G^{(2)}(\lambda,u) = \frac{\lambda - 1 - 3\lambda u + \lambda u\epsilon(1-4u) + (1 - \lambda u - \epsilon(1-4u))c^*(u)}{(1-4u)(\lambda - 1 - 2\lambda u - \lambda^2u^2)} \tag{16}$$

and hence

$$(\lambda-1)Z^{(2)}_t - 2(3\lambda-2)Z^{(2)}_{t-1} - \lambda(\lambda-8)Z^{(2)}_{t-2} + 4\lambda^2 Z^{(2)}_{t-3} = (1-\epsilon)C_t - (\lambda - 4\epsilon)C_{t-1} \tag{17}$$

In order to find the generating function for three walk configurations we will first look for a recurrence relation of a form similar to that of (17).

3 Three walks.

In the case of three walks the bubble chaining technique is insufficient to give an explicit formula and the results of the following sections are obtained by fitting a recurrence relation to computer generated coefficients. The coefficients are easily generated using the partial difference equations

$$\begin{aligned}
Z_0(x,x,x_3) &= Z_0(x_1,x,x) = \epsilon \\
Z_0(x_1,x_2,x_3) &= 1 \\
Z_t(x,x,x_3) &= \lambda Z_{t-1}(x-1,x+1,x_3-1) + \lambda Z_{t-1}(x-1,x+1,x_3+1) \\
Z_t(x_1,x,x) &= \lambda Z_{t-1}(x_1-1,x-1,x+1) + \lambda Z_{t-1}(x_1+1,x-1,x+1) \\
Z_t(x_1,x_2,x_3) &= \sum_{\delta_1=\pm1}\sum_{\delta_2=\pm1}\sum_{\delta_2=\pm1} Z_{t-1}(x_1+\delta_1, x_2+\delta_2, x_3+\delta_3) \\
Z^{(3)}_t(\lambda) &= Z_t(0,2,4)
\end{aligned} \tag{18}$$

where $Z_t(x_1, x_2, x_3)$ is the partition function for walks of length t which start at positions $x_1 \leq x_2 \leq x_3$.

The chosen form of the recurrence relation was suggested by that for two walkers but it was found necessary to include a factor 2^t on the right-hand side multiplying the linear combination of Catalan numbers. The motivation for this factor was the fact that the walks only interact in pairs so that, at any step, one of the walks can move in either direction without interacting. It is found that

$$2(\lambda-1)(\lambda-2)Z_t - 2\lambda(\lambda^2+3\lambda-8)Z_{t-1} + 16\lambda^2 Z_{t-2} + 8\lambda^4 Z_{t-3} = 2^t R_t \tag{19}$$

where

$$R_t = (1-\epsilon)(4C_{t+1} - \lambda^2 C_{t-1}) - (\lambda - 4\epsilon)(4C_t - \lambda^2 C_{t-2}). \tag{20}$$

This relation has not been proven but is almost certainly exact since far more terms were generated than were required to determine the coefficients and these were in agreement with the relation.

Let $Z_t = 2^t Y_t$ then

$$2(\lambda-1)(\lambda-2)Y_t - \lambda(\lambda^2+3\lambda-8)Y_{t-1} + 4\lambda^2 Y_{t-2} + \lambda^4 Y_{t-3} = R_t. \tag{21}$$

and with the initial conditions $Y_0 = 1$, $Y_1 = 2(1+\epsilon)$ and $Y_2 = 5+\frac{3}{2}\lambda+3\epsilon+\frac{1}{2}\epsilon\lambda$ the generating function is found to be $G^{(3)}(\lambda, u) = Y(\lambda, 2u)$ where

$$Y(\lambda, u) = \frac{uf(\lambda, u) + (4 - u^2\lambda^2)\,(1 - \lambda u - \epsilon(1-4u))\,c^*(u)}{u\,(2\,(\lambda-2) - u\lambda^2)\,(\lambda - 1 - 2u\lambda - u^2\lambda^2)} \tag{22}$$

and

$$f(\lambda, u) = 2\,(\lambda\,(\lambda-3) + 2\epsilon) - \lambda u\,(\lambda\,(\lambda-1) - 4\,(\lambda-3)\;\epsilon) + \lambda^2 u^2\,(\lambda\;(1-\epsilon) - 4\epsilon) \tag{23}$$

Notice that one of the quadratic denominator factors of (22) is the same as that for two walkers and we may therefore use (15) to replace it by a factor which is linear in λ. A similar replacement is possible for the other factor and we find

$$Y(\lambda, u) = \frac{\epsilon u(2+\lambda u) + (2 - \lambda u - \epsilon(2 - 8u + \lambda u - \lambda u^2))c^*(u)}{2u(1-\lambda u c^*(u))(1 - \lambda u(1+c^*(u))/2)} \tag{24}$$

where the numerator has also been reduced in degree from cubic to linear. Now [3]

$$(1+c^*(u))^j = \sum_{s=0}^{\infty} B_{2s+j-1,j-1} u^s \tag{25}$$

and expanding (24) directly in powers of λ and u would give an expression for $z^{(3)}_{t,i}$ involving products of Ballot numbers. However first rewriting $Y(\lambda,u)$ in the form

$$Y(\lambda,u) = Y_1(\lambda,u) + Y_2(\lambda,u) \tag{26}$$

where

$$Y_1(\lambda,u) = \frac{2(1-\epsilon) + 8\epsilon\, u - (1+\epsilon-3\epsilon\, u)u\lambda + \epsilon\, u^3\lambda^2}{u^2\lambda(1-\lambda u)(1-\lambda u c^*(u))} \tag{27}$$

and

$$Y_2(\lambda,u) = \frac{-2(1-\epsilon) - 8\epsilon\, u + 2(1+\epsilon\, u)u\lambda - \frac{1}{2}(1+\epsilon)u^2\lambda^2}{u^2\lambda(1-\lambda u)(1-\lambda u(1+c^*(u))/2)} \tag{28}$$

produces a linear combination of Ballot numbers. Thus defining

$$X_{i,1}(u) \equiv u^i \sum_{j=0}^{i} c^*(u)^j \qquad \text{and} \qquad X_{i,2}(u) \equiv u^i \sum_{j=0}^{i} 2^{-j}(1+c^*(u))^j \tag{29}$$

the generating function for configurations with exactly i osculations is therefore

$$G^{(3)}_i(u) = Y_{i,1}(2u) + Y_{i,2}(2u) \tag{30}$$

where

$$Y_{i,1}(u) = \frac{2(1-\epsilon)+8\epsilon\, u}{u^2} X_{i+1,1}(u) - \frac{1+\epsilon-3\epsilon\, u}{u} X_{i,1}(u) + \epsilon\, u X_{i-1,1}(u) \tag{31}$$

and

$$Y_{i,2}(u) = \frac{-2(1-\epsilon)-8\epsilon\, u}{u^2} X_{i+1,2}(u) + \frac{2(1+\epsilon\, u)}{u} X_{i,2}(u) - \frac{1+\epsilon}{2} X_{i-1,2}(u) \tag{32}$$

and further defining

$$b_{t,i,1} \equiv \sum_{j=0}^{i} B_{2t-2i-1,2j-1} \qquad \text{and} \qquad b_{t,i,2} \equiv \sum_{j=0}^{i} 2^{-j} B_{2t-2i+j-1,j-1} \tag{33}$$

gives the number of $t-$step configurations with i osculations as

$$z^{(3)}_{t,i} = 2^t(y_{t,i,1} + y_{t,i,2}) \tag{34}$$

where

$$y_{t,i,1} = 2(1-\epsilon)b_{t+2,i+1,1} + 8\epsilon\, b_{t+1,i+1,1} - (1+\epsilon)b_{t+1,i,1} + 3\epsilon\, b_{t,i,1} + \epsilon\, b_{t-1,i-1,1} \tag{35}$$

and

$$y_{t,i,2} = -2(1-\epsilon)b_{t+2,i+1,2} - 8\epsilon\, b_{t+1,i+1,2} + 2b_{t+1,i,2} + 2\epsilon\, b_{t,i,2} - \frac{1+\epsilon}{2} b_{t,i-1,2} \tag{36}$$

When $i = 0$ this collapses to the simpler formula for vicious walker configurations

$$z^{(3)}_{t,0} = 2^t C_{t+1} + \epsilon\, 2^t (4C_t - C_{t+1}) \tag{37}$$

Setting $\lambda = 1$ in (22) gives the total number of osculating configurations

$$G^{(3)}(1,u) = \frac{2u(u-1+\epsilon(1-5u)) + (1-u)(1-2u-\epsilon(1-8u))c^*(2u)}{4u^2(1+u)} \tag{38}$$

$$= \frac{\phi(u) - (1-u)(1-2u-\epsilon(1-8u))\sqrt{1-8u}}{16u^3(1+u)} \tag{39}$$

where $\phi(u) = (1-u)(1-6u) - \epsilon(1-13u+36u^2+8u^3)$. This agrees with the result of [6] when $\epsilon = 1$.

4 Critical behaviour

(a)$\lambda < 4$

Notice that $c^*(u)$ increases monotonically from 0 to 1 as u goes from 0 to the singular point, $u = \frac{1}{4}$, of $c^*(u)$ so that in this region of λ and u the denominators of both two and three walk functions are strictly positive. As $u \to \frac{1}{4}$ from below

$$G^{(2)}(\lambda,u) = \frac{4}{4-\lambda}\left[\frac{2}{\sqrt{1-4u}} + \epsilon - 2\frac{4+\lambda}{4-\lambda} + O((1-4u)^{\frac{1}{2}})\right] \tag{40}$$

$$G^{(3)}(\lambda,u) = \frac{4}{(4-\lambda)^2}\left[2(8-\lambda) + \epsilon(4-\lambda) - \frac{2(64-\lambda^2)\sqrt{1-8u}}{4-\lambda} + O(1-8u)\right]. \tag{41}$$

$$G^{(2)}_{\epsilon}(\lambda,u) = \frac{4}{4-\lambda}\left[1 - \frac{8\sqrt{1-4u}}{4-\lambda} + O(1-4u)\right] \tag{42}$$

$$G^{(3)}_{\epsilon}(\lambda,u) = \frac{4}{4-\lambda}\left[1 - (64-4\lambda-\lambda^2)v^2 + 8(64-\lambda^2)v^3 + O(v^4)\right] \tag{43}$$

where $v = (1-8u)^{\frac{1}{2}}/(4-\lambda)$.

(b)$\lambda > 4$

If and only if $\lambda > 4$, the denominator of $G^{(2)}(\lambda, u)$ has a simple zero at

$$u_c^{(2)}(\lambda) = 1/\mu^{(2)}(\lambda) = \frac{\sqrt{\lambda}-1}{\lambda} < \frac{1}{4} \qquad (44)$$

and therefore the dominant singularity of $G^{(2)}(\lambda, u)$ is a simple pole at this position.

Notice that $Y(u)$ also has a pole at this position which means that there is a pole in $G^{(3)}(\lambda, u)$ at $\frac{1}{2}u_c^{(2)}(\lambda)$. However there is a second pole which is closer to the origin arising from the other denominator in (24). The dominant singularity of $G^{(3)}(\lambda, u)$ is therefore a pole at

$$u_c^{(3)}(\lambda) = 1/\mu^{(3)}(\lambda) = \frac{\lambda-2}{\lambda^2} \qquad (45)$$

(c) $\lambda = 4$

Setting $\lambda = 4$ in (16) gives generating function for two walks at the collapse transition as

$$\begin{aligned} G^{(2)}(4,u) &= \frac{1}{1-4u} + \frac{(\epsilon-1)(4u - c^*(u))}{(1-4u)(3+4u)} &(46)\\ &= \frac{1}{1-4u} + \frac{\epsilon-1}{2u(3+4u)}\left[\frac{1}{\sqrt{1-4u}} - 1 - 2u\right] &(47)\end{aligned}$$

Notice that when $\epsilon = 1$ the generating function is that for independent walkers. This is the case for any number of walkers as explained earlier. When $\epsilon \neq 1$ there is a confluent singularity with $\gamma = \frac{1}{2}$ which is the dominant singularity of $G_\epsilon^{(2)}(4, u)$.

Similar behaviour is found for three walkers except that $u_c = \frac{1}{8}$ instead of $\frac{1}{4}$.

$$\begin{aligned} G^{(3)}(4,u) &= \frac{1}{1-8u} + \frac{(\epsilon-1)(2u(1+16u) - (1-16u^2)c^*(2u))}{2u(1-8u)(3+8u)} &(48)\\ &= \frac{1}{1-8u} + \frac{\epsilon-1}{8u^2(3+8u)}\left[\frac{1-16u^2}{\sqrt{1-8u}} - 1 - 4u - 8u^2\right] &(49)\end{aligned}$$

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