SWITCHING NETWORKS FOR BIDIRECTIONAL TELEPHONE SYSTEMS

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Abstract

We consider two-stage bidirectional switching networks which have a minimum number of switches. Results on the structure of such networks in terms of the number of switches per crosswire are established.

1. *Introduction*

Newbury and Raby [2] have considered two-stage switching arrangements in bidirectional telephone systems. In this kind of network each subscriber line is connected via switches to some (or all) of a set of crosswires. Thus a connection between two subscribers is made along one crosswire via two switches. We call such an arrangement of subscriber lines, crosswires and switches connecting them, a two-stage bidirectional switching network.

Such a network may be represented as an incidence structure (P,B,S) where $P = \{ P_1, \ldots, P_m \}$ is a set labelling the subscriber lines, $B = \{ x_1, \ldots, x_b \}$ is a set labelling the crosswires and $S \subseteq P \times B$ with $(P_i, x_j) \in S$ if and only if subscriber line P_i is connected to crosswire x_j by a switch. We only consider networks in which each crosswire has at least two switches on it.

Since pairs of subscriber lines must be connected via distinct crosswires and we would like to be able to connect as many subscribers in pairs as possible, we also require that $b \geq [m/2]$. If $m = 2n + \delta$, where δ is 0 or 1, and $b = n + t$, where $t \geq 0$, we use $2BSN(n,t,\delta)$ to denote such a two-stage bidirectional switching network. The two-stage bidirectional switching networks with 2n subscriber lines

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and n crosswires denoted by 2BSN(n) in [1] are here denoted by 2BSN(n,0,0).

Of interest is the minimum number of switches required to ensure that for any n disjoint pairs of subscriber lines there is an assignment of distinct crosswires to the pairs such that each pair may be connected via the crosswire assigned to it. A 2BSN(n,t,δ) with this property is called rearrangeably non-blocking (or rearrangeable) and is denoted R2BSN(n,t,δ). In this paper we consider R2BSN(n,t,δ) with $t = 0$ and $t = 1$, and show that the minimum number of switches requirement puts strong conditions on the number of switches that the individual crosswires may contain.

2. *R2BSN(n,0,δ)*

We extend the results of [1] to cover the case δ

LEMMA 2.1. *Let* D = (P,B,S) *be an R2BSN(n,0,δ). Then for each crosswire in* B *there are at most n-1 subscriber lines not connected to it by a switch.*

Proof. Suppose $x_j \in B$ is not connected to n subscriber lines. Without loss of generality we may assume these are $P_1, \ldots, P_n$. Then $(P_1,P_{n+1}), \ldots, (P_n,P_{2n})$ is a collection of n disjoint pairs of subscriber lines such that crosswire x_j does not connect any of them. Since B contains only n crosswires these n disjoint pairs cannot be connected via distinct crosswires. But D is rearrangeable, so no such x_j exists. □

LEMMA 2.2. *Let* D = (P,B,S) *be an R2BSN(n,0,δ). Then there is at most one crosswire in* B *which is not connected to n-1 subscriber lines by a switch.*

Proof. Suppose there are two such crosswires, x_1,x_2 say Without loss of generality we may assume that x_1 is not connected to $P_1, \ldots, P_{n-1}$ and x_2 is not connected to $P_1, \ldots, P_s, P_n, \ldots, P_{2n-2-s}$, where $0 \leq s \leq n-1$. Then

$\{P_1, P_{2n-1-s}\}$, ..., $\{P_s, P_{2n-2}\}$ and $\{P_{s+1}, P_n\}$, ..., $\{P_{n-1}, P_{2n-2-s}\}$ form a collection of n-1 disjoint pairs of subscriber lines such that neither x_1 nor x_2 connects any of them. Since B contains only n crosswires these disjoint pairs cannot be connected via distinct crosswires. But D is rearrangeable, and so there can exist at most one such crosswire. □

PROPOSITION 2.3. *Let* D = (P,B,S) *be an* $R2BSN(n,0,\delta)$. *Then* $|S| \geq n^2 + (2+\delta)n - 1$. *If equality holds then exactly one crosswire contains* $n + 1 + \delta$ *switches, and the other* $n-1$ *crosswires each contain* $n + 2 + \delta$ *switches.*

Proof. By Lemma 2.1 each crosswire contains at least $2n + \delta - (n-1) = n + 1 + \delta$ switches, and by Lemma 2.2 at most one has exactly $n + 1 + \delta$ switches. Thus
$|S| \geq (n+1+\delta) + (n-1)(n+2+\delta) = n^2 + (2+\delta)n - 1$. □

In [1] an R2BSN(n,0,0) with $n^2 + 2n - 1$ switches is constructed for every n. The following result shows that there also exist R2BSN(n,0,1) with $n^2 + 3n - 1$ switches for every n.

PROPOSITION 2.4. *Let* D = (P,B,S) *be an* R2BSN(n,0,0). *Let* $P' = P \cup \{P_{2n+1}\}$, $B' = B$, *and* $S' = S \cup \{(P_{2n+1}, x_j) : x_j \in B\}$. *Then* D' = (P',B',S') *is an* R2BSN(n,0,1) *with* $|S'| = n^2+3n-1$.

Proof. Let A_1, .., A_n be n disjoint pairs of subscriber lines in P'. First suppose $P_{2n+1} \notin A_1 \cup \ldots \cup A_n$. Then A_1, ..., A_n are n disjoint pairs of subscriber lines in P and so there are n distinct crosswires in B = B' which connect them.

Now suppose $P_{2n+1} \in A_1 \cup \ldots \cup A_n$. Without loss of generality we may assume that $A_1 = \{P_1, P_{2n+1}\}$ and that $P' - (A_1 \cup \ldots \cup A_n) = \{P_{2n}\}$. Put $A_1' = \{P_1, P_{2n}\}$. Then A_1', A_2, ..., A_n are n disjoint pairs of subscriber lines in P. Hence they are connected via distinct crosswires. Since the crosswire connecting P_1 and P_{2n} also connects P_1 and

P_{2n+1}, the n disjoint pairs $A_1, \ldots, A_n$ are also connected via distinct crosswires. It follows that D' is rearrangeable and so is an R2BSN(n,0,1). □

3. *R2BSN(n,1,δ)*

We consider the structure of an R2BSN(n,1,δ) having at most $n^2 + (2+\delta)n - 1$ switches, i.e. the minimum number of switches for an R2BSN(n,0,δ).

LEMMA 3.1. *Let* D = (P,B,S) *be an R2BSN(n,1,δ). Then there is at most one crosswire in* B *which is not connected to n subscriber lines by a switch.*

Proof. Suppose there are two such crosswires. Without loss of generality we may suppose that x_1 is not connected to $P_1, \ldots, P_n$ and x_2 is not connected to $P_1, \ldots, P_s, P_{n+1}, \ldots, P_{2n-s}$ where $0 \le s \le n$. Then $\{P_1,P_{2n+1-s}\}, \ldots, \{P_s,P_{2n}\}$ and $\{P_{s+1},P_{n+1}\}, \ldots, \{P_n,P_{2n-s}\}$ are n disjoint pairs of subscriber lines none of which are connected via x_1 or x_2. Since B contains only n+1 crosswires, these n disjoint pairs cannot be connected via distinct crosswires. But D is rearrangeable, so there is at most one such crosswire. □

An R2BSN(n,1,δ) in which every crosswire contains at least n+1+δ switches has at least $(n+1)(n+1+\delta) = n^2 + (2+\delta)n + 1+\delta$ switches. Thus, in an R2BSN(n,1,δ) with at most $n^2 + (2+\delta)n - 1$ switches, some crosswire, x_1 say, is connected to $s \le n-1$ subscriber lines. Moreover there must be at least s+1 other crosswires which contain exactly n+1+δ switches.

LEMMA 3.2. *Let* D = (P,B,S) *be an R2BSN(n,1,δ) with at most $n^2 + (2+\delta)n - 1$ switches. Suppose crosswire x_1 contains $s \le n-1$ switches, and two crosswires, each containing n+1+δ switches are connected to r common subscriber lines. Then $3+\delta \le r \le n+\delta$ and $s \ge \min(r+1+\delta, n+2+\delta-r)$.*

Proof. Put $u = n + 1 + \delta$. We note that at least $s+1 \geq 3$ crosswires contain u switches. Without loss of generality suppose that x_2 is connected to $P_1, \ldots, P_u$ and x_3 is connected to $P_1, \ldots, P_r, P_{u+1}, \ldots, P_{2u-r}$. We first show that $3 + \delta \leq r \leq n + \delta$.

As $2u-r = 2n + 2 + 2\delta - r \leq 2n + \delta$, we have $r \geq 2 + \delta$. If $r = 2 + \delta$ then for any permutation f of $\{ u+1, \ldots, 2n+\delta \}$ the n pairs $\{P_1,P_2\}$, $\{P_{3+\delta},P_{f(u+1)}\}$, .. $\{P_u,P_{f(2n+\delta)}\}$ are disjoint, and only the pair $\{P_1,P_2\}$ is connected via x_2 or x_3. Since D is rearrangeable the crosswire x_1 must connect one of the pairs $\{P_i,P_{f(n-1+i)}\}$. Since this is true for every f it follows that x_1 is connected to at least n of the subscriber lines $P_{3+\delta}, \ldots, P_{2n+\delta}$. But x_1 contains only $s \leq n-1$ switches. Hence $r \geq 3 + \delta$.

Now $r \leq u = n + 1 + \delta$. If $r = n + 1 + \delta$ then for any $(2+\delta)$-subset A of $\{1,\ldots,u\}$ and any bijective mapping $f : \{1,\ldots,u\}-A \rightarrow \{u+1,\ldots,2n+\delta\}$ the pairs $\{P_i,P_j\}$, $i,j \in A$, and $\{P_s,P_{f(s)}\}$, $s \in \{1,\ldots,u\}-A$, are disjoint, and only the pair $\{P_i,P_j\}$ is connected via x_2 or x_3. It follows that x_1 connects one of the pairs $\{P_s,P_{f(s)}\}$ for every A and f, and hence contains at least $n + 2 + \delta$ switches. But x_1 contains only $s \leq n-1$ switches and hence $r \leq n + \delta$.

Now suppose $3 + \delta \leq r \leq n + \delta$. Then for any $(2+\delta)$-subset A of $\{1,\ldots,r\}$ and bijective mappings $f : \{1,\ldots,r\}-A \rightarrow \{2u-r+1,\ldots,2n+\delta\}$ and $g : \{r+1,\ldots,u\} \rightarrow \{u+1,\ldots,2u-r\}$, the n pairs $\{P_i,P_j\}$, $i,j \in A$, $\{P_s,P_{f(s)}\}$, $s \in \{1,\ldots,r\}-A$, and $\{P_s,P_{g(s)}\}$, $s \in \{r+1,\ldots,u\}$, are disjoint, and only the pair $\{P_i,P_j\}$ are connected via x_2 or x_3. Hence for every A, f and g, the crosswire x_1 connects one of the pairs $\{P_s,P_{f(s)}\}$ or one of the pairs $\{P_s,P_{g(s)}\}$. It follows that x_1 is either connected to at least $r+1+\delta$ of the subscriber lines $P_1, \ldots, P_r, P_{2u-r+1}, \ldots, P_{2n+\delta}$, or to at least

$u-r+1 = n+2+\delta-r$ of $P_{r+1}, \ldots, P_{2u-r}$. Thus $s \geq \min(r+1+\delta, n+2+\delta-r)$. □

Lemma 3.2 says that in an R2BSN$(n,1,\delta)$ with at most $n^2 + (2+\delta)n - 1$ switches, in which x_1 contains $s \leq n-1$ switches, any two of the s+1 or more crosswires which contain $n+1+\delta$ switches are connected either to at most $s-1$ common subscriber lines or to at least $n+2+\delta-s$ common subscriber lines. An exhaustive search has shown that no such R2BSN$(n,1,\delta)$ exists for $n \leq 6$. The following questions remain to be answered:

1. Does there exist an R2BSN$(n,1,\delta)$ with fewer switches than the minimal R2BSN$(n,0,\delta)$ (which contains $n^2 + (2+\delta)n - 1$ switches)?

2. Does an R2BSN(n,t,δ) with the minimum number of switches have $t = 0$?

References

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