A Characterisation of Anti-Löwner Functions

Koenraad M.R. Audenaert
Department of Mathematics,
Royal Holloway, University of London,
Egham TW20 0EX, United Kingdom

April 28, 2011

Abstract

According to a celebrated result by Löwner, a real-valued function f is operator monotone if and only if its Löwner matrix, which is the matrix of divided differences $L_f = \left(\frac{f(x_i)-f(x_j)}{x_i-x_j}\right)_{i,j=1}^N$, is positive semidefinite for every integer $N > 0$ and any choice of $x_1, x_2, \ldots, x_N$. In this paper we answer a question of R. Bhatia, who asked for a characterisation of real-valued functions g defined on $(0, +\infty)$ for which the matrix of divided sums $K_g = \left(\frac{g(x_i)+g(x_j)}{x_i+x_j}\right)_{i,j=1}^N$, which we call its anti-Löwner matrix, is positive semidefinite for every integer $N > 0$ and any choice of $x_1, x_2, \ldots, x_N \in (0, +\infty)$. Such functions, which we call anti-Löwner functions, have applications in the theory of Lyapunov-type equations.

1 Introduction

A real-valued function defined on an interval (a, b) is called *matrix monotone of order* N if for any pair A, B of $N \times N$ Hermitian matrices with spectrum in (a, b) the implication $A \leq B \Longrightarrow f(A) \leq f(B)$ holds, i.e. f preserves the positive semidefinite ordering. A function is called *operator monotone* if it is matrix monotone of every order.

One of the central objects in the theory of matrix monotone functions is the so-called Löwner matrix. Given any integer $N > 1$, and any set of N finite, distinct real numbers x_i in (a, b), one constructs a Löwner matrix of f as the $N \times N$ matrix L_f of divided differences

$$L_f := \left(\frac{f(x_i) - f(x_j)}{x_i - x_j}\right)_{i,j=1}^N.$$

For the diagonal elements, $i = j$, a limit has to be taken, so that the diagonal elements are given by the first derivatives $f'(x_i)$. (A necessary condition for f being matrix monotone of order at least 2 is that it should be continuous, in fact even continuously differentiable ([3] p. 79), hence its first derivative should exist. For $N = 1$, this is strictly speaking not needed, but matrix monotonicity then reduces to ordinary monotonicity anyway.) According to a celebrated result by Löwner, f is a matrix monotone function on (a, b) of order N if and only if any $N \times N$ Löwner matrix L_f is positive semidefinite, for any choice of x_i in (a, b). For a thorough introduction to matrix monotone functions we refer to the monograph [3], and to [1] for a more concise introduction.

In ([2], p. 195) R. Bhatia raised the question whether there is a good characterisation of real-valued functions $g(x)$ defined on (a, b), with $a \geq 0$, for which every matrix of the form

$$K_g := \left(\frac{g(x_i) + g(x_j)}{x_i + x_j} \right)_{i,j=1}^{N},$$

is positive semidefinite, with x_i distinct real numbers in (a, b). That is, K_g is akin to a Löwner matrix, but has the minus signs replaced by plus signs. In this paper, we'll call these matrices *anti-Löwner matrices*, and functions for which all $N \times N$ anti-Löwner matrices are positive semidefinite will be called *anti-Löwner functions of order* N. Likewise, we call functions *anti-Löwner functions* if they satisfy this positivity criterion for all values of N.

It goes without saying that to be anti-Löwner g must first of all be non-negative, as can be seen from the trivial case $N = 1$. For $N = 1$ this is already the complete answer; to avoid trivialities we will henceforth assume $N \geq 2$. It is also straightforward to show that for $N \geq 2$, g must be continuous, similar to matrix monotone functions of order $N \geq 2$; see Proposition 1 below.

It has already been known for some time that every non-negative operator monotone function on $(0, +\infty)$ is an anti-Löwner function, and so is every non-negative operator monotone decreasing function [5]. This easily follows (see, e.g. Theorem 1 in [5]) from exploiting the well-known integral representation [1]

$$f(x) = \alpha + \beta x + \int_0^\infty \frac{x}{t + x} d\mu(t) \tag{1}$$

for non-negative operator monotone functions on $(0, +\infty)$, with $\alpha, \beta \geq 0$[1] and μ a positive Borel measure on $(0, +\infty)$ such that the given integral converges. The statement for non-negative operator monotone decreasing functions follows easily from this by noting that the function $f(x)$ is anti-Löwner if and only if $1/f(x)$ is anti-Löwner too.

2 Main results

In this paper, we obtain a complete answer to Bhatia's question:

Theorem 2.1 *Let g be a real-valued function g, defined and finite on (a, b), with $0 \leq a < b$. Let N be any integer, at least 2. If g is an anti-Löwner function*

[1] Note that for negative α, f is also operator monotone, but positivity of f requires $\alpha \geq 0$.

of order $2N$ on (a,b) then $x \mapsto g(\sqrt{x})\sqrt{x}$ is a non-negative matrix monotone function of order N on (a^2,b^2), and $x \mapsto g(\sqrt{x})/\sqrt{x}$ is a non-negative matrix monotone decreasing function of order N on (a^2,b^2).

Theorem 2.1 has applications in the study of Lyapunov-type equations and also answers a question by Kwong [5], who studied conditions on the function g such that the solution X of equation $AX + XA = g(A)B + Bg(A)$ is positive definite for all positive definite A and B. Kwong pointed out in [4] that it suffices to consider matrices A that are diagonal, with B equal to the all-ones matrix ($B_{ij} = 1$). In that case the solution of the equation reduces to X being an anti-Löwner matrix with the x_i equal to the diagonal elements of A. Thus, our Theorem 2.1 also yields the answer to Kwong's question.

As a direct corollary of Theorem 2.1, we immediately get an integral representation for anti-Löwner functions $g(x)$ (of all orders) on $(0,+\infty)$. From equation (1) we obtain $g(\sqrt{x}) = \alpha/\sqrt{x} + \beta\sqrt{x} + \int_0^\infty \frac{\sqrt{x}}{t+x} d\mu(t)$, hence

$$g(x) = \frac{\alpha}{x} + \beta x + \int_0^\infty \frac{x}{t+x^2} d\mu(t), \tag{2}$$

with $\alpha, \beta \geq 0$ and μ a positive Borel measure such that the integral exists.

Within this restricted setting, the sufficiency part of Theorem 2.1 is easy to prove, as it suffices to check each of the terms in the integral representation (2). To wit, one only needs to prove that the functions $g(x) = x$ and $g(x) = x/(t+x^2)$ (for $t \geq 0$) are anti-Löwner, as all other functions concerned are positive linear combinations of these extremal functions. This is trivial for $g(x) = x$, because then $K_g = (1)_{i,j}$, which is clearly positive semidefinite (and rank 1). Secondly, for $g(x) = x/(t+x^2)$, we have

$$\begin{aligned}
K_g &= \left(\frac{x_i/(t+x_i^2) + x_j/(t+x_j^2)}{x_i + x_j} \right)_{i,j} \\
&= \left(\frac{x_i(t+x_j^2) + x_j(t+x_i^2)}{(t+x_i^2)(x_i+x_j)(t+x_j^2)} \right)_{i,j} \\
&= \left(\frac{t + x_i x_j}{(t+x_i^2)(t+x_j^2)} \right)_{i,j},
\end{aligned}$$

which is congruent to the matrix $(t + x_i x_j)_{i,j}$ and therefore positive semidefinite as well (and in general rank 2).

The hard part is to prove necessity, i.e. that there are no other anti-Löwner functions than those with the given integral representation (2). Furthermore, there seems to be no obvious approach even to the sufficiency part in the more general setting of fixed N where no integral representation is known. An important observation that shows the way out, however, is hidden in the very statement of Theorem 2.1, as it hints at a one-to-one correspondence between anti-Löwner functions and non-negative operator monotone functions. This is no coincidence, and our method of proof will exploit an even deeper correspondence between Löwner matrices and anti-Löwner matrices, which is made apparent in Theorem 2.2 below. This is good news, as there will be no need to develop from scratch a completely new theory in parallel with Löwner's.

Theorem 2.2 *Let N be any integer and $x_1, \ldots, x_N$ a sequence of distinct positive real numbers contained in the interval (a, b), $0 \leq a < b$. For any continuous real-valued function g defined on (a, b), let L and K be its Löwner and anti-Löwner matrix of order N on the given points $x_1, \ldots, x_N$, respectively, and let K_ij be the matrix*

$$K_{ij} = \left[\frac{g(x_k + i\epsilon) + g(x_l + j\epsilon)}{(x_k + i\epsilon) + (x_l + j\epsilon)} \right]_{k,l=1}^{N}$$

and let L_{ij} be the matrix

$$L_{ij} = \left[\frac{g(x_k + i\epsilon) - g(x_l + j\epsilon)}{(x_k + i\epsilon) - (x_l + j\epsilon)} \right]_{k,l=1}^{N}.$$

Then the following are equivalent:

1. *the 2×2 block matrix $\begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$ is positive semidefinite;*
2. *the 2×2 block matrix $\begin{pmatrix} K_{00} & L_{01} \\ L_{10} & K_{11} \end{pmatrix}$ is positive semidefinite.*

In the remainder of this paper we present the proofs of these theorems.

3 Proofs

We start with a simple, but nevertheless essential proposition.

Proposition 1 *Let g be a positive real-valued function on (a, b), with $0 \leq a < b$. If g is an anti-Löwner function of order at least 2, then g is continuous.*

Proof. This follows from consideration of the 2×2 anti-Löwner matrices in the points $x_1 = x, x_2 = x + \epsilon$ $(0 \leq a < x < b)$ and letting ϵ tend to 0. Positive semidefiniteness of the anti-Löwner matrix requires non-negativity of its determinant:

$$g(x)g(x + \epsilon)/x(x + \epsilon) - (g(x) + g(x + \epsilon))^2/(2x + \epsilon)^2 \geq 0.$$

After some calculation, one finds that this requires $|(g(x+\epsilon)-g(x))/\epsilon| \leq g(x)/x$ for all $\epsilon > 0$, whence the derivative of g should exist and be bounded on any bounded closed interval in (a, b). □

The main technical result on which our proof is based is the following proposition.

Proposition 2 *Fix an integer N. Let $g = (g_1, \ldots, g_N)$ and $x = (x_1, \ldots, x_N)$ be positive vectors, where all x_i are distinct, and $s = (s_1, \ldots, s_N)$ a real vector with $s_i = \pm 1$. Then the sign of* $\det Z_N$*, where*

$$Z_N = \left(\frac{s_i g_i + s_j g_j}{s_i x_i + s_j x_j} \right)_{i,j=1}^{N},$$

is independent of the signs of the s_i's.

As an illustration of this proposition, we will prove the easiest non-trivial case $N = 2$ (the case $N = 1$ is trivial as s_1 cancels out entirely). The given determinant is

$$\begin{aligned}
\det Z_2 &= \det \begin{pmatrix} \frac{g_1}{x_1} & \frac{s_1 g_1 + s_2 g_2}{s_1 x_1 + s_2 x_2} \\ \frac{s_1 g_1 + s_2 g_2}{s_1 x_1 + s_2 x_2} & \frac{g_2}{x_2} \end{pmatrix} \\
&= \frac{g_1 g_2}{x_1 x_2} - \frac{(s_1 g_1 + s_2 g_2)^2}{(s_1 x_1 + s_2 x_2)^2} \\
&= \frac{g_1 g_2 (s_1^2 x_1^2 + s_2^2 x_2^2) - x_1 x_2 (s_1^2 g_1^2 + s_2^2 g_2^2)}{x_1 x_2 (s_1 x_1 + s_2 x_2)^2} \\
&= \frac{g_1 g_2 (x_1^2 + x_2^2) - x_1 x_2 (g_1^2 + g_2^2)}{x_1 x_2 (s_1 x_1 + s_2 x_2)^2}.
\end{aligned}$$

One sees that the numerator is independent of the signs of the s_i's, while the denominator is always positive. Hence, the sign of this determinant is independent of the signs of the s_i's.

For small values of N one can easily verify that the determinant can always be written as a rational function where the numerator is a polynomial in which the s_i's only appear to even powers, and where the denominator is always positive. This observation provided the inspiration for the following simple proof of Proposition 2 (for every value of N).

Proof of Proposition 2.
Clearly, once we prove that the sign of $\det Z_N$ does not change under a single sign change of s_i, the general statement of the proposition follows, by changing the signs of the s_i's one by one. W.l.o.g. we consider sign changes of s_1 only.

The idea of the proof is to apply a *partial* Gaussian elimination on Z_N, only bringing its first column in upper-triangular form. For each $i > 1$ we subtract $\frac{x_1}{g_1} \frac{s_1 g_1 + g_i}{s_1 x_1 + x_i}$ times row 1 from row i. As is well-known, this operation does not change the determinant. The resulting matrix is of the form

$$Z'_N = \begin{pmatrix} \frac{g_1}{x_1} & b \\ 0 & X \end{pmatrix}$$

where b is the first row of Z_N (except its element $(1,1)$) and X is an $(N-1) \times (N-1)$ matrix with elements $(i, j > 1)$

$$\begin{aligned}
X_{i,j} &= \frac{g_i + g_j}{x_i + x_j} - \frac{x_1}{g_1} \frac{s_1 g_1 + g_i}{s_1 x_1 + x_i} \frac{s_1 g_1 + g_j}{s_1 x_1 + x_j} \\
&= \frac{g_1 (g_i + g_j)(x_1^2 + x_i x_j) - x_1 (x_i + x_j)(g_1^2 + g_i g_j)}{g_1 (x_i + x_j)(s_1 x_1 + x_i)(s_1 x_1 + x_j)}.
\end{aligned} \tag{3}$$

In the last line we have used the fact that $s_1^2 = 1$.

From expression (3) it is clear that X can be written as a matrix product $X = DYD$, where D is a diagonal matrix with diagonal elements $1/(s_1 x_1 + x_i)$ $(i > 1)$, and where Y is independent of s_1. It follows that the determinant of Z_N is given by

$$\det Z_N = \frac{g_1}{x_1} \det(DYD) = \frac{g_1}{x_1} \det(Y) \det(D)^2.$$

As the only factor that depends on the sign of s_1 appears to even power, and is therefore non-negative, we have proven that the sign of $\det Z_N$ does not depend on the sign of s_1.

In a similar way, we can show that the sign of $\det Z_N$ does not depend on any of the signs of the s_i's. This ends the proof. □

Proof of Theorem 2.2. It is a simple corollary of Proposition 2 that, under the conditions stated, the positive semidefiniteness of Z_N for a given choice of signs of the s_i's implies positive semidefiniteness for any other choice. Indeed, according to Sylvester's criterion, a symmetric matrix is positive semidefinite if and only if all its principal minors are non-negative. In the case of Z_N, the principal $k \times k$ minors are determinants of the form $\det Z_k$ $(k = 1, \ldots, N)$ and according to Proposition 2, the signs of these determinants are independent of the signs of the s_i's appearing in them.

Consider now, in particular, the case where N is even, say $N = 2n$, and the x_i are given by

$$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) = (y_1, \ldots, y_n, y_1 + \epsilon, \ldots, y_n + \epsilon)$$

for any positive ϵ small enough such that no two x_i ever become equal when ϵ tends to 0. Let also $g_i = g(x_i)$.

We will consider two choices for the s_i. Firstly, we set all $s_i = +1$. We then get the matrix

$$K' = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}.$$

Secondly, with $s_i = +1$ for $i \leq n$ and $s_i = -1$ for $i > n$, we instead get

$$K'' = \begin{pmatrix} K_{00} & L_{01} \\ L_{10} & K_{11} \end{pmatrix}.$$

As, according to Proposition 2, these matrices have the same signature (same signs of the corresponding principal minors) this yields the equivalence of Theorem 2.2. □

It is now an easy matter to prove Theorem 2.1.

Proof of Theorem 2.1.

Let N be a fixed integer, at least 2. By Proposition 1, if g is an anti-Löwner function of order at least 2, then g is continuous. Conversely, if the function

$x \mapsto g(\sqrt{x})\sqrt{x}$ is a non-negative matrix monotone function of order N, then surely g must be continuous too. Thus, in any case, Theorem 2.2 applies to g.

Let g be an anti-Löwner function of order $2N$. Thus K' is positive. In the limit $\epsilon \to 0$ we then find that $K_g + L_g$ and $K_g - L_g$ are positive, due to Theorem 2.2. A simple calculation shows that $K_g + L_g$ is equal to

$$\begin{aligned} K_g + L_g &= \left(\frac{g(x_i) + g(x_j)}{x_i + x_j} + \frac{g(x_i) - g(x_j)}{x_i - x_j} \right)_{i,j=1}^{N} \\ &= 2 \left(\frac{x_i g(x_i) - x_j g(x_j)}{x_i^2 - x_j^2} \right)_{i,j=1}^{N}, \end{aligned}$$

which is (up to an irrelevant factor of 2) the Löwner matrix of the function $x \mapsto g(\sqrt{x})\sqrt{x}$ in the points x_i^2. Hence, the function $x \mapsto g(\sqrt{x})\sqrt{x}$ is a non-negative matrix monotone function of order N on (a^2, b^2).

In a similar way we find

$$K_g - L_g = -2 \left(x_i \, \frac{g(x_i)/x_i - g(x_j)/x_j}{x_i^2 - x_j^2} \, x_j \right)_{i,j=1}^{N},$$

which shows that the function $x \mapsto g(\sqrt{x})/\sqrt{x}$ is a non-negative matrix monotone *decreasing* function of order N on (a^2, b^2). □

Acknowledgments

The author gratefully acknowledges the hospitality of the University of Ulm, Germany, and of the Institut Mittag-Leffler, Stockholm, where parts of this work have been done. Also many thanks to R. Bhatia and F. Hiai, for comments on an earlier version of the manuscript.

References

[1] R. Bhatia, *Matrix Analysis*, Springer, Heidelberg (1997).

[2] R. Bhatia, *Positive Definite Matrices*, Princeton University Press, Princeton (2007).

[3] W.F. Donoghue, Jr., *Monotone matrix functions and analytic continuation*, Springer, Heidelberg (1974).

[4] M.K. Kwong, "On the definiteness of the solution of a certain matrix equation", Lin. Alg. Appl. **108**, 177–198 (1988).

[5] M.K. Kwong, "Some results on matrix monotone functions", Lin. Alg. Appl. **118**, 129–153 (1989).