Breakdown of the Chiral Luttinger Liquid in One Dimension

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We have developed a fermionic boot-strap method to solve a class of chiral one-dimensional fermion
models. Using this scheme, we show that Luttinger liquid behavior in a gas of four interacting chiral
Majorana fermions is highly sensitive to the velocity degeneracy. Upon changing the velocity of one
chiral fermion, a sharp bound (or antibound) state splits off from the original Luttinger liquid continuum,
cutting off the x-ray singularity to form a broad incoherent excitation with a lifetime that grows linearly
with frequency.

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The anomalous normal state behavior discovered in the
cuprate superconductors has stimulated enormous interest
in the possibility of new kinds of electronic fluid that
might provide an alternative to Fermi liquid behavior. The
classic model for non-Fermi liquid behavior is provided
by the one-dimensional electron gas, where the generic
fixed point behavior is a Luttinger liquid [1]. This is
due to two special 1D features: a Fermi surface which
consists of just two points \( \pm k_F \) where the electrons interact
very strongly and, second, the special kinematics of one
dimension, whereby energy and momentum conservation
impose a single constraint on scattering processes near the
Fermi surface, giving rise to a qualitative enhancement in
scattering phase space. These two factors cause the
electron to lose its eigenstate status to the collective spin
and charge density bosonic modes.

In this Letter, we introduce a generalization of the
Tomonaga Luttinger model, written

\[
H = \int dx \left[ -i \sum_{a=0}^{3} v_a \Psi^{(a)}(x) \partial_x \Psi^{(a)}(x) + g \Psi^{(0)}(x) \Psi^{(1)}(x) \Psi^{(2)}(x) \Psi^{(3)}(x) \right],
\]

where the \( \Psi^{(a)} \) \( [a = (0, 1, 2, 3)] \) represent four real (Majorana) fermions such that \( \Psi^{(a)}(x) = \Psi^{(a)}(x) \). The
fermions are chiral (right movers, say): this is a crucial property that ensures the system stays gapless, and allows
for exact solutions in a number of cases. In the special case where all velocities are the same, this is the chiral
Luttinger model with a SO(4) symmetry, where the four
Majorana modes can be associated with the spin up and
down, electron and hole excitations of the Fermi surface.

Here, we focus on the particular case of the model (1)
where three of the velocities are equal: \( v_{1,2,3} = v \neq v_0 \),
giving rise to a Hamiltonian with a reduced SO(3) sym-
metry. We shall show that once we break the degener-
acy of the velocities, a qualitatively new type of behavior
develops. This model is physically motivated in two ways:

(i) The transport phenomenology of the cuprates [2]
suggests that electrons near the Fermi surface might di-
vide up into two Majorana modes with different scattering
rates and dispersion. To date, this kind of behavior has
been realized only in impurity models [3] and their in-
finite dimensional generalization [4]. We shall show that
by breaking the velocity degeneracy of the original chi-
ral Luttinger model, we obtain a one-dimensional realiza-
tion of this behavior: a sharp Majorana mode intimately
coexisting with an incoherent continuum of excitations.

(ii) Frahm et al. [5] have recently proposed that the low
energy Hamiltonian of an integrable spin-1 Heisenberg
chain doped with mobile spin-1/2 holes is given by (1),
with one Majorana fermion \( \Psi^{(0)} \) describing a slow mov-
ing excitation coming from the dopant, interacting with
three rapidly moving Majorana fermions that describe the
spin-1 excitations of the spin chain. Such doped spin-
chain models may be relevant to certain experimental sys-
tems such as \( \mathrm{Y}_2-x \mathrm{Ca}_x \mathrm{BaNiO}_3 \) [6].

Whereas the SO(4) model can be treated by bosoniza-
tion [1,7], by changing the velocity of a single Major-
a ferion we introduce a nonquadratic term into the
bosonized Hamiltonian that precludes a simple separation
in terms of Gaussian spin and charge bosons. We shall
show, using a new fermionic approach to the problem, that
when we break the degeneracy of the velocities, the x-ray
catastrophe associated with the Luttinger liquid behavior is
cut off by the velocity shift. The “hornlike” feature in the
spectral weight of the Luttinger liquid is then split into a
sharp bound (or antibound) state that coexists with an inco-
herent spin-charge decoupled continuum, with a scattering
rate linear in frequency. We summarize these results in
Fig. 1.

Our approach is based on the observation that for these
models, the renormalized skeleton self-energy, containing
full propagators, but no vertex corrections (Fig. 2), is
exact, so that

\[
\Sigma_{a}(x, \tau) = g^2 G_b(x, \tau) G_c(x, \tau) G_d(x, \tau),
\]

where the \( G_a \) are the exact, interacting Greens functions
and \( \{a, b, c, d\} \) is a cyclic permutation of \( \{0, 1, 2, 3\} \). These
equations close with the usual relations,

\[
\Sigma_{a}(k, \omega) = (i \omega - v_a k) - G_{a}(k, \omega)^{-1}
\]

\( (a = 0, 1, 2, 3) \).
Equations (2) and (3) together define a boot-strap method to solve the problem. Provided that we have a minimal SO(3) symmetry, then the three current densities \(j^a(x) = -i\epsilon_{abc}\Psi^{(b)}(x)\Psi^{(c)}(x)\) \([a, b, c \in (1, 2, 3)]\) are conserved classically. Following Dzyaloshinskii and Larkin (1974) \([8–10]\), the continuity equation guarantees that the \(N\) point connected current-current correlation functions vanish for \(N > 2\) \([x_i = (x_i, \tau_i)]\).

\[
\langle j^a(x_1)j^a(x_2)\cdots j^a(x_N) \rangle_C = 0, \quad (N > 2).
\]

(4)

For the noninteracting system, this result leads to the “loop cancellation theorem”: for the amplitude associated with a closed fermion loop with \(N > 2\) conserved current insertions, the sum over all possible permutations of \(\{x_i\}\) of the current operators must give zero \([8–10]\). In \(d = 1\), this result depends on having a linear or quadratic dispersion. In \(d > 1\), this is true only in the asymptotic limit of small momentum and energy transfer. Dzyaloshinskii and Larkin used this theorem to eliminate all diagrams that contain such closed loops, considerably simplifying the vertex function and polarization bubbles.

We use the loop cancellation theorem in a new way, to show that the vertex corrections to the skeleton self-energy (SSE) identically vanish. To illustrate the idea, consider the self-energy of the singlet Majorana mode in the SO(3) model. The Feynman diagrams contributing to the skeleton self-energy are constructed by combining loops with two current insertions. We illustrate this using the fourth order diagram in Fig. 3(iv), but it holds to all orders in perturbation theory. Nonskeleton contributions to the self-energy involve diagrams with loops containing more than two current insertions. In these diagrams, the sum over all permutations of the current insertions into the loops is automatically zero, as illustrated to order \(g^4\) in Figs. 3(i)–3(iii). By generalizing these results to all higher order graphs, we can show that the vertex corrections to the self-energy \(\Sigma_0\) of the singlet Majorana fermion cancel, leaving the fully renormalized SSE as the only remaining contribution. To complete our proof, we now use the full Kadanoff-Baym free energy functional \(F[G] = -T[\text{Tr} \ln G^{-1}] + Y[G]\), where \(Y[G]\) is the sum of all skeleton diagrams \([11]\). Now \(\delta F/G/\delta G_0 = 0\) generates the equations for the self-energies, and, in particular, \(\delta F[G]/\delta G_0\) must generate the skeleton self-energy \(\Sigma_0\). This requires that the Kadanoff-Baym free energy functional truncates at the leading skeleton diagram,

\[
F = -T[\text{Tr} \ln G^{-1}] + \text{Tr}[\Sigma G].
\]

(5)

Finally, by differentiating the free energy functional with respect to the Greens functions \(G_{1,2,3}\) of the triplet Majorana fermions, we can show that each triplet self-energy is also given by the corresponding skeleton self-energy.

We now apply this result, using the limiting case of the SO(4) model to check the validity of our results. Our equations are dramatically simplified by seeking solutions to (2) which satisfy a scaling form

\[
G_0(x, \tau) = \frac{1}{2\pi i x} G_0(\tau/ix). 
\]

(6)

This form is motivated by the observation that chirality prevents space from acquiring an anomalous dimension. Under a Fourier transform, this scaling form is self-dual,

\[
\frac{1}{2\pi i x} G_0(\tau/ix) \leftrightarrow_{F.T.} \frac{1}{i \omega} G_0(k/i\omega),
\]

(7)

\[\ldots(i)\leftrightarrow(ii)+(iii) = 0 \quad (\text{Loop Cancellation Theorem}).\]

FIG. 3. Order \(g^4\) diagrams for the SO(3) model, demonstrating how the closed-loop theorem works. Dotted lines indicate bare propagator for the singlet Majorana fermion \(\Psi^{(0)}\). Full lines indicate bare propagator for the triplet Majorana fermions. Bold line highlights the loop with 2 or 4 current insertions.
where the same function $G_a$ appears on both sides. Inserting Eq. (7) into (3) and Fourier transforming,

$$
\Sigma_a(x, \tau) = -\frac{1}{2\pi i x} \frac{d^2}{du^2} \left[ 1 - v_a u - 1/G_a(u) \right]_{u=\tau/\pi}.
$$

(8)

Since the bare Green function scaling form is $1/G_a^0(u) = 1 - v_a u$, it does not contribute to the self-energy. Combining Eqs. (2) and (8),

$$
\frac{d^2}{du^2} \left[ G_a(u) \right]^{-1} = - \left( g/2\pi \right)^2 G_b(u) G_c(u) G_d(u),
$$

(9)

where $\{a, b, c, d\}$ are cyclic permutations of $\{0, 1, 2, 3\}$. The boundary conditions are $G_a(0) = 1$ and $G_a'(0) = v_a$, derived from the physical requirement that at high frequencies, the fermions are free particles, moving with the bare velocity $v_a$. For the SO(4) model, where $G_a(u) = G(u)$ ($a = 0, \ldots, 3$), Eq. (9) reduces to a single differential equation, for which the solution is

$$
G(x, \tau) = \frac{1}{2\pi i x} \left[ 1 - v_+ \tau / ix \right]^{-1/2} \left[ 1 - v_- \tau / ix \right]^{-1/2},
$$

(10)

where $v_+ = v \pm (g/2\pi)$ and $v$ is the bare velocity. Identical results are obtained by bosonization, confirming that the skeleton self-energy is exact for the Luttinger model.

In the SO(4) model, the electron spectral weight displays two classic x-ray singularities associated with the decay of the electron into a spinon and holon continuum (Fig. 1) [7]. We now show that if $\Delta \nu = \nu - \nu_0$ is finite, one of these x-ray edge singularities is completely eliminated. If $\nu_0 < \nu$, we find that low velocity “Horn,” originally with velocity $v_-\,\ldots$, develops a sharp bound-state pole in the singlet channel, and a broad incoherent excitation in the triplet channel with a lifetime growing linearly in energy. If $\nu_0 > \nu$, the high velocity “horn” splits off a singlet antibound state and the triplet channel develops a high-energy incoherent excitation (Fig. 5). The sharp bound state in the singlet channel develops once a velocity difference is introduced, because energy and momentum conservation now provide distinct constraints to scattering [unlike in the SO(4) model], leading to much less phase space for $\Psi^{(0)}$ to decay into.

To see this, we must analyze Eq. (9) for the SO(3) case,

$$
\frac{d^2}{du^2} G_3^{-1} = - \left( g/2\pi \right)^2 (G_3)^2 G_0,
$$

$$
\frac{d^2}{du^2} G_0^{-1} = - \left( g/2\pi \right)^2 (G_3)^3.
$$

(11)

A very convenient way to discuss these equations is to map them onto a central force problem. If we write $r = (G_3^{-1}, G_0^{-1})$, $F = -(gG_3/2\pi)^2 (G_0, G_3)$, then $\dot{r} = F$. By inspection, $r \times \dot{r} = \Delta \nu \, r$, is a constant. If we use polar coordinates $(G_3^{-1}, G_0^{-1}) = r(\cos \theta, \sin \theta)$, the equations for the Green function resemble the motion of a fictitious particle under the influence of an anisotropic central force,

$$
\frac{\text{\textbf{}} - \Delta \nu^2}{r^3} = - \left( g/2\pi \right)^2 \frac{1}{r^3 \cos^2 \theta \sin \theta},
$$

$$
\frac{r^2 \dot{\theta}}{r} = \Delta \nu.
$$

(12)

The velocity difference $\Delta \nu = \nu - \nu_0$ provides a repulsive centrifugal force. The boundary conditions mean that the “particle” starts out at $r(0) = \sqrt{2}, \theta(0) = \pi/4$. When $\Delta \nu = 0$, the particle falls directly into the origin, and both $G_3$ and $G_0$ diverge with x-ray singularities. However, once $\Delta \nu$ is finite, the orbit does not pass through the origin at $u = 1/v_+$ or $1/v_-$, thereby eliminating the associated x-ray singularity in the spectral function. The quantity $\xi = \nu - \nu_0/g$ plays the role of a coupling constant, and approximate analytic solutions are possible in the limiting cases of small and large $\xi$.

Suppose first that $\Delta \nu < 0$. In this case, $\theta \to 0$ at some finite “time” $u = 1/\nu_0$, at which $r = C$ and $\dot{\theta} = \Delta \nu/C^2$. For $u \sim 1/\nu_0^*$, it follows that $(r, \theta) = [C, \theta(u - 1/\nu_0^*)]$, from which we can read off the following asymptotics:

$$
G_3(u)^{-1} \sim C,
$$

$$
G_0(u)^{-1} \sim \left( 1 - u \nu_0^* \right) / Z, \quad Z = C \nu_0^* / |\Delta \nu|.
$$

(13)

(14)

Thus when one Majorana fermion moves faster than the others, an antibound state with spectral weight $Z$, moving with velocity $\nu_0^*$, splits off above the continuum. For $|\Delta \nu| \gg \nu_0^*$ interactions can be ignored, so $\nu_0^* \to \nu_0$, and $Z \to 1$. For $|\Delta \nu| \ll \nu_0^*$, the “motion” of the fictitious particle emulates that of the SO(4) model until the angle $\theta$ approaches zero. We may estimate $\nu_0^*$ and $C$ by integrating Eq. (12) with the approximation $r(u) \approx r(u_0) = (2(1 - v_+ u)(1 - v_- u))^{1/2}$ is the SO(4) model, the electron spectral weight displays two classic x-ray singularities associated with the decay of the electron into a spinon and holon continuum (Fig. 1) [7]. We now show that if $\Delta \nu = \nu - \nu_0$ is finite, one of these x-ray edge singularities is completely eliminated. If $\nu_0 < \nu$, we find that low velocity “Horn,” originally with velocity $v_-\,\ldots$, develops a sharp bound-state pole in the singlet channel, and a broad incoherent excitation in the triplet channel with a lifetime growing linearly in energy. If $\nu_0 > \nu$, the high velocity “horn” splits off a singlet antibound state and the triplet channel develops a high-energy incoherent excitation (Fig. 5). The sharp bound state in the singlet channel develops once a velocity difference is introduced, because energy and momentum conservation now provide distinct constraints to scattering [unlike in the SO(4) model], leading to much less phase space for $\Psi^{(0)}$ to decay into.

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solution:
\[
-\frac{\pi}{4} = \int_0^{1/v_0} \frac{\Delta v}{r^2(u)} \, du, \quad C \approx \hat{r}(1/v_0).
\]

This estimate gives (for \(|\Delta v| \ll |g|/2\pi\))
\[
v_0 = v + \frac{g}{\pi} \exp - \left| \frac{g}{2\Delta v} \right|, \tag{16}
\]
\[
Z = \left| \frac{\sqrt{2}g}{\pi \Delta v} \right| \exp - \left| \frac{g}{4\Delta v} \right|. \tag{17}
\]

indicating that the formation of the sharp antibound state is nonperturbative in the velocity difference. These results can be generalized to the case where \(\Delta v > 0\) by replacing \(v_+ \rightarrow v_-, \ g \rightarrow -g\). To illustrate these results further, we have carried out numerical solutions of the differential equations (11) for intermediate values of the coupling constant \(\zeta\), using a standard adaptive integration routine. Results are summarized in Figs. 5 and 6.

In summary, we have demonstrated that by breaking the velocity degeneracy of a system of interacting chiral fermions we restrict the scattering phase space in a way which causes a sharp bound or antibound state to split off from the spin-charge continuum, leading to a system with two qualitatively distinct spectral peaks and scattering rates. This is a significant departure from the Luttinger liquid scenario and demonstrating a new class of one-dimensional fixed point behavior.

This new fixed point exhibits properties in common with both Luttinger and Fermi liquids, and is perhaps closest in character to the marginal Fermi liquid phenomenology introduced in the context of cuprate metals [12]. Like the Fermi liquid, there is a sharp quasiparticle bound state, but this coexists with a Luttinger liquidlike continuum which is bounded by two extremal velocities.

Had we chosen to change two Majorana velocities at the same time, so that \(v_0 = v_1\) and \(v_2 = v_3\), we would have reduced the symmetry still further, to an \(SO(2) \times SO(2)\) symmetry. In this case, the Hamiltonian is readily bosonized, giving the results
\[
G_3(x, \tau) = \frac{1}{2\pi i x} \left[ 1 - \frac{v_+ - \tau}{i x} \right]^{-1/2} \gamma \left[ 1 - \frac{v - \tau}{i x} \right]^{-1/2},
\]
\[
v_\pm = \frac{1}{2} \{v_0 + v_3 \pm [(v_3 - v_0)^2 + (g/\pi)^2]^{1/2}\}, \tag{18}
\]
and \(\gamma = \frac{1}{2} (v_3 - v_0) [(v_3 - v_0)^2 + (g/\pi)^2]^{-1/2}\). Curiously, this result may also be obtained as a solution of Eq. (9), even though, as far as we can see, the closed-loop cancellation is not sufficient in the case of the \(SO(2) \times SO(2)\) model to cancel all vertex corrections. This suggests that a more general cancellation principle is at work, and that the range of validity of our solution may extend to models with a still smaller \(SO(2)\) symmetry. To date, we have not been able to prove this result.

Our work raises the question whether this kind of non-Fermi liquid behavior might survive in dimensions higher than one. In higher dimensions energy conservation and momentum conservation are distinct constraints on scattering phase space, and the Luttinger liquid reverts to a Fermi liquid, at least for short-range interactions [9,13]. By contrast, the unusual properties of the \(SO(3)\) model have reduced reliance on the coincidence between momentum and energy conservation in scattering processes. In very large dimensions [4], the two lifetime behavior persists in the \(SO(3)\) model, but undergoes a crossover to a Fermi liquid asymptotically. The case of small, but finite dimensions is however, still open.

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![Fig. 5 Quasiparticle weight Z of Ψ(0) in the SO(3) model.](image)

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