CONFORMAL CHANGEPOINT DETECTION IN CONTINUOUS MODEL SITUATIONS

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Changepoint detection

The presence of a changepoint is a natural case where IID is violated. In this work we move from the binary observations [1, 2] to continuous ones and consider the following cases:

- A Gaussian distribution $N(\mu, \sigma)$ changes one of its parameters (mean μ or standard deviation σ).
- An exponential distribution $\mathrm{Exp}(\lambda)$ changes its rate λ.
- The "almost uniform" distribution $\mathrm{AU}(c)$ on $[0, 1]$ with the CDF $F_1(y) = y^c$ and parameter $c > 0$ changes to its "reflection" $F_2(y) = 1 - (1 - y)^c$.

The sequence length is $N = 10,000$ and the change point is $T = 5,000$.

Theoretical benchmark

Let d_1 and d_2 be the pre-change and post-change probability density functions, respectively. As our benchmark, we will use the likelihood ratio:

$$\frac{\Pi_{i=1}^{T} d_1(y_i) \Pi_{i=T+1}^{N} d_2(y_i)}{\Pi_{i=1}^{N} \left(\frac{T}{N} d_1(y_i) + \frac{N-T}{N} d_2(y_i)\right)}$$

No exchangeability martingale can exceed it [2].

Conformal Test Martingales

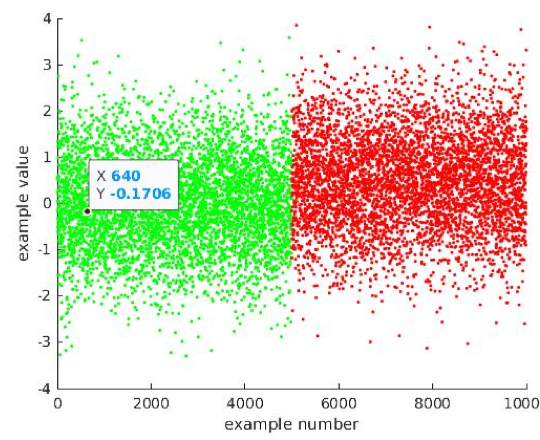
Informally, a conformal Test Martingale (CTM) is the capital of a gambler betting against IID with CP p-values. It is a nonnegative process with initial value 1 that is a martingale under any IID distribution. Each CTM is determined by a conformity measure and a betting function.

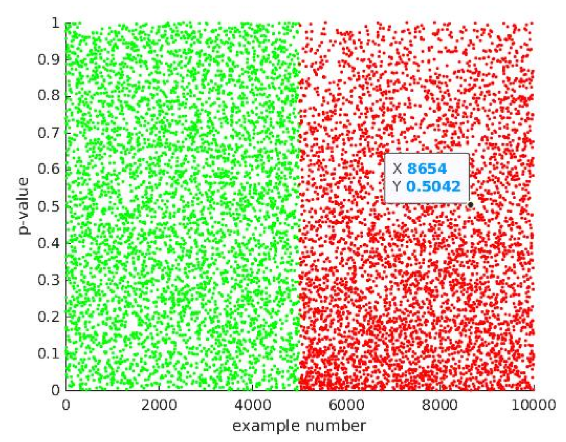
The conformity score of the ith observation y_i is defined as $\log d_1(y_i) - \log d_2(y_i)$ (the Neyman-Pearson statistic on the log scale).

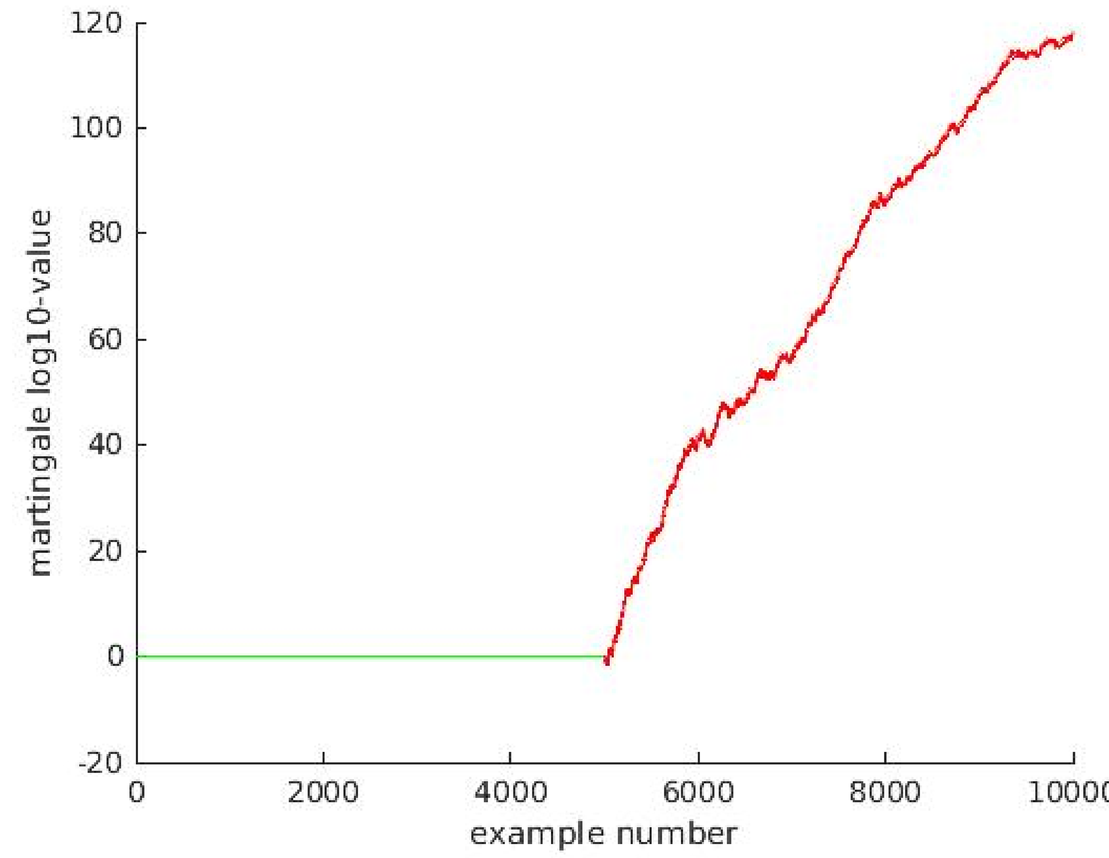
The betting function is 1 before the change and is calculated in two steps after the change.

- For each time step we calculate an empirical probability density function f for the conformal p-values using $R = 5000$ simulations from the same true stochastic mechanism varying random seeds.
- The density function f is forced to be monotonically decreasing by applying isotonic regression.

This suggestion for choosing the betting function can be justified by the statement in [3] that the optimal (in a natural sense) betting function coincides with the probability density function of the p-values.







A CTM path for $N(0,1) \to N(0.5,1)$.

pre-change	post-change	benchmark	CTM
$N(0,1)$	$N(0.5,1)$	130.8 (10.8)	126.6 (11.3)
$N(0,1)$	$N(0.2,1)$	21.3 (4.4)	18.5 (5.2)
$N(0,1)$	$N(0.1,1)$	5.3 (2.2)	3.0 (3.2)
$N(0,1)$	$N(0,1.5)$	154.3 (7.9)	150.1 (8.3)
$N(0,1)$	$N(0,1.1)$	8.8 (2.3)	6.2 (2.4)
$N(0,1)$	$N(0,0.9)$	12.3 (2.8)	9.7 (3.6)
$N(0,1)$	$N(0,0.7)$	125.8 (8.6)	121.8 (9.5)
Exp(1)	Exp(0.7)	65.2 (7.6)	61.6 (8.1)
Exp(1)	Exp(0.9)	5.6 (2.3)	3.4 (3.2)
AU(0.7)	reflected	196.3 (12.8)	191.5 (13.4)
AU(0.9)	reflected	19.1 (4.2)	16.3 (5.1)

The results are presented as decimal logarithms of the final values of the CTMs, averaged over 50 random seeds, with standard deviations of those logarithms in parentheses. This table shows that the gap between the performance of conformal testing and that of the benchmark is not excessive. So, the conformal approach to testing the IID assumption is not limited in its potential.

The dependence on the number R of simulations:

pre-change	post-change	R	final value
$N(0,1)$	$N(0.5,1)$	500	109.7 (12.6)
$N(0,1)$	$N(0.5,1)$	5000	126.6 (11.3)
$N(0,1)$	$N(0.5,1)$	50000	128.5 (10.3)
$N(0,1)$	$N(0.5,1)$	benchmark	129.5 (10.2)
AU(0.9)	reflected	500	4.1 (6.9)
AU(0.9)	reflected	5000	16.3 (5.1)
AU(0.9)	reflected	50000	18.0 (4.0)
AU(0.9)	reflected	benchmark	18.5 (3.9)

Conclusion

Our experiments support the hypothesis of CTM being a universal approach for testing martingales, as far as the knowledge of the data-generating mechanism can be translated into the language of nonconformity measures for CTM.

References

[1] Vladimir Vovk. Testing randomness online, On-line Compression Modelling project (New Series), `http://alrw.net`, Working Paper 24, June 2019.

[2] Vladimir Vovk. Conformal testing in a binary model situation. On-line Compression Modelling Project (New Series) `http://alrw.net`, Working Paper 33, 2021.

[3] Valentina Fedorova, Alex Gammerman, Ilia Nouretdinov, and Vladimir Vovk. Plug-in martingales for testing exchangeability on-line, On-line Compression Modelling project (New Series), `http://alrw.net`, Working Paper 4, April 2012.