

Proximity and Remoteness in Directed and Undirected Graphs

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Abstract

Let D be a strongly connected digraph. The average distance $\bar{\sigma}(v)$ of a vertex v of D is the arithmetic mean of the distances from v to all other vertices of D . The remoteness $\rho(D)$ and proximity $\pi(D)$ of D are the maximum and the minimum of the average distances of the vertices of D , respectively. We obtain sharp upper and lower bounds on $\pi(D)$ and $\rho(D)$ as a function of the order n of D and describe the extreme digraphs for all the bounds. We also obtain such bounds for strong tournaments. We show that for a strong tournament T , we have $\pi(T) = \rho(T)$ if and only if T is regular. Due to this result, one may conjecture that every strong digraph D with $\pi(D) = \rho(D)$ is regular. We present an infinite family of non-regular strong digraphs D such that $\pi(D) = \rho(D)$. We describe such a family for undirected graphs as well.

Keywords: average distance, strongly connected oriented graph, tournament, regular graph, regular digraph

1 Introduction

In this paper we consider only finite graphs and digraphs without loops or parallel edges and arcs. We shall assume the reader is familiar with the standard terminology on graphs and digraphs and refer to [2, 3] for terminology not discussed here.

Let $D = (V(D), A(D))$ be a strongly connected (often shortened to “strong”) digraph with $n \geq 2$ vertices (D is *strong* if for every pair u, v of vertices in D there are a path from u to v and a path from v to u). We call $n = |V(D)|$ the *order* of D and $m = |A(D)|$ the *size* of D . The *distance* $d(u, v)$ from vertex u to vertex v in D is the length of a shortest dipath from u to v . The *distance* of a vertex $u \in V(D)$ is defined as $\sigma(u) = \sum_{v \in V(D)} d(u, v)$, and the *average distance* of a vertex as $\bar{\sigma}(u) = \frac{\sigma(u)}{n-1}$. The *proximity* $\pi(D)$ and the *remoteness* $\rho(D)$ are $\min\{\bar{\sigma}(u) \mid u \in V(D)\}$ and $\max\{\bar{\sigma}(u) \mid u \in V(D)\}$, respectively. These parameters were introduced independently by Zelinka [10] and Aouchiche and Hansen [1] for undirected graphs and studied in several papers, see e.g. [1, 4, 5, 6, 7, 10]. We could not find any literature pertaining to proximity and remoteness for directed graphs. In this paper, we extend the study of these two distance measures from undirected graphs to directed graphs. We also present a new result on the topic for undirected graphs.

For additional terminology and notation, see the end of this section.

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There are several results in the literature on proximity and remoteness. Several authors have also investigated the relationships between these two distance measures and with other graph parameters such as diameter and radius. Zelinka [10] and Aouchiche and Hansen [1] independently showed that in a connected graph G of order n , the proximity and the remoteness are bounded from above as follows.

Theorem 1.1. *Let G be a connected graph of order $n \geq 2$. Then*

$$1 \leq \pi(G) \leq \begin{cases} \frac{n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{n+1}{4} + \frac{1}{4(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

The lower bound holds with equality if and only if G has a vertex of degree $n-1$. The upper bound holds with equality if and only if G is a path or a cycle. Also, $1 \leq \rho(G) \leq n/2$. The lower bound holds with equality if and only if G is a complete graph. The upper bound holds with equality if and only if G is a path.

In Section 2, we prove Theorem 2.1, which is an analog of Theorem 1.1 for strong digraphs. Note that the upper bound for proximity in Theorems 2.1 is different from the one in Theorem 1.1.

The difference between remoteness and proximity for graphs with just given order was bounded by Aouchiche and Hansen [1] as follows.

Theorem 1.2. *Let G be a connected graph on $n \geq 3$ vertices with remoteness $\rho(G)$ and proximity $\pi(G)$. Then*

$$\rho(G) - \pi(G) \leq \begin{cases} \frac{n-1}{4} & \text{if } n \text{ is odd,} \\ \frac{n-1}{4} + \frac{1}{4(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

Equality holds if and only if G is a graph obtained from a path $P_{\lfloor \frac{n}{2} \rfloor}$ and any connected graph H on $\lfloor \frac{n}{2} + 1 \rfloor$ vertices by a coalescence of an endpoint of the path with any vertex of H .

In Section 2, we obtain Theorem 2.2, which is an analog of Theorem 1.2 for strong digraphs. Note that the upper bound on the difference between remoteness and proximity given in Theorem 2.2 is different from that in Theorem 1.2. Theorem 2.2 describes all strong digraphs D for which the upper bound on $\rho(D) - \pi(D)$ holds with equality. A trivial lower bound for $\rho(D) - \pi(D)$ is 0. Neither Theorem 1.2 nor Theorem 2.2 provide characterizations on when $\rho(H) = \pi(H)$, where H is a graph and digraph, respectively. While it seems to be hard to characterize¹ all digraphs D with the property $\rho(D) = \pi(D)$, we start research in this direction in Section 3. We prove that a tournament satisfies such a property if and only if it is regular. To show this result we first prove an analog of Theorem 2.1 for strong tournaments, which may be of independent interest. One may conjecture that every strong digraph D with $\rho(D) = \pi(D)$ is regular. However, we demonstrate that this is already false for bipartite tournaments. In Section 4, we show that this is also false for undirected graphs.

We conclude our paper in Section 5 with a discussion about distance parameters for directed and undirected graphs. For more research on proximity and remoteness and related distance parameters for undirected graphs, see, e.g., [1, 4, 6, 7].

Additional Terminology and Notation. The *complement* \overline{D} of a digraph D is a digraph with $V(\overline{D}) = V(D)$ and $A(\overline{D}) = \{uv \mid u \neq v \in V(D), uv \notin A(D)\}$. A digraph D is *complete* if its complement has no arcs. If v is a vertex in a digraph D , then the *out-neighbourhood* of v is $N^+(v) = \{u \in V(D) \mid vu \in A(D)\}$ and *in-neighbourhood* of v is $N^-(v) = \{u \in V(D) \mid uv \in A(D)\}$, and the out-degree $d_D^+(v)$ and in-degree $d_D^-(v)$ of v are $|N^+(v)|$ and $|N^-(v)|$, respectively. We denote by $\Delta^+(D)$ and $\Delta^-(D)$ the maximum out-degree and maximum in-degree of D respectively, similarly, by $\delta^+(D)$ and $\delta^-(D)$ the minimum out-degree and

¹It appears to be interesting open questions to characterize such strong digraphs and connected graphs.

minimum in-degree of D , respectively. The maximum semi-degree and the minimum semi-degree of D are $\Delta^0(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ and $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$ respectively. A digraph D is *regular* if $\delta^0(D) = \Delta^0(D)$.

The *eccentricity* $\text{ecc}(u)$ of a vertex u in a digraph D is the distance from u to a vertex farthest from u , the *radius* $\text{rad}(D)$ and *diameter* $\text{diam}(D)$ are the minimum and maximum of all eccentricities of D , respectively. For a nonnegative integer p , let $[p] = \{i \in \mathbb{Z} \mid 1 \leq i \leq p\}$. Thus, $[0] = \emptyset$.

If i is a nonnegative integer and $v \in V(D)$, then $N^{+i}(v)$ is the set of all vertices at distance i from v , and n_i is its cardinality. In particular, $N^{+0}(v) = \{v\}$ and $N^{+1}(v) = N^+(v)$. Observe that we have that $n_i \geq 1$ if and only if $0 \leq i \leq \text{ecc}(v)$. Let p be a positive integer. A vertex u of a digraph D is called a *p-king* if $V(D) = \bigcup_{i=0}^p N^{+i}(u)$.

We make use of the notion of *distance degree* $X(v)$ of v defined in [7] as the sequence $(n_0, n_1, \dots, n_{\text{ecc}(v)})$. For an arbitrary sequence $X = (x_0, x_1, \dots, x_\ell)$ of integers, let $g(X) = \sum_{i=0}^{\ell} ix_i$. Observe that $\sigma(v) = g(X(v))$.

An *orientation* of a graph G is a digraph obtained from G by replacing each edge by exactly one of the two possible arcs. We call an orientation of a complete graph and complete k -partite graph a *tournament* and a *k-partite tournament*, respectively. A 2-partite tournament is also called a *bipartite tournament*.

2 Bounds on Proximity and Remoteness for Strong Digraphs

Theorem 2.1. *Let D be a strong digraph on $n \geq 3$ vertices. Then*

$$1 \leq \pi(D) \leq \frac{n}{2}.$$

The lower bound holds with equality if and only if D has a vertex of out-degree $n - 1$. The upper bound holds with equality if and only if D is a dicycle. And,

$$1 \leq \rho(D) \leq \frac{n}{2}.$$

The lower bound holds with equality if and only if D is a complete digraph. The upper bound holds with equality if and only if D is strong and contains a Hamiltonian dipath $v_1v_2 \dots v_n$ such that $\{v_iv_j \mid 2 \leq i+1 < j \leq n\} \subseteq A(\overline{D})$.

Proof. The lower bounds and extreme digraphs for them are trivial. For the upper bounds, since $\pi(D) \leq \rho(D)$, it suffices to prove only the upper bound for remoteness. Let u be an arbitrary vertex of D . It suffices to show that $\sigma(u) \leq \binom{n}{2}$. Let $X(u) = (n_0, n_1, \dots, n_d)$ and $v \in N^{+d}(u)$, where $d = \text{ecc}(u)$. Then there is a dipath from u to v implying that $n_i \geq 1$ for every $i \in [d]$ (clearly $n_0 = 1$). Let $X = X(u)$. Observe that $d \leq n - 1$ and if $d = n - 1$ then $n_i = 1$ for each $i \in [d]$ implying that $\sigma(u) = g(X) = \binom{n}{2}$. If $d < n - 1$ then there is a maximum i such that $n_i > 1$. Observe that for $X' = (n_0, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_d, 1)$, we have $g(X) < g(X')$. Replacing X by X' and constructing a new X' if d is still smaller than $n - 1$, we will end up with $X = (1, \dots, 1)$ implying that $\sigma(u) < \binom{n}{2}$.

Observe that the upper bound of $\pi(D)$ is reached if D is a dicycle. Now let D be a strong digraph which is not a dicycle and let u be a vertex such that $d = d_D^+(u) = \Delta^+(D) \geq 2$. Then $(n - 1)\pi(D) \leq \sigma(u) = g(X(u))$. Observe that $X(u) = (1, d, n_2, \dots, n_{\text{ecc}(u)})$ and $\text{ecc}(u) \leq n - d$. Using the same argument as in the paragraph above, we conclude that $g(X(u)) < g(Y)$, where $Y = (1, y_1, \dots, y_{n-1})$ and $y_i = 1$ for each $i \in [n - 1]$. Thus, $(n - 1)\pi(D) < \binom{n}{2}$ and $\pi(D) < n/2$.

The upper bound of $\rho(D)$ is reached if and only if D is strong and there is a vertex v_1 such that $\text{ecc}(v_1) = n - 1$. The last condition is equivalent to D containing a Hamiltonian dipath $v_1v_2 \dots v_n$ such that $\{v_iv_j \mid 2 \leq i+1 < j \leq n\} \subseteq A(\overline{D})$ (as otherwise $\text{ecc}(v_1) < n - 1$). \square

Theorem 2.1 allows us to easily prove the following:

Theorem 2.2. *Let D be a strong digraph, then $\rho(D) - \pi(D) \leq \frac{n}{2} - 1$. The upper bound holds with equality if and only if D contains a Hamiltonian dipath $v_1v_2 \dots v_n$ such that $\{v_iv_j \mid 2 \leq i+1 < j \leq n\} \subseteq A(\overline{D})$ and at least one of the vertices v_{n-1}, v_n has out-degree $n-1$.*

Proof. The upper bound follows from the lower bound on $\pi(D)$ and upper bound on $\rho(D)$ in Theorem 2.1. Observe that if $\rho(D) - \pi(D) = \frac{n}{2} - 1$, then $\rho(D) = \frac{n}{2}$ and $\pi(D) = 1$. By Theorem 2.1, this is equivalent to D containing a Hamiltonian dipath $v_1v_2 \dots v_n$ such that $\{v_iv_j \mid 2 \leq i+1 < j \leq n\} \subseteq A(\overline{D})$ and at least one of the vertices v_{n-1}, v_n has out-degree $n-1$. \square

3 Digraphs D with $\rho(D) = \pi(D)$

We call a tournament T *regular* if $d^+(u) = d^-(u) = \frac{n-1}{2}$ for every $u \in V(T)$. This can happen only when n is odd. A tournament is *almost regular* if n is even and $d^+(u) = \frac{n}{2}$ or $d^+(u) = \frac{n-2}{2}$ for every $u \in V(T)$.

This section has two parts: the first proves that for a strong tournament T , $\rho(T) = \pi(T)$ if and only if T is regular, and the second provides an infinite family of strong bipartite tournaments T for which $\rho(T) \neq \pi(T)$.

3.1 Tournaments

We will use the following well-known result.

Proposition 3.1. *[9] Every tournament contains a 2-king, moreover, every vertex with maximum out-degree is a 2-king.*

The next theorem is an analog of Theorem 2.1 for tournaments and allows us to easily show that for a strong tournament T , $\rho(T) = \pi(T)$ if and only if T is regular.

Theorem 3.1. *Let D be a strong tournament on n vertices, then*

$$\frac{n}{n-1} \leq \pi(D) \leq \begin{cases} \frac{3}{2} & \text{if } n \text{ is odd,} \\ \frac{3}{2} - \frac{1}{2(n-1)} & \text{if } n \text{ is even} \end{cases}$$

The lower bound holds with equality if and only if $\Delta^+(D) = n-2$. The upper bound holds with equality if and only if D is a regular or almost regular tournament. And,

$$\frac{n}{2} \geq \rho(D) \geq \begin{cases} \frac{3}{2} & \text{if } n \text{ is odd,} \\ \frac{3}{2} + \frac{1}{2(n-1)} & \text{if } n \text{ is even} \end{cases}$$

The lower bound holds with equality if and only if D is a regular or almost regular tournament. The upper bound holds with equality if and only if D is isomorphic to the tournament T_n with $V(T_n) = \{v_1, \dots, v_n\}$ and $A(T_n) = \{v_iv_{i+1} \mid i \in [n-1]\} \cup \{v_jv_i \mid 2 \leq i+1 < j \leq n\}$.

Proof. Since D is strong, $\Delta^+(D) \leq n-2$. Thus, $\frac{n}{n-1} \leq \pi(D)$. We have $\pi(D) = \frac{n}{n-1}$ if and only if there are vertices u, v, w such that $ux \in A(D)$ for all $x \in V(D) \setminus \{u, v\}$, $vu \in A(D)$ but $wv \in A(D)$, which if and only if $\Delta^+(D) = n-2$ since by Proposition 3.1 a vertex of out-degree $n-2$ is a 2-king.

For the upper bound on $\pi(D)$, consider a vertex u of D such that $d^+(u) = \Delta^+(D)$. Then, by Proposition 3.1, u is a 2-king of D . Since the average out-degree is $\frac{n-1}{2}$, we have that

$$d^+(u) \geq \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Now,

$$\begin{aligned} \sigma(u) &= \sum_{i=0}^d in_i \\ &= \Delta^+(D) + 2(n - \Delta^+(D) - 1) \\ &\leq \begin{cases} \frac{3}{2}(n-1) & \text{if } n \text{ is odd,} \\ \frac{3}{2}n - 2 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

And taking the average completes the proof to this bound. Observe that the equality holds only when

$$d^+(u) = \Delta^+(D) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The upper bound of $\rho(D)$ and the uniqueness (up to isomorphism) of a tournament D with $\rho(D) = n/2$ follow directly from the corresponding bound in Theorem 2.1 and the characterization of digraphs H for which $\rho(H) = n/2$.

For the lower bound of $\rho(D)$, consider a vertex u of D such that $d^+(u) = \delta^+(D)$. We have that

$$d^+(u) \leq \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n-2}{2} & \text{if } n \text{ is even.} \end{cases}$$

Now,

$$\begin{aligned} \sigma(u) &= \sum_{i=0}^d in_i \\ &\geq \delta^+(D) + 2(n - \delta^+(D) - 1) \\ &\geq \begin{cases} \frac{3}{2}(n-1) & \text{if } n \text{ is odd,} \\ \frac{3}{2}n - 1 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

And taking the average completes the proof to this bound. Observe that the equality holds only when

$$d^+(u) = \delta^+(D) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n-2}{2} & \text{if } n \text{ is even.} \end{cases}$$

Thus the lower bound holds with equality if and only if D is a regular or almost regular tournament. \square

Theorem 3.1 allows us to easily obtain such a characterization of strong tournaments T with $\rho(T) = \pi(T)$.

Theorem 3.2. *For any strong tournament T , we have $\rho(T) = \pi(T)$ if and only if T is a strong regular tournament.*

Proof. Since $\pi(D) \leq \rho(D)$ for every strong digraph and by the upper bounds on $\pi(H)$ and lower bounds for $\rho(H)$ for strong tournaments in Theorem 3.1 (including the characterizations of when the bounds hold), we have that for a strong tournament T of order n , $\rho(T) = \pi(T)$ if and only if T is regular. \square

3.2 Bipartite Tournaments

Due to Theorem 3.2, one may conjecture that every digraph D with $\rho(D) = \pi(D)$ is regular. However, in this subsection we will show that this is not true and such counterexamples can be found already among bipartite tournaments. To obtain such counterexamples, we will first study some properties of strong bipartite tournaments.

In what follows, let $T = T[A, B]$ be a strong bipartite tournament with partite sets A and B of sizes n and m , respectively.

Definition 3.1. A vertex x of a digraph D is called *bad* if the out-neighborhood of x is a proper subset of the out-neighborhood of another vertex of D . A vertex z is *good* if it is not bad. A bipartite tournament is called *bad* if it has a bad vertex, otherwise it is called *good*.

Definition 3.2. For a vertex v , we denote by $M(v)$ the set of vertices with the same out-neighborhood as v , i.e., $M(v) = \{u | N^+(u) = N^+(v)\}$; and we denote $|M(v)|$ by $\mu(v)$.

Lemma 3.3. *If $T = T[A, B]$ is a bad strong bipartite tournament, then $\pi(T) \neq \rho(T)$.*

Proof. Let vertices u and v be such that $N^+(u) \subset N^+(v)$. Then there is a vertex $w \in N^+(v) \setminus N^+(u)$. Note that for every $x \in (A \cup B) \setminus \{u, v, w\}$, we have $d(v, x) \leq d(u, x)$. Since vwu is a path in T , we have $2 = d(v, u) \leq d(u, v)$. Finally, $1 = d(v, w) < d(u, w)$. Thus, $\sigma(v) < \sigma(u)$ implying $\pi(T) \neq \rho(T)$. \square

Thus, we may restrict ourselves only to good strong bipartite tournaments.

The following result is Lemma 1 in [8].

Proposition 3.4. *Let T be a bipartite tournament without vertices of in-degree 0 and let x be a good vertex. Then $d(x, y) \leq 3$ for every $y \in V(D) \setminus M(x)$.*

Lemma 3.5. *Let $T = T[A, B]$ be a good strong bipartite tournament. Then every vertex of T is a 4-king.*

Proof. Let x be an arbitrary vertex and, without loss of generality, let $x \in A$. By Lemma 3.4, $d(x, y) \leq 3$ for every $y \in V(D) \setminus M(x)$. Let $z \in M(x)$. Since T is strong, there is an arc yz for some $y \in B$. Hence $d(x, u) \leq 4$ for every $u \in A \cup B$. \square

Lemma 3.6. *Let $T = T[A, B]$ be good, $\rho(T) = \pi(T)$ and $u, v \in V(T)$ such that $u \neq v$. If both u and v are in A or in B , then*

$$\mu(u) - d^+(u) = \mu(v) - d^+(v) \tag{i}$$

If $v \in A$ and $u \in B$ then

$$2(\mu(v) - d^+(v)) + |B| = 2(\mu(u) - d^+(u)) + |A| \tag{ii}$$

Proof. For every vertex $v \in A$, $\sigma(v) = \sum_{i=0}^4 in_i = d^+(v) + 2(n - \mu(v)) + 3(m - d^+(v)) + 4(\mu(v) - 1) = 2(\mu(v) - d^+(v)) + 2n + 3m - 4$. Similarly, if $v \in B$ then $\sigma(v) = 2(\mu(v) - d^+(v)) + 2m + 3n - 4$. The results of the lemma follow from $\sigma(u) = \sigma(v)$ when u and v are from the same partite set of T and when they are from different partite sets of T . \square

Lemmas 3.5 and 3.6 imply the following:

Corollary 3.7. *For a strong bipartite tournament T , we have $\rho(T) = \pi(T)$ if and only if T is good and there is a constant c such that for every $v \in A$ and $u \in B$,*

$$2(\mu(v) - d^+(v)) + m = 2(\mu(u) - d^+(u)) + n = c$$

In particular, for a strong bipartite tournament T with $n = m$, we have $\rho(T) = \pi(T)$ if and only if T is good and $d^+(u) - \mu(u)$ is the same for every vertex u .

To help us find bipartite tournaments T with $\rho(T) = \pi(T)$, one can use the following:

Corollary 3.8. *Let T be a good bipartite tournament and c a constant such that $\mu(x) = c$ for every $x \in V(T)$. Then $\rho(T) = \pi(T)$ if and only if $d^+(v) = m/2$ and $d^+(u) = n/2$ for every $v \in A$ and $u \in B$.*

Proof. Suppose that $\rho(T) = \pi(T)$. By (i) and $\mu(x) = c$ for every $x \in V(T)$, there exist constants c' and c'' such that $d^+(v) = c'$ and $d^+(u) = c''$ for every $v \in A$ and $u \in B$. Since T has nm arcs, $c'n + c''m = nm$. By (ii) and $\mu(x) = c$ for every $x \in V(T)$, we have $m - 2c' = n - 2c''$ implying $c'' = c' - \frac{m-n}{2}$. After substitution of c'' by $c' - \frac{m-n}{2}$ in $c'n + c''m = nm$ and simplification, we obtain $c' = m/2$. This and $m - 2c' = n - 2c''$ imply $c'' = n/2$.

Now suppose that $d^+(v) = m/2$ and $d^+(u) = n/2$ for every $v \in A$ and $u \in B$. Since $\mu(x) = c$ for every $x \in V(T)$ and T is good, by Corollary 3.7, we have $\rho(T) = \pi(T)$. \square

The following result shows, in particular, that there are non-regular digraphs D for which $\rho(D) = \pi(D)$.

Theorem 3.3. *For both $|A| = |B|$ and $|A| \neq |B|$, there is an infinite number of bipartite tournaments T with $\rho(T) = \pi(T)$.*

Proof. For the case of $|A| = |B|$, we can use the following simple bipartite tournament T . Let n be even and let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ be partitions of A and B into subsets of size $n/2$. Let $N^+(v_i) = B_i$ for every $v_i \in A_i$ for $i = 1, 2$. By Corollary 3.8, $\rho(T) = \pi(T)$.

For the case of $|A| \neq |B|$, first consider the following bipartite tournament T_1 . Let $A = \{1, 2, 3, 4\}$, $B = \{1', 2', 3', 4', 5', 6'\}$, and $N^+(1) = \{1', 2', 3'\}$, $N^+(2) = \{1', 4', 5'\}$, $N^+(3) = \{2', 4', 6'\}$, $N^+(4) = \{3', 5', 6'\}$. Observe that $d^+(v) = 3$ and $\mu(v) = 1$ for every $v \in A$, $d^+(u) = 2$ and $\mu(u) = 1$ for every $u \in B$. Thus, by Corollary 3.8, T_1 has $\rho(T_1) = \pi(T_1)$. Now we obtain a new bipartite tournament T_t from T_1 by replacing every vertex x in T by $t \geq 2$ vertices x_1, \dots, x_t such that $x_i y_j \in A(T_t)$ if and only if $xy \in A(T)$. Let A_t and B_t be the partite sets of T_t , $t \geq 1$. Observe that for every $t \geq 1$, T_t is good, $\mu(x) = t$ for every $x \in V(T_t)$, and $d^+(v) = |B_t|/2$ and $d^+(u) = |A_t|/2$ for every $v \in A$ and $u \in B$. Thus, by Corollary 3.8 $\rho(T) = \pi(T)$. \square

4 Undirected Graphs G with $\rho(G) = \pi(G)$

Using computer search, we found two non-regular graphs G for which $\rho(G) = \pi(G)$. The graphs are depicted in Figures 1 and 2. Using the first of the graphs, we will prove the following:

Theorem 4.1. *There is an infinite number of non-regular graphs G with $\rho(G) = \pi(G)$.*

Proof. Consider the following non-regular graph G (see Fig. 1) with $V(G) = \{v_i \mid i \in \{0\} \cup [8]\}$ and

$$E(G) = \{v_0v_3, v_0v_6, v_0v_7, v_1v_4, v_1v_6, v_1v_8, v_2v_5, v_2v_7, v_2v_8, v_3v_6, v_3v_7, v_4v_6, v_4v_8, v_5v_7, v_5v_8\}.$$

Observe that $X(v_0) = X(v_1) = \dots = X(v_5) = (1, 3, 4, 1)$ and $X(v_6) = X(v_7) = X(v_8) = (1, 4, 2, 2)$. Thus, $\sigma(v_i) = 22$ for every $i \in \{0\} \cup [8]$ implying that $\rho(G) = \pi(G)$. Now obtain a new graph G_t from G by replacing every vertex x in G by $t \geq 2$ vertices x_1, \dots, x_t such that $x_i y_j \in E(G_t)$ if and only if $xy \in E(G)$. Hence for any pair $x_i, x_j \in E(G_t)$ we have that $d(x_i, x_j) = 2$. Thus, $\sigma(x_i) = 22 + 2(t - 1)$ for every vertex x_i of G_t implying that $\rho(G_t) = \pi(G_t)$. \square

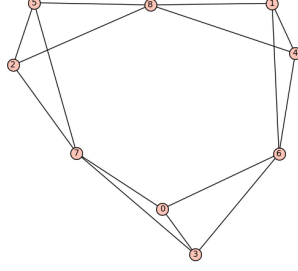


Figure 1: A non-regular graph G of order 9 with $\rho(G) = \pi(G)$

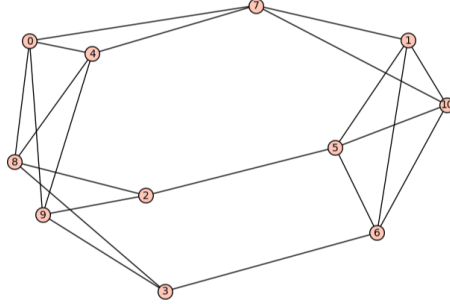


Figure 2: A non-regular graph G of order 11 with $\rho(G) = \pi(G)$

5 Discussion

Recall that the *radius* $\text{rad}(D)$ and *diameter* $\text{diam}(D)$ are the minimum and maximum of all eccentricities of D , respectively. The following inequalities hold by definition for both connected graphs and strong digraphs:

$$\pi(D) \leq \rho(D), \quad \text{rad}(D) \leq \text{diam}(D), \quad \pi(D) \leq \text{rad}(D), \quad \rho(D) \leq \text{diam}(D), \quad 1 \leq \text{diam}(D) \leq n - 1.$$

For strong digraphs, the parameters $\text{rad}(D)$ and $\rho(D)$ are incomparable as we can see from the following two examples.

(i) Let $c \in [n - 1]$ be arbitrary. Denote by D_c the strong digraph with vertices v_0, v_1, \dots, v_{n-1} and arc set

$$A(D_c) = \{v_0v_j \mid j \in [n - 1]\} \cup \{v_pv_{p+1} \mid p \in [n - 2]\} \cup \{v_{n-1}v_1, v_cv_0\}.$$

It is not hard to see that $\text{rad}(D_c) = 1$ and $\rho(D_c) = \frac{n}{2}$, hence, $\text{rad}(D_c) < \rho(D_c)$ for $n \geq 3$.

(ii) If D is a dicycle \vec{C}_n on n vertices, we have that $\text{rad}(D) = n - 1$ and $\rho(D) = \frac{n}{2}$, hence, $\text{rad}(D) > \rho(D)$ for $n \geq 3$.

The following results hold for connected graphs but do not necessarily for strong digraphs.

1. For a connected graph G , $\text{diam}(G) \leq 2\text{rad}(G)$. A simple example showing that this inequality does not hold for strong digraphs is when D is the digraph D_c defined above. Then $\text{rad}(D) = 1$ and $\text{diam}(D) = n - 1$, hence, $\text{diam}(D) > 2\text{rad}(D)$.
2. For a connected graph G , if v is a vertex of G such that $\text{ecc}(v) = \text{rad}(G) = r$, then for each $i \in [r - 1]$, there exist at least two vertices at distance i from v . A counterexample to this observation for strong digraphs is when D is a dicycle. Then $\text{ecc}(v) = \text{rad}(G) = n - 1$ for every vertex $v \in V(D)$, and for every vertex $v \in V(D)$ there is only one vertex of distance $i \in [n - 2]$ from v .

3. For a connected graph G of order n , if $n \geq 2$ then $1 \leq \text{rad}(G) \leq \lfloor \frac{n}{2} \rfloor$. As a counterexample for strong digraphs, again consider D to be a dicycle on n vertices. Indeed, $\text{rad}(D) = n - 1 > \lfloor \frac{n}{2} \rfloor$ for $n \geq 3$.

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