NUCLEARITY OF SEMIGROUP C*-ALGEBRAS

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Abstract. We study the semigroup C*-algebra of a positive cone P of a weakly quasi-lattice ordered group. That is, P is a subsemigroup of a discrete group G with $P \cap P^{-1} = \{e\}$ and such that any two elements of P with a common upper bound in P also have a least upper bound. We find sufficient conditions for the semigroup C*-algebra of P to be nuclear. These conditions involve the idea of a generalised length function, called a "controlled map", into an amenable group. Here we give a new definition of a controlled map and discuss examples from different sources. We apply our main result to establish nuclearity for semigroup C*-algebras of a class of one-relator semigroups, motivated by a recent work of Li, Omland and Spielberg. This includes all the Baumslag–Solitar semigroups. We also analyse semidirect products of weakly quasi-lattice ordered groups and use our theorem in examples to prove nuclearity of the semigroup C*-algebra. Moreover, we prove that the graph product of weak quasi-lattices is again a weak quasi-lattice, and show that the corresponding semigroup C*-algebra is nuclear when the underlying groups are amenable.

1. Introduction

C*-algebras associated to semigroups provide a rich class of examples of C*-algebras. There are interesting connections of these algebras with number theory [11, 18, 21] that were sparked by Cuntz's study of the C*-algebra of an $ax+b$-semigroup over the natural numbers [10]. See also [6] for recent developments in this subject. Further interesting examples include C*-algebras of positive cones of right-angled Artin groups [8, 9, 12] and of Baumslag–Solitar groups [7, 33], C*-algebras of self-similar actions [4, 19] and least common multiple (LCM) semigroups [3, 5, 32].

In [20], Li introduced semigroup C*-algebras that generalise the ones of Nica attached to quasi-lattice ordered groups. Li's construction takes into account the structure of constructible right ideals of the underlying semigroup. These are all the principal ideals together with the emptyset if P is a left-cancellative and right LCM semigroup. In this case, the full semigroup C*-algebra $C^*(P)$ is universal for a class of isometric representations of P satisfying a relation called Nica covariance. The reduced semigroup C*-algebra $C^*_r(P)$ is then generated by the left-regular representation of P on $\ell^2(P)$.

In this paper, we study the C*-algebras of semigroups that are positive cones of weakly quasi-lattice ordered groups. These sit between quasi-lattice ordered groups

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and right LCM semigroups. Following Nica, we carry the group in our notation, and write $C^*(G, P)$ for the semigroup C^*-algebra associated to the positive cone P of the weak quasi-lattice order (G, P). The group is crucial in our proofs and this notation helps to make our assumptions clear throughout the text, but we observe that $C^*(G, P)$ depends only on P.

As in [27], we say that (G, P) is amenable if the left-regular representation induces an isomorphism between the full and reduced semigroup C^*-algebras. If G is amenable as a group, then (G, P) is amenable. In fact, $C^*(G, P)$ is nuclear in this case [24, Corollary 6.45]. In general, it is not known whether amenability of the (weak) quasi-lattice order (G, P) implies nuclearity of its C^*-algebra. In addition, nuclearity of $C^*(G, P)$ implies that of its boundary quotient semigroup C^*-algebra. Since this latter is quite often purely infinite and simple [9, 25], nuclearity results for semigroup C^*-algebras combined with [36] may give new examples of C^*-algebras that are classifiable by K-theory. Our main purpose in this article is to analyse nuclearity of $C^*(G, P)$ and to contribute to the existing literature with new examples.

A sufficient condition for amenability of quasi-lattice orders was established by Laca and Raeburn in [17, Proposition 6.6]. This condition involves the existence of a generalised length function, now called a "controlled map", from G into an amenable group. The controlled maps from [17] have proved very useful. They were used in [8] to show that any graph product of a family of quasi-lattice orders in which the underlying groups are amenable is again an amenable quasi-lattice order. The existence of a controlled map into an amenable group implies that the partial action of G on any closed invariant subset of the Nica spectrum is amenable [9, Theorem 4.7] and that $C^*(G, P)$ is nuclear [23, Corollary 8.3]. Spielberg used ideas of a similar flavour to give sufficient conditions for nuclearity of the C^*-algebra associated to a finitely aligned category of paths (see [34, Definition 9.5 and Theorem 9.8]).

A controlled map as originally introduced by Laca and Raeburn in [17] is an order-preserving group homomorphism ϕ between quasi-lattice ordered groups that preserves the least upper bound of any two elements of P when it exists, and such that $\ker \phi \cap P$ is trivial. In [15], an Huef, Raeburn and Tolich introduced a version of a controlled map whose restriction to the positive cone allows a nontrivial kernel, and used it to prove an amenability result along the lines of [17]. A key example of a controlled map in the sense of [15] comes from the height map on a Baumslag–Solitar group. Recall that, for $c, d \geq 1$, the Baumslag–Solitar group $\mathrm{BS}(c, d)$ is defined by

$$\mathrm{BS}(c, d) = \langle a, b : ab^c = b^d a \rangle.$$

Spielberg proved that $(\mathrm{BS}(c, d), \mathrm{BS}(c, d)^+)$ is quasi-lattice ordered, where $\mathrm{BS}(c, d)^+$ is the unital subsemigroup of $\mathrm{BS}(c, d)$ generated by a and b [33]. The "height map" from $\mathrm{BS}(c, d)$ to $\mathbb{Z}$ is the group homomorphism that sends a to 1 and b to 0. Its kernel when restricted to $\mathrm{BS}(c, d)^+$ is nontrivial. It is the unital semigroup generated by b. Nevertheless, the height map is controlled in the sense of [15]. Because the kernel of the height map gives an amenable quasi-lattice order, it follows from [15] that $C^*(\mathrm{BS}(c, d), \mathrm{BS}(c, d)^+)$ is amenable.

Here we present a more general definition of controlled maps. The main advantage of this new definition is that it also allows us to treat examples of weak quasi-lattices which have an infinite descending chain

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq \cdots .$$

Our motivating example is the weak quasi-lattice order coming from the Baumslag–Solitar semigroup

$$\mathrm{BS}(c,-d)^+ = \langle a, b : b^d a b^c = a\rangle^+$$

and the corresponding Baumslag–Solitar group $\mathrm{BS}(c,-d)$ [33], where $c, d \geq 1$. The height map from $\mathrm{BS}(c,-d)$ to $\mathbb{Z}$ is not controlled in the sense of an Huef, Raeburn and Tolich because it is constant on the infinite descending chain

$$a > \cdots > b^{nd}a > b^{(n+1)d}a > \cdots .$$

This sequence is not bounded below in $\mathrm{BS}(c,-d)^+$ by an element of height 1. However, the height map in this example is controlled in our new sense (see Definition 3.6). Our main theorem is:

Theorem 1.1. *Let (G,P) and (K,Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon (G,P) \to (K,Q)$ is a controlled map in the sense of Definition* 3.6*, that K is amenable and that $\mathrm{C}^*(\ker\mu, \ker\mu \cap P)$ is nuclear. Then $\mathrm{C}^*(G,P)$ is nuclear. In particular, (G,P) is amenable.*

The proof involves a detailed analysis of the structure of a fixed-point algebra associated to a coaction of K on $\mathrm{C}^*(G,P)$, obtained from μ. This fixed-point algebra is in general much larger than $\mathrm{C}^*(\ker\mu, \ker\mu\cap P)$, but we prove that it is also nuclear. Then we apply [29, Corollary 2.17]: if a C*-algebra B carries a coaction of a discrete amenable group for which the fixed-point algebra is nuclear, then B is nuclear, too.

Recently, Li, Omland and Spielberg studied semigroup C*-algebras and their boundary quotients for right LCM one-relator semigroups. Their main result concerning nuclearity is [25, Corollary 3.10]. Under certain assumptions on the defining relator, they provided a graph model for the boundary quotient semigroup C*-algebra, and concluded that both the semigroup C*-algebra and its boundary quotient are nuclear. Motivated by [25], we also look at one-relator semigroups. We obtain nuclearity of the semigroup C*-algebras associated to a class of weakly quasi-lattice ordered groups coming from HNN extensions, where the controlled maps involved are also height maps. This contains all the Baumslag–Solitar semigroups as particular cases. Our nuclearity results for one-relator semigroups have no overlap with those in [25], and may be used in combination with [25, Corollary 3.5] and [36] to produce examples of C*-algebras that are classifiable by K-theory.

Weak quasi-lattice orders are in general not well-behaved under semidirect products. In other words, the semidirect product of two weakly quasi-lattice ordered groups may not be weakly quasi-lattice ordered. We illustrate this with Example 5.20. However, we show that particular examples of semidirect products are naturally weakly quasi-lattice ordered. These mostly involve actions on free groups $\mathbb{F}$. We then use controlled maps and nuclearity of $\mathrm{C}^*(\mathbb{F},\mathbb{F}^+)$ to obtain nuclearity for the corresponding semigroup C*-algebras.

We also include graph products of weak quasi-lattices among our examples. We first show that the graph product of weakly quasi-lattice ordered groups is again weakly quasi-lattice ordered, with a natural positive cone. To do so, we prove a result along the same lines of [8, Proposition 13]. Our proof is a little different though: that the graph product of quasi-lattice orders is again quasi-lattice ordered was implicitly used in the proof of [8, Proposition 13] (see [8, Theorem 10]). When the underlying groups are amenable, we apply our result to obtain nuclearity of the semigroup C*-algebra associated to the graph product of weak quasi-lattices.

After we submitted this article for publication, we learned that Fountain and Kambites prove that a graph product of left LCM monoids is again a left LCM monoid [14, Theorem 2.6]. Our Theorem 5.25 proves this result for the particular case of weak quasi-lattice orders, and also gives an iterative procedure for computing the least upper bound for elements in a graph product. This procedure is crucial for finding a controlled map, and so we have retained Theorem 5.25 in order to prove our nuclearity result in Corollary 5.27. We thank Nadia Larsen for bringing [14] to our attention.

Outline. We start in §2 with some background material. In §3 we discuss the old and new definitions of controlled maps, and illustrate the need for greater generality with examples. We also answer in the negative a question, by Marcelo Laca, whether a positive cone of a weak quasi-lattice order can be embedded in group such that this new pair is quasi-lattice ordered. This question was motivated by Scarparo's example (Example 3.3). In §4 we introduce the coaction built from a controlled map, analyse the structure of the fixed-point algebra, and then prove our main theorem (Theorem 4.3). In §5 we present our classes of examples. In §6, we prove an amenability result which would be of interest when $(\ker\mu, \ker\mu\cap P)$ is amenable but $\mathrm{C}^*(\ker\mu, \ker\mu\cap P)$ is not known to be nuclear. We provide in the appendix a characterisation of nuclearity for $\mathrm{C}^*(G,P)$ through faithfulness of conditional expectations.

2. Background

Let G be a discrete group with a unital subsemigroup P such that $P\cap P^{-1}=\{e\}$. There is a partial order on G defined by $x\leq y$ if and only if $x^{-1}y\in P$. It is easy to see that $\leq$ is left-invariant. In [27], Nica defined the pair (G,P) to be *quasi-lattice ordered* if the following is satisfied:

(QL) any $n\geq 1$ and $x_1,x_2,\ldots,x_n\in G$ which have common upper bounds in P also have a least common upper bound in P.

We denote the least common upper bound of $x_1,\ldots,x_n$ by $x_1\vee\cdots\vee x_n$ if it exists. In this case, we sometimes write $x_1\vee\cdots\vee x_n<\infty$. We then write $x_1\vee\cdots\vee x_n=\infty$ if $x_1,\ldots,x_n$ have no upper bound in P. As already observed by Nica, (G,P) is quasi-lattice ordered if and only if

(QL1) any $x\in PP^{-1}$ has a least upper bound in P; and
(QL2) any $x,y\in P$ with a common upper bound have a least common upper bound.

In fact, (QL) and (QL1) are equivalent. See [8, Lemma 7] for this and further equivalent formulations of (QL). Only (QL2) is needed to define the C^*-algebras studied by Nica in [27].

In this paper, we will consider pairs (G,P) as above satisfying only (QL2). We will use the terminology of Exel in [13, Definition 32.1(v)]:

Definition 2.1. Let G be a discrete group with a unital subsemigroup P such that $P\cap P^{-1}=\{e\}$ and with the induced left-invariant order on G. If (G,P) satisfies (QL2), then we say that (G,P) is a *weakly quasi-lattice ordered group* or that (G,P) is a *weak quasi-lattice*.

To make clear the relationship to right LCM semigroups, we note that we can rephrase (QL2) as follows:

(QL2') for any $x,y\in P$ with $xP\cap yP\neq\emptyset$ there exists a unique $z\in P$ such that $xP\cap yP=zP$.

In right LCM semigroups, the z may not be unique.

Consider $\ell^2(P)$ with orthonormal basis $\{e_x : x \in P\}$. For each $x \in P$, there is an isometry T_x on $\ell^2(P)$ such that $T_x e_y = e_{xy}$ for $y \in P$. Because P is left-cancellative, $T : P \to B(\ell^2(P))$ is an isometric representation of P; it is called the *Toeplitz representation* (and sometimes the left-regular representation). Nica observed that

$$T_x T_x^* T_y T_y^* = \begin{cases} T_{x\vee y} T_{x\vee y}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases} \tag{2.1}$$

The *Toeplitz algebra* is the C*-algebra $\mathcal{T}(G,P)$ generated by the Toeplitz representation, that is,

$$\mathcal{T}(G,P) := \mathrm{C}^*(\{T_x : x \in P\}) \subseteq B(\ell^2(P)).$$

An isometric representation of P satisfying (2.1) is called *Nica covariant*. There is a C*-algebra, denoted $\mathrm{C}^*(G,P)$, that is universal for Nica-covariant representations. It follows from Nica covariance that if w is the universal Nica covariant representation in $\mathrm{C}^*(G,P)$, then

$$\mathrm{C}^*(G,P) = \overline{\mathrm{span}}\{w_x w_y^* : x, y \in P\}.$$

Notice that both $\mathrm{C}^*(G,P)$ and $\mathcal{T}(G,P)$ are defined using only (QL2).

The Toeplitz representation induces a surjection

$$\pi_T : \mathrm{C}^*(G,P) \to \mathcal{T}(G,P).$$

Following Nica, we say that (G,P) is *amenable* if π_T is an isomorphism. The usual approach to prove the amenability of (G,P) involves a conditional expectation.

Let B be a C*-subalgebra of a C*-algebra C. Then $E\colon C \to B$ is a *conditional expectation* if E is linear, positive and contractive such that $E(b) = b$, $E(bc) = bE(c)$ and $E(cb) = E(c)b$ for all $b \in B$ and $c \in C$. Equivalently, $E\colon C \to B$ is a linear contractive map with $E(b) = b$ for all $b \in B$ by [35] (see also [1, Theorem II.6.10.2]). A conditional expectation is called *faithful* if $E(c^*c) = 0$ implies $c = 0$.

Nica proves that there is a conditional expectation

$$E\colon \mathrm{C}^*(G,P) \to \overline{\mathrm{span}}\{w_x w_x^* : x \in P\} \text{ such that } E(w_x w_y^*) = \begin{cases} w_x w_x^* & \text{if } x = y \\ 0 & \text{else.} \end{cases}$$

By [27, Proposition on page 34], a quasi-lattice (G,P) is amenable if and only if E is faithful. Again, Nica only needed (QL2) to define the amenability of (G,P) and to characterise it with the conditional expectation E. Moreover, by Proposition A.1, nuclearity of $\mathrm{C}^*(G,P)$ can also be characterised in terms of faithfulness of the canonical conditional expectation

$$E_{A,\max}\colon A \otimes_{\max} \mathrm{C}^*(G,P) \to A \otimes_{\max} \overline{\mathrm{span}}\{w_x w_x^* : x \in P\}$$

for all unital C*-algebras A.

3. Controlled maps – then and now

The first definition of a controlled map $\mu\colon (G,P) \to (K,Q)$ between quasi-lattice ordered groups appeared in [17, Proposition 4.2]; the existence of such a map into an amenable group K implies that (G,P) is amenable in the sense of Nica [27]. Because this first definition asked that $\ker \mu \cap P = \{e\}$, it does not apply to the quasi-lattices associated to the Baumslag-Solitar groups which were shown to be amenable in [7]. The more general notion of controlled maps of [15, Definition 3.1] allows for a nontrivial kernel, and applies to the "height map" of most, but not all, quasi-lattice ordered groups associated to the Baumslag-Solitar groups (see Examples 3.4 and 3.5

below). In Definition 3.6 below we give the most general definition of controlled maps to date: in particular it allows the infinite descending chains appearing in Example 3.5.

Here we start with Definition 3.1, a slight modification of [15, Definition 3.1] which allows weak quasi-lattices ([15, Definition 3.1] could have been stated in this generality).

Definition 3.1. Let (G, P) and (K, Q) be weak quasi-lattices. Suppose that $\mu\colon G \to K$ is an order-preserving group homomorphism. Then μ is *controlled*, in the sense of an Huef, Raeburn and Tolich, if it has the following properties:

(a) For $x, y \in P$ with $x \vee y < \infty$, we have $\mu(x) \vee \mu(y) = \mu(x \vee y)$.

(b) Let $q \in Q$. Set $\Sigma_q = \{\sigma \in \mu^{-1}(q) \cap P : \sigma \text{ is minimal}\}$.

   (i) If $x \in \mu^{-1}(q) \cap P$, then there exists $\sigma \in \Sigma_q$ such that $\sigma \leq x$.

   (ii) If $\sigma, \tau \in \Sigma_q$, then $\sigma \vee \tau < \infty \Rightarrow \sigma = \tau$.

Example 3.2. Let $\mathbb{F}$ be the free group on a countable set of generators and let $\mathbb{F}^+$ be the unital free semigroup, that is $\mathbb{F}^+$ is the smallest subsemigroup containing the identity e and the generators. Then $(\mathbb{F}, \mathbb{F}^+)$ is a quasi-lattice [27, §2.2]. Let $\mu\colon \mathbb{F} \to \mathbb{Z}$ be the homomorphism that sends each generator to 1. To see that $\mu\colon (\mathbb{F}, \mathbb{F}^+) \to (\mathbb{Z}, \mathbb{N})$ is a controlled map in the sense of Definition 3.1, we note that if $x \in \mathbb{F}^+$, then $\mu(x)$ is precisely the length of x. Suppose that $x, y \in \mathbb{F}^+$ with $x \vee y < \infty$. Then either $x = yp$ or $y = xq$ for some $p, q \in \mathbb{F}^+$. Say $x = yp$. Then $\mu(x) \vee \mu(y) = \mu(x) = \mu(x \vee y)$, giving (a). For (b) we observe that for each $n \in \mathbb{N}$ we have $\Sigma_n = \mu^{-1}(n) \cap \mathbb{F}^+$, and that no two distinct elements of Σ_n are comparable. Thus μ is controlled in the sense of Definition 3.1. Here $\ker \mu \cap \mathbb{F}^+ = \{e\}$, and hence μ is a controlled map in the sense of Laca and Raeburn [17, Proposition 4.2] as well.

Using Example 3.2 it is easy to construct an example of a weak quasi-lattice with a controlled map. This example of a weak quasi-lattice which is not a quasi-lattice was obtained by Scarparo in [31] and is discussed on page 249 of [13].

Example 3.3. Consider the free group $\mathbb{F}$ on 2 generators a and b and let $\mathbb{P} = \{e, bw : w \in \mathbb{F}^+\}$. Then $(\mathbb{F}, \mathbb{P})$ is a weak quasi-lattice. Again, let $\mu\colon \mathbb{F} \to \mathbb{Z}$ be the homomorphism that sends each generator to 1. Then μ is controlled in the sense of Definition 3.1 with $\Sigma_n = \mu^{-1}(n) \cap \mathbb{P}$. Again $\ker \mu \cap \mathbb{P} = \{e\}$.

Exel observed on page 249 of [13] that $\mathbb{P} = \{e, bw : w \in \mathbb{F}^+\}$ is isomorphic to the positive cone in the free group $\mathbb{F}_\infty$ on infinitely many generators, and hence $(\mathbb{F}_\infty, \mathbb{P})$ is a quasi-lattice ordered group. Prompted by a question by Marcelo Laca, we show in Proposition 3.10 below that the positive cone P of the Baumslag-Solitar group discussed in Example 3.9 cannot be embedded in a group G such that (G, P) is a quasi-lattice ordered group.

Example 3.4. Let $c, d \geq 1$ and consider the Baumslag-Solitar group

$$G = \langle a, b : ab^c = b^d a \rangle.$$

Let P be the unital subsemigroup generated by a and b. Then (G, P) is quasi-lattice ordered by [33, Theorem 2.11]. Let $\mu\colon G \to \mathbb{Z}$ be the "height map", that is, the homomorphism such that $\mu(a) = 1$ and $\mu(b) = 0$. We will show that μ is a controlled map in the sense of Definition 3.1.

To see that μ is order preserving, suppose that $x \leq y \in G$. Then $x^{-1}y \in P$ and $\mu(y) - \mu(x) = \mu(x^{-1}y) \in \mu(P) = \mathbb{N}$. Thus $\mu(x) \leq \mu(y)$ in $\mathbb{Z}$.

Let $x, y \in P$ with $x \vee y < \infty$. Say $\mu(y) \geq \mu(x)$. If $\mu(y) > \mu(x)$, then by [7, Lemma 3.4] there exists $t \in \mathbb{N}$ such that $x \vee y = yb^t$. So $\mu(x \vee y) = \mu(y) = \mu(x) \vee \mu(y)$. If $\mu(y) = \mu(x)$, then by [7, Lemma 3.4] there exists $t \in \mathbb{N}$ such that either $x \vee y = yb^t$ or $x \vee y = xb^t$. This also implies $\mu(x \vee y) = \mu(x) \vee \mu(y)$.

For (b) we use the normal form for elements of P to see that

$$\mu^{-1}(q) \cap P = \{b^{s_0}ab^{s_1} \cdots b^{s_{q-1}}ab^{s_q} : 0 \leq s_0, \dots, s_{q-1} < d, s_q \in \mathbb{N}\} \quad \text{and}$$
$$\Sigma_q = \{b^{s_0}ab^{s_1} \cdots b^{s_{q-1}}a : 0 \leq s_0, \dots, s_{q-1} < d\}.$$

Looking at the two sets, (bi) clearly holds. Let $x, y \in \Sigma_q$ with $x \vee y < \infty$. Then, by [7, Lemma 3.4], either $x \vee y = yb^t$ or $x \vee y = xb^t$ for some t, and since $x, y \in \Sigma_q$ we get that $x = y$. This gives (bii).

The next example, provided by Ilija Tolich, shows how (bi) in Definition 3.1 can fail.

Example 3.5. Let $d \geq 1$ and consider the Baumslag-Solitar group

$$G = \langle a, b : ab = b^{-d}a \rangle.$$

This fits into case (BS3) in [33] with $c = 1$. Let P be the subsemigroup generated by a and b. By [33, Lemma 2.12], $PP^{-1} = P \cup P^{-1}$, and it follows easily that (G, P) is a quasi-lattice[1].

Let $\mu\colon G \to \mathbb{Z}$ be the height map as in the previous example. Then $\mu^{-1}(1) \cap P$ consists of elements of the form b^iab^j for some $i, j \in \mathbb{N}$, and using the relation $ab = b^{-d}a$ we can write that as $b^m a$ for some $m \in \mathbb{Z}$. We have $b^m a \leq b^n a$ if and only if d divides $m - n$ and $n \leq m$. So for $0 \leq k < d$ we have the infinite chains

$$\dots \leq b^{3d+k}a \leq b^{2d+k}a \leq b^{d+k}a \leq b^k a \leq b^{-1d+k}a \leq b^{-2d+k}a \leq \dots$$

partioning $\mu^{-1}(1) \cap P$. Thus $\Sigma_1 = \emptyset$, and then (bi) in Definition 3.1 implies that μ is not a controlled map.

Definition 3.6. Let (G, P) and (K, Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon G \to K$ is an order-preserving group homomorphism. We say that μ is *controlled* if it has the following properties:

(a) For $x, y \in P$ with $x \vee y < \infty$, we have $\mu(x) \vee \mu(y) = \mu(x \vee y)$.

(b) For all $q \in Q$, there exist a set Λ_q and a family of sets $\{S_\lambda : \lambda \in \Lambda_q\}$ such that:

  (i) $\mu^{-1}(q) \cap P$ is the disjoint union $\bigsqcup_{\lambda \in \Lambda_q} S_\lambda$; and

  (ii) for each $\lambda \in \Lambda_q$, there is a decreasing sequence $(s_n^\lambda)_{n \in \mathbb{N}}$ in S_λ such that

$$S_\lambda = \bigcup_{n \in \mathbb{N}} \{x \in \mu^{-1}(q) \cap P \colon x \geq s_n^\lambda\}.$$

Lemma 3.7. *Suppose that $\mu\colon G \to K$ is a controlled map in the sense of Definition* 3.6. *Let $q \in Q$ and $\lambda_1 \neq \lambda_2 \in \Lambda_q$. Then $p_1 \vee p_2 = \infty$ for all $p_1 \in S_{\lambda_1}$ and $p_2 \in S_{\lambda_2}$.*

[1] At the end of the proof of [33, Lemma 2.12], it is stated that (G, P) is "totally ordered, hence quasi-lattice ordered", but (G, P) is not totally ordered. For example, $b^m a \leq b^n a$ if and only if d divides $m - n$ and $n \leq m$. In particular, if d does not divide $m - n$, then $b^n a$ and $b^m a$ are not comparable.

Proof. Looking for a contradiction, we suppose that there exist $x \in S_{\lambda_1}$ and $y \in S_{\lambda_2}$ such that $x \vee y < \infty$. Then $\mu(x \vee y) = \mu(x) \vee \mu(y) = q$ gives $x \vee y \in \mu^{-1}(q) \cap P$. Also, there exist $m, n \in \mathbb{N}$ such that $x \geq s_m^{\lambda_1}$ and $y \geq s_n^{\lambda_2}$. But now $x \vee y \geq x \geq s_m^{\lambda_1}$ implies that $x \vee y \in S_{\lambda_1}$. Similarly, $x \vee y \in S_{\lambda_2}$. This contradicts that $\mu^{-1}(q) \cap P$ is the disjoint union $\bigsqcup_{\lambda \in \Lambda_q} S_\lambda$. □

Lemma 3.8. *Let (G, P) and (K, Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon G \to K$ is an order-preserving group homomorphism. If μ is a controlled map in the sense of Definition* 3.1*, then it is controlled in the sense of Definition* 3.6*.*

Proof. Let μ be a controlled map in the sense of Definition 3.1. We need to verify item (b) of Definition 3.6. Fix $q \in Q$. Set $\Lambda_q = \Sigma_q$. For each $\lambda \in \Lambda_q$, set

$$S_\lambda = \{x \in \mu^{-1}(q) \cap P\colon x \geq \lambda\}.$$

Suppose that $S_{\lambda_1} \cap S_{\lambda_2} \neq \emptyset$ and let $x \in S_{\lambda_1} \cap S_{\lambda_2}$. Then $x \geq \lambda_i$ and hence $\lambda_1 \vee \lambda_2 < \infty$. Since $\lambda_1, \lambda_2 \in \Sigma_q$, this implies $\lambda_1 = \lambda_2$ by item (bii) of Definition 3.1. Together with (bi) of Definition 3.1 we get that

$$\mu^{-1}(q) \cap P = \bigsqcup_{\lambda \in \Lambda_q} S_\lambda$$

which is item (bi) of Definition 3.6. For item (bii) of Definition 3.6 we just take the sequence $(s_n^\lambda)_{n \in \mathbb{N}}$ to be the constant sequence $(\lambda)_{n \in \mathbb{N}}$. □

Example 3.9. Let $c, d \geq 1$ and consider the Baumslag–Solitar group

$$G = \langle a, b\colon ab^c = b^{-d}a \rangle.$$

Let P be the (unital) subsemigroup of G generated by a and b. Then (G, P) is a weakly quasi-lattice ordered group by [33, Proposition 2.10] and it is a quasi-lattice ordered group if and only if $c = 1$ by [33, Lemma 2.12].

Let $\mu\colon G \to \mathbb{Z}$ be the height map as in Examples 3.4 and 3.5. Since $\mu(P) \subseteq \mathbb{N}$, it follows that μ is order-preserving. We will show that μ is a controlled map from (G, P) to $(\mathbb{Z}, \mathbb{N})$ in the sense of Definition 3.6.

To do this, we will use the fact that every $x \in P$ with height $\mu(x) = q$ has a unique normal form

$$x = b^{s_0} a b^{s_1} \cdots b^{s_{q-1}} a b^{s_q} \tag{3.1}$$

where $0 \leq s_0, \ldots, s_{q-1} < d$, $s_q \in \mathbb{Z}$ and $\mu(x) = 0 \implies s_q \geq 0$ (see [33, Proposition 2.1 and Remark 2.4]).

Let $x, y \in P$ with $x \vee y < \infty$. We claim that x and y are comparable. To see this, we consider x, y in normal form (3.1), say

$$x = b^{s_0} a b^{s_1} \cdots b^{s_{q-1}} a b^{s_q} \text{ and } y = b^{t_0} a b^{t_1} \cdots b^{t_{r-1}} a b^{t_r}$$

Without loss of generality, suppose that $\mu(x) \leq \mu(y)$ (otherwise switch them). Then $s_i = t_i$ by [33, Proposition 2.10] for $i \leq q - 1$, and hence

$$x^{-1}y = b^{t_q - s_q} a b^{t_{q+1}} a \cdots b^{t_{r-1}} a b^{t_r}.$$

If $t_q - s_q \geq 0$, then clearly $x^{-1}y \in P$ and $x \leq y$. Suppose that $t_q - s_q < 0$. Choose $n \in \mathbb{N}$ such that $dn \geq s_q - t_q$. Then using the relation $a = b^d a b^c$ repeatedly we get that

$$x^{-1}y = b^{nd + t_q - s_q} a b^{nc} b^{t_{q+1}} a \cdots b^{t_{r-1}} a b^{t_r}.$$

Since $b^{nd + t_q - s_q} a b^{nc} \in P$ and $b^{t_{q+1}} a \cdots b^{t_{r-1}} a b^{t_r} \in P$, so is $x^{-1}y$, and $x \leq y$ as claimed. Now $\mu(x \vee y) = \mu(y) = \mu(x) \vee \mu(y)$ and μ satisfies (a) of Definition 3.6.

For (b), we start by observing that $\mu^{-1}(0)\cap P = \{b^n \colon n \in \mathbb{N}\}$. Then set $\Lambda_0 \coloneqq \{e\}$, $S_e \coloneqq \mu^{-1}(0) \cap P$ and $s_n^e \coloneqq e$ for all n.

Let $q \in \mathbb{N} \setminus \{0\}$. Using the normal form and the observation in [33, Lemma 2.6] that $x \in P$ and $\mu(x) > 0$ implies $xb^m \in P$ for all $m \in \mathbb{Z}$, we deduce that

$$\mu^{-1}(q) \cap P = \{b^{s_0}ab^{s_1}\cdots b^{s_{q-1}}ab^m \colon 0 \le s_0, \dots, s_{q-1} < d, m \in \mathbb{Z}\}.$$

Set $\Lambda_q \coloneqq [0, d)^q \cap \mathbb{N}^q$. For each $\lambda = (s_0, s_1, \dots, s_{q-1}) \in \Lambda_q$, set

$$S_\lambda \coloneqq \{b^{s_0}ab^{s_1}\cdots b^{s_{q-1}}ab^m \colon m \in \mathbb{Z}\}.$$

Then $S_\lambda \cap S_{\lambda'} = \emptyset$ if $\lambda \neq \lambda'$ by uniqueness of the normal form (3.1). This gives (bi).

For (bii), let $\lambda = (s_0, s_1, \dots, s_{q-1}) \in \Lambda_q$ and $n \in \mathbb{N}$, and set

$$s_n^\lambda \coloneqq b^{s_0}ab^{s_1}\cdots b^{s_{q-1}}ab^{-n}.$$

If $m \le n$, then $(s_n^\lambda)^{-1}s_m^\lambda = b^{n-m} \in P$, and $(s_n^\lambda)_{n\in\mathbb{N}}$ is a decreasing sequence in S_λ. It remains to verify that

$$S_\lambda = \bigcup_{n\in\mathbb{N}} \{x \in \mu^{-1}(q) \cap P \colon x \ge s_n^\lambda\}. \tag{3.2}$$

Consider $x = b^{s_0}ab^{s_1}\cdots b^{s_{q-1}}ab^m \in S_\lambda$ where $m \in \mathbb{Z}$. Choose $n \in \mathbb{N}$ such that $n + m \ge 0$. Then $(s_n^\lambda)^{-1}x = b^{n+m} \in P$. On the other hand, let $x \in \mu^{-1}(q) \cap P$ and suppose that $x \ge s_n^\lambda$ for some $n \in \mathbb{N}$. Then $(s_n^\lambda)^{-1}x \in P$. Since $\mu((s_n^\lambda)^{-1}x) = 0$ we get $(s_n^\lambda)^{-1}x = b^m$ for some $m \in \mathbb{N}$. Thus $x = s_n^\lambda b^m = b^{s_0}ab^{s_1}\cdots b^{s_{q-1}}ab^{m+n} \in S_\lambda$. This proves (3.2). Thus μ is controlled in the sense of Definition 3.6.

Prompted by the weak quasi-lattice of Example 3.3, Marcelo Laca asked the following question: let (G, P) be a weakly quasi-lattice ordered group. Can P be always embedded in a group H such that (H, P) is quasi-lattice ordered? Here we show by example that the answer to this question is "no".

Proposition 3.10. *Let $c > 1$ and $d \ge 1$ such that c does not divide d. Consider the Baumslag–Solitar group*

$$G = \langle a, b \colon ab^c = b^{-d}a\rangle.$$

Let P be the unital subsemigroup of G generated by a and b. Then there is no embedding of P in a group H such that (H, P) is quasi-lattice ordered.

Proof. Suppose that there were a group H and an embedding of P in H. We identify P with its copy in H. To show that (H, P) is not quasi-lattice ordered, we will show that

$$h = a(ab^d)^{-1} \in PP^{-1}$$

does not have a least upper bound in P.

For all $n \in \mathbb{N}$ we have

$$h^{-1}b^{nd}a = ab^d(a^{-1}b^{nd}a) = ab^db^{-nc} = ab^{-nc}b^d = b^{nd}ab^d \in P.$$

Thus $h \le b^{nd}a$ for all $n \in \mathbb{N}$, and we have an infinite descending chain

$$(h \le)\cdots \le b^{(n+1)d}a \le b^{nd}a \le \cdots.$$

We will show that no element $x \in P$ satisfies

$$h \le x \le b^{nd}a \text{ for all } n \in \mathbb{N}.$$

Assume, looking for a contradiction, that there exists $x \in P$ such that $h \le x \le b^{nd}a$ for all $n \in \mathbb{N}$. Since $e \le x \le b^{nd}a$ and the height map $\mu \colon G \to \mathbb{Z}$ preserves the order, we have either $\mu(x) = 0$ or 1. We will consider these two cases separately.

First, suppose that $\mu(x) = 0$. Then $x = b^k$ for some $k \in \mathbb{N}$ by the normal form for elements of P. Since $h \leq x$, we have $x = hp$ for some $p \in P$. In particular, $\mu(p) \geq 0$. We claim that $\mu(p) = 0$. Assume, looking for a contradiction, that $\mu(p) > 0$. Then

$$h^{-1} = px^{-1} = pb^{-k},$$

and the normal form for elements of G with positive height implies that $h^{-1} \in P$. Now

$$h^{-1} = pb^{-k} \iff ab^d = pb^{-k}a,$$

and it makes sense to apply μ to the elements ab^d, $pb^{-k}a$, pb^{-k} and a of P. But then $1 = \mu(ab^d) = \mu(pb^{-k}a) = \mu(pb^{-k}) + \mu(a) > 1$, a contradiction. It follows that $\mu(p) = 0$, as claimed. Again, using the normal form, we have $p = b^l$ for some $l \in \mathbb{N}$. Now $ab^d = b^l b^{-k} a = b^{l-k}a$, which implies that $ab^{-d} = b^{k-l}a$. But this is impossible since $c \nmid d$.

Second, suppose that $\mu(x) = 1$. Since $x \leq b^{nd}a$ for each $n \in \mathbb{N}$, there exists $p_n \in P$ such that we have $b^{nd}a = xp_n$. Then $\mu(p_n) = 0$. By the normal form, one has $x = b^{s_0}ab^{s_1}$ with $0 \leq s_0 < d$ and $s_1 \in \mathbb{Z}$ and $p_n = b^{\ell_n}$ for some $\ell_n \in \mathbb{N}$. Now

$$b^{nd}a = xp_n \iff b^{nd}a = b^{s_0}ab^{s_1}b^{\ell_n} \iff ab^{-nc} = b^{s_0}ab^{s_1+l_n}.$$

This implies that $s_0 = 0$ and $-nc = s_1 + l_n$ for all $n \in \mathbb{N}$ by uniqueness of normal forms. But now $\ell_n = -nc - s_1 \to -\infty$ as $n \to \infty$, contradicting that $\ell_n \in \mathbb{N}$.

We conclude that no $x \in P$ satisfies $h \leq x \leq b^{-nd}a$ for all $n \in \mathbb{N}$. In particular, h has no least upper bound in P. Thus (H, P) is not quasi-lattice ordered. □

Lemma 3.11. *Let (G, P) and (K, Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon G \to K$ is an order-preserving group homomorphism.*

(a) *If (G, P) is a quasi-lattice ordered group, then so is $(\ker\mu, \ker\mu \cap P)$.*
(b) *Suppose that for $x, y \in P$ with $x \vee y < \infty$, we have $\mu(x) \vee \mu(y) = \mu(x \vee y)$. Then $(\ker\mu, \ker\mu \cap P)$ is a weakly quasi-lattice ordered group.*

Proof. This is what was actually proved in [15, Lemma 3.3] though its statement consists only of item (a). □

4. Nuclearity

Let (G, P) be a weakly quasi-lattice ordered group. Suppose that there are a discrete group K and a group homomorphism $\mu\colon G \to K$. We write u for the universal unitary representation generating $C^*(K)$. By Lemma 3.4 of [15] there is a nondegenerate[2] injective coaction $\delta_\mu\colon C^*(G, P) \to C^*(G, P) \otimes_{\min} C^*(K)$ such that

$$\delta_\mu(w_p) = w_p \otimes u_{\mu(p)} \quad \text{for } p \in P. \tag{4.1}$$

Let τ be the trace on $C^*(K)$ such that $\tau(u_k)$ is 1 if $k = e$ and 0 otherwise. By [15, Lemma 3.5],

$$\Psi_\mu = (\mathrm{id} \otimes \tau) \circ \delta_\mu \tag{4.2}$$

is a conditional expectation on $C^*(G, P)$ such that

$$\Psi_\mu(w_p w_q^*) = \begin{cases} w_p w_q^* & \text{if } \mu(p) = \mu(q) \\ 0 & \text{else,} \end{cases}$$

[2]Full coactions of discrete groups may not be nondegenerate – see the Erratum [16]. While nondegeneracy of δ_μ is not mentioned in the statement of [15, Lemma 3.4], it is in fact proved there.

and the range of Ψ_μ is

$$\overline{\operatorname{span}}\{w_p w_q^* : p, q \in P, \mu(p) = \mu(q)\}.$$

Moreover, if K is amenable, then Ψ_μ is faithful.

Proposition 4.1. *Let (G, P) be a weak quasi-lattice. Suppose that there exist an amenable group K and a homomorphism $\mu\colon G \to K$. Then $\mathrm{C}^*(G, P)$ is nuclear if and only if $\overline{\operatorname{span}}\{w_p w_q^* : p, q \in P$ and $\mu(p) = \mu(q)\}$ is nuclear.*

Proof. Let δ_μ be the coaction defined by (4.1). The proposition follows from [29, Corollory 2.17] after observing that $\overline{\operatorname{span}}\{w_p w_q^* : p, q \in P, \mu(p) = \mu(q)\}$ is the fixed-point algebra $\mathrm{C}^*(G, P)^{\delta_\mu}$ of the cosystem $(\mathrm{C}^*(G, P), K, \delta_\mu)$. (The amenability of K is only needed for the direction $\mathrm{C}^*(G, P)^{\delta_\mu}$ nuclear implies $\mathrm{C}^*(G, P)$ nuclear.)

For the observation, we recall from [29, Lemma 1.3] that the fixed-point algebra $\mathrm{C}^*(G, P)^{\delta_\mu}$ of $(\mathrm{C}^*(G, P), K, \delta_\mu)$ is

$$\mathrm{C}^*(G, P)^{\delta_\mu} = \{a \in \mathrm{C}^*(G, P) : \delta_\mu(a) = a \otimes u_e\}.$$

Let $p, q \in P$ such that $\mu(p) = \mu(q)$. Then $\delta_\mu(w_p w_q^*) = w_p w_q^* \otimes u_{\mu(p)} u_{\mu(q)}^* = w_p w_q^* \otimes u_e$, and it follows that $\overline{\operatorname{span}}\{w_p w_q^* : p, q \in P, \mu(p) = \mu(q)\} \subseteq \mathrm{C}^*(G, P)^{\delta_\mu}$.

For the other inclusion, let $a \in \mathrm{C}^*(G, P)^{\delta_\mu}$. Let Ψ_μ be the conditional expectation at (4.2). Then

$$\Psi_\mu(a) = (\operatorname{id} \otimes \tau) \circ \delta_\mu(a) = \operatorname{id} \otimes \tau(a \otimes u_e) = a,$$

and so a is in the range $\overline{\operatorname{span}}\{w_p w_q^* : p, q \in P, \mu(p) = \mu(q)\}$ of Ψ_μ. Thus

$$\overline{\operatorname{span}}\{w_p w_q^* : p, q \in P, \mu(p) = \mu(q)\} = \mathrm{C}^*(G, P)^{\delta_\mu}$$

as needed. □

Lemma 4.2. *Let (G, P) be a weak quasi-lattice. Suppose that there exist a group K and a homomorphism $\mu\colon G \to K$ such that $(\ker \mu, \ker \mu \cap P)$ is weakly quasi-lattice ordered.*

(a) *If $(\ker \mu, \ker \mu \cap P)$ is amenable, then $\mathrm{C}^*(\ker \mu, \ker \mu \cap P)$ is isomorphic to the C^*-subalgebra*

$$B_e := \overline{\operatorname{span}}\{w_\alpha w_\beta^* : \alpha, \beta \in P, \mu(\alpha) = e = \mu(\beta)\}$$

of $\mathrm{C}^*(G, P)$; the isomorphism is induced by the restriction of the universal representation $w : P \to \mathrm{C}^*(G, P)$ to $\ker \mu \cap P$.

(b) *If B_e is nuclear, then $\mathrm{C}^*(\ker \mu, \ker \mu \cap P)$ is nuclear. In particular, B_e and $\mathrm{C}^*(\ker \mu, \ker \mu \cap P)$ are isomorphic and $(\ker \mu, \ker \mu \cap P)$ is amenable.*

Proof. The restriction $w|$ of $w : P \to \mathrm{C}^*(G, P)$ to $\ker \mu \cap P$ is a Nica covariant representation of $\ker \mu \cap P$, and hence induces a representation $\pi_{w|} : \mathrm{C}^*(\ker \mu, \ker \mu \cap P) \to B_e$ which is surjective. Let T and S be the Toeplitz representations of P and $\ker \mu \cap P$ on $\ell^2(P)$ and $\ell^2(\ker \mu \cap P)$, respectively. We view $\ell^2(\ker \mu \cap P)$ as a closed subspace of $\ell^2(P)$. Since $\ell^2(\ker \mu \cap P)$ is invariant under B_e, we obtain a representation π_T^μ of B_e on $\ell^2(\ker \mu \cap P)$ by restriction. Then $\pi_T^\mu \circ \pi_{w|} = \pi_S$.

For (a), suppose that $(\ker \mu, \ker \mu \cap P)$ is amenable. Then π_S is injective and then $\pi_{w|}$ is the required isomorphism.

For (b), suppose that B_e is nuclear. Since π_T^μ is surjective, $\mathcal{T}(\ker \mu, \ker \mu \cap P)$ is a quotient of B_e and hence is nuclear as well. We claim that $\ker \mu \cap P$ satisfies the independence condition of [24, Definition 6.30]. It then follows from [24, Theorem 6.44] that $\mathrm{C}^*(\ker \mu, \ker \mu \cap P)$ is nuclear and that $(\ker \mu, \ker \mu \cap P)$ is amenable. Then item (a) completes the proof of (b).

To see that $\ker\mu \cap P$ satisfies the independence condition and that [24, Theorem 6.44] applies we need to do some reconciling. In [24], the full C*-algebra $C^*(Q)$ of a left-cancellative semigroup Q is defined as a full inverse semigroup C*-algebra (see also [28]). It is pointed out at the end of §6.6 of [24] that this is the same as the C*-algebra $C^*_S(Q)$ introduced in [20]. This $C^*_S(Q)$ is a quotient of a C*-algebra $C^*(Q)$ defined in [20, Definition 3.2]. If (K, Q) is a weak quasi-lattice, then the latter $C^*(Q)$ is the C*-algebra $C^*(K, Q)$ universal for Nica-covariant representations that we are studying in this paper (see [20, p. 4313]). Further, on page 4327 of [20], Li shows that if (K, Q) is a weak quasi-lattice, then $C^*(Q)$ and $C^*_S(Q)$ are isomorphic. If (K, Q) is a weak quasi-lattice, then Q satisfies the independence condition (see Definition 6.20 and Lemmas 6.31 and 6.32 of [24]). Thus [24, Theorem 6.4] applies as claimed. □

It follows from Lemma 4.2(b) that we can restate Theorem 1.1 as:

Theorem 4.3. *Let (G, P) and (K, Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon (G, P) \to (K, Q)$ is a controlled map in the sense of Definition 3.6, that K is amenable and that the C*-subalgebra*

$$B_e = \overline{\operatorname{span}}\{w_\alpha w_\beta^* : \alpha, \beta \in P, \mu(\alpha) = e = \mu(\beta)\}$$

of $C^*(G, P)$ is nuclear. Then $C^*(G, P)$ is nuclear. In particular, (G, P) is amenable.

Theorem 4.3 extends [23, Corollary 8.3] in two ways: our notion of controlled maps is weaker and we only assume that (G, P) and (K, Q) are weak quasi-lattices. Before we prove the theorem, we observe that the hypothesis that the C*-subalgebra B_e is nuclear is often easily seen to hold. It is equivalent to nuclearity of $C^*(\ker\mu, \ker\mu \cap P)$ by Lemma 4.2 since this has B_e as a quotient.

Corollary 4.4. *Let (G, P) and (K, Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon (G, P) \to (K, Q)$ is a controlled map in the sense of Definition 3.6. If the group generated by $\ker\mu \cap P$ is amenable, then $C^*(\ker\mu, \ker\mu \cap P)$ is nuclear.*

Proof. Let G' be the subgroup of G generated by $P' := \ker\mu \cap P$. Then the identity map on (G', P') is a controlled map. Here $\mathbb{C} = C^*(\ker \mathrm{id} \cap P') \cong B_e(G', P')$ is nuclear. If G' is amenable, Theorem 4.3 implies that $C^*(\ker\mu, \ker\mu \cap P)$ is nuclear. □

In view of the new definition of controlled maps, to prove Theorem 4.3 we need to re-analyse the structure of the fixed-point algebra

$$C^*(G, P)^{\delta_\mu} = \overline{\operatorname{span}}\{w_p w_q^* : p, q \in P \text{ and } \mu(p) = \mu(q)\}.$$

For this, we start with the analogues of [15, Lemmas 3.6 and 3.9].

Proposition 4.5. *Let (G, P) and (K, Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon (G, P) \to (K, Q)$ is a controlled map in the sense of Definition 3.6.*

(a) *Let $k \in Q$, let F be a finite subset of Λ_k and let $n \in \mathbb{N}$. Set*

$$B_k := \overline{\operatorname{span}}\{w_p w_q^* : p, q \in P, \mu(p) = k = \mu(q)\},$$
$$B_{k,n} := \overline{\operatorname{span}}\{w_{s_n^\lambda} d w_{s_n^\rho}^* : \lambda, \rho \in \Lambda_k \text{ and } d \in B_e\},$$
$$B_{k,n,F} := \operatorname{span}\{w_{s_n^\lambda} d w_{s_n^\rho}^* : \lambda, \rho \in F \text{ and } d \in B_e\}.$$

Then

(i) *B_k, $B_{k,n}$ and $B_{k,n,F}$ are C*-subalgebras of $C^*(G, P)^{\delta_\mu}$;*
(ii) *$B_{k,n,F}$ is isomorphic to $M_F(\mathbb{C}) \otimes B_e$;*

(iii) $B_{k,n} = \varinjlim B_{k,n,F}$ *is isomorphic to* $\mathcal{K}(\ell^2(\Lambda_k)) \otimes B_e$*;*
(iv) $B_{k,n} \subseteq B_{k,n+1}$ *and* $B_k = \varinjlim B_{k,n}$.

(b) *Let $\mathcal{I}$ be the set of all finite sets $I \subseteq Q$ that are closed under $\vee$ in the sense that $s, t \in I$ and $s \vee t < \infty$ implies that $s \vee t \in I$. Let $I \in \mathcal{I}$ and set*

$$C_I := \overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q) \in I\}.$$

Then
(i) C_I *is a* C^**-subalgebra of* $\mathrm{C}^*(G,P)^{\delta_\mu}$*;*
(ii) $C_I = \sum_{k\in I} B_k$[3]*; and*
(iii) $\mathrm{C}^*(G,P)^{\delta_\mu} = \varinjlim_{I\in\mathcal{I}} C_I$.

Proof. To see that B_k is a C*-subalgebra, take spanning elements $w_p w_q^*$ and $w_r w_s^* \in B_k$. Then, using Nica covariance, we deduce that

$$w_p w_q^* w_r w_s^* = \begin{cases} w_{pq^{-1}(q\vee r)} w^*_{sr^{-1}(q\vee r)} & \text{if } q \vee r < \infty \\ 0 & \text{else.} \end{cases}$$

Suppose that $q \vee r < \infty$. Then $w_p w_q^* w_r w_s^* \neq 0$. Since μ is a $\vee$-preserving homomorphism we have

$$\mu(pq^{-1}(q \vee r)) = \mu(p)\mu(q)^{-1}\big(\mu(q) \vee \mu(r)\big) = kk^{-1}k = k$$

and $\mu(sr^{-1}(q \vee r)) = k$ similarly. Thus $w_p w_q^* w_r w_s^* \in B_k$ and it follows that B_k is a C*-subalgebra of $\mathrm{C}^*(G,P)^{\delta_\mu}$. Similarly, using Nica covariance and Lemma 3.7, we have

$$\begin{aligned}&(w_{s_n^\lambda} w_\alpha w_\beta^* w^*_{s_n^\rho})(w_{s_n^{\lambda'}} w_\sigma w_\tau^* w^*_{s_n^{\rho'}}) \\ &\quad= \begin{cases} w_{s_n^\lambda} w_{\alpha\beta^{-1}(\beta\vee\sigma)} w^*_{\tau\sigma^{-1}(\beta\vee\sigma)} w^*_{s_n^{\rho'}} & \text{if } \rho = \lambda' \text{ and } \beta \vee \sigma < \infty \\ 0 & \text{else.} \end{cases}\end{aligned}$$

Since $\mu(\alpha\beta^{-1}(\beta \vee \sigma)) = e = \mu(\tau\sigma^{-1}(\beta \vee \sigma))$ it follows that $B_{k,n}$ is a C*-subalgebra of $\mathrm{C}^*(G,P)^{\delta_\mu}$.

Next we show that $B_{k,n,F}$ is isomorphic to $M_F(\mathbb{C}) \otimes B_e$. This will complete the proof of (ai) as well as giving (aii). Let $\lambda, \lambda', \rho, \rho' \in F$. Then

$$w_{s_n^\lambda} w^*_{s_n^\rho} w_{s_n^{\lambda'}} w^*_{s_n^{\rho'}} = \begin{cases} w_{s_n^\lambda} w^*_{s_n^{\rho'}} & \text{if } \rho = \lambda' \\ 0 & \text{else} \end{cases}$$

using Lemma 3.7. Thus $\{w_{s_n^\lambda} w^*_{s_n^\rho} : \lambda, \rho \in F\}$ is a set of matrix units in $B_{k,n}$. This gives an injective homomorphism $\theta\colon M_F(\mathbb{C}) \to \overline{B_{k,n,F}}$ that maps the matrix units $\{E_{\lambda,\rho}\colon \lambda, \rho \in F\}$ in $M_F(\mathbb{C})$ to $\{w_{s_n^\lambda} w^*_{s_n^\rho} : \lambda, \rho \in F\}$ in $B_{k,n,F}$. It is easy to check that the formula

$$\psi(d) = \sum_{\lambda\in F} w_{s_n^\lambda} d w^*_{s_n^\lambda}$$

gives an injective homomorphism $\psi : B_e \to \overline{B_{k,n,F}}$ such that

$$\theta(E_{\lambda,\rho})\psi(d) = w_{s_n^\lambda} d w^*_{s_n^\rho} = \psi(d)\theta(E_{\lambda,\rho}).$$

It follows that the ranges of θ and ψ commute. Now the universal property of the maximal tensor product gives a homomorphism $\theta \otimes \psi$ of $M_F(\mathbb{C}) \otimes B_e$ into $\overline{B_{k,n,F}}$.

[3]That $C_I = \sum_{k\in I} B_k$ was also asserted in [15, Lemma 3.9], but the proof given there is not correct; the mistake is in the statement that "the finite span of closed subalgebras is closed".

By [30, Theorem B.18]

$$M_F(\mathbb{C}) \otimes B_e = \operatorname{span}\{E_{\lambda,\rho} \otimes d : \lambda, \rho \in F \text{ and } d \in B_e\},$$

with no closure. So the range of $\theta \otimes \psi$ is spanned by $\theta(E_{\lambda,\rho})\psi(d) = w_{s_n^\lambda} d w_{s_n^\rho}^*$, and hence the range is $B_{k,n,F}$. (In particular, $B_{k,n,F}$ is closed.) Thus $B_{k,n,F}$ is isomorphic to $M_F(\mathbb{C}) \otimes B_e$.

Now direct the finite subsets of Λ_k by inclusion. Consider a spanning element $w_{s_n^\lambda} d w_{s_n^\rho}^* \in B_{k,n}$. Take $F = \{\lambda, \rho\}$. Then $w_{s_n^\lambda} d w_{s_n^\rho}^* \in B_{k,n,F}$. Thus $B_{k,n} = \varinjlim_F B_{k,n,F}$. It follows that $B_{k,n}$ is isomorphic to $\mathcal{K}(\ell^2(\Lambda_k)) \otimes B_e$, giving (aiii).

To see that $B_{k,n} \subseteq B_{k,n+1}$, consider a spanning element $w_{s_n^\lambda} w_\alpha w_\beta^* w_{s_n^\rho}^* \in B_{k,n}$. We have $s_{n+1}^\lambda \leq s_n^\lambda$, that is, there exists $\sigma \in P$ such that $s_n^\lambda = s_{n+1}^\lambda \sigma$. Notice that $\mu(\sigma) = e$ because $\mu(s_n^\lambda) = k = \mu(s_{n+1}^\lambda)$. Similarly, there exists $\tau \in \ker \mu \cap P$ such that $s_n^\rho = s_{n+1}^\rho \tau$. Now $w_{s_n^\lambda} w_\alpha w_\beta^* w_{s_n^\rho}^* = w_{s_{n+1}^\lambda} w_{\sigma\alpha} w_{\tau\beta}^* w_{s_{n+1}^\rho}^* \in B_{k,n+1}$. Thus $B_{k,n} \subseteq B_{k,n+1}$

Now take a spanning element $w_p w_q^* \in B_k$. By item (bii) of Definition 3.6 there exist $\lambda, \rho \in \Lambda_k$ and $m, n \in \mathbb{N}$ such that $s_m^\lambda \leq p$ and $s_n^\rho \leq q$. Say $m \leq n$. Then $s_n^\lambda \leq s_m^\lambda \leq p$. So there exist $\alpha, \beta \in \ker \mu \cap P$ such that $p = s_n^\lambda \alpha$ and $q = s_n^\rho \beta$. Now $w_p w_q^* = w_{s_n^\lambda} w_\alpha w_\beta^* w_{s_n^\rho}^* \in B_{k,n}$. Thus $B_k = \varinjlim B_{k,n}$, giving (aiv).

To see that C_I is a C*-subalgebra, let $p, q, r, s \in P$ such that $\mu(p) = \mu(q) \in I$ and $\mu(r) = \mu(s) \in I$. Then

$$w_p w_q^* w_r w_s^* = \begin{cases} w_{pq^{-1}(q\vee r)} w_{sr^{-1}(q\vee r)}^* & \text{if } q \vee r < \infty \\ 0 & \text{else.} \end{cases}$$

Suppose that $w_p w_q^* w_r w_s^* \neq 0$. Then $q \vee r < \infty$. Since μ is a controlled map we have

$$\mu(pq^{-1}(q \vee r)) = \mu(p)\mu(q)^{-1}\mu(q \vee r) = \mu(q) \vee \mu(r) = \mu(sr^{-1}(q \vee r)).$$

Since I is closed under $\vee$ we have $\mu(q) \vee \mu(r) \in I$, and hence

$$w_p w_q^* w_r w_s^* = w_{pq^{-1}(q\vee r)} w_{sr^{-1}(q\vee r)}^* \in C_I.$$

It follows that C_I is a C*-subalgebra. This gives (bi).

Let $I \in \mathcal{I}$. We prove item (bii) by induction on $|I|$. If $|I| = 1$, then $I = \{k\}$ for some $k \in Q$ and $C_I = B_k$. Now let $n \geq 1$ and assume: if $J \in \mathcal{I}$ and $|J| = n$, then $C_J = \sum_{k \in J} B_k$.

Suppose that $|I| = n + 1$. Let m be a minimal element of I and let $J = I \setminus \{m\}$. Then $J \in \mathcal{I}$ and $|J| = n$. By the induction hypothesis,

$$C_J = \sum_{k \in J} B_k.$$

Let $k \in J$. We claim that $B_k B_m \subseteq C_J$. To see the claim, let $p, q \in \mu^{-1}(k) \cap P$ and $r, s \in \mu^{-1}(m) \cap P$. Then the product of the spanning elements is

$$w_p w_q^* w_r w_s = \begin{cases} w_{pq^{-1}(q\vee r)} w_{sr^{-1}(q\vee r)}^* & \text{if } q \vee r < \infty \\ 0 & \text{otherwise.} \end{cases}$$

If $q \vee r = \infty$, then $w_p w_q^* w_r w_s = 0 \in B_k$. So suppose that $q \vee r < \infty$. Then $k \vee m = \mu(q) \vee \mu(r) < \infty$. Since m is minimal in I and $k \neq m$, we have $m < k \vee m \in I$. Thus $k \vee m \in J$. Now

$$\mu\big(pq^{-1}(q \vee r)\big) = \mu(p)\mu(q)^{-1}\big(\mu(q) \vee \mu(r)\big) = \mu(q) \vee \mu(r) = k \vee m \in J.$$

Similarly, $\mu(sr^{-1}(q \vee r)) \in J$. Thus $w_{pq^{-1}(q\vee r)} w^*_{sr^{-1}(q\vee r)} \in C_J$, and the claim follows.

The claim implies that

$$C_J B_m = \Big(\sum_{k\in J} B_k\Big) B_m = \sum_{k\in J} B_k B_m \subseteq C_J.$$

Similarly, $B_m C_J \subseteq C_J$. It follows that C_J is a closed two-sided ideal in the C*-algebra C_I. Clearly C_I is generated by C_J and the C*-subalgebra B_m of C_I. By [26, Theorem 3.1.7] we get that $C_I = C_J + B_m$, which proves item (bii).

Finally, for item (biii), we note that $\mathcal{I}$ is partially ordered by inclusion and is directed: if $I_1, I_2 \in \mathcal{I}$, then they are both contained in $I_1 \cup I_2 \cup (I_1 \vee I_2) \in \mathcal{I}$. Take a spanning element $w_p w_q^* \in \mathrm{C}^*(G,P)^{\delta_\mu}$. Then $\mu(p) = \mu(q)$, and $w_p w_q^* \in C_I$ where $I = \{\mu(p)\}$. Thus $\mathrm{C}^*(G,P)^{\delta_\mu} = \varinjlim_{I\in\mathcal{I}} C_I$. □

The next step is to prove the following lemma from which we can then deduce that the C_I's are nuclear provided that the B_k's are nuclear for all $k \in Q$.

Lemma 4.6. *Suppose that B is a nuclear* C**-subalgebra of a* C**-algebra C and $I \subseteq C$ is a nuclear ideal. Then $B + I$ is a nuclear* C**-algebra.*

Proof. Notice that $B + I$ is a C*-subalgebra of C by [26, Theorem 3.1.7]. The canonical inclusion $I \subseteq B + I$ together with the quotient map $B + I \to (B + I)/I$ produce a short exact sequence

$$0 \to I \to B + I \to (B + I)/I \to 0.$$

Now $(B+I)/I$ is nuclear because it is isomorphic to $B/(B\cap I)$ and nuclearity passes to quotients. Since I is nuclear as well and an extension of a nuclear C*-algebra by a nuclear ideal is again nuclear, we conclude that $B + I$ is also nuclear. □

Proof of Theorem 4.3. By Proposition 4.1, it suffices to show that the fixed-point algebra

$$\mathrm{C}^*(G,P)^{\delta_\mu} = \overline{\mathrm{span}}\{w_p w_q^* : p, q \in P \text{ and } \mu(p) = \mu(q)\}$$

is nuclear. To do this, we go back to the analysis of the structure of $\mathrm{C}^*(G,P)^{\delta_\mu}$ in Proposition 4.5. Direct limits of nuclear C*-algebras are nuclear. Since $\mathrm{C}^*(G,P)^{\delta_\mu} = \varinjlim_{I\in\mathcal{I}} C_I$, it suffices to show that each C_I is nuclear. We will prove this by induction on $|I|$. For $|I| = 1$, we have $C_I = B_k$ for some $k \in I$. Because B_k is a direct limit $\varinjlim_{n\in\mathbb{N}} B_{k,n}$, it suffices to show that each $B_{k,n}$ is nuclear. But $B_{k,n}$ is isomorphic to $\mathcal{K}(\ell^2(\Lambda_k)) \otimes B_e$. Since B_e is nuclear by assumption, each $B_{k,n}$ is nuclear. Thus B_k is nuclear, as needed.

Now fix $n \in \mathbb{N}$, $n > 1$ and suppose that C_I is nuclear for all $I \in \mathcal{I}$ with $|I| = n$. Let $I \in \mathcal{I}$ with $|I| = n+1$. Take a minimal element $m \in I$ and consider $J = I \setminus \{m\}$. Then $|J| = n$ and so C_J is nuclear by the induction hypothesis. As in the proof of Proposition 4.5, C_J is an ideal in C_I and $C_I = B_m + C_J$. Since B_m is also nuclear, Lemma 4.6 implies that C_I is nuclear as desired.

It follows that $\mathrm{C}^*(G,P)^{\delta_\mu} = \varinjlim_{I\in\mathcal{I}} C_I$ is nuclear as well. Thus $\mathrm{C}^*(G,P)$ is nuclear.

It follows from the nuclearity of $\mathrm{C}^*(G,P)$ that (G,P) is amenable [24, Theorem 6.42]. (That [24, Theorem 6.42] applies was discussed in the proof of Lemma 4.2.) □

5. Examples

5.1. Examples from §3.

Example 5.1. Let $\mathbb{F}$ be the free group on a countable set of generators and let $\mathbb{F}^+$ be the unital free semigroup. Also let $\mu\colon \mathbb{F} \to \mathbb{Z}$ be the homomorphism that sends each generator to 1. We observed in Example 3.2 that μ is a controlled map into an abelian group with $\ker\mu \cap \mathbb{F}^+ = \{e\}$. Since $\mathrm{C}^*(\ker\mu, \ker\mu \cap \mathbb{F}^+) \cong \mathbb{C}$ is nuclear, Theorem 1.1 implies that $\mathrm{C}^*(\mathbb{F}, \mathbb{F}^+)$ is nuclear. (Nica already observed that $(\mathbb{F}, \mathbb{F}^+)$ is amenable.)

Example 5.2. Consider the subsemigroup $\mathbb{P} = \{1, bw : w \in \mathbb{F}^+\}$ of $\mathbb{F}$ from Example 3.3, and let $\mu\colon \mathbb{F} \to \mathbb{Z}$ be the homomorphism that sends each generator to 1. Then μ is a controlled map into an abelian group with $\ker\mu \cap \mathbb{P} = \{e\}$. Thus $\mathrm{C}^*(\mathbb{F}, \mathbb{P})$ is nuclear and $(\mathbb{F}, \mathbb{P})$ is amenable by Theorem 1.1.

We already know from [7, Corollary A.7] that the quasi-lattice ordered group (G, P) associated to the Baumslag-Solitar group of Example 3.4 is amenable. We can now use Theorem 1.1 to see that $\mathrm{C}^*(G, P)$ is nuclear for all Baumslag-Solitar groups. Together with Proposition 3.10, this shows that Theorem 1.1 gives new examples of nuclear semigroup C*-algebras that do not come from quasi-lattice orders.

Example 5.3. Consider the Baumslag-Solitar groups $G = \langle a, b : ab^c = b^d a\rangle$ where either 1) $c, d \geq 1$ or 2) $c \geq 1$ and $d \leq -1$. Let P be the semigroup generated by a and b. We showed in Examples 3.4 and 3.9 that the height map $\mu\colon G \to \mathbb{Z}$ such that $\mu(a) = 1$ and $\mu(b) = 0$ is a controlled map into an abelian group. Here

$$\ker\mu \cap P = \{b^t : t \in \mathbb{N}\}$$

is isomorphic to $\mathbb{N}$. It follows that the subgroup of G generated by $\ker\mu \cap P$ is isomorphic to $\mathbb{Z}$ and hence amenable. Thus $\mathrm{C}^*(\ker\mu, \ker\mu \cap P)$ is nuclear by Corollary 4.4. Now Theorem 1.1 implies that $\mathrm{C}^*(G, P)$ is nuclear and that (G, P) is amenable.

5.2. HNN extensions, controlled maps and one-relator groups.

Here we consider certain weak quasi-lattice orders arising from HNN extensions of free groups. We were particularly inspired by the analysis of one-relator groups with positive presentation due to Li–Omland–Spielberg [25] and the work of an Huef–Raeburn–Tolich on HNN extensions [15] of quasi-lattice orders. Recall that a one-relator group is a group $G = \langle \Sigma; r\rangle$ with generating set Σ and a single defining relator r. We say that the presentation $(\Sigma; r)$ is *positive* if r is a relation of the form $u = v$, where u and v are nonempty words in Σ. In [25], Li, Omland and Spielberg analysed when the semigroup $P = \langle \Sigma; u = v\rangle^+$ defined by the same presentation is right LCM and provided sufficient conditions on the presentation $(\Sigma; u = v)$ for the pair (G, P) to be a weak quasi-lattice. Under some conditions on the words u and v, they found a graph model for the boundary quotient $\partial\mathrm{C}^*(G, P)$ if $|\Sigma| \geq 3$ and concluded, in particular, that $\mathrm{C}^*(G, P)$ and $\partial\mathrm{C}^*(G, P)$ are nuclear [25, Corollary 3.10]. Here we treat two different classes of presentations $(\Sigma; u = v)$, which are not covered by the results of [25]. If $|\Sigma| = 2$, we obtain nuclearity of the semigroup C*-algebras of Baumslag–Solitar semigroups as a particular case. More precisely, we will prove the following:

Corollary 5.4. *Let S be a nonempty set and let $\Sigma = S \sqcup \{\sigma\}$. Let u and w be nonempty words in S and let $G = \langle \Sigma; u\sigma = \sigma w\rangle$. Let $P = \langle \Sigma; u\sigma = \sigma w\rangle^+$ be the* *semigroup defined by the same relation. Then (G,P) is an amenable quasi-lattice ordered group and $\mathrm{C}^*(G,P)$ is nuclear.*

Corollary 5.5. *Let S be a nonempty set and let $\Sigma = S \sqcup \{\sigma\}$. Let u and w be nonempty words in S and let $G = \langle \Sigma; u\sigma w = \sigma\rangle$. Let $P = \langle \Sigma; u\sigma w = \sigma\rangle^+$ be the semigroup defined by the same relation. Then (G,P) is an amenable weakly quasi-lattice ordered group and $\mathrm{C}^*(G,P)$ is nuclear.*

We should observe that the proof of Corollary 5.4 is essentially done in [15]. However, we will include it here since our ideas to prove Corollary 5.5 are also based on HNN extensions.

We briefly introduce the definition of a Higman–Neumann–Neumann extension (HNN extension, for short). Let H be a group, and let A and B be sugroups of H with an isomorphism $\phi\colon A \to B$. The *HNN extension* of H relative to A, B and ϕ is the group

$$H^* := \langle H, t : t^{-1}at = \phi(a), a \in A\rangle.$$

The group H is the *base* of H^* and t is the *stable letter*. We refer the reader to [22] for further details on this construction. Following [15], we let $\theta\colon H^* \to \mathbb{Z}$ be the *height* map. That is, θ is the group homomorphism that vanishes on H and maps t to 1. We let X and Y be complete sets of left coset representatives for H/A and H/B, respectively. Every element $h^* \in H^*$ has a unique representation of the form

$$h^* = h_0 t^{\epsilon_1} h_1 t^{\epsilon_2} \cdots h_{k-1} t^{\epsilon_k} h_k \qquad (k \geq 0), \tag{5.1}$$

where $\epsilon_i = \pm 1$, $h_{i-1} \in X$ if $\epsilon_i = 1$ and $h_{i-1} \in Y$ if $\epsilon_i = -1$, and $h_k \in H$. The representation (5.1) is the (right) *normal form* of h^* relative to X and Y (see [22, Theorem 2.1]).

Let $\mathbb{F}$ be the free group on a set of generators S and let u and w be nonempty words in S. We consider the subgroups of $\mathbb{F}$ given by $A := \langle u\rangle$ and $B := \langle w\rangle$. Our goal is to analyse the HNN extensions determined by the group isomorphisms $\phi^+, \phi^-\colon A \to B$ given by $u \mapsto w$ and $u \mapsto w^{-1}$, respectively. We begin with an appropriate choice of left coset representatives for $\mathbb{F}/A$ and $\mathbb{F}/B$.

Lemma 5.6. *Let $\mathbb{F}$ be the free group on a set of generators S and let $A = \langle u\rangle$ be the subgroup of $\mathbb{F}$ generated by a nonempty word u in S. Define*

$$\Omega_u := \{\alpha \in \mathbb{F}^+ : \alpha = e \text{ or } \alpha \text{ is a nontrivial right divisor of } u\}.$$

(a) *For $\alpha \in \Omega_u \setminus \{e\}$, let X_α be the set of words h in $\mathbb{F}$ for which the largest common suffix with u in their reduced form is α so that, in particular, $h \cdot u$ involves no cancellation. Also, let X_e be the set of words $h \in \mathbb{F}$ for which the products $h \cdot u$ and $h \cdot u^{-1}$ involve no cancellation. Then*

$$X := \bigsqcup_{\alpha \in \Omega_u} X_\alpha$$

is a complete set of left coset representatives for $\mathbb{F}/A$.

(b) *Suppose that $h \in X$ satisfies $hA \cap \mathbb{F}^+ \neq \emptyset$. Then $h \in \mathbb{F}^+$ and $h \leq_{\mathbb{F}^+} q$ for all $q \in hA \cap \mathbb{F}^+$.*

Proof. We first establish (a). Let $\alpha, \beta \in \Omega_u$. Clearly $X_\alpha \cap X_\beta = \emptyset$ if $\alpha \neq \beta$. Now take $h \in X_\alpha$ and $h' \in X_\beta$ and suppose that $hA = h'A$. Then $h^{-1}h' \in A$ and we may assume, without loss of generality, that

$$h = h'u^m, \qquad m \geq 0.$$

We claim that $m = 0$. This follows because $h'u$ is in reduced form, and hence there are no cancellations. This implies that h has u as a suffix, contradicting that $h \in X_\alpha$. Hence we must have $m = 0$, which gives $h = h'$.

Now let $h' \in \mathbb{F}$ be arbitrary. We will prove that $h'A = hA$ for some $h \in X$. We can write $h' = h_0u^m$, where $m \in \mathbb{Z}$, h_0 does not end in a nontrivial power of u and the product $h_0 \cdot u^m$ is in reduced form. So $h'A = h_0A$. If the product $h_0 \cdot u^{-1}$ is not reduced, then $h_0 \in X_\alpha \subseteq X$ for some nontrivial proper right divisor α of u. Suppose that $h_0 \cdot u^{-1}$ is reduced. If $h_0 \cdot u$ is also reduced, then $h_0 \in X_e \subseteq X$. Otherwise, i.e. if h_0u is not reduced, there is a largest proper nontrivial prefix β of u such that β^{-1} is a suffix of h_0 in its reduced form. Set $\alpha = \beta^{-1}u$. Let $h_1 \in \mathbb{F}$ be such that $h_0 = h_1\beta^{-1}$, where the product involves no cancellation. Then $h_0 = h_1(\alpha u^{-1})$ and $h_0A = h_1\alpha A$. Since the product $h_1 \cdot \beta^{-1}$ involves no cancellation, α is indeed the largest common suffix of u and $h_1\alpha$. Thus $h_1\alpha \in X_\alpha$ and $h'A = h_0A = h_1\alpha A$ as desired.

To prove (b), let $h \in X$ with $hA \cap \mathbb{F}^+ \neq \emptyset$. Take $q \in hA \cap \mathbb{F}^+$. We will show that $e \leq h \leq q$ in $(\mathbb{F}, \mathbb{F}^+)$. Let $m \in \mathbb{Z}$ be such that $h^{-1}q = u^m$. We must have $m \geq 0$ because the product $q \cdot u$ involves no cancellation and h does not end in a nontrivial power of u in its reduced form. So $q = hu^m$ with $m \geq 0$. Since $h \in X$, the product $h \cdot u^m$ involves no cancellation. Therefore, $h \in \mathbb{F}^+$ and $h \leq_{\mathbb{F}^+} q$ as asserted. □

5.2.1. *Case $u\sigma = \sigma w$.* We analyse the HNN extension of $\mathbb{F}$ relative to $A = \langle u \rangle$, $B = \langle w \rangle$ and the isomorphism $\phi^+ \colon A \to B$ that sends u to w. We begin by showing that $(\mathbb{F}^*, \mathbb{F}^{+*})$ is quasi-lattice ordered by applying Lemma 5.6 and [15, Theorem 4.1], which gives the following three sufficient conditions for and HNN-extension (H^*, P^*) as above to be quasi-lattice ordered:

(1) $\phi(A \cap P) = B \cap P$.
(2) Every left coset $hA \in H/A$ such that $hA \cap P \neq \emptyset$ has a minimal coset representative $p \in P$.
(3) for every $x, y \in B$ such that $x \vee y < \infty$, we have $x \vee y \in B$.

Proposition 5.7. *Let $\mathbb{F}$ be the free group on a set of generators S. Let A and B be the subgroups of $\mathbb{F}$ generated by the nonempty words u and w in S, respectively. Let $\phi^+ \colon A \to B$ be the isomorphism that sends u to w. Then $(\mathbb{F}^*, \mathbb{F}^{+*})$ is quasi-lattice ordered, where $\mathbb{F}^*$ is the HNN extension of $\mathbb{F}$ relative to A, B and ϕ^+, and $\mathbb{F}^{+*}$ denotes the subsemigroup of $\mathbb{F}^*$ generated by $S \cup \{t\}$.*

Proof. In order to prove that $(\mathbb{F}^*, \mathbb{F}^{+*})$ is quasi-lattice ordered, we will apply [15, Theorem 4.1]. Let X and Y be the complete sets of left coset representatives for $\mathbb{F}/A$ and $\mathbb{F}/B$, respectively, as in Lemma 5.6. Clearly, $\phi^+(A \cap \mathbb{F}^+) = B \cap \mathbb{F}^+$, which is condition (1) of [15, Theorem 4.1]. Condition (3) of [15, Theorem 4.1] is trivially true in our setting. Now suppose that $h \in X$ satisfies $hA \cap \mathbb{F}^+ \neq \emptyset$. It follows from Lemma 5.6 that $h \in \mathbb{F}^+$ and $h \leq_{\mathbb{F}^+} q$ whenever $q \in hA \cap \mathbb{F}^+$. This gives precisely condition (2) of [15, Theorem 4.1]. Therefore, $(\mathbb{F}^+, \mathbb{F}^{+*})$ is quasi-lattice ordered by [15, Theorem 4.1]. □

Before proving the next proposition, we introduce some notation. Let H^* be the HNN extension of H relative to A, B and $\phi \colon A \to B$. Let $h^* \in H^*$ with normal form

$$h^* = h_0t^{\epsilon_1}h_1t^{\epsilon_2} \dots h_{k-1}t^{\epsilon_k}h_k \qquad (k \geq 0).$$

The *stem* of h^*, denoted by $\mathrm{stem}(h^*)$, is the element $h_0t^{\epsilon_1}h_1t^{\epsilon_2}\dots h_{k-1}t^{\epsilon_k}$. As in [15, Theorem 5.1], we will use that the height map $\theta\colon \mathbb{F}^* \to \mathbb{Z}$ is controlled to deduce nuclearity of $\mathrm{C}^*(\mathbb{F}^*, \mathbb{F}^{+*})$.

Proposition 5.8. *Let* $(\mathbb{F}^*, \mathbb{F}^{+*})$ *be as in Proposition* 5.7. *Then the height map* $\theta\colon \mathbb{F}^* \to \mathbb{Z}$ *is controlled and, in particular,* $\mathrm{C}^*(\mathbb{F}^*, \mathbb{F}^{+*})$ *is nuclear.*

Proof. That the height map is controlled is already established in [15, Theorem 5.1] and so we will only indicate here how the proof goes.

Let $x, y \in \mathbb{F}^{+*}$ be such that $x \vee y < \infty$. By [15, Lemma 5.5], if $\theta(x) < \theta(y)$, there is $r \in \mathbb{F}^+$ with $x \vee y = \mathrm{stem}(y)r$, and so $\theta(x \vee y) = \theta(y) = \max\{\theta(x), \theta(y)\}$. If $\theta(x) = \theta(y)$, then $\mathrm{stem}(x) = \mathrm{stem}(y)$ and, in this case,

$$x \vee y = \mathrm{stem}(y)(p \vee_{\mathbb{F}^+} q),$$

where $x = \mathrm{stem}(x)p$ and $y = \mathrm{stem}(y)q$. Hence $\theta(x \vee y) = \theta(x) \vee \theta(y)$. Given $k \in \mathbb{N}$, the set of minimal elements $\Sigma_k \subseteq \mathbb{F}^{+*}$ is then given by

$$\{x \in \theta^{-1}(k) \cap \mathbb{F}^{+*} : x = \mathrm{stem}(x)\}.$$

It is then easy to see that θ is controlled. Since $\ker\theta \cap \mathbb{F}^{+*} = \mathbb{F}^+$ and $\mathrm{C}^*(\mathbb{F}, \mathbb{F}^+)$ is nuclear (see Example 5.1), Theorem 1.1 implies that $\mathrm{C}^*(\mathbb{F}^*, \mathbb{F}^{+*})$ is nuclear. This completes the proof of the statement. □

Remark 5.9. Although here we are only considering HNN extensions of free groups, a version of [15, Theorem 5.1] in which the hypothesis "(G, P) is amenable" is replaced by "$\mathrm{C}^*(G, P)$ is nuclear" also follows using that the height map is controlled.

Rewriting Proposition 5.8 in terms of one-relator groups with positive presentation gives Corollary 5.4:

Proof of Corollary 5.4. Let $(\mathbb{F}^*, \mathbb{F}^{+*})$ be the quasi-lattice ordered group constructed as in Proposition 5.7. We obtain an isomorphism $\mathbb{F}^* \cong G$ by identifying generators $t \mapsto \sigma$, $s \mapsto s$. In particular, this restricts to an isomorphism $\mathbb{F}^{+*} \cong P$. So (G, P) is quasi-lattice ordered. That $\mathrm{C}^*(G, P)$ is nuclear then follows from Proposition 5.8. □

5.2.2. *Case $u\sigma w = \sigma$.* We consider the HNN extension of $\mathbb{F}$ relative to A, B and the group isomorphism $\phi^-\colon A \to B$ which sends u to w^{-1}. In this case, we cannot apply [15, Theorem 4.1] because $\phi^-(A \cap \mathbb{F}^+) \nsubseteq B \cap \mathbb{F}^+$. However, we can directly prove that $(\mathbb{F}^*, \mathbb{F}^{+*})$ is weakly quasi-lattice ordered. We will need the following technical lemma.

Lemma 5.10. *Let* $\mathbb{F}$ *be the free group on* S. *Let* A *and* B *be the subgroups of* $\mathbb{F}$ *generated by the nonempty words* u *and* w*, respectively, and let* $\phi^-\colon A \to B$ *be the isomorphism that sends* u *to* w^{-1}. *Let* $\mathbb{F}^*$ *be the corresponding HNN extension and let* $h^* \in \mathbb{F}^*$ *with normal form*

$$h^* = h_0t^{\epsilon_1}h_1t^{\epsilon_2}\dots h_{k-1}t^{\epsilon_k}h_k$$

relative to the complete set of representatives X *and* Y *for* $\mathbb{F}/A$ *and* $\mathbb{F}/B$ *as in Lemma* 5.6. *Then* $h^* \in \mathbb{F}^{+*}$ *if and only if*

(a) $\epsilon_i = 1$ *for all* i, $h_0 \in \mathbb{F}^+$ *and* $h_k = w^m q$ *for some* $m \in \mathbb{Z}$ *and* $q \in \mathbb{F}^+$*;*
(b) *for* $2 \leq i \leq k$, *the double coset*

$$Bh_{i-1}A = \{w^m h_{i-1} u^n : m, n \in \mathbb{Z}\}$$

has a representative in $\mathbb{F}^+$.

Proof. Suppose that $h^* \in \mathbb{F}^{+*}$. Then there exist $l \geq 0$ and $p_0, p_1, \dots, p_l \in \mathbb{F}^+$ such that

$$h^* = p_0 t p_1 t \cdots p_{l-1} t p_l.$$

We can compute the normal form of h^* by working from left to right using its representation $p_0 t p_1 t \cdots p_{l-1} t p_l$ and the relation $ut = tw^{-1}$. No factor t^{-1} appears in the normal form of $p_0 t p_1 t \cdots p_{l-1} t p_l$, from where we deduce that $l = k$ and $\epsilon_i = 1$ for all $1 \leq i \leq k$.

The representative of the left coset $p_0 A$ is $h_0 \in X$. Since $p_0 \in h_0 A \cap \mathbb{F}^+$, it follows from item (b) of Lemma 5.6 that $h_0 \in \mathbb{F}^+$ and $h_0 \leq_{\mathbb{F}^+} p_0$. We write $p_0 = h_0 u^{m_0}$, where $m_0 \geq 0$. Next, $h_1 \in X$ is the representative of the left coset $w^{-m_0} p_1$. So there exists $n_1 \in \mathbb{Z}$ such that $h_1 = w^{-m_0} p_1 u^{n_1}$. Then

$$w^{m_0} h_1 u^{-n_1} = p_1 \in B h_1 A \cap \mathbb{F}^+.$$

Thus the double coset Bh_1A has representative p_1 in $\mathbb{F}^+$. Similarly, we conclude that the double coset $Bh_{i-1}A$ has a representative in $\mathbb{F}^+$ for all $2 \leq i \leq k$. That h_k has the form $w^{m_k} p_k$ for some $m_k \in \mathbb{Z}$ follows because $u^{-m_k} t p_k = t w^{m_k} p_k$.

Conversely, let $h^* = h_0 t^{\epsilon_1} h_1 t^{\epsilon_2} \cdots t^{\epsilon_{k-1}} h_{k-1} t^{\epsilon_k} h_k$ and suppose that conditions (a) and (b) in the statement hold. Let $\theta\colon \mathbb{F}^* \to \mathbb{Z}$ be the height map. Our proof will be by induction on $\theta(h^*) = k$.

If $k = 0$, then $h^* = h_0 \in \mathbb{F}^+ \subseteq \mathbb{F}^{+*}$. If $k = 1$, let $p_1 \in \mathbb{F}^+$ and $m_1 \in \mathbb{Z}$ be such that $h_1 = w^{m_1} p_1$. If $m_1 \geq 0$, we clearly have

$$h^* = h_0 t h_1 = h_0 t w^{m_1} p_1 \in \mathbb{F}^{+*}.$$

In case $m_1 < 0$, the relation $ut = tw^{-1}$ tells us that $h^* \in \mathbb{F}^{+*}$.

Fix $k \geq 2$ and suppose that $h'^* \in \mathbb{F}^{+*}$ whenever its normal form satisfies (a) and (b), and $\theta(h'^*) \leq k - 1$. Let $m_k \in \mathbb{Z}$ and $p_k \in \mathbb{F}^+$ be such that $h_k = w^{m_k} p_k$. Using that the double coset $Bh_{k-1}A$ has a representative $p_{k-1} \in \mathbb{F}^+$, we can find $m_{k-1}, n_k \in \mathbb{Z}$ with

$$p_{k-1} = w^{m_{k-1}} h_{k-1} u^{n_{k-1}}.$$

By the induction hypothesis, we conclude that

$$h_0 t h_1 t \cdots t w^{-m_{k-1}} p_{k-1}$$

lies in $\mathbb{F}^{+*}$. We claim that $u^{-n_{k-1}} t w^{m_k}$ belongs to $\mathbb{F}^{+*}$ as well. Indeed, if $-n_{k-1}, m_k \geq 0$ we are done. Otherwise, we use the relations $u^{-1}t = tw$ and $ut = tw^{-1}$ to write $u^{-n_{k-1}} t w^{m_k}$ as a product of t and a nonnegative power of u or w. Thus one has $u^{-n_{k-1}} t w^{m_k} \in \mathbb{F}^{+*}$ and so

$$h^* = h_0 t h_1 t \cdots t h_{k-1} t h_k = h_0 t h_1 t \cdots t w^{-m_{k-1}} p_{k-1} u^{-n_{k-1}} t w^{m_k} p_k$$

lies in $\mathbb{F}^{+*}$, as desired. □

Next we will prove that $(\mathbb{F}^*, \mathbb{F}^{+*})$ is weakly quasi-lattice ordered.

Proposition 5.11. *Let $\mathbb{F}^*$ be the HNN extension of $\mathbb{F}$ relative to A, B and ϕ^- as in Lemma* 5.10. *Then $(\mathbb{F}^*, \mathbb{F}^{+*})$ is weakly quasi-lattice ordered. Let $\theta\colon \mathbb{F}^* \to \mathbb{Z}$ be the height map. If $x, y \in \mathbb{F}^{+*}$ and $x \vee y < \infty$, then x and y are comparable and*

$$\theta(x \vee y) = \max\{\theta(x), \theta(y)\}.$$

Proof. Let $x, y \in \mathbb{F}^{+*}$ and suppose that there is a common upper bound in $\mathbb{F}^{+*}$ for x and y.

Case 1. Suppose that $\theta(x) = \theta(y) =: k$. Let X and Y be the usual sets of left coset representatives for $\mathbb{F}/A$ and $\mathbb{F}/B$ obtained from Lemma 5.6, respectively. By

Lemma 5.10, we can write $x = \mathrm{stem}(x)w^m p$ and $y = \mathrm{stem}(y)w^n q$, where $m, n \in \mathbb{Z}$ and $p, q \in \mathbb{F}^+$. Without loss of generality, assume $n \geq m$. Since x and y have an upper bound in $\mathbb{F}^{+*}$, there are $r, s \in \mathbb{F}^{+*}$ with $xr = ys$. When putting xr and ys into normal form using the procedure as in Lemma 5.10, we work from left to right, and hence do not change $\mathrm{stem}(x)$ or $\mathrm{stem}(y)$. Thus $\mathrm{stem}(x) = \mathrm{stem}(y)$ by uniqueness of the normal form. Applying Lemma 5.10 (b) to the normal forms of xr and ys, we also deduce that there are $r_0, s_0 \in \mathbb{F}^+$ with

$$w^m p r_0 A = w^n q s_0 A.$$

In particular, p and $w^{n-m}q$ have an upper bound in $\mathbb{F}^+$. This implies that p and $w^{n-m}q$ are comparable in $\mathbb{F}^+$. If $w^{n-m}q \leq_{\mathbb{F}^+} p$, we have

$$y = \mathrm{stem}(y)w^n q = \mathrm{stem}(x)w^m w^{n-m} q \leq x.$$

If $p \leq w^{n-m}q$, then

$$x = \mathrm{stem}(x)w^m p = \mathrm{stem}(y)w^m p \leq y.$$

So x and y are comparable and hence $x \vee y = \max\{x, y\}$. Thus $\theta(x \vee y) = \theta(x) = \theta(y)$.

Case 2. Suppose that $\theta(x) \neq \theta(y)$. Without loss of generality, we assume that $\theta(x) < \theta(y)$. We shall prove that $x < y$. Let

$$y = h_0 t \cdots t h_{\theta(x)} t \cdots t h_{\theta(y)}$$

be the normal form of y relative to the complete sets of left coset representatives X and Y. Since there exists an upper bound in $\mathbb{F}^{+*}$ for x and y, it follows that

$$y = \mathrm{stem}(x) h_{\theta(x)} t \dots t h_{\theta(y)}.$$

Let $m \in \mathbb{Z}$ and $p \in \mathbb{F}^+$ be such that $x = \mathrm{stem}(x)w^m p$. Since there exist $r, s \in \mathbb{F}^{+*}$ such that $xr = ys$, we can find $r_0 \in \mathbb{F}^+ \cap X$ with $w^m p r_0 A = h_{\theta(x)} A$. Then there is $n \in \mathbb{Z}$ with $w^m p r_0 = h_{\theta(x)} u^n$. Thus

$$x^{-1} y = r_0 u^{-n} h_{\theta(x)}^{-1} \mathrm{stem}(x)^{-1} y = r_0 t w^n h_{\theta(x)+1} t \cdots t h_{\theta(y)}.$$

If $\theta(y) = \theta(x) + 1$, we have $x^{-1}y = r_0 t w^n h_{\theta(x)+1} \in \mathbb{F}^{+*}$ by Lemma 5.10, because $y \in \mathbb{F}^{+*}$. Suppose $\theta(x) + 1 < \theta(y)$. Observe that the double coset $B w^n h_{\theta(x)+1} A$ coincides with $B h_{\theta(x)+1} A$. On the other hand, $B w^n h_{\theta(x)+1} A$ equals the double coset of the representative of the left coset $w^n h_{\theta(x)+1} A$. It follows that this latter double coset has a representative in $\mathbb{F}^+$ because $B h_{\theta(x)+1} A$ does so. Proceeding with this argument for all $\theta(x) + 1 \leq i < \theta(y)$ and using Lemma 5.10 again, we deduce that $x^{-1}y \in \mathbb{F}^{+*}$. So $x < y$. Thus $y = x \vee y$ and $\theta(x \vee y) = \theta(y) = \max\{\theta(x), \theta(y)\}$. ☐

Proposition 5.12. *Let* $(\mathbb{F}^*, \mathbb{F}^{+*})$ *be as in Proposition* 5.11. *Then the height map* $\theta\colon \mathbb{F}^* \to \mathbb{Z}$ *is controlled and, in particular,* $\mathrm{C}^*(\mathbb{F}^*, \mathbb{F}^{+*})$ *is nuclear.*

Proof. By Proposition 5.11, we have $\theta(x \vee y) = \theta(x) \vee \theta(y)$ whenever $x, y \in \mathbb{F}^{+*}$ have an upper bound in $\mathbb{F}^{+*}$. We will show that θ also satisfies item (b) of Definition 3.6.

Let $q \in \mathbb{N}$. If $q = 0$, we take $\Lambda_0 = \{e\}$. So

$$S_e := \theta^{-1}(0) \cap \mathbb{F}^{+*} = \{x \in \theta^{-1}(0) \cap \mathbb{F}^{+*} : x \geq e\} = \mathbb{F}^+$$

and the required sequence $(s_n^e)_{n \in \mathbb{N}} \subseteq S_e$ is just the constant sequence $s_n^e = e$ for all $n \in \mathbb{N}$.

Assume $q \geq 1$. Set

$${}_wX := \{h \in X : BhA = BpA \text{ for some } p \in \mathbb{F}^+\}.$$

Put $\Lambda_q = (X \cap \mathbb{F}^+) \times ({}_wX)^{q-1}$ (in case $q = 1$, we naturally identify $({}_wX)^{q-1}$ with the unit set). For $\lambda = (h_0, h_1, \ldots, h_{q-1}) \in \Lambda_q$, we let

$$S_\lambda := \{x \in \theta^{-1}(q) \cap \mathbb{F}^{+*} : \mathrm{stem}(x) = h_0 t h_1 t \cdots t h_{q-1} t\}.$$

Then

$$\theta^{-1}(q) \cap \mathbb{F}^{+*} = \bigsqcup_{\lambda \in \Lambda_q} S_\lambda.$$

The required decreasing sequence associated to $\lambda = (h_0, h_1, \ldots, h_{q-1})$ is the sequence whose n-th term is

$$s_n^\lambda := h_0 t \cdots t h_{q-1} t w^{-n}.$$

We claim that

$$S_\lambda = \bigcup_{n \in \mathbb{N}} \{x \in \theta^{-1}(q) \cap \mathbb{F}^{+*} : x \geq s_n^\lambda\}.$$

Indeed, given $x \in S_\lambda$, there are $m \in \mathbb{Z}$ and $p \in \mathbb{F}^+$ with $x = \mathrm{stem}(x) w^m p$ by Lemma 5.10. Take $n = |m|$. Then

$$(s_n^\lambda)^{-1} x = w^{|m|} w^m p \in \mathbb{F}^+ \subseteq \mathbb{F}^{+*}.$$

This proves our claim. We conclude that θ is controlled in the sense of Definition 3.6.

Because $\ker \theta \cap \mathbb{F}^{+*} = \mathbb{F}^+$ and $\mathrm{C}^*(\mathbb{F}, \mathbb{F}^+)$ is nuclear, we deduce from Theorem 1.1 that $\mathrm{C}^*(\mathbb{F}^*, \mathbb{F}^{+*})$ is also nuclear. □

Remark 5.13. Observe that the height map $\theta\colon \mathbb{F}^* \to \mathbb{Z}$ above is not controlled in the sense of Definition 3.1. Indeed, the infinite descending sequence

$$\cdots \leq t w^{-(n+1)} \leq t w^{-n} \leq \cdots$$

is not bounded below by any element $x \in \theta^{-1}(1) \cap \mathbb{F}^{+*}$.

The proof of Corollary 5.5 now follows as in Corollary 5.4.

5.3. **Semidirect products of weakly quasi-lattice ordered groups.** In this subsection, we analyse the semidirect product of weak quasi-lattice orders. We provide examples in which the semidirect product is again weakly quasi-lattice ordered with a natural positive cone, and study nuclearity of the corresponding semigroup C*-algebra using controlled maps. Most of our examples involve free groups.

Let (N, L) and (H, Q) be weak quasi-lattice orders. Suppose that $\varphi\colon H \leq \mathrm{Aut}(N)$ is a subgroup of automorphisms of N, so that we can form the semidirect product

$$G = N \rtimes_\varphi H,$$

where $\varphi\colon H \to \mathrm{Aut}(N)$, $h \mapsto \varphi_h$ is the corresponding action of H on N. Recall how the semidirect product is defined: the underlying set is the Cartesian product $N \times H$ of N and H, and multiplication is given by

$$(n, h)(n', h') := (n\varphi_h(n'), hh').$$

Our first goal is to find sufficient conditions for the semidirect product $G = N \rtimes_\varphi H$ to be weakly quasi-lattice ordered as well, with positive cone $P = L \rtimes_\varphi Q$.

Lemma 5.14. *Let (N, L) and (H, Q) be as above and suppose that for all $q \in Q$*

$$\varphi_q(L) \subseteq L.$$

Then

$$P = L \rtimes_\varphi Q$$

is a generating subsemigroup of G such that $P \cap P^{-1} = \{e\}$.

In this case we have a partial order on G given by $g_1 \leq_P g_2$ if and only if $g_1^{-1}g_2 \in P$, and we write:

$$(G, P) = (N, L) \rtimes_\varphi (H, Q).$$

Proof. That P generates G follows directly from the construction. Now suppose $(l, q) \in P \cap P^{-1}$. Since $(l, q)^{-1} = (\varphi_{q^{-1}}(l^{-1}), q^{-1})$ and $Q \cap Q^{-1} = \{e_q\}$, we must have $q = e_Q$. Hence $\varphi_{q^{-1}}(l^{-1}) = l^{-1} \in L \cap L^{-1}$ and so $l = e_L$. □

We will invoke the following fact several times in this subsection.

Lemma 5.15. *Let (N, L) and (H, Q) be weak quasi-lattices with $H \leq \mathrm{Aut}(N)$ such that $\varphi_q(L) \subseteq L$ for all $q \in Q$, and let $(G, P) = (N, L) \rtimes_\varphi (H, Q)$ be the semidirect product. If $(l, q) \leq_P (m, r)$ with $l, m \in L$ and $q, r \in Q$, then $l \leq_L m$ and $q \leq_Q r$.*

Proof. Assume that $(l, q) \leq_P (m, r)$. This entails

$$(l, q)^{-1}(m, r) = (\varphi_{q^{-1}}(l^{-1})\varphi_{q^{-1}}(m), q^{-1}r) \in (L, Q).$$

Hence $q \leq_Q r$. Also, $\varphi_{q^{-1}}(l^{-1})\varphi_{q^{-1}}(m) = \varphi_{q^{-1}}(l^{-1}m) \in L$, which implies that $l^{-1}m \in \varphi_q(L) \subseteq L$. Hence $l \leq_L m$. □

Lemma 5.16. *Let (N, L) and (H, Q) be weak quasi-lattices with $H \leq \mathrm{Aut}(N)$. Suppose that $\varphi_q(L) = L$ for all $q \in Q$. Then the semidirect product*

$$(G, P) = (N, L) \rtimes_\varphi (H, Q)$$

is weakly quasi-lattice ordered. In this case,

$$(l, q) \vee_P (m, r) = (l \vee_L m, q \vee_Q r).$$

Proof. Suppose that $(l, q), (m, r) \in P$ have a common upper bound. By Lemma 5.15 and the fact that (N, L) and (H, Q) are weak quasi-lattices, we have $l \vee_L m < \infty$ and $q \vee_Q r < \infty$. We claim that $(l \vee_L m, q \vee_Q r) \in P$ is the least upper bound of (l, q) and (m, r). Indeed, we have that

$$\begin{aligned}(l, q)^{-1}(l \vee_L m, q \vee_Q r) &= (\varphi_{q^{-1}}(l^{-1}), q^{-1})(l \vee_L m, q \vee_Q r) \\ &= (\varphi_{q^{-1}}(l^{-1}(l \vee_L m)), q^{-1}(q \vee_Q r)).\end{aligned}$$

This belongs to L because $l \leq_L l \vee_L m$, $q \leq_Q q \vee_Q r$ and $\varphi_{q^{-1}}(L) = \varphi_{q^{-1}}(\varphi_q(L)) = L$. Similarly, $(m, r) \leq_P (l \vee_L m, q \vee_Q r)$.

Now let $(n, s) \in P$ be a common upper bound for (l, q) and (m, r). All we have to show is that $(l \vee_L m, q \vee_Q r) \leq_P (n, s)$. By Lemma 5.15, we must have $l \vee_L m \leq_L n$ and $q \vee_Q r \leq_Q s$. Using that $\varphi_{(q \vee_Q r)^{-1}}(L) = L$, we obtain $(l \vee_L m, q \vee_Q r) \leq_P (n, s)$. So $(l \vee_L m, q \vee_Q r)$ is the least upper bound of (l, q) and (m, r) as claimed. □

The assumption $\varphi_q(L) = L$ for all $q \in Q$ is fairly strong, as we will see later in examples. However, in the situation of the previous lemma, we may use a natural controlled map to deduce nuclearity of $C^*(N \rtimes_\varphi H, L \rtimes_\varphi Q)$.

Proposition 5.17. *Let (N, L) and (H, Q) be weak quasi-lattices with $H \leq \mathrm{Aut}(N)$. Let*

$$\mu\colon \begin{array}{ccc} N \rtimes_\varphi H & \to & H \\ (n, h) & \mapsto & h \end{array}$$

be the natural surjection. Suppose that $x, y \in P = L \rtimes_\varphi Q$ have a least upper bound $x \vee_P y$ in P. Then $\mu(x \vee_P y) = \mu(x) \vee_Q \mu(y)$. If $\varphi_q(L) = L$ for all $q \in Q$, then μ is controlled in the sense of Definition 3.1.

Proof. Let $(G, P) = (N, L) \rtimes (H, Q)$. Let $(l, q), (m, r) \in P$ be such that $(l, q) \vee_P (m, r)$ exists. Write $(l, q) \vee_P (m, r) = (n, s)$. In order to prove the first part of the statement, it suffices to show that $s = q \vee_Q r$. We have that $q \vee_Q r$ exists and $q, r \leq_Q q \vee_Q r \leq_Q s$. Also, both $\varphi_{q^{-1}}(l^{-1}n)$ and $\varphi_{r^{-1}}(m^{-1}n)$ belong to L. Hence $(n, q \vee_Q r)$ is a common upper bound for (l, q) and (m, r), which gives $(n, s) \leq_P (n, q \vee_Q r)$. Thus $s \leq_Q q \vee_Q r$ and so $s = q \vee_Q r$ as desired.

Suppose that $\varphi_q(L) = L$ for all $q \in Q$. So (G, P) is weakly quasi-lattice ordered by Lemma 5.16. We have proven that μ satisfies item (a) of Definition 3.1. In order to show that μ also satisfies (b), we observe that for all $q \in Q$,

$$\mu^{-1}(q) \cap P = \{(l, q) : l \in L\}.$$

So the required set of minimal elements is $\Sigma_q = \{(e, q)\}$, where e denotes the unit element of N. This verifies item (b) of Definition 3.1. □

We now consider examples (and non-examples) coming from free-by-cyclic groups $\mathbb{F} \rtimes \mathbb{Z}$.

Example 5.18. Let $\mathbb{F} = \langle a, b\rangle$ be the free group on two generators. We let $\mathbb{Z}$ act on $\mathbb{F}$ via the automorphism $\varphi\colon \mathbb{F} \to \mathbb{F}$ given by $\varphi(a) = b$ and $\varphi(b) = a$. In this case we have $\varphi(\mathbb{F}^+) = \mathbb{F}^+$ and so $(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (\mathbb{Z}, \mathbb{N})$ is a weak quasi-lattice. By Proposition 5.17, the canonical projection $\mu\colon \mathbb{F} \rtimes_\varphi \mathbb{Z} \to \mathbb{Z}$ is controlled. Since $(\ker \mu) \cap (\mathbb{F}^+ \rtimes_\varphi \mathbb{N}) = \mathbb{F}^+$, Theorem 4.3 tells us that $\mathrm{C}^*(\mathbb{F} \rtimes_\varphi \mathbb{Z}, \mathbb{F}^+ \rtimes_\varphi \mathbb{N})$ is nuclear.

Example 5.19. More generally, if $\mathbb{F}$ is the free group on n generators and φ acts by permuting generators, then $\varphi(\mathbb{F}^+) = \mathbb{F}^+$ and $(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (\mathbb{Z}, \mathbb{N})$ is weakly quasi-lattice ordered. By the same reasoning as above we conclude that $\mathrm{C}^*(\mathbb{F} \rtimes_\varphi \mathbb{Z}, \mathbb{F}^+ \rtimes_\varphi \mathbb{N})$ is nuclear.

Next, we give an example of a free-by-cyclic group $\mathbb{F} \rtimes_\varphi \mathbb{Z}$ with $\varphi(\mathbb{F}^+) \subsetneq \mathbb{F}^+$ and such that $(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (\mathbb{Z}, \mathbb{N})$ is not a weak quasi-lattice order.

Example 5.20. Let $\mathbb{F}$ be freely generated by a and b, and let φ be defined by $\varphi(a) = ba$, $\varphi(b) = b^2a$. Then $(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (\mathbb{Z}, \mathbb{N})$ is not weakly quasi-lattice ordered. To see this, notice that φ^2 is the automorphism of $\mathbb{F}$ given by

$$\varphi^2(a) = b^2aba, \qquad \varphi^2(b) = b^2ab^2aba.$$

Take $p = (a, 2)$ and $q = (ab, 1)$. Thus $p, q \in P = (\mathbb{F}^+, \mathbb{N})$ have $n_1 = (ab^2aba, 2)$ and $n_2 = (ab^2ab^2aba, 2)$ as common upper bounds in P. To see that p and q have no least upper bound, suppose that (s, m) is less than n_1 and n_2 in P and greater than p, q. Then $m = 2$ and $ab \leq_{\mathbb{F}^+} s \leq_{\mathbb{F}^+} ab^2ab$ by Lemma 5.15. In particular, $s^{-1}(ab^2aba) \notin \varphi^2(\mathbb{F}^+)$, contradicting the assumption that $(s, m) \leq_P n_1$.

In view of Example 5.20, we now present some sufficient conditions for a semidirect product $(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (H, Q)$ to be weakly quasi-lattice ordered.

Proposition 5.21. *Let $\mathbb{F}$ be a free group on the set of generators S and let $(\mathbb{F}, \mathbb{F}^+)$ be the quasi-lattice ordered group obtained from the free semigroup $\mathbb{F}^+$ on S. Let (H, Q) be a weak quasi-lattice with $H \leq \mathrm{Aut}(\mathbb{F})$ such that if $q, r \in Q$ and $q \vee_Q r < \infty$, then q and r are comparable. Suppose further that, for all $q \in Q$, we have $\varphi_q(\mathbb{F}^+) \subseteq \mathbb{F}^+$ and that $\varphi_q(x)$ and $\varphi_q(y)$ have no common divisors whenever $x, y \in S$ are different generators of $\mathbb{F}$. Then*

$$(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (H, Q)$$

is a weak quasi-lattice.

Proof. Let $l, m \in \mathbb{F}^+$, $q, r \in Q$ be such that (l, q) and (m, r) have a common upper bound in $(\mathbb{F}^+, Q)$. In particular, $q \vee_Q r < \infty$ and $l \vee_{\mathbb{F}^+} m < \infty$. We may suppose, without loss of generality, that l is a prefix of m. Say $(l, q), (m, r) \leq_P (n, s)$. Then

$$(\varphi_{q^{-1}}(l^{-1}n), q^{-1}s), (\varphi_{r^{-1}}(m^{-1}n), r^{-1}s) \in P.$$

Thus there exist $v, v' \in \mathbb{F}^+$ such that

$$\varphi_q(v) = l^{-1}n, \qquad \varphi_r(v') = m^{-1}n,$$

which gives $l\varphi_q(v) = m\varphi_r(v')$. It follows that the set

$$W := \{\alpha \in \varphi_r(\mathbb{F}^+) : l^{-1}m\alpha \in \varphi_q(\mathbb{F}^+)\}$$

is nonempty and bounded below by e. We claim that W has a unique minimal element. Indeed, if $l^{-1}m \in \varphi_q(\mathbb{F}^+)$, then e is the unique minimal element of W. Now suppose that $l^{-1}m \notin \varphi_q(\mathbb{F}^+)$ and let w, w' be minimal elements of W. Let $p, p' \in \mathbb{F}^+$ with $\varphi_q(p) = l^{-1}mw$ and $\varphi_q(p') = l^{-1}mw'$. Since $l^{-1}m$ is a prefix of both $\varphi_q(p)$ and $\varphi_q(p')$, our assumption that different generators have relatively prime images under φ_q implies that the first letters of p and p' coincide. Suppose that s_1 is the first letter of p and p'. If $\varphi_q(s_1) <_{\mathbb{F}^+} l^{-1}m$, then the first letters of $s_1^{-1}p$ and $s_1^{-1}p'$ must coincide. Proceeding with this argument, we can find a largest prefix u of p and p' so that $\varphi_q(u) <_{\mathbb{F}^+} l^{-1}m$. Again the first letter of $u^{-1}p$, say t, must coincide with the first letter of $u^{-1}p'$ because $\varphi_q(u^{-1})l^{-1}m$ is a nontrivial prefix of both $\varphi_q(u^{-1}p)$ and $\varphi_q(u^{-1}p')$. We also have $l^{-1}m <_{\mathbb{F}^+} \varphi_q(ut)$ because of our assumption that $l^{-1}m \notin \varphi_q(\mathbb{F}^+)$. Now $(l^{-1}m)^{-1}\varphi_q(ut)$ is a prefix of both w and w'. Using that different elements of S have relatively prime images under φ_r, we deduce that there are $t_1, v, v' \in \mathbb{F}^+$ with $w = \varphi_r(t_1v)$, $w' = \varphi_r(t_1v')$. If $\varphi_r(t_1) = (l^{-1}m)^{-1}\varphi_q(ut)$, then $v = v' = e$ and $w = w'$. Otherwise, we have either $\varphi_r(t_1) <_{\mathbb{F}^+} (l^{-1}m)^{-1}\varphi_q(ut)$ or $(l^{-1}m)^{-1}\varphi_q(ut) <_{\mathbb{F}^+} \varphi_r(t_1)$. The latter inequality requires the first letter of v be equal to the first letter of v', while the former inequality implies that the first letters of $(ut)^{-1}p$ and $(ut)^{-1}p'$ coincide. Employing this argument finitely many times, we will arrive at $p = p'$ or $v = v'$. Therefore $w = w'$ and this proves our claim that W has a unique minimal element.

Finally, we prove that $(mw, q \vee r)$ is the least upper bound of (l, q) and (m, r). Let (n, s) be an upper bound for (l, q) and (m, r). Then $s \geq q \vee r$. Moreover, $l^{-1}n = l^{-1}mm^{-1}n \in \varphi_q(\mathbb{F}^+)$ and $m^{-1}n \in \varphi_r(\mathbb{F}^+)$. Hence $m^{-1}n \in W$ and it follows that $w^{-1}(m^{-1}n) \in \mathbb{F}^+$. Let $p' \in \mathbb{F}^+$ be such that $m^{-1}n = \varphi_r(p')$. Thus $\varphi_r(p^{-1}p') \in \mathbb{F}^+$. However, using our hypothesis again we deduce that $\varphi_r(p^{-1}p') \in \mathbb{F}^+$ if and only if $p^{-1}p' \in \mathbb{F}^+$. So $(mw)^{-1}n = w^{-1}m^{-1}n \in \varphi_r(\mathbb{F}^+)$. By a similar argument, $(mw)^{-1}n \in \varphi_q(\mathbb{F}^+)$. Here we invoke the fact that if $q \vee r < \infty$ then $q \vee r = q$ or $q \vee r = r$ to guarantee that $(mw, q \vee r) \leq (n, s)$ in $(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (H, Q)$. □

We will now illustrate how Proposition 5.21 can be used to produce examples of weak quasi-lattices as well as controlled maps.

Example 5.22. Let $\varphi\colon \mathbb{F} \to \mathbb{F}$ be given by $\varphi(a) = ab$ and $\varphi(b) = b$. Here $a \notin \varphi(\mathbb{F}^+)$, and so Lemma 5.16 does not apply to this case. Nevertheless, $(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (\mathbb{Z}, \mathbb{N})$ is still a weak quasi-lattice by Proposition 5.21, because $\varphi(a) = ab$ and $\varphi(b) = b$ are relatively prime.

We will prove that $C^*(\mathbb{F}^+ \rtimes_\varphi \mathbb{N})$ is nuclear using Theorem 4.3. To do so, consider the homomorphism $\sigma_a\colon \mathbb{F} \to \mathbb{Z}$ that sends a to 1 and b to 0. Since φ does not change the exponent sum of a in a word, this induces a homomorphism $\mu\colon \mathbb{F} \rtimes_\varphi \mathbb{Z} \to \mathbb{Z}^2$

defined by $\mu(w,n) := (\sigma_a(w), n)$. We will see that

$$\mu\colon (\mathbb{F} \rtimes_\varphi \mathbb{Z}, \mathbb{F}^+ \rtimes_\varphi \mathbb{N}) \to (\mathbb{Z}^2, \mathbb{N}^2)$$

is controlled.

We begin by showing that $\mu(x \vee_P y) = \mu(x) \vee \mu(y)$ when it exists. Let $x = (p, m)$ and $y = (q, n)$ with a common bound (s, l) in $P = \mathbb{F}^+ \rtimes_\varphi \mathbb{N}$. In particular, p and q have a common upper bound in $\mathbb{F}^+$ and we may suppose, without loss of generality, that $p \leq_{\mathbb{F}^+} q$. That is, p is a prefix of q. Since $q \leq_{\mathbb{F}^+} s$ we have $p^{-1}s = p^{-1}qq^{-1}s \in \varphi_m(\mathbb{F}^+)$. Thus $p^{-1}q$ has reduced form

$$b^{i_0}ab^{i_1}a \cdots ab^{i_k}, \qquad (k \geq 0)$$

where $i_0, i_k \in \mathbb{N}$ and $i_j \geq m$ for all $j \neq 0, k$. It follows that the least upper bound for (p, m) and (q, n) is given by

$$(p, m) \vee_P (q, n) = \begin{cases} (q, m \vee_{\mathbb{N}} n), & \text{if } k = 0 \text{ or } i_k \geq m \\ (qb^{m-i_k}, m \vee_{\mathbb{N}} n) & \text{else.} \end{cases}$$

From this we deduce that $\mu(x \vee_P y) = \mu(x) \vee \mu(y)$, which is item (a) of Definition 3.1.

To see that μ satisfies item (b) of Definition 3.1, take $(m, n) \in \mathbb{N}^2$. Observe that $\sigma_a(p)$ is precisely the exponent sum of a in $p \in \mathbb{F}$. Then $\mu^{-1}(m, n) \cap P$ consists of all elements of the form $(p, n) \in \mathbb{F}^+ \rtimes_\varphi \mathbb{N}$ for which the exponent sum $\sigma_a(p)$ is m. Set

$$\Sigma_{(m,n)} := \{(p, n) \in \mathbb{F}^+ \rtimes_\varphi \mathbb{N} : \sigma_a(p) = m \text{ and } b \text{ is not a suffix of } p\}.$$

Since $b = \varphi_n^{-1}(b) \in \varphi_n^{-1}(\mathbb{F}^+)$, every $(q, n) \in \mu^{-1}(m, n) \cap P$ can be written as the product $(p, n) \cdot (b^k, 0)$ for some $(p, n) \in \Sigma_{(m,n)}$ and $k \in \mathbb{N}$. So $(p, n) \leq_P (q, n)$. Moreover, if $(p, n), (q, n) \in \Sigma_{(m,n)}$ and $p \neq q$, then $p \vee_{\mathbb{F}^+} q = \infty$, because $\sigma_a(p^{-1}q) = 0$ and both p and q end in a and so cannot be prefixes of one another. Hence $(p, n) \vee_P (q, n) = \infty$ and we conclude that μ is controlled.

Since $(\ker \mu) \cap (\mathbb{F}^+ \rtimes_\varphi \mathbb{N}) = \langle b \rangle \times \{0\}$, it follows from Theorem 4.3 that $\mathrm{C}^*(\mathbb{F} \rtimes_\varphi \mathbb{Z}, \mathbb{F}^+ \rtimes_\varphi \mathbb{N})$ is nuclear.

Example 5.23. Let $c, d \geq 1$ such that $c + d$ is even. We will construct an action of the Baumslag–Solitar group $\mathrm{BS}(c, -d) = \langle a, b \colon ab^c = b^{-d}a \rangle$ on the free group $\mathbb{F}$ with generators $\{x, y, z\}$.

Let $\varphi_a, \varphi_b \colon \mathbb{F} \to \mathbb{F}$ be given by

$$\begin{aligned} \varphi_a(x) &= xy, & \varphi_a(y) &= y, & \varphi_a(z) &= zy; \\ \varphi_b(x) &= z, & \varphi_b(y) &= y, & \varphi_b(z) &= x. \end{aligned}$$

Then φ_a, φ_b are commuting automorphisms of $\mathbb{F}$, and $\varphi_b^2 = \mathrm{id}_{\mathbb{F}}$. Since $c + d$ is even, $\varphi_b^{c+d} = \mathrm{id}_{\mathbb{F}}$. Thus φ_a, φ_b induce an action

$$\varphi\colon \mathrm{BS}(c, -d) \to \mathrm{Aut}(\mathbb{F})$$

satisfying $\varphi_p(\mathbb{F}^+) \subseteq \mathbb{F}^+$ for all $p \in \mathrm{BS}(c, -d)^+$. Notice that $\varphi_p(x), \varphi_p(y)$ and $\varphi_p(z)$ are relatively prime. Hence

$$(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (\mathrm{BS}(c, -d), \mathrm{BS}(c, -d)^+)$$

is weakly quasi-lattice ordered by Proposition 5.21.

We will now show that $\mathrm{C}^*(\mathbb{F}^+ \rtimes_\varphi \mathrm{BS}(c, -d)^+)$ is nuclear. Consider the homomorphism $\sigma\colon \mathbb{F} \to \mathbb{Z}$ that sends both x and z to 1 and y to 0. Let $\theta\colon \mathrm{BS}(c, -d) \to \mathbb{Z}$ be the height map. Set $\mu(u, w) := (\sigma(u), \theta(w))$. Then

$$\mu\colon \mathbb{F} \rtimes_\varphi \mathrm{BS}(c, -d) \to \mathbb{Z}^2$$

is a group homomorphism because φ does not change the exponent sum $\sigma(u) = \sigma_x(u) + \sigma_z(u)$ in a word u. We claim that μ is controlled. Let $(r,p),(s,q) \in P = \mathbb{F}^+ \rtimes_\varphi \mathrm{BS}(c,-d)^+$ with a common upper bound in P. We must have $r \vee_{\mathbb{F}^+} s < \infty$. Without loss of generality, assume that r is a prefix of s. If $\theta(p) = 0$, then $(r,p) \vee_P (s,q) = (s, p \vee q)$ and so

$$\mu((r,p) \vee_P (s,q)) = \mu(r,p) \vee_{\mathbb{Z}^2} \mu(s,q).$$

In case $\theta(p) > 0$, the same reasoning as in Example 5.22 gives a least $m \in \mathbb{N}$ so that $r^{-1}sy^m \in \varphi_p(\mathbb{F}^+)$. Then $(r,p) \vee_P (s,q) = (sy^m, p \vee q)$ and so

$$\mu((r,p) \vee_P (s,q)) = \mu(r,p) \vee_{\mathbb{Z}^2} \mu(s,q).$$

It remains to show that μ satisfies item (b) of Definition 3.6. First, for $m \in \mathbb{N}$, we set

$$\Sigma_m \coloneqq \{r \in \mathbb{F}^+ : \sigma(r) = m \text{ and } y \text{ is not a suffix of } r\}.$$

Then $r \vee_{\mathbb{F}^+} s = \infty$ whenever $r, s \in \Sigma_m$ and $r \neq s$. Second, because the height map $\theta\colon \mathrm{BS}(c,-d) \to \mathbb{Z}$ is controlled, for all $k \in \mathbb{N}$, there exist a set Λ_k and a decreasing sequence $(s_n^\lambda)_{n\in\mathbb{N}} \subseteq \mu^{-1}(k) \cap \mathrm{BS}(c,-d)^+$ associated to each $\lambda \in \Lambda_k$ so that the conditions of Definition 3.6 hold. Finally, for each $(m,k) \in \mathbb{N}^2$ we set

$$\Lambda_{(m,k)} \coloneqq \Sigma_m \times \Lambda_k.$$

Thus $\gamma_1, \gamma_2 \in \Lambda_{(m,k)}$ and $\gamma_1 \neq \gamma_2$ entails $\gamma_1 \vee_P \gamma_2 = \infty$. The decreasing sequence associated to $(r,\lambda) \in \Sigma_m \times \Lambda_k$ has n-th term (r, s_n^λ). Given $(s,p) \in P$ with $\mu(s,p) = (m,k)$, write $s = ry^l$, where $r \in \mathbb{F}^+$ ends in x or z and $l \in \mathbb{N}$. Because $y = \varphi_q(y) \in \varphi_q(\mathbb{F}^+)$ for all $q \in \mathrm{BS}(c,-d)^+$, it follows that there are $n \in \mathbb{N}$ and $\lambda \in \Lambda_k$ such that $(r, s_n^\lambda) \leq_P (s,p)$. This completes the proof that μ also satisfies item (b) of Definition 3.6 and, therefore, is a controlled map.

Now $(\ker \mu) \cap (\mathbb{F}^+ \rtimes_\varphi \mathrm{BS}(c,-d)^+) = \langle y \rangle \times \langle b \rangle$. Hence $\mathrm{C}^*(\mathbb{F} \rtimes_\varphi \mathrm{BS}(c,-d), \mathbb{F}^+ \rtimes_\varphi \mathrm{BS}(c,-d)^+)$ is nuclear by Theorem 1.1.

We finish this section by presenting an example with a slightly different flavour. In particular, Lemma 5.16 and Proposition 5.21 do not apply in this case and we have to show the existence of least upper bounds directly.

Example 5.24. Let $\mathbb{F} = \langle a, b \rangle$ be the free group on two generators and let φ be the automorphism given by $\varphi(a) = ba$ and $\varphi(b) = b$. Then the semidirect product $(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (\mathbb{Z}, \mathbb{N})$ is weakly quasi-lattice ordered. To see this, let $(p,m),(q,n) \in P = \mathbb{F}^+ \rtimes_\varphi \mathbb{N}$ with an upper bound in P. This implies $p \vee_{\mathbb{F}^+} q < \infty$ and we can assume, say, $p \leq q$. Also, there is $r \in \mathbb{F}^+$ such that $p^{-1}r = p^{-1}qq^{-1}r \in \varphi_m(\mathbb{F}^+)$. Hence $p^{-1}q$ must have reduced form

$$b^{i_0}ab^{i_1}a \cdots ab^{i_k}, \qquad (k \geq 0)$$

where $i_k \in \mathbb{N}$ and $i_j \geq m$ for all $0 \leq j \leq k-1$. From this we deduce that $p^{-1}q$ itself lies in $\varphi_m(\mathbb{F}^+)$. Thus $(p,m) \vee_P (q,n) = (q, m \vee n)$ and this shows that $(\mathbb{F}, \mathbb{F}^+) \rtimes_\varphi (\mathbb{Z}, \mathbb{N})$ is weakly quasi-lattice ordered as claimed.

We will now see that the canonical surjection

$$\mu\colon \mathbb{F} \rtimes_\varphi \mathbb{Z} \twoheadrightarrow \mathbb{Z}$$

is controlled. By Proposition 5.17, $\mu(x \vee_P y) = \mu(x) \vee_{\mathbb{Z}} \mu(y)$ holds for all $x, y \in P$ such that $x \vee_P y < \infty$. So we need to establish condition (b) of Definition 3.1. Let

$n \in \mathbb{N}$. If $n = 0$, then $\mu^{-1}(0) \cap P = \mathbb{F}^+ \times \{0\}$ and we set $\Sigma_0 := \{(e_{\mathbb{F}^+}, 0)\}$. That is, Σ_0 is the unit set given by the identity of $\mathbb{F}^+ \rtimes_\varphi \mathbb{N}$. Suppose $n > 0$. We let

$$\Sigma_n := \{(\sigma, n) \in \mu^{-1}(n) \cap P : \sigma = \tau a, \tau \in \mathbb{F}^+ \text{ and } b^n \text{ is not a suffix of } \tau\}.$$

We prove that $\sigma_1 = \sigma_2$ whenever $(\sigma_1, n), (\sigma_2, n) \in \Sigma_n$ and $(\sigma_1, n) \vee_P (\sigma_2, n) < \infty$. Indeed, $(\sigma_1, n) \vee_P (\sigma_2, n) < \infty$ entails $\sigma_1 \vee_{\mathbb{F}^+} \sigma_2 < \infty$. So we may assume that σ_1 is a prefix of σ_2. Also, there exists $r \in \mathbb{F}^+$ such that $\sigma_1^{-1}\sigma_2$ is a prefix of $\varphi_n(r)$. In particular, $\sigma_1^{-1}\sigma_2 \neq e_{\mathbb{F}^+}$ would imply that $\sigma_1^{-1}\sigma_2$ ends in $b^n a$ because a is a suffix of σ_2. Since $\sigma_1^{-1}\sigma_2 \in \mathbb{F}^+$ is also a suffix of σ_2, we must have $\sigma_1 = \sigma_2$, as desired.

Now let $(x, n) \in P$. Write $x = \sigma u$, where $(\sigma, n) \in \Sigma_n$ and $u \in \varphi_n(\mathbb{F}^+)$. More precisely, u is the tail of x in $\varphi_n(\mathbb{F}^+)$. Thus $(x, n) \geq_P (\sigma, n)$ and this completes the proof that μ is controlled.

Finally, because $\ker \mu \cap P = \mathbb{F}^+ \times \{0\}$, Theorem 1.1 tells us that $C^*(\mathbb{F} \rtimes_\varphi \mathbb{Z}, \mathbb{F}^+ \rtimes_\varphi \mathbb{N})$ is also nuclear.

5.4. **Graph products.** Here we briefly introduce notation from [8] that we need. Let Γ denote a graph with vertex set Λ and edge set $E(\Gamma) = \{\{I, J\} : I, J \in \Lambda, I \neq J\}$. If $\{I, J\} \in E(\Gamma)$, we say that I and J are *adjacent*. Let $\{G_I\}_{I \in \Lambda}$ be a family of groups. Then the graph product

$$G := \Gamma_{I \in \Lambda} G_I$$

is the quotient of the free product $*_\Lambda G_I$ by the smallest normal subgroup containing $x_1 x_2 x_1^{-1} x_2^{-2}$ for all pairs $x_1 \in G_I$ and $x_2 \in G_J$ where I and J are adjacent. A generating set for G is $\sqcup_{I \in \Lambda} G_I \setminus \{e\}$. Given a generator x, we write $I(x)$ for the unique vertex I such that $x \in G_I$.

An *expression* for an element of $x \in G$ is a word $x_1 x_2 \dots x_l$ in the generators which equals x. The graph product relations allow modification of an expression by replacing a subexpression $x_i x_{i+1}$ with $x_{i+1} x_i$ if $I(x_i)$ is adjacent to $I(x_{i+1})$; this replacement is called a *shuffle*. If an expression contains a subexpression $x_i x_{i+1}$ with $I(x_i) = I(x_{i+1})$, then we obtain a shorter expression for x by *amalgamating* the subexpression $x_i x_{i+1}$ of length two into the subexpression $\hat{x}_i = x_i x_{i+1}$ of length one. An expression is called *reduced* if its length cannot be reduced by finitely many shuffles and amalgamation. The *length* $l(x)$ of x is the length of any reduced expression equal to x. An *initial vertex* on x is a vertex $I \in \Lambda$ such that $x_i \in G_I$ is a generator in a reduced expression $x_1 \dots x_{l(x)}$ for x and $I(x_i)$ is adjacent to $I(x_j)$ for $j < i$. If I is an initial vertex on x corresponding to the generator x_i, we write x_I for x_i; if I is not an initial vertex on x we set $x_I = e$.

Now suppose that each G_I is partially ordered with positive cone P_I. A reduced expression $x_1 x_2 \dots x_{l(x)}$ for x is *positive* if $x_i \in P_{I(x_i)}$ for $1 \leq i \leq l$. We say that x is *positive* if it has a reduced expression which is positive (and then all reduced expressions of x are positive). We let P denote the subsemigroup of G consisting of positive elements of G. Then (G, P) is a partially ordered group.

Theorem 10 of [8] says that a graph product of quasi-lattice ordered groups is a quasi-lattice ordered group, and then [8, Proposition 13] finds an effective algorithm to compute a least upper bound when it exists. We now prove similar results for weakly quasi-lattice ordered groups. Our proof is very different to those in [8]. The proof of [8, Theorem 10] used the characterisation of quasi-lattice ordered group from item (iv) of [8, Lemma 7] which does not apply to a weak quasi-lattice. The algorithm of [8, Proposition 13] is based on the equality (5.2) below, and to prove it [8, Theorem 10] is used.

Theorem 5.25. *A graph product $(G,P)=\Gamma_{I\in\Lambda}(G_I,P_I)$ of weakly quasi-lattice ordered groups is a weakly quasi-lattice ordered group. In particular, if $x,y\in P$ have a common upper bound in P and I is any vertex in Λ, then with $x=x_Ix'$ and $y=y_Iy'$ we have*

$$x\vee y=(x_I\vee y_I)(x'\vee y'), \tag{5.2}$$

and the vertices on $x'\vee y'$ are vertices on x' and y'.

Proof. We start by proving that (G,P) is weakly quasi-lattice ordered. For $n\in\mathbb{N}$ we set

$$P_n=\{(x,y)\in P\times P\colon l(x)+l(y)=n \text{ and } x,y \text{ have a common upper bound}\}.$$

Notice that if $(x,y)\in P_0$, then $x=y=e$ and $x\vee y=e$; trivially, all the vertices on $x\vee y$ are vertices on x and y. We will prove the result by induction on n. For $n\geq 0$, assume that if $(x,y)\in P_n$, then $x\vee y$ exists and the vertices on $x\vee y$ are the vertices on x and y.

Let $(x,y)\in P_{n+1}$. Then there exists a common upper bound $z\in P$ for x and y. Let $I\in\Lambda$ be an initial vertex on x or y. We write $x=x_Ix'$, $y=y_Iy'$ and $z=z_Iz'$. [8, Lemma 11] applies to a graph product of partially ordered groups; applying it to $x\leq z$ gives

(i) $x_I\leq z_I$, and
(ii) either $x_I=z_I$ or the vertices on x' are adjacent to I.

It follows that $x_I\leq z_I$ and $y_I\leq z_I$. Since (G_I,P_I) is a weakly quasi-lattice ordered group we have $x_I\vee y_I\leq z_I$.

We claim that z' is a common upper bound for x' and y'. If $x_I=z_I$, then

$$x\leq z\Longrightarrow x_Ix'\leq z_Iz'\Longrightarrow x'\leq z'$$

using left invariance.

Next suppose that $x_I<z_I$. Then the vertices on x' are adjacent to I. In particular, I is not a vertex on x', as otherwise we could amalgamate and thus shorten the expression. Since $x\leq z$ we have

$$x^{-1}z=(x')^{-1}x_I^{-1}z_Iz'=x_I^{-1}z_I(x')^{-1}z'\in P.$$

There exists $v_I\in P_I$ and v' in P such that $v:=x^{-1}z=v_Iv'$. Let $w_1\dots w_{l(w)}$ be a reduced expression for $(x')^{-1}z'$. We now want to compare the two expressions

$$x_I^{-1}z_Iw_1\dots w_{l(w)}=v_Iv'$$

for $x^{-1}z$. For $1\leq i\leq l(w)$, if $I(w_i)=I$, then w_i is in P_I because I is not a vertex on $(x')^{-1}$. Equating $(x^{-1}z)_I$ in both expressions, we see that $x_I^{-1}z_I\leq v_I$. Now $w_1\dots w_{l(w)}=(x_I^{-1}z_I)^{-1}v_Iv'\in P$. Thus $(x')^{-1}z'\in P$ and $x'\leq z'$, as claimed.

It follows from a similar argument that also $y'\leq z'$.

Since I is an initial vertex on x or y we have $l(x')+l(y')<l(x)+l(y)$. By the induction hypothesis, $x'\vee y'$ exists and the vertices on $x'\vee y'$ are the vertices on x' and y'.

We claim that the right-hand-side

$$(x_I\vee y_I)(x'\vee y')$$

of (5.2) is a least upper bound for x and y. We first show that $x,y\leq(x_I\vee y_I)(x'\vee y')$. We again consider the two cases arising from [8, Lemma 11]: firstly, if $x_I=z_I$, then $x_I\vee y_I=z_I=x_I$. Secondly, if $x_I<z_I$ then I is adjacent to the vertices of x'. Thus

$$x^{-1}(x_I\vee y_I)(x'\vee y')=(x')^{-1}x_I^{-1}(x_I\vee y_I)(x'\vee y')$$

$$= \begin{cases} (x')^{-1}(x' \vee y') & \text{if } x_I = z_I \\ x_I^{-1}(x_I \vee y_I)(x')^{-1}(x' \vee y') & \text{if } x_I < z_I, \end{cases}$$

which in both cases is in P. Thus $x \leq (x_I \vee y_I)(x' \vee y')$, and similarly $y \leq (x_I \vee y_I)(x' \vee y')$. Thus $(x_I \vee y_I)(x' \vee y')$ is a common upper bound for x and y, as claimed.

Next, suppose that $w \in P$ is a common upper bound for x and y. We need to show that $w \geq (x_I \vee y_I)(x' \vee y')$. Write $w = w_I w'$. Since $x \leq w$, by [8, Lemma 11], either $x_I = w_I$ or I is adjacent to every vertex on x', and similarly for y. As above, we have $x', y' \leq w'$.

First, suppose that $x_I = w_I$ or $y_I = w_I$. Then

$$\begin{aligned}\big((x_I \vee y_I)(x' \vee y')\big)^{-1} w &= (x' \vee y')^{-1}(x_I \vee y_I)^{-1} w_I w' \\ &= (x' \vee y')^{-1} w_I^{-1} w_I w' = (x' \vee y')^{-1} w' \in P.\end{aligned}$$

Thus $(x_I \vee y_I)(x' \vee y') \leq w$ as needed.

Second, suppose that $x_I < w_I$ and $y_I < w_I$. Recall that the vertices on x' and y' are adjacent to I, and hence so are the vertices on $x' \vee y'$ and $(x' \vee y')^{-1}$. Thus

$$\begin{aligned}\big((x_I \vee y_I)(x' \vee y')\big)^{-1} w &= (x' \vee y')^{-1}\big((x_I \vee y_I)^{-1} w_I\big) w' \\ &= \big((x_I \vee y_I)^{-1} w_I\big)\big((x' \vee y')^{-1} w'\big);\end{aligned}$$

this is in P because both $(x_I \vee y_I)^{-1} w_I$ and $(x' \vee y')^{-1} w'$ are in P. Thus $(x_I \vee y_I)(x' \vee y') \leq w$ and hence is a least upper bound for x and y, as claimed. We have proved that (G, P) is a weak quasi-lattice, and that if I is an initial vertex on x or y, then (5.2) holds. On the other hand, if I is not an initial vertex of x or y, then $x_I = e = y_I$, and (5.2) holds trivially. □

Let $\Gamma_{I \in \Lambda}(G_I, P_I)$ be a graph product of partially ordered groups (G_I, P_I). By the universal property of the free product there is a unique group homomorphism $\phi : *_\Lambda G_I \to \oplus_{I \in \Lambda} G_I$ extending the inclusion of G_I in $*_\Lambda G_I$. Since $\phi(G_I)$ and $\phi(G_J)$ commute if $I \neq J$, it follows that ϕ factors through the quotient $\Gamma_{I \in \Lambda} G_I$. Thus we have a homomorphism

$$\phi : \Gamma_{I \in \Lambda}(G_I, P_I) \to \bigoplus_{I \in \Lambda}(G_I, P_I), \tag{5.3}$$

also denoted by ϕ, which extends the inclusion of G_I in $\Gamma_{I \in \Lambda} G_I$.

Let $I \in \Lambda$ and for $x \in P$ write $x = x_I x'$. Then $\phi(x) = \phi(x_I)\phi(x') = x_I \phi(x')$ and $\phi(x)_I = x_I \phi(x')_I$.

Now suppose that $(G, P) = \Gamma_{I \in \Lambda}(G_I, P_I)$ is a graph product of partially ordered groups (G_I, P_I). It is proved in [8, Proposition 19] that if (G, P) is a quasi-lattice ordered group, then ϕ is a controlled map with $\ker \phi \cap P = \{e\}$. We now prove the analogous result when (G, P) is only a weakly quasi-lattice ordered group.

Proposition 5.26. *Let $(G, P) = \Gamma_{I \in \Lambda}(G_I, P_I)$ be a graph product of weakly quasi-lattice ordered groups. Let $\phi : \Gamma_{I \in \Lambda}(G_I, P_I) \to \bigoplus_{I \in \Lambda}(G_I, P_I)$ be the homomorphism defined at* (5.3). *Let $x, y \in P$ such that $x \vee y < \infty$. Then*

(a) *$\phi(x) \vee \phi(y) = \phi(x \vee y)$, and*
(b) *$\phi(x) = \phi(y) \implies x = y$.*

In particular, ϕ is a controlled map.

Proof. We first observe that ϕ is order-preserving. Suppose that $g, h \in G$ with $g \leq h$. Then $h^{-1} g \in P$ and $\phi(h)^{-1}\phi(g) = \phi(h^{-1}g) \subseteq \phi(P) = \oplus P_I$. Thus $\phi(g) \leq \phi(h)$.

For (a) we induct on $l(x)+l(y)$. When $l(x)+l(y)=0$ we have $x=y=e$ and $\phi(x)\vee\phi(y)=e=\phi(e)=\phi(x\vee y)$. Let $n\geq 0$ and assume that if $v,w\in P$ with $v\vee w<\infty$ and $l(v)+l(w)\leq n$, then $\phi(v)\vee\phi(w)=\phi(v\vee w)$.

Suppose that $x,y\in P$ with $x\vee y<\infty$ and $l(x)+l(y)=n+1$. Let $I\in\Lambda$ be an initial vertex on x. Write $x=x_Ix'$ and $y=y_Iy'$, and notice that $l(x')<l(x)$. We again consider the cases arising from [8, Lemma 11]. First, suppose that $x_I<x_I\vee y_I$ and that $y_I<x_I\vee y_I$. Then the vertices on x' and y' are adjacent to I. We have

$$\begin{aligned}\phi(x\vee y)&=\phi\big((x_I\vee y_I)(x'\vee y')\big)\quad\text{(by Theorem 5.25)}\\&=\phi(x_I\vee y_I)\phi(x'\vee y')\\&=\phi(x_I\vee y_I)\big(\phi(x')\vee\phi(y')\big)\quad\text{(by the induction hypothesis)}\\&=\big(\phi(x_I)\vee\phi(y_I)\big)\big(\phi(x')\vee\phi(y')\big).\end{aligned}$$

Since the vertices on $\phi(x')$ and $\phi(y')$ are the vertices on x' and y', and I is adjacent to these, we get

$$\begin{aligned}\phi(x\vee y)&=\big(\phi(x_I)\vee\phi(y_I)\big)\vee\big(\phi(x')\vee\phi(y')\big)\\&=\big(\phi(x_I)\vee\phi(x')\big)\vee\big(\phi(y_I)\vee\phi(y')\big)\\&=\phi(x_I)\phi(x')\vee\phi(y_I)\phi(y')\\&=\phi(x)\vee\phi(y).\end{aligned}$$

Second, suppose that $x_I=x_I\vee y_I$ or that $y_I=x_I\vee y_I$. We assume, without loss of generality, that $x_I=x_I\vee y_I$. If $y_I\neq e$, then $l(y')<l(y)$ and

$$\begin{aligned}\phi(x\vee y)&=\phi(x_Ix'\vee y_Iy')\\&=\phi\big(y_I(y_I^{-1}x_Ix'\vee y')\big)\quad\text{(by left invariance)}\\&=\phi(y_I)\phi(y_I^{-1}x_Ix'\vee y')\\&=\phi(y_I)\big(\phi(y_I^{-1}x_Ix')\vee\phi(y')\big)\quad\text{(by the induction hypothesis)}\\&=\phi(x_Ix')\vee\phi(y_Iy')\\&=\phi(x)\vee\phi(y).\end{aligned}$$

If $y_I=e$, then $y=y'$. It follows from [8, Lemma 11] applied to $y\leq x\vee y$ that I is adjacent to the vertices of y. Now

$$\begin{aligned}\phi(x\vee y)&=\phi\big((x_I\vee y_I)(x'\vee y')\big)\\&=\phi\big((x_I)(x'\vee y)\big)\\&=\phi(x_I)\big(\phi(x')\vee\phi(y)\big)\quad\text{(by the induction hypothesis, since } l(x')<l(x))\\&=\phi(x_I)\phi(x')\vee\phi(x_I)\phi(y)\\&=\phi(x)\vee\phi(x_I)\vee\phi(y)\\&=\phi(x)\vee\phi(y)\end{aligned}$$

since $\phi(x_I)\leq\phi(x)$. We have now proved (a).

For (b), suppose that $\phi(x)=\phi(y)$. Write $x\vee y=xu=yv$ where $u,v\in P$. By (a) we have

$$\phi(x)=\phi(x)\vee\phi(y)=\phi(x\vee y)=\phi(x)\phi(u).$$

Thus $\phi(u)=e$. But $u_I\leq\phi(u)_I$ for all $I\in\Lambda$. So $\phi(u)=e$ implies $u=e$. Similarly, $v=e$ and hence $x=y$. This gives (b), and it follows that ϕ is a controlled map. □

Corollary 5.27. *Let $(G,P) = \Gamma_{I\in\Lambda}(G_I,P_I)$ be a graph product of weakly quasi-lattice ordered groups. Suppose that G_I is an amenable group for every $I \in \Lambda$. Then (G,P) is amenable as a weakly quasi-lattice ordered group and $\mathrm{C}^*(G,P)$ is nuclear.*

Proof. Since each G_I is amenable, so is the direct sum $\oplus G_I$. Now by Proposition 5.26, ϕ is a controlled map into an amenable group. Since $\ker\phi \cap P = \{e\}$, $\mathrm{C}^*(G,P)$ is nuclear and (G,P) is amenable by Theorem 1.1. □

6. Amenability

Recall from §2 that (G,P) is amenable in the sense of Nica if the Toeplitz representation $T : P \to B(\ell^2(P))$ induces an isomorphism $\pi_T : \mathrm{C}^*(G,P) \to \mathcal{T}(G,P)$. We can now extend the amenability theorem of [15] to weak quasi-lattices admitting a controlled map in the sense of Definition 3.6. The proof ideas are the ones from [15, Theorem 3.2] adjusted to the new definition. Since nuclearity of $\mathrm{C}^*(G,P)$ implies that (G,P) is amenable by [24, Theorem 6.42], Theorem 6.1 is of interest only when we don't already know that $\mathrm{C}^*(G,P)$ is nuclear.

Theorem 6.1. *Let (G,P) and (K,Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon (G,P) \to (K,Q)$ is a controlled map in the sense of Definition 3.6, that K is amenable and that $(\ker\mu, \ker\mu\cap P)$ is amenable. Then (G,P) is amenable.*

For $k \in Q$ we consider the subspaces

$$H_k := \overline{\operatorname{span}}\{e_{s_n^\lambda\alpha} \colon \lambda \in \Lambda_k, n \in \mathbb{N}, \alpha \in \ker\mu \cap P\}.$$

Notice that $H_k = \ell^2(\mu^{-1}(k) \cap P)$, and in particular, that $H_e = \ell^2(\ker\mu \cap P)$.

Lemma 6.2. *Let (G,P) and (K,Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon (G,P) \to (K,Q)$ is a controlled map in the sense of Definition 3.6. Suppose that $(\ker\mu, \ker\mu\cap P)$ is amenable. Then $\pi_T(\cdot)|_{H_k}$ is isometric on B_k for all $k \in Q$.*

Proof. We start with $k = e$. Let $\alpha, \beta \in \ker\mu \cap P$. Then $\mu(\alpha\beta) = e = \mu(\alpha^{-1}\beta)$. Thus $T_\alpha| : H_e \to H_e$ with $T_\alpha\epsilon_\beta = \epsilon_{\alpha\beta} = S_\alpha\epsilon_\beta$ and $T_\alpha^*\epsilon_\beta = S_\alpha^*\epsilon_\beta$. Thus

$$\pi_T(\cdot)|_{H_e} : B_e \to \mathcal{T}(\ker\mu, \ker\mu\cap P)$$

is well-defined. We write S for the Toeplitz representation of $\ker\mu\cap P$ on $H_e = \ell^2(\ker\mu\cap P)$. Since $\big(\ker\mu, \ker\mu\cap P\big)$ is amenable, π_S is faithful, and the restriction of w to $\ker\mu\cap P$ induces an isomorphism $\pi_{w|} : \mathrm{C}^*(\ker\mu, \ker\mu\cap P) \to B_e$ by Lemma 4.2(a). We have $\pi_S = \pi_T(\cdot)|_{H_e} \circ \pi_{w|}$. Now $\pi_T(\cdot)|_{H_e}$ has to be faithful on B_e as well.

Next we consider $k \neq e$. By Proposition 4.5, $B_k = \varinjlim B_{k,n}$ and $B_{k,n} = \varinjlim B_{k,n,F}$. Thus it suffices to show that $\pi_T(\cdot)|_{H_k}$ is isometric on $B_{k,n,F}$ for $n \in \mathbb{N}$ and finite subsets F of Λ_k.

We start by showing that H_k is invariant for $\pi_T(B_{k,n,F})$ so that $\pi_T(\cdot)|_{H_k}$ makes sense. We have

$$\begin{aligned} B_{k,n,F} &= \operatorname{span}\{w_{s_n^\lambda} D w_{s_n^\rho}^* \colon \lambda,\rho \in F \text{ and } D \in B_e\} \\ &= \overline{\operatorname{span}}\{w_{s_n^\lambda\alpha} w_{s_n^\rho\beta}^* \colon \lambda,\rho \in F \text{ and } \alpha,\beta \in \ker\mu\cap P\}. \end{aligned} \tag{6.1}$$

For a spanning element $w_{s_n^\lambda\alpha} w_{s_n^\rho\beta}^*$ of $B_{k,n,F}$ and a spanning element $e_{s_m^\sigma\gamma}$ of H_k we have

$$\pi_T(w_{s_n^\lambda\alpha} w_{s_n^\rho\beta}^*) e_{s_m^\sigma\gamma} = \begin{cases} e_{s_n^\lambda\alpha(s_n^\rho\beta)^{-1}s_m^\sigma\gamma} & \text{if } s_n^\rho\beta \le s_m^\sigma\gamma \text{ (and then } \rho = \sigma) \\ 0 & \text{else.} \end{cases}$$

Since $\mu\big(\alpha(s_n^\rho\beta)^{-1}s_m^\sigma\gamma\big) = e$ we see that $\pi_T(w_{s_n^\lambda\alpha}w^*_{s_n^\rho\beta})e_{s_m^\sigma\gamma} \in H_k$. It follows that H_k is invariant for $\pi_T(\cdot)$.

Now suppose that $a \in B_{k,n,F}$ and $\pi_T(a)|_{H_k} = 0$. Then $a = \sum_{\lambda,\rho\in F} w_{s_n^\lambda}a_{\lambda,\rho}w_{s_n^\rho}$ for some $a_{\lambda,\rho} \in B_e$. (Here we use the property that there is no closure at (6.1).) Fix $\eta, \xi \in F$. Then

$$T^*_{s_n^\eta}\pi_T(a)T_{s_n^\xi} = \pi_T(w^*_{s_n^\eta}aw_{s_n^\xi}) = \pi_T(a_{\eta,\xi}).$$

Since $T_{s_n^\xi}$ is an isometry from H_e to H_k and $\pi_T(a)|_{H_k} = 0$, it follows that $\pi_T(a)T_{s_n^\xi}|_{H_e} = 0$. Thus $\pi_T(a_{\eta,\xi})|_{H_e} = 0$. But we proved that $\pi_T(\cdot)|_{H_e}$ is faithful on B_e above. Thus $a_{\eta,\xi} = 0$. It follows that $a = 0$. Thus $\pi_T(\cdot)|_{H_k}$ is faithful on $B_{k,n,F}$, and hence is isometric on $B_{k,n,F}$. □

Proposition 6.3. *Let (G,P) and (K,Q) be weakly quasi-lattice ordered groups. Suppose that $\mu\colon (G,P) \to (K,Q)$ is a controlled map in the sense of Definition* 3.6. *If $(\ker\mu, \ker\mu\cap P)$ is amenable, then π_T is faithful on $\mathrm{C}^*(G,P)^{\delta_\mu}$.*

Proof. By Proposition 4.5, $\mathrm{C}^*(G,P)^{\delta_\mu} = \varinjlim_{I\in\mathcal{I}} C_I$. So it suffices to show that π_T is isometric on each C_I. Let $a \in C_I$ such that $\pi_T(a) = 0$. Since $C_I = \sum_{k\in I} B_k$, there exist $a_k \in B_k$ such that $a = \sum_{k\in I} a_k$. Then $0 = \sum_{k\in I}\pi_T(a_k)$.

We claim that $k \not\leq l$ implies that $\pi_T(B_k)H_l = \{0\}$. We prove the contrapositive. Suppose that $\pi(B_k)H_l \neq \{0\}$. Then there exists $q \in \mu^{-1}(k)\cap P$ and a spanning vector $e_{s_n^\lambda\alpha} \in H_l$ such that $\pi_T(w_q^*)e_{s_n^\lambda\alpha} \neq 0$. We have

$$\pi_T(w_q^*)e_{s_n^\lambda\alpha} = \begin{cases} e_{q^{-1}s_n^\lambda\alpha} & \text{if } q \leq s_n^\lambda\alpha \\ 0 & \text{else.}\end{cases}$$

Here $s_n^\lambda \in \mu^{-1}(l)\cap P$ and $\alpha \in \ker\mu\cap P$. So $\pi_T(B_k)H_l \neq \{0\}$ implies $k = \mu(q) \leq \mu(s_n^\lambda\alpha) = \mu(s_n^\lambda) = l$. This proves the claim.

Now let l_1 be a minimal element in I. Then

$$0 = \big(\sum_{k\in I}\pi_T(a_k)\big)H_{l_1} = \pi_T(a_{l_1})H_{l_1}.$$

Since $(\ker\mu, \ker\mu\cap P)$ is amenable, Lemma 6.2 implies that $\pi_T(\cdot)|_{H_{l_1}}$ is isometric on B_{l_1}. Thus $a_{l_1} = 0$. Now let l_2 be a minimal element of $I \setminus \{l_1\}$. Repeat the above argument to get $a_{l_2} = 0$. Since I is finite, we conclude that $a = 0$. Thus π_T is faithful on C_I, and hence is isometric on C_I. It follows that π_T is isometric on $\mathrm{C}^*(G,P)^{\delta_\mu}$. □

Proof of Theorem 6.1. We need to show that the conditional expectation

$$E : \mathrm{C}^*(G,P) \to \overline{\operatorname{span}}\{w_pw_p^*\colon p\in P\}$$

is faithful. Since K is amenable, the conditional expectation $\Psi_\mu\colon \mathrm{C}^*(G,P) \to \mathrm{C}^*(G,P)^{\delta_\mu}$ is faithful by [15, Lemma 3.5]. We claim that it suffices to show that E is faithful on $\mathrm{C}^*(G,P)^{\delta_\mu}$. To see this, let $a \in \mathrm{C}^*(G,P)$ and suppose that $E(a^*a) = 0$. We observe that $E = E\circ\Psi_\mu$. Since Ψ_μ is positive, there exists $b \in \mathrm{C}^*(G,P)^{\delta_\mu}$ such that $\Psi_\mu(a^*a) = b^*b$. Then $0 = E(a^*a) = E\circ\Psi_\mu(a^*a) = E(b^*b)$. If E is faithful on $\mathrm{C}^*(G,P)^{\delta_\mu}$, we get $b = 0$. Then $\Psi_\mu(a^*a) = 0$ implies $a = 0$ because Ψ_μ is faithful. This proves the claim.

Let P_p be the orthogonal projection onto the subspace $\operatorname{span}\{e_p\}$. There is a conditional expectation Δ on $B(\ell^2(P))$ such that $\Delta(T) = \sum_{p\in P} P_pTP_p$. It is easy to see that Δ is faithful and, by computing on generators, that

$$\Delta\circ\pi_T = \pi_T\circ E.$$

Now suppose that $b \in \mathrm{C}^*(G,P)^{\delta_\mu}$ such that $E(b^*b) = 0$. Then $0 = \pi_T(E(b^*b)) = \Delta(\pi_T(b^*b))$. Since Δ is faithful, $\pi_T(b^*b) = 0$. By Proposition 6.3 the Toeplitz representation is faithful on $\mathrm{C}^*(G,P)^{\delta_\mu}$, and hence $b^*b = 0$. Thus $b = 0$, and E is faithful on $\mathrm{C}^*(G,P)^{\delta_\mu}$. As mentioned above, this implies that E is faithful. □

Appendix A. Characterising nuclearity using the conditional expectation

Proposition A.1. *Let (G,P) be a weakly quasi-lattice ordered group. Then $\mathrm{C}^*(G,P)$ is nuclear if and only if, for every unital C^*-algebra A, the conditional expectation $E_{A,\max} : A \otimes_{\max} \mathrm{C}^*(G,P) \to A \otimes \overline{\operatorname{span}}\{w_p w_p^* : p \in P\}$ is faithful.*

For the proof of Proposition A.1 we need the following lemma.

Lemma A.2. *Let A and C be C^*-algebras, and let B be a C^*-subalgebra of C. Suppose that $E : C \to B$ is a conditional expectation. Then $\mathrm{id} \odot E : A \odot C \to A \odot B$ extends to conditional expectations*

$$E_{A,\min} : A \otimes_{\min} C \to A \otimes_{\min} B \text{ and } E_{A,\max} : A \otimes_{\max} C \to A \otimes_{\max} B.$$

If E is faithful, then so is $E_{A,\min}$.

Proof. Since E is a conditional expectation, it is a completely positive contractive map by [2, Theorem 1.5.10]. Now both id and E are completely positive maps, and so by [2, Theorem 3.5.3], $\mathrm{id} \odot E$ extends to completely positive maps $E_{A,\min}$ and $E_{A,\max}$ of norm 1. Then $E_{A,\min}$ and $E_{A,\max}$ are linear idempotents of norm 1 that act as the identity map on $A \otimes_{\min} B$ and $A \otimes_{\max} B$, respectively, and hence are conditional expectations.

Now suppose that E is faithful. We will show that $E_{A,\min}$ is faithful. Let X be the standard A-A imprimitivity bimodule. Let $Y := C$ be the right B-module with action $y \cdot b := yb$ for all $y \in Y$ and $b \in B$. Since E is faithful, $\langle y_1\,,\,y_2\rangle := E(y_1^* y_2)$ is a B-valued inner product and turns Y into a right Hilbert B-module. View Y as a $\mathcal{K}(Y)$-B imprimitivity bimodule. By [30, Proposition 3.36] $X \odot Y$ completes to give an $A \otimes \mathcal{K}(Y)$-$A \otimes_{\min} B$ imprimitivity bimodule with right inner product characterised by $\langle x_1 \otimes y_1\,,\,x_2 \otimes y_2\rangle = \langle x_1\,,\,x_2\rangle \otimes \langle y_1\,,\,y_2\rangle$ for $x_1, x_2 \in A$ and $y_1, y_2 \in C$. In particular,

$$\begin{aligned}\langle x_1 \otimes y_1\,,\,x_2 \otimes y_2\rangle = x_1^* x_2 \otimes E(y_1^* y_2) &= \mathrm{id} \otimes E(x_1^* x_2 \otimes y_1^* y_2)\\ &= E_{A,\min}(x_1^* x_2 \otimes y_1^* y_2).\end{aligned}$$

It follows that for $z \in A \otimes_{\min} C$ we have $\langle z\,,\,z\rangle = E_{A,\min}(z^* z)$. Thus $E_{A,\min}(z^* z) = 0$ implies $z = 0$, and $E_{A,\min}$ is faithful. □

Proof of Proposition A.1. Let A be a unital C^*-algebra and write $q_A : A \otimes_{\max} \mathrm{C}^*(P) \to A \otimes_{\min} \mathrm{C}^*(P)$ for the quotient map. By Lemma A.2, $\mathrm{id} \odot E$ extends to conditional expectations

$$\begin{aligned}E_{A,\max} &: A \otimes_{\max} \mathrm{C}^*(G,P) \to A \otimes \overline{\operatorname{span}}\{w_p w_p^* : p \in P\}\\ E_{A,\min} &: A \otimes_{\min} \mathrm{C}^*(G,P) \to A \otimes \overline{\operatorname{span}}\{w_p w_p^* : p \in P\},\end{aligned}$$

and $E_{A,\min}$ is faithful if E is. We have $E_{A,\max} = E_{A,\min} \circ q_A$.

First, suppose that $\mathrm{C}^*(G,P)$ is nuclear and fix a unital C^*-algebra A. Since (G,P) is a weak quasi-lattice, it follows from [24, Theorem 6.44] that (G,P) is amenable. Thus E is faithful, and then so is $E_{A,\min}$ by Lemma A.2. Since $\mathrm{C}^*(G,P)$ is nuclear, q_A is injective. Thus $E_{A,\max} = E_{A,\min} \circ q_A$ is faithful.

Second, suppose that, for every unital C*-algebra A, the expectation $E_{A,\max}$ is faithful. Since $E_{A,\max} = E_{A,\min} \circ q_A$ we must have that q_A injective. Thus $A \otimes_{\max} \mathrm{C}^*(G,P)$ is isomorphic to $A \otimes_{\min} \mathrm{C}^*(G,P)$, and $\mathrm{C}^*(G,P)$ is nuclear by [30, Lemma B.42]. □

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