

# Arc-disjoint Strong Spanning Subdigraphs of Semicomplete Compositions\*

Jørgen Bang-Jensen<sup>1</sup> Gregory Gutin<sup>2</sup> Anders Yeo<sup>1</sup>

<sup>1</sup> IMADA, University of Southern Denmark  
Odense, Denmark

jbj@imada.sdu.dk, andersyeo@gmail.com

<sup>2</sup> Department of Computer Science  
Royal Holloway, University of London  
Egham, Surrey, TW20 0EX, UK  
g.gutin@rhul.ac.uk

## Abstract

A **strong arc decomposition** of a digraph  $D = (V, A)$  is a decomposition of its arc set  $A$  into two disjoint subsets  $A_1$  and  $A_2$  such that both of the spanning subdigraphs  $D_1 = (V, A_1)$  and  $D_2 = (V, A_2)$  are strong. Let  $T$  be a digraph with  $t$  vertices  $u_1, \dots, u_t$  and let  $H_1, \dots, H_t$  be digraphs such that  $H_i$  has vertices  $u_{i,j_i}$ ,  $1 \leq j_i \leq n_i$ . Then the composition  $Q = T[H_1, \dots, H_t]$  is a digraph with vertex set  $\bigcup_{i=1}^t V(H_i) = \{u_{i,j_i} \mid 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$  and arc set

$$\left( \bigcup_{i=1}^t A(H_i) \right) \cup \left( \bigcup_{u_i u_p \in A(T)} \{u_{i,j_i} u_{p,q_p} \mid 1 \leq j_i \leq n_i, 1 \leq q_p \leq n_p\} \right).$$

We obtain a characterization of digraph compositions  $Q = T[H_1, \dots, H_t]$  which have a strong arc decomposition when  $T$  is a semicomplete digraph and each  $H_i$  is an arbitrary digraph. Our characterization generalizes a characterization by Bang-Jensen and Yeo (2003) of semicomplete digraphs with a strong arc decomposition and solves an open problem by Sun, Gutin and Ai (2019) on strong arc decompositions of digraph compositions  $Q = T[H_1, \dots, H_t]$  in which  $T$  is semicomplete and each  $H_i$  is arbitrary. Our proofs are constructive and imply the existence of a polynomial algorithm for constructing a strong arc decomposition of a digraph  $Q = T[H_1, \dots, H_t]$ , with  $T$  semicomplete, whenever such a decomposition exists.

**Keywords:** strong spanning subdigraph; decomposition into strong spanning subdigraphs; semicomplete digraph; digraph composition.

**AMS subject classification (2010):** 05C20, 05C70, 05C76, 05C85.

---

\*Research supported by the Danish research council under grant number DFF-7014-00037B.

# 1 Introduction

We refer the reader to [1,2] for graph theoretical notation and terminology not given here. A **digraph** is not allowed to have parallel arcs or loops. A **directed multigraph**  $D = (V, A)$  can have parallel arcs, i.e.  $A$  is a multiset. A directed multigraph  $D = (V, A)$  is **strongly connected** (or **strong**) if there exists a path from  $x$  to  $y$  and a path from  $y$  to  $x$  in  $D$  for every pair of distinct vertices  $x, y$  of  $D$ . A directed multigraph  $D$  is  **$k$ -arc-strong** if  $D - X$  is strong for every subset  $X \subseteq A$  of size at most  $k - 1$ .

A directed multigraph  $D = (V, A)$  has a **strong arc decomposition** if  $A$  can be partitioned into disjoint subsets  $A_1$  and  $A_2$  such that both  $(V, A_1)$  and  $(V, A_2)$  are strong [6]. A directed multigraph  $D$  is **semicomplete** if there is at least one arc between any pair of distinct vertices in  $D$ . In particular, a tournament is semicomplete digraph with just one arc between any pair of distinct vertices. (A semicomplete digraph can have two arcs between a pair  $x, y$  of distinct vertices:  $xy$  and  $yx$ .)

Bang-Jensen and Yeo [7] proved that it is NP-complete to decide whether a digraph has a strong arc decomposition. They also characterized semicomplete digraphs with a strong arc decomposition. Note that every digraph with a strong arc decomposition must be 2-arc-strong.

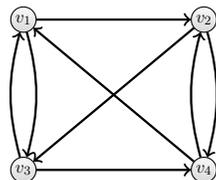


Figure 1: Digraph  $S_4$

**Theorem 1.1** [7] *A 2-arc-strong semicomplete digraph  $D$  has a strong arc decomposition if and only if  $D$  is not isomorphic to  $S_4$ , where  $S_4$  is obtained from the complete digraph with four vertices by deleting the arcs of a cycle of length four (see Figure 1). Furthermore, a strong arc decomposition of  $D$  can be obtained in polynomial time when it exists.*

The following result by Bang-Jensen and Huang extends Theorem 1.1 to locally semicomplete digraphs. A digraph is **locally semicomplete** if every two vertices with a common out- or in-neighbour have an arc between them. Clearly, the class of locally semicomplete digraphs is a generalization of semicomplete digraphs. The **square** of a directed cycle  $v_1v_2 \dots v_nv_1$  is obtained by adding an arc from  $v_i$  to  $v_{i+2}$  for every  $i \in [n]$ , where  $v_{n+1} = v_1$  and  $v_{n+2} = v_2$ .

**Theorem 1.2** [6] *A 2-arc-strong locally semicomplete digraph  $D$  has a strong arc decomposition if and only if  $D$  is not the square of an even cycle.*

Let  $T$  be a digraph with  $t$  vertices  $u_1, \dots, u_t$  and let  $H_1, \dots, H_t$  be digraphs such that  $H_i$  has vertex set  $\{u_{i,j_i} | 1 \leq j_i \leq n_i\}$ . Then the **composition**  $Q = T[H_1, \dots, H_t]$  is a digraph with vertex set  $\cup_{i=1}^t V(H_i)$  and arc

set

$$\left( \bigcup_{i=1}^t A(H_i) \right) \cup \left( \bigcup_{u_i u_p \in A(T)} \{u_{ij_i} u_{pq_p} \mid 1 \leq j_i \leq n_i, 1 \leq q_p \leq n_p\} \right).$$

We say that a composition  $Q = T[H_1, \dots, H_t]$  is a **semicomplete composition** if  $T$  is semicomplete. In the important special case when each  $H_i$  has no arc we say that  $Q$  is an **extension** of  $T$ . In particular the class of **extended semicomplete digraphs** consists of all digraphs that are extensions of a semicomplete digraph, that is, of the form  $Q = T[\overline{K}_{n_1}, \dots, \overline{K}_{n_t}]$  where  $T$  is a semicomplete digraph and  $\overline{K}_r$  is a digraph on  $r$  vertices and no arcs.

Recently, Sun, Gutin and Ai [12] proved the following characterization of a subset of semicomplete compositions with a strong arc decomposition, where  $\vec{C}_3$  is a directed cycle on three vertices and  $\vec{P}_2$  is a directed path on two vertices (that is, it is just an arc).

**Theorem 1.3** [12] *Let  $T$  be a strong semicomplete digraph on  $t \geq 2$  vertices and let  $H_1, \dots, H_t$  be arbitrary digraphs, each with at least two vertices. Then  $Q = T[H_1, \dots, H_t]$  has a strong arc decomposition if and only if  $Q$  is not isomorphic to one of the following three digraphs:  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \vec{P}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$ .*

**Remark 1.1** *Note that all three exceptions in Theorem 1.3 are extended semicomplete digraphs (the middle one is an extension of the unique strong tournament  $T_4^s$  on four vertices, see Figure 2).*

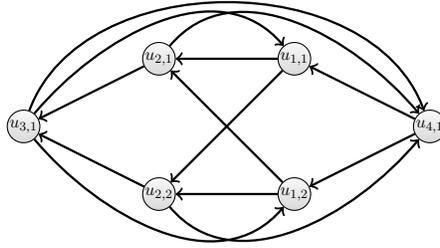


Figure 2:  $T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1]$

In this paper, solving an open problem in [12], we obtain a characterization of **all** semicomplete compositions with a strong arc decomposition. Note that a digraph with a strong arc decomposition is 2-arc-strong. Our characterization is as follows:

**Theorem 1.4** *Let  $T$  be a strong semicomplete digraph on  $t \geq 2$  vertices and let  $H_1, \dots, H_t$  be arbitrary digraphs. Then  $D = T[H_1, \dots, H_t]$  has a strong arc decomposition if and only if  $D$  is 2-arc-strong and is not isomorphic to one of the following four digraphs:  $S_4$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \vec{P}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$ .*

It is remarkable that all the four exemptions in this theorem are simply the union of the exemptions in Theorems 1.1 and 1.3. However, we see no simple way to prove our theorem by a direct reduction to Theorems 1.1 and 1.3 and do not believe that such a reduction exists. Note that the digraphs covered by Theorem 1.4 but not by Theorem 1.3 are all semicomplete decompositions in which at least one  $H_i$  has just one vertex. Having just one vertex in some  $H_i$ 's makes the strong arc decomposition problem on semicomplete digraphs much more complicated than the case when all  $H_i$ 's have at least two vertices since in the latter case the semicomplete composition has more symmetries (i.e., automorphisms) that can be exploited in the proofs. Theorem 1.1 covers just a special subcase of the former case and its proof in [7] is not easier than that of Theorem 1.3 in [12].

Apart from Theorems 1.1 and 1.3 used in our proof of Theorem 1.4, we apply several other results including Edmonds' branching theorem, the existence of nice vertex decompositions proved in [4] (for details see the next section) and an extension of Theorem 1.1 to directed multigraphs (Theorem 3.3) proved in this paper. Interestingly, the extension of Theorem 1.1 has three further exceptions.

Note that the class of strong semicomplete compositions is a generalization of strong quasi-transitive digraphs by the following recursive characterization of quasi-transitive digraphs by Bang-Jensen and Huang [5]. A digraph  $D = (V, A)$  is **quasi-transitive (transitive)** if for any triple  $x, y, z$  of distinct vertices of  $D$ ,  $xy, yz \in A$  implies that there is an arc between  $x$  and  $z$  (from  $x$  to  $z$ ). Clearly, the class of quasi-transitive digraphs is a generalization of semicomplete digraphs. For a recent overview of quasi-transitive digraphs and their generalization, see [10].

**Theorem 1.5** [5] *Let  $D$  be a quasi-transitive digraph.*

- (a) *If  $D$  is strong, then there exists a strong semicomplete digraph  $S$  with  $s$  vertices and quasi-transitive digraphs  $Q_1, Q_2, \dots, Q_s$  such that  $Q_i$  is either a vertex or is non-strong and  $D = S[Q_1, Q_2, \dots, Q_s]$ .*
- (b) *If  $D$  is not strong, then there exist a transitive oriented graph  $T$  with  $t$  vertices and strong quasi-transitive digraphs  $H_1, H_2, \dots, H_t$  such that  $D = T[H_1, H_2, \dots, H_t]$ .*

Theorem 1.4 implies a characterization of quasi-transitive digraphs with a strong arc decomposition (this solves another open question in [12]). In fact the following follows immediately from Theorems 1.5 and 1.4 (observe also that all four exceptions in Theorem 1.4 are quasi-transitive digraphs).

**Theorem 1.6** *Let  $D$  be a quasi-transitive digraph.  $D$  has a strong arc decomposition if and only if  $D$  is 2-arc-strong and is not isomorphic to one of the following four digraphs:  $S_4$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$ ,  $\vec{C}_3[\vec{P}_2, \overline{K}_2, \overline{K}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$ .*

To see that strong quasi-transitive digraphs form a relatively small subset of strong semicomplete compositions, note that the Hamiltonicity problem is polynomial-time solvable for quasi-transitive digraphs [11], but NP-complete

for strong semicomplete compositions. Indeed, consider a digraph  $H$  with vertex set  $U$  and an extra vertex  $x$ . Construct a new digraph  $D$  by adding to  $H$  all arcs between  $U$  and  $x$ . Observe that  $D$  is a strong semicomplete composition and it has a Hamilton cycle if and only if  $H$  has a Hamilton path.

The paper is organized as follows. The next section provides additional terminology and notation and a number of results used later in the paper. We prove an extension of Theorem 1.1 to directed multigraphs in Section 3. In Section 4 we prove a lemma which simplifies our further proofs: every 2-arc-strong semicomplete composition containing a cut-vertex has a strong arc decomposition. Our main result, Theorem 1.4, is proved in Section 5. However, the proof of Theorem 1.4 uses our main technical result, Theorem 5.1, which is proved in Section 6. We complete the paper in Section 7, where we briefly discuss some open problems.

## 2 Additional Terminology, Notation and Results

Let  $D = (V, A)$  be a directed multigraph. The **multiplicity**,  $\mu(x, y)$  of an arc  $xy$  in  $D$  is the number of copies of  $xy$  in  $D$ . An arc is **single** (**double**, respectively) if it is of multiplicity 1 (2, respectively). For  $S \subset V$  such that  $S \neq \emptyset$ , let  $(S, \bar{S})_D$  be the set of arcs of  $D$  with tails in  $S$  and heads in  $\bar{S}$ . The sets of tails in  $S$  and heads in  $\bar{S}$  of arcs in  $(S, \bar{S})_D$  are denoted by  $N_D^+(S)$  and  $N_D^-(\bar{S})$ , respectively. The cardinalities of  $N_D^-(\bar{S})$  and  $N_D^+(S)$  are denoted by  $d_D^-(\bar{S})$  and  $d_D^+(S)$ , respectively.

If  $X$  and  $Y$  are disjoint vertex sets in a digraph, then we use the notation  $X \rightarrow Y$  to denote that  $xy$  is an arc for every choice of  $x \in X, y \in Y$ . For a non-empty subset  $X$  of  $V$ , the subdigraph of  $D$  induced by  $X$  is denoted by  $D\langle X \rangle$ . Let  $P = x_1x_2 \dots x_p$  be a path in  $D$ . For  $1 \leq i \leq j \leq p$ ,  $P[x_i, x_j] = x_ix_{i+1} \dots x_j$  denotes the subpath of  $P$  from  $x_i$  to  $x_j$ . An arc  $uv$  of a digraph  $D = (V, A)$  is a **cut-arc** if  $D - uv$  is not strongly connected. A vertex  $v \in V$  is a **cut-vertex** if  $D - v$  is not strongly connected.

A **vertex decomposition of a digraph**  $D$  is a partition  $(S_1, \dots, S_p)$ ,  $p \geq 1$ , of its vertex set. The **index** of vertex  $v$  in the decomposition, denoted by  $\text{ind}(v)$ , is the integer  $i$  such that  $v \in S_i$ . An arc  $uv$  is **forward** if  $\text{ind}(u) < \text{ind}(v)$ , **backward** if  $\text{ind}(u) > \text{ind}(v)$ . A vertex decomposition  $(S_1, \dots, S_p)$  is **strong** if  $D\langle S_i \rangle$  is strong for all  $1 \leq i \leq p$ . A **nice vertex decomposition** of a digraph  $D$  is a strong decomposition such that the set of cut-arcs of  $D$  is exactly the set of backward arcs.

**Theorem 2.1** [4] *Every strong semicomplete digraph  $S$  of order at least 4 admits a nice decomposition  $(S_1, S_2, \dots, S_p)$  and this decomposition is unique. The sets  $S_i, i \in [p]$  are exactly the strong components of the digraph  $D$  that we obtain from  $S$  by deleting all cut-arcs.*

**Proposition 2.2** [4] *Let  $(S_1, \dots, S_p)$  be a nice decomposition of a strong semicomplete digraph  $D$ . The following properties hold:*

- (i) *If  $u_1v_1$  and  $u_2v_2$  are two cut-arcs, then  $\text{ind}(u_1) \neq \text{ind}(u_2)$  and  $\text{ind}(v_1) \neq \text{ind}(v_2)$ .*

(ii) If  $\text{ind}(u_1) < \text{ind}(u_2)$  then  $\text{ind}(v_1) < \text{ind}(v_2)$ .

The following simple lemma sometimes allows one to reduce the number of digraphs under consideration in proofs of results on strong arc decompositions.

**Lemma 2.3** [12] *Let  $D = Q[H_1, \dots, H_t]$ , where  $Q$  is an arbitrary digraph and every  $H_i$  has no arcs. If an induced subdigraph  $D'$  of  $D$  with at least one vertex in each  $H_i$  has a strong arc decomposition, then so has  $D$ .*

**Lemma 2.4** *Let  $D = R[\overline{K}_{n_1}, \dots, \overline{K}_{n_r}]$  be an extension of a digraph  $R$ . If  $D$  is 2-arc-strong and some  $n_i$  is larger than 2, then the digraph  $D'$  obtained from  $D$  by deleting a vertex from  $\overline{K}_{n_i}$  is also 2-arc-strong.*

**Proof:** Let  $x$  be the vertex that we deleted from  $H_i = \overline{K}_{n_i}$  and let  $y, z$  be two other vertices of  $H_i$ . Suppose that  $D'$  is not 2-arc-strong. Then there exists a vertex partition  $(X, \overline{X})$  of  $V(D')$  so that there is at most one arc from  $X$  to  $\overline{X}$  in  $D'$ . As  $D$  is 2-arc-strong this implies that  $x$  has an out-neighbour  $w^+$  in  $\overline{X}$  and an in-neighbour  $w^-$  in  $X$ . However now  $w^-yw^+$  and  $w^-zw^+$  are two arc-disjoint paths from  $X$  to  $\overline{X}$  in  $D'$ , contradicting the fact that there is at most one arc from  $X$  to  $\overline{X}$  in  $D'$ . Hence  $D'$  is 2-arc-strong.  $\square$

An **out-branching** (**in-branching**, resp.)  **$B$  rooted** at vertex  $z$  in a directed multigraph  $D$  is a spanning subdigraph, which is an oriented tree such that only  $z$  has in-degree (out-degree, resp.) zero. A vertex of an out-branching (in-branching, resp.) is called a **leaf** if its out-degree (in-degree, resp.) equals zero. We will use the following result called Edmonds' branching theorem.

**Theorem 2.5** [9] *A directed multigraph  $D = (V, A)$  with a vertex  $z$ , has  $k$  arc-disjoint out-branchings rooted at  $z$  if and only if  $d^-(X) \geq k$  for all non-empty  $X \subseteq V \setminus \{z\}$ .*

Note that, by Menger's theorem, the condition of Theorem 2.5 is equivalent to the existence of  $k$  arc-disjoint paths from  $z$  to any vertex  $x \in V \setminus \{z\}$ .

### 3 Extending Theorem 1.1 to Semicomplete Directed Multigraphs

If the arc  $xy$  in a directed multigraph  $D$  has multiplicity  $\mu(x, y) \geq 3$ , we may delete  $\mu(x, y) - 2$  copies  $xy$  of  $D$  and the resulting directed multigraph has a strong arc decomposition if and only if so has  $D$ . Thus, we may assume that all directed multigraphs considered in this paper have no arcs of multiplicity 3 or more.

Recall the semicomplete digraph  $S_4$  from Theorem 1.1. Without loss of generality, we may assume that  $V(S_4) = \{v_1, v_2, v_3, v_4\}$  and  $A(S_4) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3, v_3v_1, v_2v_4, v_4v_2\}$ . We call the cycle  $v_1v_2v_3v_4v_1$  the **Hamilton cycle** of  $S_4$  and cycles  $v_1v_3v_1$  and  $v_2v_4v_2$  **2-cycles** of  $S_4$ .

To prove Theorem 3.3, we will use the following lemma.

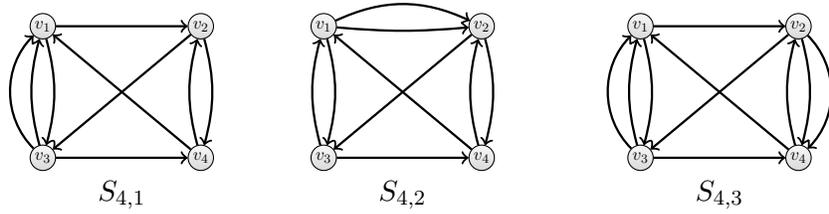


Figure 3: The digraphs  $S_{4,1}$ ,  $S_{4,2}$ ,  $S_{4,3}$

**Lemma 3.1** *Let  $D$  be a directed multigraph with no arcs of multiplicity more than 2 and let  $D$  contain  $S_4$  as a spanning subdigraph. Then  $D$  has no strong arc decomposition if and only if  $D$  is isomorphic to one of the following exceptional digraphs:*

- $S_4$ .
- A directed multigraph obtained from  $S_4$  by adding a copy of an arc in  $S_4$  (isomorphic to  $S_{4,1}$  or  $S_{4,2}$ , see Figure 3).
- A directed multigraph obtained from  $S_4$  by adding a copy of one arc in each of the two 2-cycles of  $S_4$  (isomorphic to  $S_{4,3}$ , see Figure 3).

**Proof:** Observe that the Hamilton cycle  $v_1v_2v_3v_4v_1$  is the only Hamilton cycle in  $S_4$  and  $|A(S_4)| = 8$ . Let  $D' \in \{S_4, S_{4,1}, S_{4,2}, S_{4,3}\}$  be arbitrary and for the sake of contradiction assume that  $D'$  has a strong arc decomposition consisting of strong subdigraphs  $D_1, D_2$  with arc sets  $A_1$  and  $A_2$ .

First consider the case when  $|A_1| = 4$ , which implies that  $A_1$  contains the arcs of the unique (up to copies of the same arc) Hamilton cycle  $v_1v_2v_3v_4v_1$ . However  $D' - \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$  is not strong for any digraph in  $\{S_4, S_{4,1}, S_{4,2}, S_{4,3}\}$ , implying that  $|A_1| \geq 5$ . Analogously  $|A_2| \geq 5$ , which implies that  $D' = S_{4,3}$  and  $|A_1| = |A_2| = 5$ .

In  $D' = S_{4,3}$  we note that  $d^+(v_4) = 2$ , so we may without loss of generality assume that  $v_4v_2 \in A_1$  and  $v_4v_1 \in A_2$  and obtain the following:

- $d^-(v_2) = 2$  implies that  $v_1v_2 \in A_2$  (as  $v_4v_2 \in A_1$ ).
- $d^+(v_1) = 2$  implies that  $v_1v_3 \in A_1$  (as  $v_1v_2 \in A_2$ ).
- $d^-(v_3) = 2$  implies that  $v_2v_3 \in A_2$  (as  $v_1v_3 \in A_1$ ).

Thus, all arcs from  $\{v_2, v_4\}$  to  $\{v_1, v_3\}$  belong to  $A_2$ , a contradiction with the assumption that  $D_1$  is strong.

Assume now that  $D$  is not isomorphic to any directed multigraph described in the statement of the lemma. We will show that  $D$  has a strong arc decomposition with subdigraphs with disjoint arc sets  $A_1$  and  $A_2$ . We have four cases, which cover all possibilities subject to isomorphism. It is not hard to check that the subdigraphs induced by both  $A_1$  and  $A_2$  given below are strong (see Figure 4).

**Case 1:**  $\mu(v_2, v_4) = \mu(v_4, v_2) = 2$ . Then let  $A_1 = \{v_1v_2, v_2v_3, v_3v_1, v_2v_4, v_4v_2\}$  and  $A_2 = \{v_1v_3, v_3v_4, v_4v_1, v_2v_4, v_4v_2\}$ .

**Case 2:**  $\mu(v_1, v_2) = \mu(v_2, v_3) = 2$ . Then let  $A_1$  contain the arcs of the Hamilton cycle of  $S_4$  and  $A_2$  the rest of the arcs of  $D$ .

**Case 3:**  $\mu(v_1, v_2) = \mu(v_3, v_4) = 2$ . Then let  $A_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_2, v_3v_1\}$  and  $A_2$  the rest of the arcs of  $D$ .

**Case 4:**  $\mu(v_1, v_2) = \mu(v_3, v_1) = 2$ . Then let  $A_1 = \{v_1v_2, v_2v_4, v_4v_1, v_1v_3, v_3v_1\}$  and  $A_2$  the rest of the arcs of  $D$ .

It is not hard to check that up to isomorphism all the exception digraphs are depicted in Figures 1 and 3.  $\square$

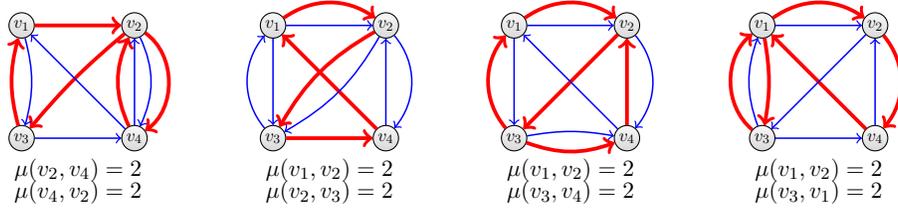


Figure 4: The strong arc decompositions given in the proof of Lemma 3.1.

The following theorem was used in [7] to prove Theorem 1.1. We will use it to prove Theorem 3.3. Note that while in [7] Theorem 3.2 was stated only for semicomplete digraphs, its proof in [7] shows that it holds also for semicomplete directed multigraphs.

**Theorem 3.2** [7] *Let  $k \geq 1$  and let  $D = (V, A)$  be a  $k$ -arc-strong semicomplete directed multigraph such that there is a set  $S \subset V$ , with  $2 \leq |S| \leq |V| - 2$  and  $|(S, V - S)_D| = k$ . Then there exist  $k$  arc-disjoint strong spanning subgraphs of  $D$  except if  $D = S_4$ .*

Now we are ready to prove the main result of this section.

**Theorem 3.3** *A 2-arc-strong semicomplete directed multigraph  $D = (V, A)$  has a strong arc decomposition if and only if it is not isomorphic to one of the exceptional digraphs depicted in Figures 1 and 3. Furthermore, a strong arc decomposition of  $D$  can be obtained in polynomial time when it exists.*

**Proof:** We are going to prove the first part of the statement by induction over  $n = |V|$  and then over the number of double arcs. The second part of the statement then follows as our proof is constructive. If there are no double arcs, then the claim follows from Theorem 1.1. If  $n = 2$  then  $D$  is a directed multigraph consisting of two vertices  $u, v$  with  $\mu(u, v) = \mu(v, u) = 2$ . Clearly this has a strong arc decomposition.

Let  $D$  be a semicomplete directed multigraph on at least 3 vertices with a double arc  $uv$ . If we can delete one copy of  $uv$  and still have a 2-arc-strong semicomplete directed multigraph  $D'$ , then the claim follows by the

induction hypothesis, so we may assume that  $D'$  is not 2-arc-strong. Hence there is a partition  $(X, V - X)$  of  $V$  with  $u \in X$  and  $v \in V - X$  so that the two copies of  $uv$  are the only arcs from  $X$  to  $V - X$ . If  $\min\{|X|, |V - X|\} \geq 2$ , then it follows from Theorem 3.2 that  $D$  has a strong arc decomposition. Hence, we may assume without loss of generality that  $X = \{u\}$ .

Let  $D^*$  be the digraph obtained from  $D$  by contracting  $\{u, v\}$  into one vertex, say  $w$  (that is remove  $\{u, v\}$  and add  $w$  such that for all  $x \in V(D) \setminus \{u, v\}$  we have  $\mu_{D^*}(w, x) = \mu_D(u, x) + \mu_D(v, x)$  and  $\mu_{D^*}(x, w) = \mu_D(x, u) + \mu_D(x, v)$ ). Clearly  $D^*$  is 2-arc-strong since any cut  $(X, V - X)$  in  $D^*$  gives a cut in  $D$  (by replacing  $w$  by  $\{u, v\}$ ) with equally many arcs across. By the induction hypothesis,  $D^*$  has a strong arc decomposition unless it has four vertices and is one of the exceptions from Lemma 3.1. However this is not the case as no vertex in these exceptions is an out-neighbour of all the other three vertices (note that  $\mu(x, w) \geq 1$  for all  $x \in V(D^*) - w$  as  $u$  is dominated by  $V(D) - \{u, v\}$ ). Hence  $D^*$  has a strong arc decomposition  $D_1, D_2$  and it remains to show that this can be modified to a strong arc decomposition of  $D$ .

Start by replacing  $w$  by  $u, v$  in each of  $D_1, D_2$  and then add a copy of  $uv$  to each of the new versions of  $D_1$  and  $D_2$  and for the  $\mu_{D^*}(x, w)$  arcs from every  $x \in V \setminus \{u, v\}$  into  $w$  in  $D^*$  let  $\mu_D(x, u)$  of these go to  $u$  and  $\mu_D(x, v)$  of these go to  $v$  (keeping them in the same  $D_i$  as they were before). Note that all arcs out of  $w$  in  $D^*$  correspond to arcs out of  $v$  in  $D$ . If we can do the above procedure such that  $u$  receives an arc into it in both  $D_1$  and  $D_2$ , then we obtain a strong arc decomposition of  $D$ . Furthermore this is always possible if there exist two vertices  $z_1, z_2$  so that  $z_i$  is an in-neighbour of  $u$  in  $D_i$  (in  $D^*$ ),  $i \in \{1, 2\}$  and either  $z_1 \neq z_2$  or  $z_1 = z_2$  and this vertex has a double arc to  $u$  in  $D$ . So we may assume that this is not the case, which implies that  $n = 3$  and  $V = \{u, v, z\}$ . By the arguments above and the fact that  $D$  is 2-arc-strong we get that  $vz$  is a double arc and there is at least one arc from  $v$  to  $u$  since  $zu$  is not a double arc. Now we get the desired strong arc decomposition by taking the two sets of arcs  $\{uv, vz, zu\}$  and  $\{uv, vu, vz, zv\}$ .

□

## 4 Semicomplete compositions containing a cut-vertex

In this section we will prove the following lemma, which will turn out to be very useful in the proofs below, and is of interest in its own right.

**Lemma 4.1** *Let  $D = T[H_1, \dots, H_t]$  be 2-arc-strong ( $t \geq 2$ ), where  $T$  is a semicomplete digraph and every  $H_i$  is an arbitrary digraph. If  $D$  contains a cut-vertex then  $D$  has a strong arc decomposition.*

**Proof:** Let  $u$  be a cut-vertex in  $D$  and let  $D' = D - u$ . Let  $(X, Y)$  be a cut in  $D'$  such that there is no arc from  $Y = V(D') - X$  to  $X$  in  $D'$ . Let  $D^* = D \setminus (X \cup \{u\})$  and let  $D^{**} = D \setminus (Y \cup \{u\})$ . Note that  $|V(D^*)| \geq 2$  and  $|V(D^{**})| \geq 2$ . We now prove the following claims.

**Claim 1.** We may assume that  $D'$  contains vertices from more than one  $H_i$ .

**Proof of Claim 1:** Suppose that  $V(D') \subseteq V(H_1)$ . We will prove that then  $D$  has a strong arc decomposition, which will complete the proof of Claim 1.

Since  $D = T[H_1, \dots, H_t]$  and  $t \geq 2$ , this implies that  $t = 2$ ,  $T$  is a 2-cycle,  $V(H_2) = \{u\}$  and for every  $v \in V(H_1)$  we have  $vu, uv \in A(D)$ . Let  $Q_1, Q_2, \dots, Q_l$  be strong components in  $D'$ , such that there is no arc from  $Q_i$  to  $Q_j$  when  $i > j$ . We will now construct a strong arc decomposition,  $(G_1, G_2)$  of  $D$  as follows.

For all  $Q_i$  with  $|V(Q_i)| \geq 2$ , do the following. Let  $x_i y_i$  be any arc in  $Q_i$ . Add the arcs  $(A(Q_i) \setminus \{x_i y_i\}) \cup \{x_i u, u y_i\}$  to  $G_1$  and add all arcs  $\{u x_i, x_i y_i, y_i u\}$  and all arcs  $\{uw, wu\}$  for all  $w \in V(Q_i) \setminus \{x_i, y_i\}$  to  $G_2$ . Note that this implies that  $G_a \langle V(Q_i) \cup \{u\} \rangle$  is strong for  $a = 1, 2$ .

Now add all arcs between different  $Q_i$ 's to  $G_1$ . Furthermore for all  $Q_i$  with  $|V(Q_i)| = 1$  assume that  $V(Q_i) = \{x_i\}$  and add  $\{x_i u, u x_i\}$  to  $G_2$ . As  $d^+(x_i) \geq 2$  and  $d^-(x_i) \geq 2$  in  $D$  we note that  $2 \leq i \leq l - 1$  and  $x_i$  has an arc into it from a  $Q_j$  with  $j < i$  and an arc out of it to a  $Q_k$  with  $k > i$  and these arcs belong to  $G_1$ . It follows that  $x_i$  therefore belongs to a path from a  $Q_a$  to a  $Q_b$  in  $G_1$ , where  $a < b$  and  $|V(Q_a)| \geq 2$  and  $|V(Q_b)| \geq 2$ . Therefore  $(G_1, G_2)$  is a strong arc decomposition in  $D$ .

**Claim 2.** We may assume that  $X \cap V(H_i) = \emptyset$  or  $Y \cap V(H_i) = \emptyset$  for all  $i \in \{1, 2, \dots, t\}$ .

**Proof of Claim 2:** Assume without loss of generality that  $X \cap V(H_1) \neq \emptyset$  and  $Y \cap V(H_1) \neq \emptyset$ . By Claim 1 either  $X$  or  $Y$  contains a vertex not in  $H_1$ . Assume without loss of generality that  $Y \setminus V(H_1) \neq \emptyset$ . Let  $X' = X \cup V(H_1) \setminus \{u\}$  and let  $Y' = Y \setminus (V(H_1) \cup \{u\})$ . Note that  $(X', Y')$  is a cut in  $D'$  with no arc from  $Y'$  to  $X'$  as otherwise there are arcs from  $Y'$  to  $H_1$ , contradicting that there is no arc from  $Y$  to  $X$ . The process of moving from  $(X, Y)$  to  $(X', Y')$  decreased the number of  $H_i$  with vertices in both  $X$  and  $Y$ , so continuing this process we will obtain that  $X \cap V(H_i) = \emptyset$  or  $Y \cap V(H_i) = \emptyset$  for all  $i \in \{1, 2, \dots, t\}$ . This completes the proof of Claim 2.

**Claim 3.** There exists two arc-disjoint out-branchings in  $D^*$  both rooted at  $u$  and there exists two arc-disjoint in-branchings in  $D^{**}$  both rooted at  $u$ .

**Proof of Claim 3:** Let  $w \in X$  be arbitrary. As  $D$  is 2-arc-strong there are 2 arc-disjoint paths,  $P_1$  and  $P_2$ , from  $u$  to  $w$  in  $D$ . We note that  $V(P_1) \subseteq V(D^*)$  and  $V(P_2) \subseteq V(D^*)$ , as any path from a vertex not in  $X$  to  $w$  goes through  $u$ . Therefore there exists two arc-disjoint paths from  $u$  to  $w$  in  $D^*$ . By Edmonds' branching theorem, Theorem 2.5, two arc-disjoint out-branchings exist in  $D^*$ , both rooted at  $u$ .

Analogously if  $w \in Y$  then there exist two arc-disjoint paths from  $w$  to  $u$  in  $D$  and therefore also in  $D^{**}$ . Again, by Theorem 2.5, there exists two arc-disjoint in-branchings in  $D^{**}$  both rooted at  $u$ . This completes the proof of Claim 3.

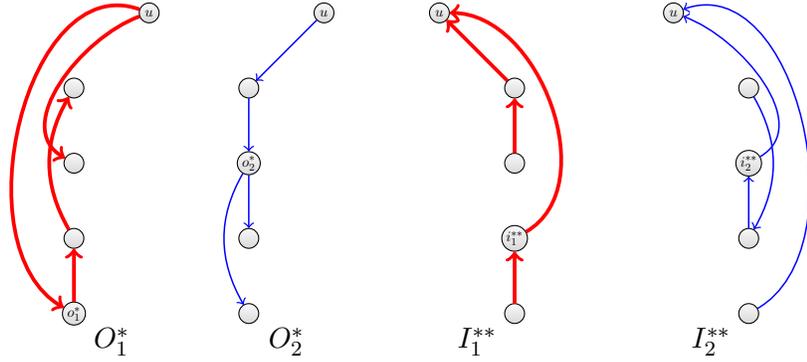


Figure 5: An example of  $O_1^*$ ,  $O_2^*$ ,  $I_1^{**}$  and  $I_2^{**}$  and  $o_1^*$ ,  $o_2^*$ ,  $i_1^{**}$  and  $i_2^{**}$ .

**Definition of  $O_1^*$ ,  $O_2^*$ ,  $I_1^{**}$  and  $I_2^{**}$ :** Let  $O_1^*$  and  $O_2^*$  be two arc-disjoint out-branchings in  $D^*$  rooted at  $u$  and let  $I_1^{**}$  and  $I_2^{**}$  be two arc-disjoint in-branchings in  $D^{**}$  rooted at  $u$  found in Claim 3.

**Claim 4.** Each branching  $O_1^*$ ,  $O_2^*$ ,  $I_1^{**}$  and  $I_2^{**}$  contains a vertex that is neither a leaf nor the root  $u$ .

Denote these vertices by  $o_1^*$ ,  $o_2^*$ ,  $i_1^{**}$  and  $i_2^{**}$ , respectively (see Figure 5).

**Proof of Claim 4:** Consider  $O_1^*$ . As  $|V(D^*)| \geq 2$  we note that there is an arc  $uv \in O_2^*$ . This implies that  $uv \notin O_1^*$  (as we do not have parallel arcs in  $D$ ). However some arc, say  $o_1^*v$ , enters  $v$  in  $O_1^*$ . Now  $o_1^*$  is not the root of  $O_1^*$  and is also not a leaf of  $O_1^*$ . The cases of  $O_2^*$ ,  $I_1^{**}$  and  $I_2^{**}$  can be proved analogously, which completes the proof of Claim 4.

**Definition of  $G_1$  and  $G_2$ :** We now define an arc decomposition of  $D$  as follows (see Figures 5 and 6 for an illustration). Let  $G_1$  contain all arcs of  $O_1^*$  and  $I_1^{**}$  and let  $G_2$  contain all arcs of  $O_2^*$  and  $I_2^{**}$ . Note that all arcs between  $o_1^*$  and  $V(D) \setminus V(D^*)$  exist (by Claim 2 and the fact that  $T$  is semicomplete) and go out of  $o_1^*$ . Add all arcs from  $o_1^*$  to  $V(D) \setminus (V(D^*) \cup \{i_2^{**}\})$  to  $G_2$ . Analogously add all arcs from  $o_2^*$  to  $V(D) \setminus (V(D^*) \cup \{i_1^{**}\})$  to  $G_1$ . Also, add all arcs from  $V(D) \setminus (V(D^{**}) \cup \{o_2^*\})$  to  $i_1^{**}$  to  $G_2$  and add all arcs from  $V(D) \setminus (V(D^{**}) \cup \{o_1^*\})$  to  $i_2^{**}$  to  $G_1$ . Any remaining arcs from  $D$  which have not been added to  $G_1$  or  $G_2$  yet can be added arbitrarily. This completes the definition of  $G_1$  and  $G_2$ .

**Claim 5:**  $(G_1, G_2)$  is a strong arc decomposition of  $D$ .

**Proof of Claim 5:** First let  $v \in X$  be arbitrary. We will now show that there exists a  $(v, u)$ -path in  $G_1$ . As  $O_1^* \subseteq G_1$ , there is a path,  $P_1$ , from  $v$  to a leaf  $l^*$  in  $O_1^*$ . By construction the arc  $l^*i_2^{**}$  belongs to  $G_1$  (as  $l^* \neq o_1^*$ , since  $l^*$  is a leaf in  $O_1^*$  and  $o_1^*$  is not a leaf). As  $I_1^{**} \subseteq G_1$ , there is a path,  $P_2$ , from  $i_2^{**}$  to  $u$  in  $I_1^{**}$ . The path  $P_1P_2$  is now the desired  $(v, u)$ -path in  $G_1$ .

Analogously we can show that there exists a  $(v, u)$ -path in  $G_2$ . Furthermore as  $O_1^*$  and  $O_2^*$  are out-branchings in  $G_1$  and  $G_2$ , respectively we can also find a  $(u, v)$ -path in both  $G_1$  and  $G_2$ .

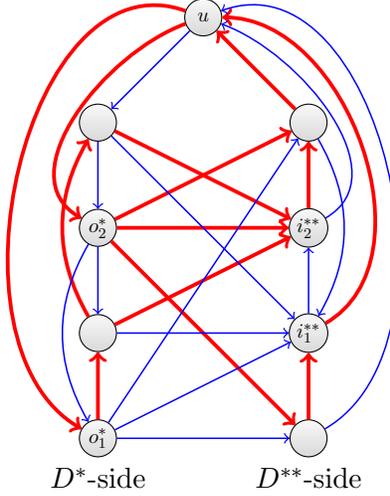


Figure 6: An illustration of  $G_1$  and  $G_2$  obtained from  $O_1^*$ ,  $O_2^*$ ,  $I_1^{**}$  and  $I_2^{**}$  seen in Figure 5. The thick arcs give us  $G_1$  and the thin arcs  $G_2$ . Note that both  $G_1$  and  $G_2$  induce strong spanning subdigraphs.

Let  $w \in Y$  be arbitrary. Analogously we can find a  $(u, w)$ -path in both  $G_1$  and  $G_2$ , by considering a path from a leaf  $l_r^{**}$  in  $G_r$  ( $r \in \{1, 2\}$ ) such that there exists a  $(l_r^{**}, w)$ -path in  $I_r^{**}$  and noting that  $o_{3-r}^* l_r^{**}$  is an arc in  $G_r$  and there exists a  $(u, o_{3-r}^*)$ -path in  $O_r^*$ . As  $I_1^{**}$  and  $I_2^{**}$  are in-branchings in  $G_1$  and  $G_2$ , respectively we can also find a  $(w, u)$ -path in both  $G_1$  and  $G_2$ .

This implies that every vertex in  $D'$  has a path to  $u$  and a path from  $u$  in  $G_1$  and in  $G_2$ , showing that  $G_1$  and  $G_2$  is a strong arc decomposition, thereby proving Claim 5 and the lemma.  $\square$

## 5 Main Results

Our main technical result is the following theorem. Recall that by Remark 1.1,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{P}_2] = T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1]$ , where  $T_4^s$  is the unique strong tournament on four vertices, see Figure 2.

**Theorem 5.1** *Let  $D = T[\overline{K}_{n_1}, \dots, \overline{K}_{n_t}]$  be a 2-arc-strong extended semi-complete digraph where  $n_i \leq 2$  for  $i \in [t]$ . If  $D$  is not isomorphic to one of the following three digraphs:  $S_4$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{P}_2] = T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1]$  then  $D$  has a strong arc decomposition.*

Before proving this theorem in the next section, we use it to prove Theorem 5.2, which is the special case of our main result, Theorem 1.4, for extended semicomplete digraphs. Theorem 5.2 and Lemma 5.3 will imply Theorem 1.4.

**Theorem 5.2** *Let  $D = T[\overline{K}_{n_1}, \dots, \overline{K}_{n_t}]$  be a 2-arc-strong extended semi-complete digraph. If  $D$  is not isomorphic to one of the following four digraphs:  $S_4$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \vec{P}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$  then  $D$  has a strong arc decomposition.*

**Proof:** Let  $\mathcal{C} = \{\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2], \vec{C}_3[\overline{K}_2, \overline{K}_2, \vec{P}_2], \vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]\}$ . We will prove the theorem by induction over  $|V(D)|$ . If  $|V(D)| \leq 3$ , then the theorem clearly holds, so the base case holds. If  $n_i \leq 2$  for all  $i = 1, 2, \dots, t$  then we are done by Theorem 5.1, so we may assume that  $n_j \geq 3$  for some  $j$ . Let  $D' = D - u_{j, n_j}$ . Since  $n_j - 1 \geq 2$ ,  $D' \neq S_4$ .

It follows from Lemma 2.4 that  $D'$  is 2-arc-strong and hence it fulfils the statement of the theorem by induction. If  $D' \notin \mathcal{C}$ , then, by induction, it has a strong arc decomposition and hence  $D$  also has a strong arc decomposition by Lemma 2.3. Hence we may assume that  $D' \in \mathcal{C}$  and consider the corresponding three cases.

**Case 1:**  $D' = \vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$ . This implies that  $D = \vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$ , and therefore  $D \in \mathcal{C}$ , which completes this case.

**Case 2:**  $D' = \vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$ . In this case  $D = \vec{C}_3[\overline{K}_2, \overline{K}_3, \overline{K}_3]$  or  $D = \vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_4]$ . These digraphs have strong arc decompositions by Theorem 1.3.

**Case 3:**  $D' = \vec{C}_3[\overline{K}_2, \overline{K}_2, \vec{P}_2]$ . As in Case 2,  $D$  has a strong arc decomposition by Theorem 1.3.

This completes the proof of the theorem.  $\square$

Now Theorem 5.2 and the following lemma imply our main result, Theorem 1.4.

**Lemma 5.3** *Let  $D = T[H_1, \dots, H_t]$  be 2-arc-strong ( $t \geq 2$ ), where  $T$  is a semicomplete digraph and every  $H_i$  is an arbitrary digraph. Then at least one of the following cases holds.*

- (a)  $D$  has a strong arc decomposition.
- (b)  $D$  is an extended semicomplete digraph.
- (c) For every  $i \in [t]$  and every arc  $e$  of  $H_i$ ,  $D - e$  is 2-arc-strong.

**Proof:** For the sake of contradiction assume that none of (a)-(c) hold and let  $D = (V, A)$ . As  $D$  is not an extended semicomplete digraph (otherwise (b) holds) there exists an arc in some  $H_i$  and since (c) does not hold we can choose  $i \in [t]$  and an arc  $e = uv$  of  $H_i$  such that the digraph  $D' = D - e$  is strong, but not 2-arc-strong. Let  $(X, V - X)$  be a cut in  $D'$  such that there is only one arc,  $xy$ , from  $X$  to  $V - X$  and note that  $u \in X$  and  $v \in V - X$  as  $D$  is 2-arc-strong. Note that either  $x \neq u$  or  $y \neq v$ , and we assume without loss of generality that  $y \neq v$ . This implies that  $v$  has no arc into it from  $X$  in  $D'$ .

First consider the case when  $X \setminus V(H_i) \neq \emptyset$ . In this case let  $Y = X \setminus V(H_i)$  and note that  $(Y, V - Y)$  is a cut in  $D$  with at most one arc from  $Y$  to  $V - Y$

(the only possible arc is the arc  $xy$  since if  $z \in Y$  has an arc to a vertex in  $V(H_i) \cap X$  then  $zv$  is an arc from  $X$  to  $V - X$ ), a contradiction to  $D$  being 2-arc-strong. We may therefore assume that  $X \setminus H_i = \emptyset$ , which is equivalent to  $X \subseteq V(H_i)$ .

If  $y \in V(H_i)$  then  $N_{D'}^+(x) \subseteq V(H_i)$  which implies that there is no arc leaving  $V(H_i)$  in  $D'$ , contradicting that  $D'$  is strong (and  $t \geq 2$ ). Therefore  $y \notin V(H_i)$  and  $N_{D'}^+(V(H_i)) = \{y\}$ . If  $V(D) \neq H_i \cup \{y\}$  then we note that  $y$  is a cut-vertex in  $D'$ , separating  $H_i$  from  $V \setminus (H_i \cup \{y\})$  in  $D'$ . In this case there is a strong arc decomposition in  $D'$  by Lemma 4.1, a contradiction to (a) not holding as clearly  $D$  has a strong arc decomposition, too).

We may therefore assume that  $V(D) = V(H_i) \cup \{y\}$ . By Lemma 4.1 we may assume that  $H_i$  is strongly connected, as  $y$  otherwise would be a cut-vertex. Let us consider the following pair  $G_1, G_2$  of disjoint spanning subdigraphs of  $D$ . The arcs of  $G_1$  are  $(A(H_i) \setminus \{uv\}) \cup \{uy, yv\}$  and the arcs of  $G_2$  are  $\{yu, uv, vy\}$  and all arcs  $\{yw, wy\}$  for all  $w \in V(H_i) \setminus \{u, v\}$ . Since both  $G_1$  and  $G_2$  are strong,  $D$  has a strong arc decomposition, contradicting the assumption that (a) does not hold, and thereby completing the proof.  $\square$

Recall the statement of Theorem 1.4.

**Theorem 1.4** *Let  $T$  be a strong semicomplete digraph on  $t \geq 2$  vertices and let  $H_1, \dots, H_t$  be arbitrary digraphs. Then  $D = T[H_1, \dots, H_t]$  has a strong arc decomposition if and only if  $D$  is 2-arc-strong and is not isomorphic to one of the following four digraphs:  $S_4$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \vec{P}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$ .*

**Proof:** By Theorems 1.1 and 1.3,  $S_4$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \vec{P}_2]$ ,  $\vec{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$  have no strong arc decompositions. Suppose that  $D$  satisfies the conditions of the theorem and yet has no strong arc decomposition. Then  $D$  satisfies either Case (b) or (c) of Lemma 5.3. However, Case (c) of Lemma 5.3 can be reduced to Case (b). Thus, Theorem 5.2 implies Theorem 1.4.  $\square$

## 6 Proof of Theorem 5.1

Before giving the proof of Theorem 5.1 we will prove the following lemma, which is needed in the proof of Theorem 5.1.

**Lemma 6.1** *Let  $D = T[H_1, \dots, H_t]$  be a 2-arc-strong extended semicomplete digraph which has no cut-vertex,  $V(T) = \{u_1, \dots, u_t\}$ ,  $|V(H_i)| \leq 2$  for  $i \in [t]$  and  $V(H_r) = \{x, y\}$ . Suppose  $D' = D - y$  is not 2-arc strong. Then there exists an index  $q \neq r$  such that one of the following holds,*

- (i)  $u_r u_q$  is a cut-arc of  $T$  and  $N^-(V(H_q)) = V(H_r)$ , or
- (ii)  $u_q u_r$  is a cut-arc of  $T$  and  $N^+(V(H_q)) = V(H_r)$ .

**Proof:** As  $D'$  is strong (since  $D$  contains no cut-vertex) but not 2-arc-strong, there is a proper subset  $S$  of  $V(D')$  such that there is exactly one arc  $uv$  from  $S$  to  $\bar{S} = V(D) - S$  in  $D'$ . Suppose first that  $x \in S$ . If  $x \neq u$  then  $(S + y, \bar{S})$  is a vertex partition of  $V(D)$  with only one arc from  $S + y$  to  $\bar{S}$ , contradicting that  $D$  is 2-arc-strong (here we used the fact that  $x$  and  $y$  have the same out-neighbours). Thus we must have  $x = u$  which implies that  $u_r u_q$  is a cut-arc of  $T$ , where  $v \in V(H_q)$ , as  $u_r u_q$  is the only arc from  $S'$  to  $V(T) - S'$  where  $S' \subset V(T)$  is the set of vertices in  $T$  that we obtain by taking  $u_j$  in  $S'$  precisely when  $V(H_j) \cap S \neq \emptyset$ .

If  $|V(H_q)| = 2$ , then we may assume that  $H_q = \{v, w\}$  for some  $w \neq v$ . In this case we must have  $w \in S$  as  $xw \in A(D)$ . Since  $v$  is not a cut-vertex we must have  $\bar{S} = \{v\}$ . Similarly if  $|V(H_q)| = 1$  then  $\bar{S} = \{v\}$ , as otherwise  $v$  is a cut-vertex in  $D$ . As  $\bar{S} = \{v\}$ , we note that  $N^-(V(H_q)) = V(H_r)$ , implying that Part (i) of the lemma holds in this case.

It is easy to see that case when  $x \in \bar{S}$  leads to Part (ii) of the lemma.  $\square$

Recall the statement of Theorem 5.1.

**Theorem 5.1** *Let  $D = T[\bar{K}_{n_1}, \dots, \bar{K}_{n_t}]$  be a 2-arc-strong extended semi-complete digraph where  $n_i \leq 2$  for  $i \in [t]$ . If  $D$  is not isomorphic to one of the following three digraphs:  $S_4$ ,  $\vec{C}_3[\bar{K}_2, \bar{K}_2, \bar{K}_2]$ ,  $\vec{C}_3[\bar{K}_2, \bar{K}_2, \vec{P}_2] = T_4^s[\bar{K}_2, \bar{K}_2, \bar{K}_1, \bar{K}_1]$  then  $D$  has a strong arc decomposition.*

**Proof:** Let  $\mathcal{D}_2 = \{S_4, \vec{C}_3[\bar{K}_2, \bar{K}_2, \bar{K}_2], \vec{C}_3[\bar{K}_2, \bar{K}_2, \vec{P}_2]\}$ ,  $D$  and  $T$  defined as in the theorem, and  $V(T) = \{u_1, \dots, u_t\}$ . For all  $i \in [t]$  let  $H_i$  denote the  $i$ 'th subdigraph in the decomposition, i.e.  $H_i = \bar{K}_{n_i}$  and denote the vertices of  $H_i$  by  $u_{i,j_i}$ ,  $1 \leq j_i \leq n_i$ . By the assumption of the theorem  $n_i \in \{1, 2\}$  for all  $i \in [t]$ . We consider the following cases.

**Case 1.**  $|V(T)| \leq 2$ .

As  $D$  is 2-arc-strong we must have  $|V(T)| = t = 2$ ,  $A(T) = \{u_1 u_2, u_2 u_1\}$  and  $|H_1| = |H_2| = 2$ . Then  $u_{1,1} u_{2,1} u_{1,2} u_{2,2} u_{1,1}$  and  $u_{1,1} u_{2,2} u_{1,2} u_{2,1} u_{1,1}$  form arc-disjoint Hamilton cycles (see Figure 7(a)), thereby proving that  $D$  has a strong arc decomposition.

**Case 2.**  $|V(T)| = 3$ .

If  $|H_1| = |H_2| = |H_3| = 2$  then we are done by Theorem 1.3, so we may without loss of generality assume that  $|H_1| = 1$ . As  $T$  is a strong semicomplete digraph, it contains a Hamilton cycle, by Camion's theorem (see [8] or [3, Theorem 2.2.6]), so we may assume that  $u_1 u_2, u_2 u_3, u_3 u_1 \in A(T)$ . As  $|H_1| = 1$  and the vertices in  $H_2$  have in-degree at least two we must then have  $u_3 u_2 \in A(T)$ .

If  $|H_2| = |H_3| = 2$ , then we note that  $D\langle H_2 \cup H_3 \rangle$  has a strong arc decomposition which can easily be extended to a strong arc decomposition of  $D$  as can be seen in Figure 7(b). We may therefore assume without loss of generality that  $|H_1| = |H_2| = 1$ . As the vertices in  $H_3$  have in-degree at least two we must then have  $u_1 u_3 \in A(T)$ .

If  $|H_3| = 2$  then note that  $u_{1,1} u_{3,1} u_{2,1} u_{3,2} u_{1,1}$  and  $u_{1,1} u_{3,2} u_{2,1} u_{3,1} u_{1,1}$  form arc-disjoint Hamilton cycles, thereby proving that  $D$  has a strong arc

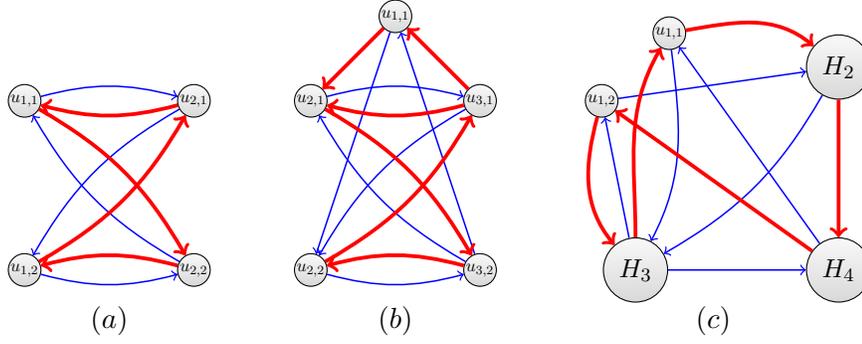


Figure 7: Strong arc decompositions of different digraphs. The red arcs form one strong spanning subdigraph and the blue arcs form the other strong spanning subdigraph. The arcs between  $\{u_{1,1}, u_{1,2}\}$  and the  $H_i$ 's in (c) indicate the direction of all arcs between these sets.

decomposition.

The only remaining case is when  $|H_1| = |H_2| = |H_3| = 1$  and  $u_2 u_1 \in A(T)$ . However in this case  $D$  has a strong arc decomposition as it consists of two 3-cycles in the opposite directions. This completes the proof of Case 2.

**Case 3.**  $T$  is 2-arc-strong and  $|V(T)| \geq 4$ .

If  $T$  is not isomorphic to  $S_4$ , then we are done by Theorem 1.1 and Lemma 2.3. Now assume that  $T$  is isomorphic to  $S_4$ . If  $|H_1| = |H_2| = |H_3| = |H_4| = 1$ , then  $D$  is isomorphic to  $S_4$  and  $D \in \mathcal{D}_2$ . By symmetry we may therefore without loss of generality assume that  $|H_1| = 2$  and a strong arc decomposition of  $D$  can be seen in Figure 7(c) (the arcs shown between all  $u_{1,i}$ 's and  $H_j$ 's and between all  $H_j$ 's and  $H_k$ 's show the direction of all arcs between them), completing the proof of Case 3.

**Case 4.**  $T$  is not 2-arc-strong and  $|V(T)| \geq 4$ .

We will now prove that  $D$  has a strong arc decomposition. Suppose this is not true and consider a minimum counterexample  $D$ .

Suppose first that  $D$  has a vertex  $z \in H_i$ , where  $|H_i| = 2$ , so that  $\hat{D} = D - z$  is 2-arc-strong. By minimality of  $D$ ,  $\hat{D}$  has a strong arc decomposition, unless it is one of the exceptions in the theorem. As we have assumed that  $|V(H_i)| \leq 2$  for  $i \in [t]$  we note that  $\hat{D}$  in this case is either  $S_4$  or  $T_4^s[\bar{K}_2, \bar{K}_2, \bar{K}_1, \bar{K}_1]$ . If  $\hat{D}$  is isomorphic to  $S_4$ , then  $T = \hat{D}$  and  $T$  is 2-arc-strong, a contradiction by the statement of Case 4. Thus, we may assume that  $\hat{D}$  is isomorphic to  $T_4^s[\bar{K}_2, \bar{K}_2, \bar{K}_1, \bar{K}_1]$ . However in this case we can find a strong arc decomposition of  $D = T_4^s[\bar{K}_2, \bar{K}_2, \bar{K}_1, \bar{K}_1]$  as seen in Figure 8. Hence  $\hat{D}$  has a strong arc decomposition and by Lemma 2.3 so has  $D$ , contradiction our assumption.

Hence we may assume below that

$$\begin{aligned} &\text{for every } H_i \text{ with } |H_i| = 2 \text{ and every } z \in V(H_i) \\ &\text{the digraph } D - z \text{ is not 2-arc-strong.} \end{aligned} \tag{1}$$

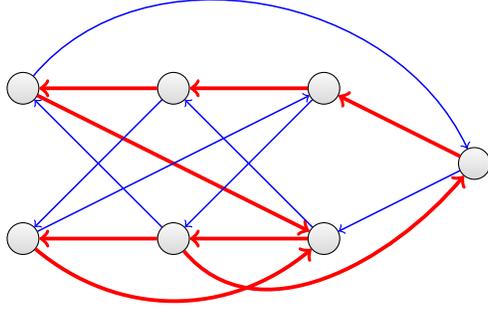


Figure 8: A strong arc decomposition of  $D = T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_2, \overline{K}_1]$

By Theorem 2.1,  $T$  has a nice vertex decomposition  $(T_1, \dots, T_p)$ . Let  $(D_1, D_2, \dots, D_p)$  be the vertex decomposition of  $D$  obtained by replacing each vertex  $u_i$  of  $T$  by the corresponding independent set  $H_i$  (so  $V(D_j) = \bigcup_{u_i \in V(T_j)} V(H_i)$ ). By our assumption on  $D$ , it has no strong arc decomposition. We now prove the following claims and use these to obtain a contradiction to the existence of  $D$ .

**Claim A.** If  $u_i u_j$  is a cut-arc in  $T$  and  $|H_j| = 1$ , then  $V(T_1) = \{u_j\}$ .

**Proof of Claim A:** Assume that  $u_i u_j$  is a cut-arc in  $T$  and  $|H_j| = 1$ . Define  $r$  such that  $u_j \in T_r$ . Let  $X = V(D_1) \cup V(D_2) \cup \dots \cup V(D_r) \setminus \{u_{j,1}\}$  and let  $Y = V(D_{r+1}) \cup \dots \cup V(D_p)$ . Note that  $Y \neq \emptyset$  as  $V(H_i) \subseteq Y$ . Also note that there is no arc from  $Y$  to  $X$  in  $D$  (as the only arcs out of  $Y$  go to  $H_j = \{u_{j,1}\}$ , because  $u_i u_j$  is a cut-arc of  $T$ ). If  $X \neq \emptyset$  then  $u_j$  is a cut-vertex in  $D$  and we are done by Lemma 4.1. So,  $X = \emptyset$  which implies that  $V(T_1) = \{u_j\}$ , completing the proof of Claim A. ■

**Claim B.** If  $u_i u_j$  is a cut-arc in  $T$  and  $|H_i| = 1$ , then  $T_p = \{u_i\}$ .

**Proof of Claim B:** This can be proved analogously to Claim A. ■

**Claim C.** If  $|H_r| = 2$ , for some  $r \in [t]$ , then  $u_r$  is incident with a cut-arc into  $T_1$  or a cut-arc out of  $T_p$  (or both).

This implies that for every  $u_q \in V(T)$  that is not incident to a cut-arc we have  $|V(H_q)| = 1$ .

**Proof of Claim C:** Let  $|H_r| = 2$ , for some  $r \in [t]$ . By Lemma 6.1 we note that there exists a  $q \neq r$  such that one of the following holds.

- (i)  $u_r u_q$  is a cut-arc of  $T$  and  $N^-(V(H_q)) = V(H_r)$ , or
- (ii)  $u_q u_r$  is a cut-arc of  $T$  and  $N^+(V(H_q)) = V(H_r)$ .

Assume without loss of generality that (i) above holds. Therefore  $u_r$  is incident with the cut-arc  $u_r u_q$  in  $T$ . As  $N^-(V(H_q)) = V(H_r)$ , we note that  $d_T^-(u_q) = 1$ . If  $u_q \in T_1$ , then we are done, so assume that this is not the case. However, as  $u_r u_q$  is a cut-arc in  $T$  observe that  $u_r \notin T_1$ , which implies

that  $u_q$  dominates  $T_1$  (as (i) holds). This implies that  $|T_1| = 1$  since there is only one arc entering  $T_1$  in  $T$ , and assuming that  $T_1 = \{u_1\}$ , we note that  $u_q u_1$  is the cut-arc into  $T_1$  in  $T$ . Therefore  $N_T^-(u_1) = \{u_q\}$ . As  $u_r u_q$  is a cut-arc in  $T$  we note that  $u_q \notin T_p$ , which by Claims A and B implies that  $|H_q| = 2$  (as  $u_q u_1$  is a cut-arc in  $T$ ).

We will now show that  $|H_1| = 1$ . For the sake of contradiction assume that  $|H_1| = 2$ , which by Lemma 6.1(ii), implies that  $N^+(H_q) = H_1$ . However this implies that  $d_T^+(u_q) = 1$  and from above  $d_T^-(u_q) = 1$ , implying that  $|V(T)| = 3$ , contradicting the fact that  $|V(T)| \geq 4$ . Therefore  $|H_1| = 1$ . As  $T_1 = \{u_1\}$  we note that  $d_T^-(u_1) = 1$ .

To summarize, we have now shown the following.

- (I)  $H_1 = \{u_{1,1}\}$ ,  $H_q = \{u_{q,1}, u_{q,2}\}$  and  $H_r = \{u_{r,1}, u_{r,2}\}$  in  $D$ .
- (II)  $u_r u_q$  and  $u_q u_1$  are cut-arcs in  $T$ .
- (III)  $N_T^-(u_1) = \{u_q\}$  and  $N_T^-(u_q) = \{u_r\}$ .

We now consider  $D' = D - u_{1,1}$ . By Lemma 4.1 we note that  $D$  contains no cut-vertices and therefore  $D'$  is strongly connected. If  $D'$  has a strong arc decomposition then this can easily be extended to a strong arc decomposition of  $D$ , as  $u_{1,1}$  has at least two arcs into  $D'$  and at least two arcs out of  $D'$ . We may therefore assume that  $D'$  has no strong arc decomposition.

If  $D'$  is 2-arc-strong, this implies that  $D'$  is one of our exceptions. In this case  $D'$  is either  $\vec{C}_3[\vec{K}_2, \vec{K}_2, \vec{K}_2]$  or  $\vec{C}_3[\vec{K}_2, \vec{K}_2, \vec{P}_2]$ , since there is an  $H_i$  with at least two vertices. If  $D' = \vec{C}_3[\vec{K}_2, \vec{K}_2, \vec{K}_2]$ , then it follows from (III) that  $D = T_4^s[\vec{K}_1, \vec{K}_2, \vec{K}_2, \vec{K}_2]$  and we obtain a good decomposition of  $D$  using the decomposition in Figure 8 and then reversing all arcs. This contradicts our assumption that  $D$  has no strong arc decomposition. Suppose now that  $D' = \vec{C}_3[\vec{K}_2, \vec{K}_2, \vec{P}_2]$ . Then  $D$  contains  $T_4^s[\vec{K}_1, \vec{K}_2, \vec{K}_2, \vec{K}_2]$  as a spanning subdigraph and hence it has a strong arc decomposition, contradiction.

So we may now assume that  $D'$  is not 2-arc-strong and we will let  $(S, \bar{S})$  be a partition of  $V(D')$  with exactly one arc from  $S$  to  $\bar{S}$ . Since  $T$  is strong,  $d_{D'}^-(u_r) \geq 1$ . We will now show that  $d_{D'}^-(u_{r,1}) = 1$  (and  $d_{D'}^-(u_{r,2}) = 1$ ). In order to do this we consider the three possible placements of the vertices  $u_{q,1}$  and  $u_{q,2}$  in the partition  $(S, \bar{S})$ .

*Case C.1:*  $\{u_{q,1}, u_{q,2}\} \subseteq S$ . If  $\bar{S}$  contains any vertex not in  $V(H_r)$  then there are at least two arcs from  $S$  to  $\bar{S}$  (coming from  $u_{q,1}$  and  $u_{q,2}$ ), a contradiction. Therefore,  $\bar{S} \subseteq H_r$ . Without loss of generality  $u_{r,1} \in \bar{S}$  and  $x u_{r,1}$  is the arc from  $S$  to  $\bar{S}$ . Considering  $u_{r,1}$  we note that in  $D'$  it only has one arc into it (from  $x$ ), implying that  $d_{D'}^-(u_{r,1}) = 1$  as desired.

*Case C.2:*  $\{u_{q,1}, u_{q,2}\} \subseteq \bar{S}$ . Adding  $u_{1,1}$  to  $\bar{S}$  we note that there is still only one arc from  $S$  to  $\bar{S}$  (as  $u_{1,1}$  only has arcs into it from  $u_{q,1}$  and  $u_{q,2}$ ), a contradiction to  $D$  being 2-arc-strong.

*Case C.3:*  $|\{u_{q,1}, u_{q,2}\} \cap S| = 1$ . Without loss of generality assume that  $u_{q,1} \in S$  and  $u_{q,2} \in \bar{S}$ . As there is only one arc from  $S$  to  $\bar{S}$  we note that

$u_{r,1}$  or  $u_{r,2}$  must belong to  $\overline{S}$ . Without loss of generality assume that  $u_{r,2} \in \overline{S}$ . For the sake of contradiction assume that  $d_{D'}^-(u_{r,1}) \geq 2$  (and therefore  $d_{D'}^-(u_{r,2}) \geq 2$ ) and let  $z_1, z_2 \in N_{D'}^-(u_{r,1})$  (and therefore  $z_1, z_2 \in N_{D'}^-(u_{r,2})$ ) be arbitrary.

If  $z_i \in S$ , then we note that  $z_i u_{r,2}$  is an arc from  $S$  to  $\overline{S}$  and if  $z_i \in \overline{S}$  then either  $z_i = u_{q,2}$  or  $u_{q,1} z_i$  is an arc from  $S$  to  $\overline{S}$  for  $i = 1, 2$ . As there is only one arc from  $S$  to  $\overline{S}$  we note that  $z_1 = u_{q,2}$  or  $z_2 = u_{q,2}$ . Without loss of generality we may assume that  $z_1 = u_{q,1}$  and  $z_2 = u_{q,2}$  (as if  $u_{q,2}$  dominates  $u_{r,1}$  then so does  $u_{q,1}$ ). However if  $u_{r,1} \in S$  then both  $u_{r,1} u_{q,2}$  and  $u_{q,1} u_{r,2}$  go from  $S$  to  $\overline{S}$  and if  $u_{r,1} \in \overline{S}$  then both  $u_{q,1} u_{r,1}$  and  $u_{q,1} u_{r,2}$  go from  $S$  to  $\overline{S}$ , a contradiction. This completes Case C.3.

We have now shown that  $d_{D'}^-(u_{r,1}) = 1$ , so we may define  $z$ , such that  $N_{D'}^-(u_{r,1}) = \{u_{z,1}\}$ . Note that  $|H_z| = 1$ . Let  $Y = H_1 \cup H_q \cup H_r \cup H_z$ . We will show that  $V(D) = Y$ , so assume for the sake of contradiction that there exists a vertex  $y \in V(D) \setminus Y$ . Then there is no path from  $y$  to  $Y \setminus \{u_{z,1}\}$  in  $D - u_{z,1}$ , as  $N_T^-(u_1) = \{u_q\}$  and  $N_T^-(u_q) = \{u_r\}$  and  $N_T^-(u_r) = \{u_1, u_z\}$  (so all arcs into  $Y \setminus \{u_{z,1}\}$  come from  $u_{z,1}$ ). This implies that  $u_{z,1}$  is a cut-vertex in  $D$ , a contradiction by Lemma 4.1. Therefore we must have  $V(D) = Y$ .

As  $|V(T)| \geq 4$ , we note that  $H_z, H_1, H_q$  and  $H_r$  are distinct. However in this case  $D$  is the exception  $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$ . This completes the proof of Claim C.  $\blacksquare$

**Claim D.**  $T$  has at most three cut-arcs.

**Proof of Claim D:** Let  $u_i u_j$  be a cut-arc in  $T$  with  $1 < j < i < p$ . By Claims A and B we note that  $|H_j| = |H_i| = 2$ . By Claim C we note that  $u_j$  is incident with a cut-arc into  $T_1$  and  $u_i$  is incident with a cut-arc from  $T_p$ . This implies that there are only these three cut-arcs in this case. Furthermore, if there is no cut-arc,  $u_i u_j$ , in  $T$  with  $1 < j < i < p$  then there are at most two cut-arcs in  $T$  (one into  $T_1$  and one out of  $T_p$ ).  $\blacksquare$

The remaining part of the proof is split into three cases, covering the number of possible cut-arcs in  $T$  according to Claim D.

**Case 4.1.**  $T$  has exactly one cut-arc  $u_p u_1$ .

Let  $T'$  be the semicomplete multigraph that we obtain by adding an extra copy of the arc  $u_p u_1$  to  $T$  (so  $T'$  has exactly one pair of parallel arcs). As  $u_p u_1$  was the only cut-arc in  $T$ , we note that  $T'$  is 2-arc strong. By the statement of Case 4 we note that  $T$  is not 2-arc-strong, and therefore not isomorphic to  $S_4$ , implying that  $T'$  is not one of the exceptions in Theorem 3.3. Therefore  $T'$  contains a strong arc decomposition  $(R_1, R_2)$ .

Let  $R'_1, R'_2$  be the arc-disjoint spanning subdigraphs of  $D$  that we obtain by replacing the vertex  $u_p$  by  $\{u_{p,1}, u_{p,2}\}$  and the vertex  $u_1$  by  $\{u_{1,1}, u_{1,2}\}$ . That is, if  $x u_p$  ( $u_p y$ ) is an arc of  $R_i$ , then  $R'_i$  contains the arcs  $x u_{p,1}, x u_{p,2}$  ( $u_{p,1} y, u_{p,2} y$ ) and analogously for arcs entering and leaving  $u_1$ . Note that  $R'_1, R'_2$  well-defined for all arcs apart from the ones from  $\{u_{p,1}, u_{p,2}\}$  to  $\{u_{1,1}, u_{1,2}\}$ , for these we let  $u_{p,1} u_{1,1}$  and  $u_{p,2} u_{1,2}$  belong to  $R'_1$  and  $u_{p,1} u_{1,2}$  and  $u_{p,2} u_{1,1}$  belong to  $R'_2$ . We will now show that  $(R'_1, R'_2)$  is a strong arc

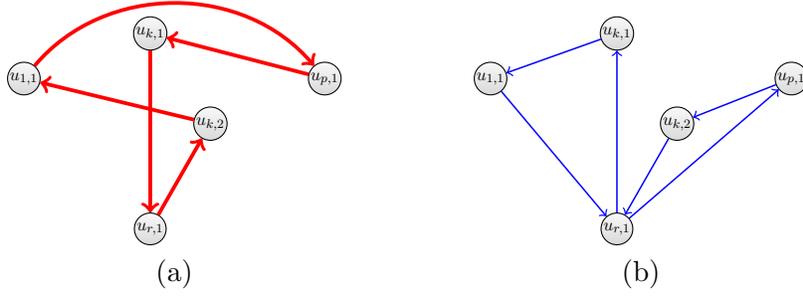
decomposition of  $D$ , contradiction the assumption on  $D$ .

First consider the case when  $|H_p| = |H_1| = 2$ . As there is a path from  $u_1$  to  $u_p$  in  $R_i$  ( $i \in [2]$ ), we note that for  $j \in [2]$  the vertex  $u_{p,j}$  can reach every vertex in  $\{u_{1,1}, u_{1,2}\}$  in  $R'_i$ , either by a direct arc, or by an arc from  $H_p$  to  $H_1$  followed by the equivalent of a  $(u_1, u_p)$ -path in  $R_i$  followed by another arc from  $H_p$  to  $H_1$ . (For example, in  $R'_1$  the vertex  $u_{p,1}$  can reach  $u_{1,2}$  via arc  $u_{p,1}u_{1,1}$  followed by a  $(u_{1,1}, u_{p,2})$ -path in  $R_1$  and finally the arc  $u_{p,2}u_{1,2}$ ). Therefore we have all the same connections in  $R'_1$  and  $R'_2$  as in  $R_1$  and  $R_2$ , completing the proof of the case when  $|H_p| = |H_1| = 2$ .

We may therefore without loss of generality assume that  $|H_1| = 1$ . As  $D$  is 2-arc-strong and  $u_{1,1}$  is not a cut-vertex we note that  $|H_p| = 2$ ,  $V(D_1) = \{u_{1,1}\}$  and there exists a vertex  $u_{y,1} \in N_D^+(u_{p,1}) \setminus \{u_{1,1}\}$ . Without loss of generality assume that  $u_p u_y \in A(R_1)$ . As  $D - u_{p,2} = T$  we can assign every arc of  $D - u_{p,2}$  to  $R'_i$  if and only if it was assigned to  $R_i$  in  $T'$ , except the arc from  $u_{p,1}$  to  $u_{1,1}$  which gets assigned to  $R'_2$ . Let  $u_x u_p$  be the last arc on a path from  $u_y$  to  $u_p$  in  $R_1$ . Now let the arcs  $u_{x,1} u_{p,2}$  and  $u_{p,2} u_{1,1}$  belong to  $R'_1$ . Now we note that  $R'_1$  is a strong spanning subdigraph of  $D$ , as there exists a path from both  $u_{p,1}$  and  $u_{p,2}$  to  $u_{1,1}$  (and therefore all paths in  $R_1$  also work for  $R'_1$ ). Adding any arc into  $u_{p,2}$ , different from  $u_{x,1} u_{p,2}$  and any arc out of  $u_{p,2}$ , different from  $u_{p,2} u_{1,1}$  to  $R'_2$  makes  $R'_2$  into a strong spanning subdigraph of  $D'$  (as it was already strong in  $D - u_{p,2}$ ). Therefore  $(R'_1, R'_2)$  is a strong arc decomposition of  $D$ .

**Case 4.2.**  $T$  has exactly two cut-arcs  $u_p u_h$  and  $u_k u_1$ . In Claim C we proved that  $|H_1| = 1$ . We can show similarly that  $|H_p| = 1$ . Since  $D$  is 2-arc-strong we note that  $|H_h| = |H_k| = 2$ . By Claim A and B we note that  $|T_1| = |T_p| = 1$ . This and the fact that  $D$  has no cut-vertex implies that  $|V(D_1)| = |V(D_p)| = 1$ . There are 3 subcases to consider:  $u_k = u_h$ ,  $u_k, u_h$  are distinct but  $u_k, u_h \in V(T_i)$  for some  $i \in [p]$  and finally the case where  $u_k \in V(T_j), u_h \in V(T_i)$  where  $i < j$  (as otherwise  $T$  is not strong).

**Case 4.2.1.**  $u_k = u_h$ . Let the index  $i$  be chosen so that  $u_k \in V(T_i)$ . As  $D$  has no cut-vertex we note that  $i = 2$  and  $p = 3$ . As  $|T_1| = |T_3| = 1$  and  $|V(T)| \geq 4$  we must have  $|T_2| \geq 2$ . Therefore the set  $W = V(D_2) - \{u_{k,1}, u_{k,2}\}$  contains at least one vertex. As  $V(T_2)$  is strong the digraph  $D'_2 = D_2 - \{u_{k,1}\}$  is strong. If  $V(T_2) = \{u_k, u_r\}$  for some  $r$  (that is,  $V(T) = \{u_1, u_k, u_r, u_p\}$ ) then the hamiltonian cycle  $u_{1,1} u_{p,1} u_{k,1} u_{r,1} u_{k,2} u_{1,1}$  (see (a) below) is arc-disjoint from the strong spanning subdigraph whose arc set is the arcs of the two paths  $u_{p,1} u_{k,2} u_{r,1} u_{k,1} u_{1,1}$  and  $u_{1,1} u_{r,1} u_{p,1}$  (see (b) below), showing that  $D$  has a strong arc decomposition, contradicting our assumption.



Suppose now that  $|V(T_2)| > 2$ . Let  $v$  be an in-neighbour of  $u_{k,1}$  in  $V(D_2)$  and let  $w \neq v$  be an out-neighbour of  $u_{k,1}$  in  $V(D_2)$ . Let  $D'_2 = D_2 - u_{k,1}$  and note that  $D'_2$  is strong. Now let the two spanning digraphs  $G_1 = (V, A_1), G_2 = (V, A_2)$  contain the following arcs (see Figure 9).

- $A_1 = \{u_{p,1}u_{k,1}, u_{k,1}u_{1,1}, u_{1,1}w, vu_{p,1}\} \cup A(D'_2)$
- $A_2 = \{u_{p,1}u_{k,2}, u_{k,2}u_{1,1}, u_{1,1}v, vu_{k,1}, u_{k,1}w, wu_{p,1}\} \cup \{u_{1,1}z, zu_{p,1} \mid z \in V(D'_2) - \{u_{k,2}, v, w\}\}$

It is easy to verify that  $G_1, G_2$  are arc-disjoint strong spanning subdigraphs of  $D$ , contradiction the assumption on  $D$ .

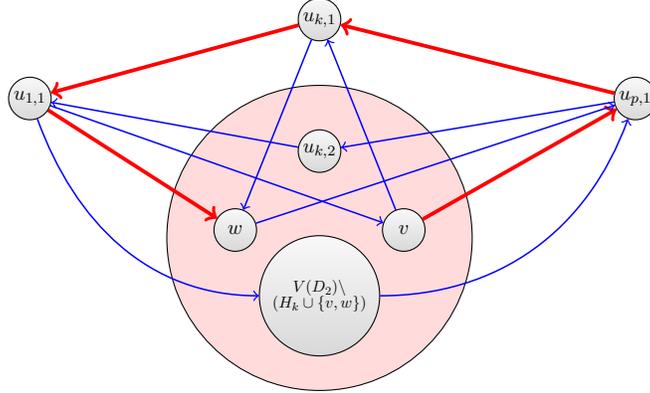


Figure 9: The strong arc decomposition used in Case 4.2.1, where all arcs within the big red circle are red arcs.

**Case 4.2.2**  $u_k$  and  $u_h$  are distinct but belong to the same  $T_i$ .

As in the proof of Case 4.2.1, we note that  $i = 2$  and  $p = 3$  (as  $D$  has no cut-vertex) and  $|V(D_1)| = |V(D_3)| = 1$ .

As  $u_p u_h$  and  $u_k u_1$  are the only cut-arcs of  $T$  it follows from Menger's theorem that there are two arc-disjoint  $(u_h, u_k)$ -paths  $P_1, P_2$  in  $T_2$ . For  $i \in [2]$  let  $A'_i$  be the arcs of  $D_2$  that correspond to  $A(P_i)$ , that is, we replace the first arc  $u_h v$  (last arc  $v' u_k$ ) of  $P_i$  by the two arcs  $u_{h,1} v, u_{h,2} v$  (respectively,  $v' u_{k,1}, v' u_{k,2}$ ). Recall that, by Claim C, we have  $|H_g| = 1$  when  $u_g$  is not incident to a cut-arc of  $T$  so  $A(P_i)$  corresponds exactly to  $A'_i$  in  $D_2$ .

We will now construct  $F_1$  and  $F_2$  as follows. Let  $X = V(D_2) \setminus (H_k \cup H_h)$ . Initially let  $F_1$  and  $F_2$  consist of the following arcs (see Figure 10):

- $A(F_1)$  initially consists of the arcs  $\{u_{p,1}u_{h,1}, u_{k,2}u_{1,1}, u_{1,1}u_{h,2}, u_{k,1}u_{p,1}\}$  and all arcs of  $A'_1$ .
- $A(F_2)$  initially consists of the arcs  $\{u_{p,1}u_{h,2}, u_{k,2}u_{p,1}, u_{1,1}u_{h,1}, u_{k,1}u_{1,1}\}$  and all arcs of  $A'_2$ .

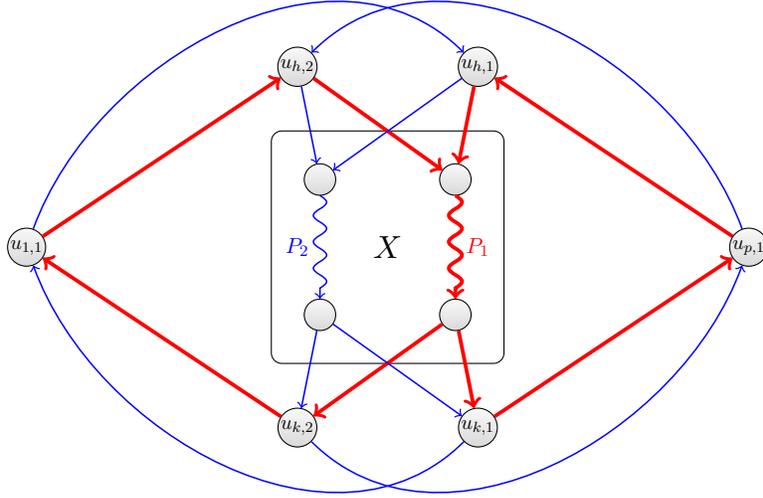


Figure 10: The initial assignments of arcs to  $F_1$  and  $F_2$  in Case 4.2.2.

Now for every vertex  $x \in X$  we add the following arcs to  $F_1$  and  $F_2$ .

- If  $x \notin V(P_1)$  then add the arcs  $u_{1,1}x$  and  $xu_{p,1}$  to  $A(F_1)$ .
- If  $x \in V(P_1)$  then add the arcs  $u_{1,1}x$  and  $xu_{p,1}$  to  $A(F_2)$ .

Finally we add all arcs not assigned to any  $F_i$  yet to  $A(F_2)$ . It is easy to check that  $F_1$  is a strong spanning subdigraph of  $D$ . In order to show that  $F_2$  is also a strong spanning subdigraph of  $D$  we consider any  $x \in X$  and will show that  $x$  has a path to and from  $\{u_{p,1}, u_{1,1}\}$  in  $F_2$ . If  $x \in V(P_1) \cup V(P_2)$ , then this is clearly the case by the construction above (as either the arcs  $u_{1,1}x$  and  $xu_{p,1}$  belong to  $F_2$  or  $x \in V(P_2)$ ). So assume that  $x \notin V(P_1) \cup V(P_2)$ . In this case any path from  $x$  to  $V(P_1) \cup V(P_2)$  in  $D_2$ , and any path to  $x$  from  $V(P_1) \cup V(P_2)$ , belongs to  $F_2$ , so we are done as all vertices in  $V(P_1) \cup V(P_2)$  have a path to and from  $\{u_{p,1}, u_{1,1}\}$  in  $F_2$ . Therefore  $(F_1, F_2)$  is a strong arc decomposition of  $D$ , contradiction our assumption on  $D$ . This completes the proof of Case 4.2.2.

**Case 4.2.3** There are indices  $1 < i < j < p$  such that  $u_h \in V(T_i)$  and  $u_k \in V(T_j)$ .

Recall that  $|V(H_1)| = |V(H_p)| = 1$  and  $|T_1| = |T_p| = 1$ , implying that we must have  $V(D_1) = \{u_{1,1}\}$  and  $V(D_p) = \{u_{p,1}\}$ . As  $D$  has no cut vertex we must furthermore have  $i = 2$  (or  $u_{1,1}$  would be a cut-vertex of  $D$ ) and  $j = p - 1$  (or  $u_{p,1}$  would be a cut-vertex of  $D$ ). If  $V(D) = \{u_{1,1}, u_{h,1}, u_{h,2}, u_{k,1}, u_{k,2}, u_{p,1}\}$ , then  $T$  has another cut-arc, namely  $u_2u_3$ , contradicting that we are in Case 4.2. Thus we can choose a vertex  $z \in V(D) - \{u_{1,1}, u_{h,1}, u_{h,2}, u_{k,1}, u_{k,2}, u_{p,1}\}$  so that  $z$  is an out-neighbour of  $u_{h,1}, u_{h,2}$  and an in-neighbour of  $u_{k,1}, u_{k,2}$ . Let  $U = \{z, u_{1,1}, u_{h,1}, u_{h,2}, u_{k,1}, u_{k,2}, u_{p,1}\}$  and note that every vertex of  $V(D) - U$  has at least two in-neighbours and at least two out-neighbours in  $U$  so it suffices to give a strong arc decomposition for  $D[U]$ .

Observe that  $D$  has a pair arc-disjoint strong spanning subdigraphs: one induced by the arcs of the 6-cycle  $u_{p,1}u_{h,1}u_{k,1}u_{1,1}u_{h,2}u_{k,2}u_{p,1}$  and the two arcs  $u_{1,1}z, zu_{p,1}$  and the other by the arcs of the 5-cycle  $u_{k,2}u_{1,1}u_{p,1}u_{h,2}zu_{k,2}$  and the arcs of the paths  $u_{1,1}u_{h,1}u_{k,2}$  and  $u_{h,2}u_{k,1}u_{p,1}$  (see Figure 11). Thus,  $D$  has a strong arc decomposition, contradiction our assumption on  $D$ .

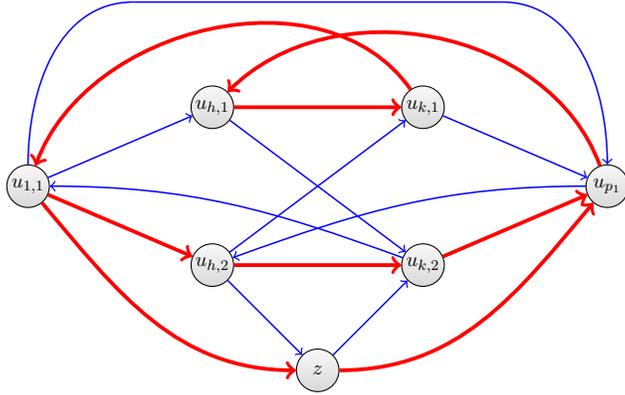


Figure 11: A pair of arc-disjoint strong spanning subdigraphs of  $D$ .

**Case 4.3.**  $T$  has three cut-arcs  $u_pu_h, u_hu_k, u_ku_1$ .

Recall that  $|T_1| = |T_p| = 1$  and  $|H_1| = |H_p| = 1$ , which implies that  $V(D_1) = \{u_{1,1}\}$  and  $V(D_p) = \{u_{p,1}\}$ . Suppose first that  $|V(D)| = 6$ . Then  $V(D) = \{u_{1,1}, u_{k,1}, u_{k,2}, u_{h,1}, u_{h,2}, u_{p,1}\}$  and either  $D$  is isomorphic to  $T_4^s[\overline{K}_2, \overline{K}_2, \overline{K}_1, \overline{K}_1] = \overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overrightarrow{P}_2]$  or at least one of the three cut-arcs are part of a 2-cycle in  $T$ . In the later case we obtain the contradiction that  $D$  has a strong arc decomposition as shown in Figure 12. Thus, we may assume that  $|V(D)| \geq 7$ . Now we can choose a vertex  $w$  which is an out-neighbour of  $u_{k,1}, u_{k,2}$  and an in-neighbour of  $u_{h,1}, u_{h,2}$ . As above it suffices to show that the subdigraph  $D^*$  induced by the vertices in  $\{w, u_{1,1}, u_{k,1}, u_{k,2}, u_{h,1}, u_{h,2}, u_{p,1}\}$  has a strong arc decomposition. This

follows from the fact that  $D^*$  has a pair of arc-disjoint strong spanning subdigraphs: the first is induced by the arcs of the 5-cycle  $u_{1,1}wu_{p,1}u_{h,2}u_{k,2}u_{1,1}$  and the path  $u_{1,1}u_{h,1}u_{k,1}u_{p,1}$  and the second is induced by the 7-cycle  $u_{1,1}u_{p,1}u_{h,1}u_{k,2}wu_{h,2}u_{k,1}u_{1,1}$  (see Figure 13).

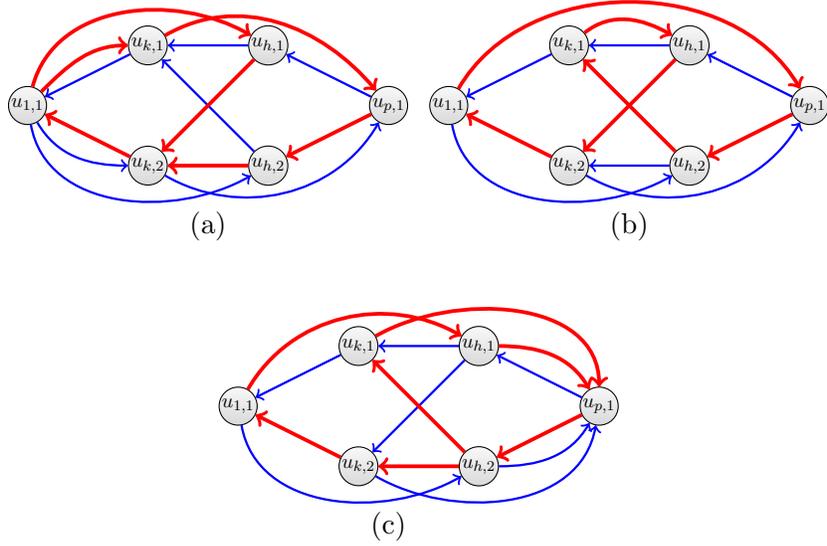


Figure 12: Arc disjoint spanning subdigraphs of  $D$  when  $|V(D)| = 6$  and one of the cut-arcs of  $T$  is in a 2-cycle. The case when  $u_k u_1$  is in a 2-cycle of  $T$  is shown in (a), the one where  $u_h u_k$  is in a 2-cycle in  $T$  is shown in (b) and finally (c) shows the case when  $u_p u_h$  is in a 2-cycle in  $T$ .

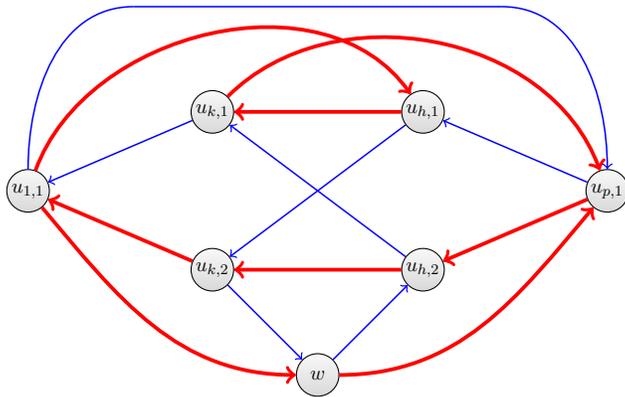


Figure 13: A pair of arc-disjoint strong spanning subdigraphs of  $D^*$ .

Thus we have shown that the counterexample  $D$  does not exist and the proof of the theorem is complete.  $\square$

## 7 Concluding remarks

All proofs in this paper are constructive and can be turned into polynomial algorithms for finding strong arc decompositions. Thus, the problem of finding a strong arc decomposition in a semicomplete composition, which has one, admits a polynomial time algorithm.

Recall that strong semicomplete compositions generalize both strong semicomplete digraphs and strong quasi-transitive digraphs. However, they do not generalize locally semicomplete digraphs and their generalizations in- and out-locally semicomplete digraphs. A digraph  $D$  is **in-locally semicomplete** (**out-locally semicomplete**, respectively) if the in-neighbourhood (out-neighbourhood, respectively) of every vertex of  $D$  induces a semicomplete digraph. (For information on in- and out-locally semicomplete digraphs, see e.g. [1] and [2, Chapter 6].)

While there is a characterization of locally semicomplete digraphs having a strong arc decomposition (see Theorem 1.2), no such a characterization is known for in-locally semicomplete digraphs<sup>1</sup> and it would be interesting to obtain such a characterization or at least establish the complexity of deciding whether an in-locally semicomplete digraph has a strong arc decomposition. Similar questions are of interest for other generalizations of semicomplete digraphs such as generalizations of quasi-transitive digraphs overviewed in [10].

**Acknowledgement** We are very grateful to the referees for providing us with a large number of suggestions to improve the presentation.

## References

- [1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, 2nd Edition, Springer, London, 2009.
- [2] J. Bang-Jensen and G. Gutin (eds.) *Classes of Directed Graphs* Springer, London, 2018.
- [3] J. Bang-Jensen and F. Havet, Tournaments and Semicomplete Digraphs, in *Classes of Directed Graphs* (J. Bang-Jensen and G. Gutin, eds.), Springer, 2018.
- [4] J. Bang-Jensen, F. Havet and A. Yeo, Spanning eulerian subdigraphs of semicomplete digraphs, submitted.
- [5] J. Bang-Jensen and J. Huang, Quasi-transitive digraphs, *J. Graph Theory*, 20(2), 1995, 141–161.
- [6] J. Bang-Jensen and J. Huang, Decomposing locally semicomplete digraphs into strong spanning subdigraphs, *J. Combin. Theory Ser. B*, 102, 2012, 701–714.

---

<sup>1</sup>Clearly, such a characterization, if it exists, could be easily transformed into that of locally out-semicomplete digraphs.

- [7] J. Bang-Jensen and A. Yeo, Decomposing  $k$ -arc-strong tournaments into strong spanning subdigraphs, *Combinatorica* 24(3), 2004, 331–349.
- [8] P. Camion, Chemins et circuits hamiltoniens des graphes complets, *Comptes Rendus de l'Académie des Sciences de Paris*, 249, 1959, 2151–2152.
- [9] J. Edmonds, Edge-disjoint branchings, in *Combinatorial Algorithms* (B. Rustin, ed.), Academic Press, 1973, 91–96.
- [10] H. Galeana-Sánchez and C. Hernández-Cruz, Quasi-transitive digraphs and their extensions, in *Classes of Directed Graphs* (J. Bang-Jensen and G. Gutin, eds.), Springer, 2018.
- [11] G. Gutin, Polynomial algorithms for finding Hamiltonian paths and cycles in quasi-transitive digraphs, *Australasian J. Combin.* 10, 1994, 231–236.
- [12] Y. Sun, G. Gutin and J. Ai, Arc-disjoint strong spanning subdigraphs in compositions and products of digraphs, *Discrete Mathematics* 342 (2019) 2297–2305.
- [13] C. Thomassen, Configurations in graphs of large minimum degree, connectivity, or chromatic number, *Annals of the New York Academy of Sciences*, 555, 1989, 402–412.