

# The permissible and the forbidden \*

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January 22, 2020

## ABSTRACT:

In many economic situations, neither prices nor authority rules. Rather, individual behavior is governed by social norms that specify what is allowed (socially acceptable) and what is forbidden (socially unacceptable). These norms can emerge in a decentralized way and can serve as a method to bring order to economic situations. The key component of our solution concept is a uniform permissible set which plays a role parallel to that of a price system in competitive equilibrium. The concept is analysed and applied to a variety of economic and social settings.

KEYWORDS: Equilibrium, General Equilibrium, Social norms, Convexity, Envy-Free.

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\*We would like to thank Martin Osborne for his kind assistance and Kemal Yildiz, Jan Werner and the referees of this journal for their comments.

## 1. Introduction

A family with  $n$  members sits down to enjoy a holiday feast. The grandparents have prepared a delicious traditional pie. All members would love to eat as much as possible from the pie. Some perhaps appreciate the pie more, others perhaps less, but no market operates around the dinner table in which family members can exchange slices of the pie for other assets and the grandparents do not conduct an auction. Instead, a norm prevails according to which each family member considers grabbing at most  $1/n$  of the pie. If the norm would be to take up to  $q > 1/n$  of the pie, a family crisis would burst. If the norm were stricter, to take at most  $q < 1/n$  of the pie, no conflict arises but the norm would be unstable since if the upper bound  $q$  would increase a bit, no conflict with feasibility arises.

The family dinner is a typical social scenario which is economic in nature: members of society face a shortage of some resources and a conflict arises over how to allocate them. It is also an example in which the resolution of the conflict is decentralized *but not by market mechanisms*. Rather, norms evolve which restrict the set of actions that each agent considers. These norms play an analogous role to competitive prices in markets. They adjust until harmony is achieved. Unlike prices, they are also subject to another force: they are relaxed if unnecessary.

The institutions we wish to model are more basic than those traditionally modeled in economics. They do not require a legal system which enforces ownership rights. Nonetheless, they do require coordination between the members of society regarding the limits of what is allowed. As Basu (2010) argues, the existence of norms that prevent individuals from taking certain actions (like stealing from other people or getting rid of rivals) is also implicitly assumed in any market model. Thus, the standard general equilibrium model is not solely a price-based model but rather is a price-and-norms-based model. In these terms, the model we present here is only norms-based.

**A description of the model and the solution concept.** The components of the model are a set of individuals, a uniform set of alternatives that each agent chooses from, a preference relation for each agent on the set of alternatives, and a societal feasibility constraint which takes the form of a set of feasible profiles.

Our proposed equilibrium concept is a set of permissible alternatives combined with a profile of choices (one for each agent) such that:

- (i) each agent's alternative is optimal from among the permissible alternatives;
- (ii) the profile is feasible; and
- (iii) there is no larger set of permissible alternatives from which a profile satisfying (i) and (ii) can be assigned.

In many models, the set of alternatives is a subset of an Euclidean space. In those cases, we impose a convexity restriction on the permissible set. This restriction reflects a simplicity requirement of a norm and we will show that it implies that the permissible set is defined by a finite number of linear inequalities. Furthermore, it is natural that if two alternatives are permissible, then everything "between them" is permissible as well. For example if a driver knows that both 40 mph and 80 mph are allowed, then he can safely conclude that all in-between speeds are as well.

The solution concept reflects a pure stability notion. Two forces make a permissible set unstable: The first modifies the permissible set in the case that the profile of (intended) choices is not feasible while the second loosens restrictions on the permissible set as long as the new profile of choices is feasible.

The equilibrium notion offers a decentralized institution for achieving harmony in a society without introducing any extraneous medium. There is no authority that sets the norms, just as there is no authority that sets prices in the market. We think about the permissible set as an expression of *social norms* which, like competitive prices, apply uniformly to all individuals in the society, so that all agents face the same choice set. This uniformity, is first and for most, is a simplicity condition. Norms, especially those that evolve without a central authority, must be simple and one aspect of simplicity is that it is applied equally to all agents. There are circumstances in which it would be natural for norms to discriminate between groups of agents (such as between seniors and non-seniors) but such a possibility is not discussed here.

We envision that without a central authority, the same invisible hand that calculates equilibrium prices so effectively is also able to determine a minimal set of forbidden alternatives for which optimal individualistic behavior is compatible. The social norm adjusts until harmony is achieved and if there are unnecessarily forbidden alternatives

such that harmony would prevail even if they were allowed, then the set of permissible alternatives will expand. While we do not provide a general dynamic process that converges to equilibrium, in several examples we demonstrate how a tatonnement-like process leads to an equilibrium.

**Some leading scenarios.**

**1. Give-and-take** Each member of a society either gives to or takes from the society. Some individuals want to contribute to society while others want to take from society. The feasibility constraint requires that what people take cannot exceed what people give. A unique equilibrium exists and is Pareto-efficient. Assuming that the sum of the ideal contributions is less than the sum of the ideal withdrawals, the equilibrium permissible set does not restrict contributions but puts a cap on what agents consider taking, so that a balance is maintained.

**2. Keeping close economy** The survival of a group depends on the ability of its members to reach one another within a certain amount of time in case of danger. Therefore, they need to live within a certain distance from each other. The members have preferences over where they will live. An equilibrium imposes minimal restrictions on the permitted locations so that individuals' choices fulfill the closeness requirement. In this economy, every Pareto-efficient profile of locations is an equilibrium outcome.

**3. Division** A total bundle is to be distributed among a group of people. Social norms determine the set of permissible bundles. In equilibrium, the total demand should not exceed the total bundle. The division of the dinner pie is an example with one good. With more than one good, the model becomes more intriguing.

A prominent equilibrium (though not necessarily unique) is defined by the egalitarian competitive equilibrium linear prices (in which all agents are initially endowed with an equal share of the total bundle). The equilibrium permissible set contains all bundles with a value that does not exceed  $1/n$  of the total bundle's value.

**A comparison to fairness notions.** This paper is not meant to be normative by any means although the equilibrium has two fairness properties:

(i) All agents face the same choice set. This property is shared with competitive equilibrium in which all agents face the same (local) trading opportunities.

(ii) An equilibrium profile is envy-free (see Foley (1965)).

These fairness properties call for a comparison of our approach with the fairness literature which deals with division economies (see for example, Thomson (2019)). We will see that our equilibrium concept is very different from well-known fairness concepts such as efficient envy-free outcomes (Varian (1974)) or the egalitarian efficient allocation notion (Pazner and Schmeidler (1978)).

In the rest of the paper, we formally define the solution concept and prove some general results about its structure. We present several existence theorems and "welfare theorems" which identify when equilibrium outcomes are efficient and find conditions under which any Pareto-efficient allocation is an equilibrium outcome. Much of what follows is devoted to an analysis of examples, some of which are traditional economic settings while others demonstrate the potential to apply the model to other types of social situations.

## 2. The Equilibrium Concept

### 2.1 Economy and Equilibrium

We start by defining the economic environment. Its ingredients are a set of agents, a set of individualistic alternatives, the agents' preference relations on the set of alternatives, and a constraint on the feasible profiles of choices:

**Definition 1** An *economy* is a tuple  $\langle N, X, \{\succsim^i\}_{i \in N}, F \rangle$  where  $N$  is a finite set of agents,  $X$  is a set of (individualistic) alternatives,  $\succsim^i$  is agent  $i$ 's preference on  $X$  and  $F$  is a subset of  $X^N$  (the set of profiles) which contains all feasible choice profiles.

A candidate for an equilibrium is a configuration consisting of a permissible set and a profile of choices:

**Definition 2** A *configuration* is a pair  $\langle Y, (y^i) \rangle$  where  $Y \subseteq X$  and  $(y^i)$  is a profile of elements in  $Y$ . We refer to  $Y$  as a *permissible set* and to  $(y^i)$  as an *outcome*.

Thus, a configuration has a structure analogous to that of competitive equilibrium. The price system that is applied uniformly to all economic agents in the competitive case is replaced here by a permissible set which uniformly binds all agents.

Our central equilibrium concept is given in the following definition which makes use of the auxiliary concept of para-equilibrium. A para-equilibrium is a configuration in which each individual maximizes his interests and in addition the profile of choices is feasible. An equilibrium is a para-equilibrium such that any expansion of the admissible set will cause a violation of feasibility if agents optimally respond to the expansion.

**Definition 3** A *para-equilibrium* is a configuration  $\langle Y, (y^i) \rangle$  satisfying:

(i) for all  $i$ ,  $y^i$  is a  $\succsim^i$ -maximal alternative in  $Y$ ; and

(ii)  $(y^i) \in F$

An *equilibrium* is a para-equilibrium such that there is no para-equilibrium  $\langle Z, (z^i) \rangle$  such that  $Z \supsetneq Y$ .

Like competitive equilibrium prices, we view the permissible set not as being chosen by some authority but rather evolving through an invisible-hand-like process with two forces: first, if the profile of intended choices from the permissible set is not feasible, alternatives are slowly added or removed to the permissible set. Second, when the profile of chosen alternatives is feasible, additional alternatives are added to the permissible set as long as harmony is not disturbed. Note that when assessing whether harmony is violated by loosened constraints, choices are not static and may adapt to the loosening.

The permissible set reflects uniform social norms. This uniformity is analogous to the uniformity of the price system in a competitive equilibrium. We view the uniformity mainly as a simplicity property. Simple social norms should not distinguish between agents based on their names or preferences. As mentioned earlier, there are situations in life where norms naturally place nonuniform constraints on agents based on additional personal information (such as allowing handicapped drivers to park in places where others would not consider) but such a possibility is not considered here.

In equilibrium, all agents face the same choice set and since all agents are rational, no one strictly desires the alternative chosen by another to his own. In other words, the profile of choices in any equilibrium is "envy-free" (see Foley (1967)):

**Definition 4** A profile  $(y^i)$  is *envy-free* if for all  $i \neq j$ ,  $y^i \succsim^i y^j$ .

A profile  $(y^i)$  is *strictly envy-free* if for all  $i \neq j$ ,  $y^i \succ^i y^j$ .

**Comment:** We have been asked how our equilibrium concept relates to Debreu (1952)'s notion of a "generalized game". First, Debreu's equilibrium notion is game-theoretic, unlike ours. A generalized game differs from a standard non-cooperative game in that the set of actions available to players is restricted by the actions of the other players. Debreu's equilibrium is a profile of actions such that each player's action is a best response *from among the set of actions that are available to him given the other players' actions*. In contrast, a fundamental feature that underlies competitive equilibrium analysis - and our concept as well - is that an agent considers all permissible alternatives, including those that are not compatible with the other agents' choices. Thus, a key difference is that Debreu's notion restricts the profile of actions that agents can consider, whereas our feasibility restriction limits the profiles of choices that are compatible.

## 2.2 Existence

Later we provide several general existence results, however not every economy has an equilibrium. For example, consider the "housing economy" in which each agent can choose a single house and feasibility requires that each house is chosen at most once. If all individuals have the same strict preferences over the houses then no equilibrium exists since, whatever the permissible set is, all agents would pick the same house, which violates feasibility. The lack of equilibrium in such an economy fits the intuition that social norms regarding "the permissible and the forbidden" do not resolve conflicts when agents have similar preferences and feasibility requires them to make different choices.

## 2.3 Efficiency

An equilibrium outcome can be Pareto-inefficient. For example, consider the "housing economy" where  $N = \{1, 2\}$ ,  $X = \{a, b, c, d, e\}$  and the agents' preferences are  $a \succ^1 b \succ^1 c \succ^1 d \succ^1 e$  and  $a \succ^2 c \succ^2 b \succ^2 e \succ^2 d$ . One para-equilibrium is  $Y = \{d, e\}$ ,  $y^1 = d$ ,  $y^2 = e$  but it is not an equilibrium. The unique equilibrium is  $Y = \{b, c, d, e\}$  with  $y^1 = b$ ,  $y^2 = c$  (the alternative  $a$  cannot be added to the permissible set). The equilibrium outcome is Pareto-inefficient because the universal favorite house is unassigned.

Although an equilibrium outcome may be Pareto-inefficient, the following proposition is a "second-best" result: The equilibrium outcomes are precisely the profiles that are Pareto-efficient *among the set of envy-free profiles*.

**Proposition 1** *A profile is an equilibrium outcome if and only if it is Pareto-efficient among all feasible envy-free profiles.*

*Proof.* Let  $\langle Y, (y^i) \rangle$  be an equilibrium. The profile  $(y^i)$  is envy-free. If it is Pareto-inefficient among the feasible envy-free profiles, then there is a feasible envy-free profile  $(z^i)$  that Pareto-dominates  $(y^i)$ . Clearly,  $\langle Y \cup \{z^1, \dots, z^n\}, (z^i) \rangle$  is a para-equilibrium and for at least one agent  $i$ ,  $z^i \succ^i y^i$  and therefore  $z^i \notin Y$ , contradicting the maximality of  $\langle Y, (y^i) \rangle$ .

Let  $(y^i)$  be Pareto-efficient among the feasible envy-free profiles. Define the permissible set  $Y = \cup_i \{y^i\} \cup \{x \mid \text{for all } i, y^i \succsim^i x\}$ . Clearly,  $\langle Y, (y^i) \rangle$  is a para-equilibrium. Suppose that  $\langle Z, (z^i) \rangle$  is a para-equilibrium with  $Z \supsetneq Y$ . Then,  $z^i \succsim^i y^i$  for all  $i$  and  $(z^i)$  is envy-free. Take an  $x \in Z - Y$ . There is an agent  $j$  for whom  $x \succ^j y^j$  and consequently,  $z^j \succsim^j x \succ^j y^j$ . Therefore,  $(z^i)$  is an envy-free profile which Pareto-dominates  $(y^i)$ , contradicting  $(y^i)$  being Pareto-efficient among the envy-free profiles. Thus, no such  $\langle Z, (z^i) \rangle$  can exist and  $\langle Y, (y^i) \rangle$  is an equilibrium.  $\square$

A condition on  $F$  which guarantees that any equilibrium outcome is overall Pareto-efficient is given in the following proposition. The condition, which we refer to as the imitation property, requires that if a profile is feasible, then so is any profile for which one agent adopts the alternative chosen by another agent instead of his own. Example E in Section 4 satisfies this condition.

**Proposition 2** *Assume that  $F$  satisfies the following imitation property: if  $a \in F$ , then any profile  $b$ , which differs from  $a$  only in that there is a unique  $i$  for which  $b^i \neq a^i$  and  $b^i = a^j$  for some  $j$ , is also in  $F$ . Then, a profile is an equilibrium outcome if and only if it is Pareto-efficient.*

*Proof.* Let  $\langle Y, (y^i) \rangle$  be an equilibrium with Pareto-inefficient outcome. Then, there is a feasible profile  $(z^i)$  which Pareto-dominates  $(y^i)$ . However, in the profile  $(z^i)$  some agents may envy others, so we use  $(z^i)$  to define an envy-free feasible profile  $(x^i)$  that also Pareto-dominates  $(y^i)$ , violating Proposition 1.

Assign  $x^1$ , a  $\succsim^1$ -maximal alternative from  $\{z^1, \dots, z^N\}$ , to agent 1. Assign  $x^2$ , a  $\succsim^2$ -maximal alternative from  $\{x^1, z^2, \dots, z^N\}$ , to agent 2, and so on to form the profile  $(x^i)$ .

In this construction: 1) the profile selected at each stage is feasible (due to the imitation property) and weakly Pareto-dominates the previous one; and 2) at stage  $j$ , no agent  $i \leq j$  envies any other agent. Therefore,  $(x^i)$  is envy-free and Pareto-dominates  $(y^i)$ .

The other direction follows from Proposition 1 because under the imitation condition on  $F$ , every Pareto-efficient profile is envy-free and therefore is efficient among the envy-free allocations.  $\square$

### 3. Examples

#### Example A: The splitting cakes economy

There are situations in life where limited quantities of goods can be distributed between agents but each agent can consume only one of the goods. The goods could be consumption goods in different locations, risky assets in a world of risk-loving agents or meat and dairy dishes in a kosher community.

Social norms attach a "quota" to each good. An equilibrium in this economy is a specification of maximal quotas for which demand does not exceed supply for each of the goods. The quotas should be low enough so that demand will not exceed the available quantity but also not too low so that they can be relaxed without the demands hitting one of the bounds. We will see that an equilibrium exists and is unique. The economy might not have an envy-free and efficient allocation, but the equilibrium will be shown to always be "almost efficient".

Formally, there are two goods in the economy, 1 and 2, with total supply  $(\alpha, \beta)$ . Each agent can choose a quantity of a single good, that is, the set of alternatives  $X$  consists of all objects of the type  $(a, 0)$  and  $(0, b)$  where  $a$  and  $b$  are non-negative real numbers. A profile is feasible if for each good the sum of the agents' consumptions of that good does not exceed its total supply. Agents have continuous and strictly monotonic preferences over  $X$ . To avoid degenerate cases, we assume that there are numbers  $a$  and  $b$  such that all agents prefer  $(a, 0)$  to  $(0, \beta)$  and  $(0, b)$  to  $(\alpha, 0)$ . Note that our analysis would apply equally to any economy of  $K$  goods where each agent consumes only one of them.

**Claim A** *In any splitting cakes economy:*

- (i) *There is a unique equilibrium set.*
- (ii) *In any equilibrium, at least one of the goods is fully consumed.*
- (iii) *In any equilibrium, if a good is not fully allocated, then the unallocated portion is not larger than the maximum allowed quantity of that good.*

*Proof.* Note that in any para-equilibrium, if at least one agent takes a portion of a certain pie, then this must be the upper bound on the permissible consumption of that pie and all agents who consume from this pie take this bound. If one of the pies is not consumed by any agent, then by the continuity of the preferences we can assume that the permissible set specifies an upper bound on its consumption. Thus, given a para-equilibrium  $\langle W, (w^i) \rangle$ , let  $a_W = \max(a : (a, 0) \in W)$  and  $b_W = \max(b : (0, b) \in W)$ .

**Step 1:** If  $\langle Y, (y^i) \rangle$  and  $\langle Z, (z^i) \rangle$  are para-equilibria, then there is a para-equilibrium with the permissible set  $Y \cup Z$ .

The claim is trivial if one of the the permissible sets offers a weakly larger portion of each pie than the other. Otherwise, without loss of generality, we have  $a_Y > a_Z$  and  $b_Z > b_Y$ . Then,  $a_{Y \cup Z} = a_Y$  and  $b_{Y \cup Z} = b_Z$ . Take the permissible set to be  $Y \cup Z$  and attach to each agent an individually optimal bundle in  $\{(a_Y, 0), (0, b_Z)\}$ . Total consumption of the first good is then bounded above by  $\#\{i : (a_Y, 0) \succsim^i (0, b_Z)\} * a_Y \leq \#\{i : (a_Y, 0) \succ^i (0, b_Y)\} * a_Y \leq \alpha$ . An analogous argument applies to the second good, and thus for each good demand does not exceed supply.

**Step 2:** Existence of an equilibrium.

Let  $a^* = \sup\{a_Y : \langle Y, (y^i) \rangle \text{ is a para-equilibrium}\}$  and similarly define  $b^*$  (obviously, para-equilibria exist). It is sufficient to show that  $M = \{(x_1, x_2) \in X : x_1 \leq a^*, x_2 \leq b^*\}$  is a permissible set in some para-equilibrium. By definition, there are sequences of para-equilibrium permissible sets  $(Y_n)$  and  $(Z_n)$  such that  $a_{Y_n} \rightarrow a^*$  and  $b_{Z_n} \rightarrow b^*$ . By Step 1,  $W_n = Y_n \cup Z_n$  is a sequence of para-equilibrium sets and of course  $(a_{W_n}, b_{W_n}) \rightarrow (a^*, b^*)$ . This sequence of para-equilibria has a subsequence in which a fixed set of agents  $Q$  choose  $(a_{W_n}, 0)$  and the remainder  $N - Q$  choose  $(0, b_{W_n})$ . It remains to be shown that  $\langle M, (m^i) \rangle$  is a para-equilibrium where  $m^i = (a^*, 0)$  if  $i \in Q$  and  $m^i = (0, b^*)$  if  $i \in N - Q$ . To verify feasibility of  $(m^i)$ , notice that  $a_{W_n} \cdot |Q| \leq \alpha$  and therefore  $a^* \cdot |Q| \leq \alpha$  and similarly for the other good. To verify individual optimality, notice that since  $(a_{W_n}, 0) \succsim^i (0, b_{W_n})$

for all  $i \in Q$  and for all  $n$ , then by continuity  $(a^*, 0) \succsim^i (0, b^*)$  for all  $i \in Q$ . Similarly,  $(0, b^*) \succsim^i (a^*, 0)$  for all  $i \in N - Q$ .

**Step 3:**  $M$  is the unique equilibrium permissible set.

Given any para-equilibrium  $\langle Y, (y^i) \rangle$ , by the definition of  $a^*$  and  $b^*$ , it is the case that  $a^* \geq a_Y$  and  $b^* \geq b_Y$ . Thus,  $Y \subseteq M$ .

**Step 4:** In an equilibrium, at least one of the goods is fully consumed.

Assume otherwise. Let  $\langle M, (y^i) \rangle$  be an equilibrium where  $k$  agents are allocated  $(a^*, 0)$  while  $N - k$  agents are allocated  $(0, b^*)$  and no good is fully consumed, that is,  $ka^* < \alpha$  and  $(N - k)b^* < \beta$  and thus  $a' = \alpha/k > a^*$  and  $b' = \beta/(N - k) > b^*$ . Define  $Y^\lambda = \{(x_1, x_2) \in X : (x_1, x_2) \leq (\lambda a' + (1 - \lambda)a^*, \lambda b^* + (1 - \lambda)b')\}$ . When  $\lambda = 0$ , at least  $N - k$  agents prefer  $(0, b_{Y^0} = b')$  to  $(a_{Y^0} = a^*, 0)$ . When  $\lambda = 1$ , at least  $k$  agents prefer  $(a_{Y^1} = a', 0)$  to  $(0, b_{Y^1} = b^*)$ . By continuity, there is some intermediate  $\lambda$  where at least  $k$  agents weakly prefer  $(\lambda a' + (1 - \lambda)a^*, 0)$  to  $(0, \lambda b^* + (1 - \lambda)b')$  and at least  $N - k$  agents weakly prefer  $(0, \lambda b^* + (1 - \lambda)b')$  to  $(\lambda a' + (1 - \lambda)a^*, 0)$ . Then,  $Y^\lambda$  is a larger para-equilibrium permissible set, contradicting step 3.

**Step 5:** For any equilibrium, if a good is not fully consumed, then its unallocated portion is not larger than each allocated portion of that good.

Suppose  $\langle M, (y^i) \rangle$  is an equilibrium where  $k$  agents choose  $(a^*, 0)$  and  $\alpha - ka^* > a^*$ . If every agent who chooses  $(0, b^*)$  strictly prefers  $(0, b^*)$  to  $(a^*, 0)$ , then  $a^*$  can be slightly increased without changing consumption patterns, thus violating the maximality of  $M$ . Otherwise, for at least one  $i$ ,  $y^i = (0, b^*)$  and  $(a^*, 0) \sim_i (0, b^*)$ . Then, modifying the equilibrium so that agent  $i$  would choose  $(a^*, 0)$  instead of  $(0, b^*)$  is an equilibrium (it is a para-equilibrium and since we started with an equilibrium there is no larger para-equilibrium) in which no good is fully consumed, contradicting Step 4.  $\square$

*Discussion:* The unique equilibrium for this economy has an intuitive structure. All agents face a quota for each good and any increase in one or both quotas will yield excess demand. The unique equilibrium can emerge through a dynamic process where the quotas are adjusted according to excess supply or demand. In the case that both goods are fully consumed, the equilibrium is Pareto-efficient. Otherwise, at most one quota of one of the goods is wasted, an inefficiency which is necessary for harmony.

**Example B: Quorum economy**

The following is a particular simple model of clubs (Buchanan (1965)) where agents have preferences on the set of clubs and in order to operate, a club needs a minimal quorum. If each agent were to choose his most beloved club, then there may be insufficiently occupied non-empty clubs. The role of the permissible set is to facilitate coordination with minimal restrictions on the agents. We will see that in such an economy efficiency is not guaranteed in equilibrium.

Formally, let  $X$  be a finite set of clubs. Each agent has preferences over  $X$  and selects one club. Feasibility requires that club  $x$  is empty or chosen by at least  $m_x \leq n$  members. Any configuration  $Y = \{x\}$  combined with all agents choosing  $x$  is a para-equilibrium. Since there are finitely many permissible sets, an equilibrium always exists.

**Claim B** *In some quorum economies, every equilibrium is Pareto-inefficient.*

*Proof.* Consider the quorum economy with  $N = 6$ ,  $X = \{a, b, c\}$ , and  $m_x = 3$  for all  $x$ . Suppose two agents have the preferences  $a \succ b \succ c$ , two have the preferences  $b \succ c \succ a$  and two have the preferences  $c \succ a \succ b$ . There is no feasible allocation with three active clubs. There is no para-equilibrium set with two clubs since four of the agents would choose one club and only two would choose the other, violating feasibility. Therefore, any equilibrium permissible set consists of one club. However, any constant profile is Pareto-inefficient since there is a different club that is strictly preferred by four agents, and there is a Pareto improvement whereby exactly three of them switch clubs.  $\square$

*Comment:* Two natural forces may operate dynamically in this example. First, a club that does not have a quorum is removed from the permissible set. Second, from time to time, a forbidden club may become socially acceptable for a while until it is determined whether it attracts a sufficient crowd without invalidating any of the existing clubs.

## 4. Euclidean Economies

### 4.1 Euclidean economy

Of special interest are the *Euclidean economies* in which the set of alternatives is embedded in a Euclidean space. In such economies, we introduce standard closedness, convexity and differentiability restrictions on the parameters of the model (the set of alternatives, the preference relations and the feasibility set).

**Definition 5** *An economy  $\langle N, X, \{\succsim^i\}_{i \in N}, F \rangle$  is a **Euclidean economy** if:*

- (i) *The set  $X$  is a closed convex subset of some Euclidean space.*
- (ii) *The preferences  $\{\succsim^i\}_{i \in N}$  are continuous and convex.*
- (iii) *The feasibility set  $F$  is closed and convex.*

*We say that a Euclidean economy is **differentiable** if the preferences are strictly convex and differentiable (differentiable preferences have differentiable utility representations or more generally satisfy the condition suggested in Rubinstein (2007)).*

### 4.2 Existence

Any Euclidean economy, when  $F$  is compact and anonymous (closed under permutations), has an equilibrium. Note that in any equilibrium, the permissible set must be closed since if  $\langle Y, (y^i) \rangle$  is a para-equilibrium then  $\langle cl(Y), (y^i) \rangle$  is as well.

**Proposition 3** *For Euclidean economies, if  $F$  is compact and closed under permutations, then an equilibrium exists.*

*Proof.* Let  $EFF$  be the set of envy-free feasible profiles. To see that  $EFF$  is not empty, start with any feasible profile. By assumption, all permutations of this profile are in  $F$  as well. The average of these permutations is a constant profile which is in  $F$  (because  $F$  is convex) and is envy-free, and thus it is in  $EFF$ .

Since each preference  $\succsim^i$  is continuous and  $X$  is a subset of a Euclidean space, there is a continuous utility function  $u^i$  representing  $\succsim^i$ . Also, by continuity of the preferences and  $F$  being compact, the set  $EFF$  (which is defined by weak inequalities) is compact. Thus, there is at least one profile  $z \in EFF$  that maximizes  $\sum_i u^i(x^i)$  over  $EFF$  and therefore  $z$  is Pareto-efficient in  $EFF$ . By Proposition 1,  $z$  is an equilibrium outcome.  $\square$

### 4.3 Efficiency

As shown in Proposition 1, equilibrium outcomes are the Pareto-efficient allocations from among the envy-free allocations, but may be overall Pareto-inefficient (Examples A and B). The next proposition states that for differentiable Euclidean economies, the gap between equilibrium outcomes and Pareto-efficient profiles is within the set of envy-free allocations with indifferences.

**Proposition 4** *In a Euclidean economy, a strictly envy-free profile is an equilibrium outcome if and only if it is overall Pareto-efficient.*

*Proof.* One direction is trivial. If an envy-free profile is Pareto-efficient, then by Proposition 1 it is an equilibrium outcome.

In the other direction, assume that  $\langle Y, (y^i) \rangle$  is an equilibrium and  $y^i \succ^i y^j$  for all  $i \neq j$ . If  $(y^i)$  is Pareto-inefficient, then there is  $(z^i)$  such that  $z^i \succsim^i y^i$  for all  $i$  and  $z^k \succ^k y^k$  for some  $k$ . By convexity of the preferences, any profile  $(\lambda z^i + (1 - \lambda)y^i)$  weakly Pareto-dominates  $(y^i)$  and is feasible because  $F$  is convex. Let  $\bar{\lambda} < 1$  be the largest  $\lambda$  for which  $\lambda z^i + (1 - \lambda)y^i \sim_i y^i$  for all  $i$ . By the continuity of the agents' preferences, for  $\varepsilon > 0$  small enough,  $((\bar{\lambda} + \varepsilon)z^i + (1 - \bar{\lambda} - \varepsilon)y^i) \succ^i ((\bar{\lambda} + \varepsilon)z^j + (1 - \bar{\lambda} - \varepsilon)y^j)$  for all  $i \neq j$ . Thus, the profile  $((\bar{\lambda} + \varepsilon)z^i + (1 - \bar{\lambda} - \varepsilon)y^i)$  is envy-free and by the definition of  $\bar{\lambda}$ , it Pareto-dominates  $(y^i)$ . Thus,  $(y^i)$  is not Pareto-efficient among the envy-free profiles, violating Proposition 1.  $\square$

## 5. Convex Equilibrium

Up to this point, we have not imposed any restrictions on the structure of the permissible set. In this section, we study Euclidean economies and require that the permissible set be convex. We first define the notion of convex equilibrium and then consider three issues: the efficiency of its outcome, its existence and the structure of the convex equilibrium permissible set.

### 5.1 Convex Equilibrium

A convex equilibrium is required to have a convex permissible set. This requirement captures a natural asymmetry between what is allowed and what is forbidden. For example, one would certainly conclude that if driving on a highway at 60 mph and at 80

mph are permitted, then driving at 70 mph is as well. On the other hand, knowing that driving at 110 mph and at 10 mph are forbidden does not lead one to believe that driving at 60 mph is forbidden. This is consistent with the intuition that forbidden actions are typically "extreme" and that the permissible set captures some "middle ground".

Another justification for requiring that the permissible set be convex is the view that the equilibrium permissible set must be describable in a "simple" way. In the Euclidean setting, a natural notion of simplicity is that the set is describable by a small number of inequalities. Proposition 7 below shows that the requirement that the permissible set be convex is equivalent to the requirement that it be defined by a small system of linear inequalities.

**Definition 6** *For Euclidean economies, a **convex para-equilibrium** is a para-equilibrium  $\langle Y, (y^i) \rangle$  such that  $Y$  is convex.*

*A **convex equilibrium** is a convex para-equilibrium  $\langle Y, (y^i) \rangle$  such that there is no other convex para-equilibrium  $\langle Z, (z^i) \rangle$  with  $Z \supsetneq Y$ .*

The notions of equilibrium and convex equilibrium are non-nested, although every convex para-equilibrium is a para-equilibrium. This is because to rule out a convex para-equilibrium being a convex equilibrium there is a need to find a larger *convex* para-equilibrium. Example D contains cases in which there are more equilibria than convex equilibria and in Example E the opposite may occur.

## 5.2 Convex Equilibrium and Efficiency

Proposition 5 states that any profile which is Pareto-efficient among the convex para-equilibrium outcomes is also a convex equilibrium outcome. This is analogous to one of the directions of Proposition 1. The other direction does not hold: in Example E, there is a convex equilibrium outcome which is Pareto-dominated by another convex para-equilibrium outcome.

**Proposition 5** *For Euclidean economies, any convex para-equilibrium outcome which is Pareto-efficient among the convex para-equilibrium outcomes, is a convex equilibrium outcome.*

*Proof.* Let  $(y^i)$  be a convex para-equilibrium outcome. Let  $P$  consist of all sets  $Y$  for which  $\langle Y, (y^i) \rangle$  is a convex para-equilibrium. Endow  $P$  with the partial order  $\supseteq$ . To show

that  $P$  has a maximal set we apply Zorn's Lemma. (Given a partially ordered set  $P$ , if every chain – a completely ordered subset of  $P$  – has an upper bound in  $P$ , then the set  $P$  has at least one maximal element.)

In order to show that any chain  $C$  of elements in  $P$  has an upper bound in  $P$ , we show that  $U$ , the union of the sets in  $C$ , is in  $P$ . The set  $U$  is convex since for any two points  $x, y$  in  $U$ , there is some  $Y \in C$  such that  $x, y \in Y$  and therefore all convex combinations of  $x$  and  $y$  are in  $Y$  and therefore in  $U$ . To see that  $\langle U, (y^i) \rangle$  is a para-equilibrium, it suffices to show that for each  $i$  the element  $y^i$  is  $\succsim^i$ -maximal in  $U$ . Suppose that there is an  $x \in U$  such that  $x \succ^i y^i$  for some  $i$ . Then, there is some  $Y \in C$  such that  $x \in Y$ , contradicting that  $\langle Y, (y^i) \rangle$  is a para-equilibrium.

Now suppose that there is a convex para-equilibrium  $\langle Z, (z^i) \rangle$  such that  $Z \supsetneq Y^*$ . It must be that  $z^i \succsim^i y^i$  for all  $i$ . Since  $(y^i)$  is Pareto-efficient [in  $P$ ] [among the convex para-equilibrium outcomes], it must be that  $z^i \sim^i y^i$  for all  $i$ . Then,  $\langle Z, (y^i) \rangle$  is a convex para-equilibrium, contradicting the maximality of  $Y^*$ .  $\square$

### 5.3 Existence of Convex Equilibrium

The following proposition demonstrates that when  $X$  is compact and  $F$  is closed under permutations, then a convex equilibrium exists.

**Proposition 6** *For Euclidean economies, if  $F$  is compact and closed under permutations, then a convex equilibrium exists.*

*Proof.* Let  $O$  be the set of convex para-equilibrium outcomes. The set  $O$  is not empty, since, as in Proposition 3, there is a constant profile  $(y^i = y^*)$  in  $F$  and thus, the pair  $\langle \{y^*\}, (y^i = y^*) \rangle$  is a convex para-equilibrium.

The set  $O$  is compact. To see this, since  $F$  is compact, it suffices to show that  $O$  is closed. Take a sequence of profiles  $(y^i)_n$  in  $O$  that converges to  $(z^i)$ . Let  $Z = \text{conv}(z^i)$ . The configuration  $\langle Z, (z^i) \rangle$  is a convex para-equilibrium since if there is an agent  $j$  such that  $\sum \lambda^i z^i \succ^j z^j$ , then by continuity, for some large enough  $n$ ,  $\sum \lambda^i y_n^i \succ^j y_n^j$ , violating that  $(y^i)_n$  is a convex para-equilibrium outcome.

Now proceed as in Proposition 3. Since  $O$  is compact, there is a profile  $z$  that is Pareto-efficient in  $O$ . By Proposition 5,  $z$  is a convex equilibrium outcome.  $\square$

## 5.4 A Structure Theorem

We turn to the most significant result in this section. Proposition 7 concerns the structure of the permissible set of convex equilibria in differentiable Euclidean economies. It states that in any convex equilibrium the permissible set is an intersection of a finite set of half-spaces, with at most one half-space per agent. Thus, as previously mentioned, Proposition 7 provides a formal basis for the assertion that the convexity of the permissible set is a simplicity requirement.

**Proposition 7** *Let  $\langle Y, (y^i) \rangle$  be a convex equilibrium in a differentiable Euclidean economy and let  $J = \{i \mid y^i \text{ is not } \succsim^i\text{-global maximum in } X\}$ . Then, there is a profile of closed half-spaces  $(H^i)_{i \in J}$ , such that  $Y = \cap_{i \in J} H^i$ .*

*Proof.* By the differentiability and strict convexity of the agents' preference relations, for every  $i \in J$  there is a unique largest closed half-space  $H^i$  containing  $y^i$  such that  $y^i$  is strictly preferred to all other elements in  $H^i$ .

First we show that  $Y \subseteq \cap_{i \in J} H^i$ : Suppose that for some  $i \in J$ , there is  $w^i \in Y \setminus H^i$ . By the differentiability and strict convexity of  $i$ 's preferences, and for small  $\varepsilon > 0$ ,  $\varepsilon w^i + (1 - \varepsilon)y^i \succ^i y^i$  and by convexity of  $Y$ ,  $\varepsilon w^i + (1 - \varepsilon)y^i \in Y$ . Therefore,  $y^i$  is not top  $\succ^i$ -ranked in  $Y$ , a contradiction.

It suffices to show that  $\langle \cap_{i \in J} H^i, (y^i) \rangle$  is a convex para-equilibrium. This follows from:

- (i) the set  $\cap_{i \in J} H^i$  is convex;
- (ii) for each agent  $k$ ,  $y^k \in Y \subseteq \cap_{i \in J} H^i$ ;
- (iii) for each  $j \notin J$ ,  $y^j$  is a global maximum and therefore is  $\succsim^j$ -maximal in  $\cap_{i \in J} H^i$ ;
- (iv) for each  $j \in J$ ,  $y^j$  is  $\succsim^j$ -maximal in  $H^j$  and therefore is  $\succsim^j$ -maximal in  $\cap_{i \in J} H^i$ .  $\square$

## 6. Convex Equilibrium: Examples

### Example C: A division economy

A leading economic problem is the division of limited resources among the members of a society. In the introduction, we discussed the case in which one "pie" (a single good) is to be divided among a group of agents. In this case, a natural equilibrium is the norm which forbids grabbing more than  $1/n^{\text{th}}$  of the pie. However, when there is more than

one good to be divided, the norm which allows an agent to take up to  $1/n^{\text{th}}$  of each good is not an equilibrium: it will be shown that there is always an equilibrium with a larger admissible set in which agents' choices may reflect their preferences regarding the tradeoffs between the goods. We now explore the existence, structure and efficiency properties of convex equilibria in this economy.

A **division economy**  $\langle N, X, \{\succsim^i\}_{i \in N}, F \rangle$  is a differentiable Euclidean economy such that:

- (i) The set  $X = \mathbb{R}_+^m$  is the set of bundles.
- (ii) Agents' preferences  $\{\succsim^i\}_{i \in N}$  are monotonic (in addition to being continuous, strictly convex and differentiable).
- (iii) There is a bundle  $e \in \mathbb{R}_{++}^m$  such that  $(x^i) \in F$  if and only if  $\sum_i x^i \leq e$ .

The next claim states that every egalitarian competitive equilibrium allocation, in which at least one agent chooses an interior bundle, is a convex equilibrium outcome and that every Pareto-efficient interior convex equilibrium profile is an egalitarian competitive equilibrium allocation. However there may also exist interior convex equilibrium outcomes that are Pareto-inefficient.

**Claim C** *In any division economy  $\langle N, X, \{\succsim^i\}_{i \in N}, F \rangle$ :*

- (i) (a) *Each egalitarian competitive equilibrium allocation is a convex equilibrium outcome. (b) If at least one agent has an interior bundle, then the permissible set equals an egalitarian competitive equilibrium budget set.*
- (ii) *If an interior profile is a Pareto-efficient convex equilibrium outcome, then it is an egalitarian competitive equilibrium allocation.*
- (iii) *There can exist a Pareto-efficient convex equilibrium outcome (where some agent receives zero of some good) that is not an egalitarian competitive equilibrium allocation.*
- (iv) *There can exist a Pareto-inefficient interior convex equilibrium outcome.*

*Proof.* (i) (a) Let  $\langle p^*, (y^i) \rangle$  be a competitive equilibrium in the exchange economy in which each agent is initially endowed with  $e/n$  and let  $Y$  be the (common) budget set. The pair  $\langle Y, (y^i) \rangle$  is a convex para-equilibrium and the profile  $(y^i)$  is Pareto-efficient. Thus, by Proposition 5,  $(y^i)$  is a convex equilibrium outcome.

(b) If  $\langle Z, (y^i) \rangle$  is a convex equilibrium, then  $Z = \bigcap_{i \in N} H^i$  (Proposition 7) where  $H^i$  is the lower half-space of  $\succsim^i$  at  $y^i$ . Furthermore, for all  $i$ ,  $Y \subseteq H^i$  and since  $y^j$  is interior for some  $j$ , then  $H^j = Y$  and thus,  $Z = \bigcap_j H^j = Y$ .

(ii) Let  $\langle Y, (y^i) \rangle$  be a convex equilibrium and  $(y^i)$  be interior. For each agent  $i$ , the chosen alternative  $y^i$  is not  $\succsim^i$ -globally maximal and thus by Proposition 7,  $Y = \bigcap_{i \in N} H^i$ . Since each  $y^i \in \mathbb{R}_{++}^m$  and the allocation is Pareto-efficient the half-spaces must be parallel (otherwise, any two agents on non-parallel half spaces could make a Pareto-improving local exchange). By monotonicity, the half-spaces must be identical and equal to  $Y = \{x | px \leq w\}$  for some positive vector  $p$  and a number  $w$ . For each  $i$ , the bundle  $y^i$  is optimal in  $Y$  and by monotonicity,  $py^i = w$ . Since  $pe = p \sum_{i \in N} y^i$ , we have  $py^i = p(e/n)$ . Thus,  $(y^i)$  is a competitive egalitarian equilibrium allocation with price vector  $p$ .

(iii) Consider the division economy (see Figure 1) with three agents,  $e = (5, 5)$  and preferences represented by the following utility functions (a slight modification of the preferences will make the preference relations strictly convex):

$$u^1(x_1, x_2) = x_1 + 5x_2$$

$$u^2(x_1, x_2) = x_1 + x_2$$

$$u^3(x_1, x_2) = 5x_1 + x_2$$

Let  $y^1 = (0, 3)$ ,  $y^2 = (2, 2)$  and  $y^3 = (3, 0)$ . The allocation  $(y^i)$  is Pareto-efficient since if  $(z^i)$  Pareto-dominates  $(y^i)$  then  $z_1^i + z_2^i \geq y_1^i + y_2^i$  for all  $i$  with at least one inequality. Thus,  $\sum_i (z_1^i + z_2^i) > \sum (y_1^i + y_2^i) = 10$  which is not feasible. Let  $Y$  be the set of all bundles below

the three indifference curves of individuals  $i = 1, 2, 3$  through  $y^i$ . The pair  $\langle Y, (y^i) \rangle$  is a convex para-equilibrium. If there were a larger convex para-equilibrium,  $\langle Z, (z^i) \rangle$ , then  $Z$  would contain an element that is not in  $Y$  and any such element is strictly preferred to  $y^i$  for at least one agent  $i$ . Thus,  $(z^i)$  would Pareto-dominate  $(y^i)$ .

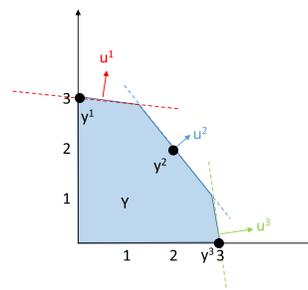


Figure 1: A convex equilibrium with a non-egalitarian Pareto-efficient assignment (Example C).

(iv) Consider the division economy (depicted in Figure 2) with two agents, two goods, total bundle  $e = (1 + \sqrt{3}, 1 + \sqrt{3})$  and utility functions  $U^1(x_1, x_2) = x_1^{3/4} x_2^{1/4}$  and  $U^2(x_1, x_2) = x_1^{1/4} x_2^{3/4}$ . The depicted allocation  $y^1 = (\sqrt{3}, 1)$  and  $y^2 = (1, \sqrt{3})$  is inefficient since for small enough  $\varepsilon > 0$  it is mutually beneficial to have agent 1 getting  $\varepsilon$  more of good 1 and  $\varepsilon$  less of good 2.

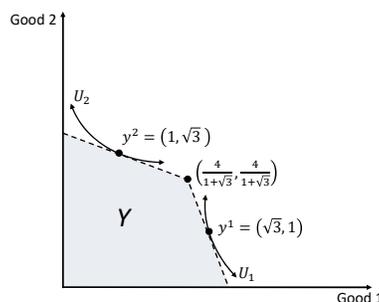


Figure 2: A Pareto-inefficient convex equilibrium.

The configuration  $\langle Y, (y^i) \rangle$  is a convex para-equilibrium. In any larger convex para-equilibrium,  $\langle Z, (z^i) \rangle$ , the convex set  $Z$  includes a bundle which is strictly better for at least one of the agents and therefore  $z^1 \neq y^1$  and  $z^2 \neq y^2$ . Given the indifference curves through  $y^1$  and  $y^2$ , agent 1 receives in  $z^1$  more of good 1 and less of good 2 and agent 2 receives in  $z^2$  less of good 1 and more of good 2. Also, agent 1 must not prefer another bundle on the line segment between  $z^1$  and  $(\frac{4}{1+\sqrt{3}}, \frac{4}{1+\sqrt{3}})$ . Therefore, the negative slope through agent 1's indifference curve at  $z^1$  (namely,  $\frac{3z_2^1}{z_1^1}$ ) is at least as large as the negative slope of the line segment (namely  $\frac{\frac{4}{1+\sqrt{3}} - z_1^1}{\frac{4}{z_1^1} - \frac{4}{1+\sqrt{3}}}$ ). Thus,  $1 + \sqrt{3} \geq \frac{3}{z_1^1} + \frac{1}{z_2^1}$ . However, at least one agent's, say agent 1's, total consumption weakly decreases, that is  $z_1^1 + z_2^1 \leq \sqrt{3} + 1$ . The only  $z^1$  that satisfies both inequalities is  $z^1 = y^1$ . A contradiction.  $\square$

*Discussion:* Our equilibrium notion is very different than Pazner and Schmeidler (1978)'s fairness notion of egalitarian efficiency: an allocation  $(x^i)$  is egalitarian efficient if it is Pareto-efficient and there is a bundle  $y^*$  such that  $x^i \sim^i y^*$  for all  $i$ . An egalitarian efficient allocation need not be envy-free and therefore is never an equilibrium outcome.

Note that a division economy differs from an exchange economy because it does not specify any initial distribution. However, the analysis of the division economy can also be applied to the exchange economy with initial distribution  $(e^i)$  as follows: Let  $X = \mathbb{R}^K$  where a member of  $X$  is interpreted as an exchange. Set  $F$  to include all profiles  $(t^i)$  such that  $\sum_i t^i = 0$  and for each agent  $i$ ,  $t^i + e^i \geq 0$ . The preferences of agent  $i$  over the set  $\{t^i \mid t^i + e^i \geq 0\}$  are derived from the preferences on the consumption bundles; all transfers that leave the agent with a negative amount of at least one good are taken to be inferior to the no-exchange option 0. Claim C then is read as (i) any competitive

equilibrium of the exchange economy is also a convex equilibrium outcome and (ii) any interior Pareto-efficient convex equilibrium outcome is a competitive equilibrium vector of transfers in the exchange economy.

**Division economy with concave preferences:** The existence of a Pareto-efficient envy-free profile may fail in the case of concave preferences (Varian, 1974) and therefore an egalitarian competitive equilibrium may not exist. In contrast, Proposition 6 shows that if preferences are continuous then a convex equilibrium exists even if neither convexity nor monotonicity hold. The constructed convex permissible set includes  $e/N$ ; however, without convexity of preferences, Proposition 7 does not apply and the permissible set is not necessarily a budget set.

This brings us to another definition: a *budget para-equilibrium* is a para-equilibrium whose permissible set is given by a budget set  $Y = \{x \in R_+^m : p \cdot x \leq I\}$  for some price vector  $p \geq 0$  and  $I \geq 0$ . A *budget equilibrium* is a budget para-equilibrium for which there is no budget para-equilibrium with a strictly larger budget set. When an egalitarian competitive equilibrium exists it is a budget equilibrium. However, we will see that in concave division economies there are budget equilibria even when there are no egalitarian competitive equilibria. The proof is based on Example A (the splitting cakes economy).

**Claim C (Concavity)** For a division economy with strictly concave preferences, there is a budget equilibrium, its permissible set is unique and the budget equilibria outcome is unique up to a permutation. In these equilibria, each agent consumes only one good, at least one good is fully consumed and any good that is not fully consumed has a leftover not larger than any agent's non-zero consumption of that good.

*Proof.* From any budget set, an agent with concave preferences always chooses a quantity of only one good. Thus, agents' behavior is identical to that in the splitting cakes economy, where agents were physically restricted to choosing a quantity of only one good. Each budget para-equilibrium corresponds to a splitting cakes para-equilibrium. Since there is a unique equilibrium permissible set in Example A, the convex hull of the cake-splitting equilibrium permissible set is the unique budget equilibrium set.  $\square$

When preferences are convex, the budget equilibria are precisely the standard egalitarian competitive equilibria and there may be a multiplicity of equilibrium budget sets. In contrast, when agents have concave preferences, a budget equilibrium exists and its budget set is unique. Thus, while in many economic settings convexity guarantees "good behavior" of the economic model, here the opposite is true...

**Example D: The give-and-take economy**

There are situations in life where a redistribution is done entirely voluntarily. Some individuals give to others and some take (for example, through a food charity) without exercising force, without commitment to get back, and without an authority who forces the redistribution. Social norms that bound the individuals' actions constitute a potential decentralized mechanism that can bring balance.

In order to formalize these ideas, we analyse the give-and-take economy first studied by Sprumont (1991) (see also Richter and Rubinstein (2015)). Let  $X = [-1, 1]$ , where a positive  $x$  represents a withdrawal of  $x$  from a social fund and a negative  $x$  represents a contribution of  $-x$ . Feasibility requires that the social fund is balanced, that is,  $(x^i) \in F$  iff  $\sum_i x^i = 0$ . All agents have strictly convex and continuous (i.e. single-peaked) preferences over  $X$  with agent  $i$ 's ideal denoted by  $peak^i$ . Obviously, if  $\sum_i peak^i = 0$  there is a unique equilibrium  $\langle Y, (peak^i) \rangle$ .

The following claim characterizes the unique convex equilibrium and shows that it is Pareto-efficient. As it turns out, the unique convex equilibrium outcome coincides with Sprumont (1991)'s Uniform Rule, which he derives through a quite different axiomatic characterization as opposed to the equilibrium approach taken here.

**Claim D** *Consider a give-and-take economy with  $\sum peak^i > 0$ . There is a unique convex equilibrium  $\langle Y, (y^i) \rangle$ . The set  $Y$  takes the form  $[-1, m]$  and  $(y^i)$  is Pareto-efficient.*

*Proof.* We first show that there is a unique  $m$  such that  $[-1, m]$  is a para-equilibrium set. If  $m < 0$  there is a surplus of giving. If  $m \geq 0$ , every agent who wants to give will select his peak, and every agent who wants to take is either at his peak or cannot reach his peak and so takes  $m$ . Let  $D(m)$  be the sum of all agents' choices given the permissible set  $[-1, m]$ . The function  $D$  is continuous,  $D(0) \leq 0$ , strictly increasing for any  $m$  smaller than  $\max\{peak^i\}$  and is constant with value  $\sum_i peak^i > 0$  for any larger  $m$ . Thus, there

is a unique  $m^* \geq 0$  for which  $D(m^*) = 0$ . The configuration of the set  $[-1, m^*]$  and the agents' optimal choices is a convex para-equilibrium. It is a convex equilibrium since for any larger para-equilibrium, its permissible set must be of the form  $[-1, m]$  where  $m > m^*$ . However, for any such  $m$ ,  $D(m) > 0$ .

In order to demonstrate the uniqueness of the convex equilibrium, it suffices to show that any convex para-equilibrium permissible set  $[x, y]$  is included in  $[-1, m^*]$ . In order for the social fund to be balanced, it must be that  $x \leq 0 \leq y$ . In equilibrium, agents who wish to give will do so at either their peak or at  $x$  if  $peak^i < x$ . Therefore, the total giving in  $[x, y]$  is not more than that in  $[-1, m^*]$ . Since the social fund is balanced, the total taking in  $[x, y]$  is also less than or equal to that in  $[-1, m^*]$ , and therefore  $y \leq m^*$ . Thus,  $[x, y] \subseteq [-1, m^*]$ .

In the convex equilibrium outcome  $(y^i)$  every agent is at or to the left of his peak. Thus, if  $(z^i)$  Pareto-dominates  $(y^i)$ , then  $y^i \leq z^i$  for all  $i$  with strict inequality for at least one agent, violating the feasibility constraint. Therefore  $(y^i)$  is Pareto-efficient.  $\square$

*Comment:* If we do not require the permissible set to be convex, then Pareto-efficiency does not necessarily result. Consider the two-agent give-and-take economy with preferences represented by the utilities depicted in Figure 3. The set  $[-1, 0]$  is the unique convex equilibrium permissible set. However, the economy has an equilibrium that is inefficient:  $Y = \{-1, 1\}$  and  $y^1 = -1, y^2 = 1$ . (Suppose that  $\langle Z, (z^i) \rangle$  is a para-equilibrium with  $Z \supsetneq Y$ . Feasibility requires that  $z^1 = -z^2$ . Agent 2 prefers 1 to 0, so  $|z^1| = |z^2| \neq 0$ . Since  $Z$  contains another alternative and agent 1 bottom-ranks 1 and  $-1$ , it is the case that  $|z^1| \neq 1$ . It is impossible that  $0 < |z^1| < 1$ , because both agents prefer  $|z^1|$  to  $-|z^1|$ .)

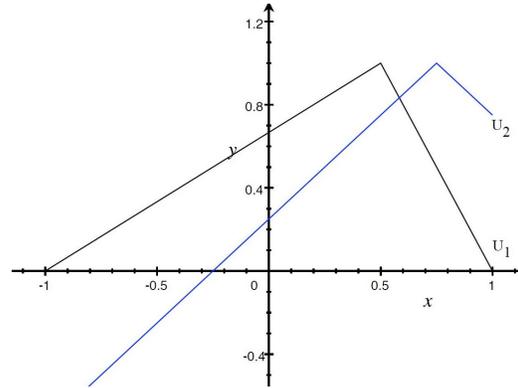


Figure 3: Convex preferences for which there is a Pareto-inefficient equilibrium (Example D).

*Discussion:* The give-and-take economy is an economic situation, in which standard market forces do not play a role. An agent can just give or take and there is no room for trade. Claim D demonstrates the effectiveness of norms as a non-market tool for achieving harmony in the absence of markets.

A process which echoes the excess-demand adjustment process for achieving Walrasian equilibrium can be adapted to the give-and-take economy. Given a convex permissible set, if there is too much taking then the lower bound on giving is relaxed and if this is not possible, then the upper bound on taking is tightened and vice versa for too much giving. Together with continuous adjustment of the agents' choices, this dynamic process ends in the convex equilibrium from any initial permissible set.

### **Example E: The keeping close economy**

Consider a society in which each member chooses a position (whether a political stance or a geographical location) and the survival of the group relies on the members "keeping close" to one another. An extreme case is where all agents need to choose the same location. The problem for society is that its members may have diverse ideal locations which may not fulfill the closeness requirement. In a centralized society, the authorities will force agents' locations. We cannot imagine how a market can resolve this problem. On the other hand, norms could evolve that determine the borders of the permissible locations, striking a balance between societal harmony and individual liberty. Each agent chooses his most desirable location within the borders and the outcome is that they live close enough. If the borders are enlarged in any way, then the resulting individual choices would not be "close enough".

Formally, a *keeping close economy* is an Euclidean economy in which  $X$  is a closed convex set of locations and  $F$  is the set of profiles for which the distance between each pair of choices is at most  $d^*$ . The set  $F$  satisfies the imitation condition defined in Proposition 2. The *consensus economy* is the keeping close economy with  $d^* = 0$ .

Note that this economy has envy-free Pareto-efficient profiles. The following process, in which each agent sequentially chooses a position not too far from his predecessors', leads to such a profile: agent 1 selects his ideal point  $x^1 = peak^1$  in  $X^1 = X$ ; each subsequent agent  $i$  selects his most preferred point  $x^i$  in  $X^i = \{x : d(x, x^j) \leq 1, \forall j < i\}$ .

In this economy, any Pareto-efficient profile is an outcome of some convex equilibrium but efficiency is not guaranteed. (In contrast, Proposition 2 implies that without the convexity requirement every equilibrium outcome is Pareto-efficient.) Nevertheless, for the one-dimensional case, all convex equilibria outcomes are Pareto-efficient.

**Claim E** *For a keeping close economy:*

(i) *Any Pareto-efficient allocation is a convex equilibrium outcome.*

(ii) *A convex equilibrium may be Pareto-inefficient.*

(iii) *If  $X \subseteq \mathbb{R}$ , then a profile is a convex equilibrium outcome if and only if it is Pareto-efficient.*

*Proof.* (i) Given a Pareto-efficient allocation  $(y^i)$ , define  $Y = \text{conv}(\{y^1, \dots, y^n\})$ . Since  $d(\sum_i \lambda^i y^i, y^j) \leq \max_i d(y^i, y^j)$ , any move by an agent to another location in  $Y$  preserves feasibility. Thus, by the Pareto efficiency of  $(y^i)$ , each  $y^i$  is  $\succsim^i$ -maximal in  $Y$ . Therefore,  $\langle Y, (y^i) \rangle$  is a convex para-equilibrium and by Proposition 5,  $(y^i)$  is also a convex equilibrium outcome.

(ii) Consider the two-agent consensus economy with  $X = \mathbb{R}^2$  and agents' preferences represented by  $U^1(x_1, x_2) = 2x_2 - (x_2 - x_1)^2$  and  $U^2(x_1, x_2) = 2x_2 - (x_2 + x_1)^2$  (see Figure 4). Let  $Y = \{(x_1, x_2) : x_2 \leq 0\}$ . From  $Y$ , both agents most prefer  $y^1 = y^2 = (0, 0)$  and thus the pair  $\langle Y, (y^i) \rangle$  is a convex para-equilibrium. If there were a larger convex para-equilibrium set  $Z$ , it would have to be of the form  $\{(x_1, x_2) : x_2 \leq z\}$  with  $z > 0$ . However, from  $Z$ , agent 1 prefers  $(z, z)$  and agent 2 prefers  $(-z, z)$ , and this profile is not in  $F$ . The equilibrium outcome is Pareto-inefficient since both agents prefer  $(0, 1)$  to  $(0, 0)$ . This example can be easily modified for any  $d^* > 0$  by setting  $Y = \{(x_1, x_2) : x_2 \leq d^*/2\}$ ,  $y^1 = (d^*/2, d^*/2)$  and  $y^2 = (-d^*/2, d^*/2)$ .

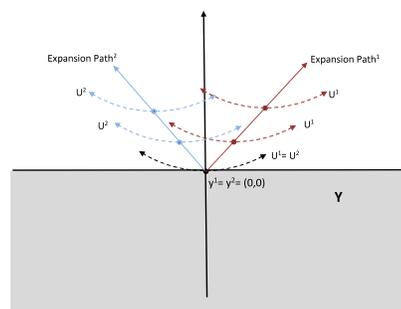


Figure 4: *Multidimensional Consensus Economy: Pareto-inefficient convex equilibrium*

(iii) Let  $l$  denote the minimum of the agents' peaks and  $r$  the maximum. For a profile  $y = (y^i)$ , define  $\underline{y} = \min_i y^i$  and  $\bar{y} = \max_i y^i$ .

One direction follows from (i) and therefore we need to show that if  $\langle Y, (y^i) \rangle$  is a convex equilibrium then  $(y^i)$  is Pareto-efficient.

If  $l - r \leq d^*$ , then the only convex equilibrium is where the permissible set is  $X$  and every agent chooses his peak, obviously a Pareto-efficient outcome.

If  $l - r > d^*$ , then it must be that for each  $i$ ,  $y^i$  is  $\succsim^i$ -maximal in  $[\underline{y}, \bar{y}]$  and  $\bar{y} - \underline{y} = d^*$ , because otherwise the set  $Y$  could be enlarged. It must be that  $\underline{y} \leq r$ , since otherwise  $\langle \{x : x \geq r\}, (y^i = r) \rangle$  is a larger convex equilibrium. Therefore,  $\bar{y} \leq r$ . Similarly  $\underline{y}, \bar{y} \geq l$ . Thus, all choices are between  $l$  and  $r$ . The outcome is Pareto-efficient. To see why note that every agent is either at his peak, at  $\underline{y}$  if his peak is to the left of  $\underline{y}$ , or at  $\bar{y}$  if his peak is to the right of  $\bar{y}$ . Therefore, only agents who choose  $\underline{y}$  or  $\bar{y}$  can improve their locations, but every improvement for an agent with a peak on one side must be at the expense of an agent whose peak is on the other side.  $\square$

*Comment:* Some natural forces might push to equilibrium. Consider the consensus economy on a one-dimensional space. One force narrows the permissible set when there is disagreement thus pushing towards consensus. If a consensus is achieved outside the interval  $[l, r]$ , then the other force extends the permissible set, pushing the consensus towards efficiency.

## 7. The Grand Project

This paper is a part of our grand vision to explore the logic of "price-like" institutions that can bring order to economic environments.

In Richter and Rubinstein (2015), we investigated one such institution. The model there consists of an economy and a set of "primitive" orderings on the set  $X$ . The set of primitive orderings is thought to be the basic language used in both the agents' formation of preferences and the equilibrium structure. Specifically, agents' preferences are required to be convex given the geometry generated by those primitive orderings. The solution concept, *primitive equilibrium*, is a "public ordering" (on  $X$ ), interpreted as a prestige ranking of the alternatives, and a profile of individual choices. An essen-

tial requirement is that the public ordering is a primitive ordering. The profile is feasible and each agent's choice is personally optimal from among the set of alternatives that are weakly less prestigious than the one assigned to the agent. Thus, in equilibrium agents make choices from *different individual choice sets*. This is fundamentally different from the current paper in which all agents choose from a common permissible set.

It is tempting to think about our current equilibrium concept as a degenerate case of primitive equilibrium by defining a public ordering such that all admissible alternatives are equally bottom-ranked, all forbidden alternatives are ranked above them and all agents are assigned bottom-ranked alternatives only. However, viewing our equilibrium in the primitive equilibrium framework is more misleading than useful, due to essential differences between the concepts:

(i) The constraint that the set of forbidden alternatives be minimal is not present in the primitive equilibrium notion. Thus, at most it captures the para-equilibrium notion.

(ii) The implied degenerate ordering is always not a primitive ordering.

(iii) In a primitive equilibrium, the "forbidden" set is convex whereas here it is required that the permissible set is convex.

A more recent paper in the grand project is Rubinstein and Wolinsky (2018). While here the invisible hand restricts the permissible set, in that paper the invisible hand systematically biases the preference relations (essentially multiplying the relevant rates of substitution by some factor). In equilibrium, the biases are such that the profile of optimal choices by the biased individuals from the entire set  $X$  is feasible. The equilibrium notion reflects pressures on the individuals' preferences to adjust to the feasibility constraint.

**To sum up**, as economists we are used to thinking about prices as the central mechanism for balancing between conflicting desires in the economy. In the absence of externalities we are amazed by the "positive features" of the price mechanism. Our grand vision is to divert attention to other social institutions which can (and do) bring harmony into a society.

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