Arc-disjoint in- and out-branchings rooted at the same vertex in compositions of digraphs

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Abstract

A digraph \(D = (V, A)\) has a good pair at a vertex \(r\) if \(D\) has a pair of arc-disjoint in- and out-branchings rooted at \(r\). Let \(T\) be a digraph with \(t\) vertices \(u_1, \ldots, u_t\) and let \(H_1, \ldots, H_t\) be digraphs such that \(H_i\) has vertices \(u_{i,j_i}, 1 \leq j_i \leq n_i\). Then the composition \(Q = T[H_1, \ldots, H_t]\) is a digraph with vertex set \(\{u_{i,j_i} | 1 \leq i \leq t, 1 \leq j_i \leq n_i\}\) and arc set

\[A(Q) = \bigcup_{i=1}^{t} A(H_i) \cup \{u_{ij_i}u_{pq} | u_{ij_i}, u_{pq} \in A(T), 1 \leq j_i \leq n_i, 1 \leq q \leq n_p\}.\]

If \(T\) is strongly connected, then \(Q\) is called a strong composition and if \(T\) is semicomplete, i.e., there is at least one arc between every pair of vertices, then \(Q\) is called a semicomplete composition.

We obtain the following result: every strong digraph composition \(Q\) in which \(n_i \geq 2\) for every \(1 \leq i \leq t\), has a good pair at every vertex of \(Q\). The condition of \(n_i \geq 2\) in this result cannot be relaxed. We characterize semicomplete compositions with a good pair, which generalizes the corresponding characterization by Bang-Jensen and Huang (J. Graph Theory, 1995) for quasi-transitive digraphs. As a result, we can decide in polynomial time whether a given semicomplete composition has a good pair rooted at a given vertex.

Keywords: branching; semicomplete digraph; digraph composition.

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1 Introduction

We use a standard digraph terminology and notation as in [4, 5]. A digraph \(D = (V, A)\) is strongly connected (or, just strong) if there exists a

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path from $x$ to $y$ and a path from $y$ to $x$ in $D$ for every pair of distinct vertices $x, y$ of $D$. An out-tree (in-tree, respectively) rooted at a vertex $r$ is an orientation of a tree such that the in-degree (out-degree, respectively) of every vertex but $r$ equals one. An out-branching (in-branching, respectively) in a digraph $D$ is a spanning subgraph of $D$ which is out-tree (in-tree, respectively). It is well-known and easy to show [4,5] that a digraph has an out-branching (in-branching, respectively) rooted at $r$ if and only if $D$ has a unique initial strong connectivity component (terminal strong connectivity component, respectively) and $r$ belongs to this component. Out-branchings and in-branchings when they exist can be found in linear-time using, say, depth-first search from the root.

Edmonds [11] characterized digraphs with $k$ arc-disjoint out-branchings rooted at a specified vertex $r$. Furthermore, there exists a polynomial algorithm for finding $k$ arc-disjoint out-branchings with a given root $r$ if they exist [4]). However, it is NP-complete to decide whether a digraph $D$ has a pair of arc-disjoint out-branching and in-branching rooted at $r$, which was proved by Thomassen, see [1]. Following [9] we will call such a pair a good pair rooted at $r$. Note that a good pair forms a strong spanning subgraph of $D$ and thus if $D$ has a good pair, then $D$ is strong. The problem of the existence of a good pair was studied for tournaments and their generalizations, and characterizations (with proofs implying polynomial-time algorithms for finding such a pair) were obtained in [1] for tournaments, [7] for quasi-transitive digraphs and [9] for locally semicomplete digraphs. Also, Bang-Jensen and Huang [7] showed that if $r$ is adjacent to every vertex of $D$ (apart from itself) then $D$ has a good pair rooted at $r$.

In this paper, we study the existence of good pairs for digraph compositions. Let $T$ be a digraph with $t$ vertices $u_1, \ldots, u_t$ and let $H_1, \ldots, H_t$ be digraphs such that $H_i$ has vertices $u_{ij}, 1 \leq j_i \leq n_i$. Then the composition $Q = T[H_1, \ldots, H_t]$ is a digraph with vertex set \{u_{ij} | 1 \leq i \leq t, 1 \leq j \leq n_i\} and arc set

$$A(Q) = \bigcup_{i=1}^t A(H_i) \cup \{u_{ij}u_{pq} | u_iu_p \in A(T), 1 \leq j \leq n_i, 1 \leq q \leq n_p\}.$$

If $T$ is strongly connected, then $Q$ is called a strong composition and if $T$ is semicomplete, i.e., there is at least one arc between every pair of vertices, then $Q$ is called a semicomplete composition.

Digraph compositions generalize some families of digraphs. In particular, semicomplete compositions generalize strong quasi-transitive digraphs as every strong quasi-transitive digraph is a strong semicomplete composition in which $H_i$ is either a one-vertex digraph or a non-strong quasi-transitive digraph. To see that strong compositions form a significant generalization of strong quasi-transitive digraphs, observe that the Hamiltonian cycle problem is polynomial-time solvable for quasi-transitive digraphs [13], but NP-complete for strong compositions (see, e.g., [6]). When $H_i$ is the same digraph $H$ for every $i \in [t]$, $Q$ is the lexicographic product of $T$ and $H$, see, e.g., [15]. While digraph compositions has been used since 1990s to study quasi-transitive digraphs and their generalizations, see, e.g., [3,4,12], the study of digraph decompositions in their own right was initiated only recently in [16].
In the next section, we obtain the following somewhat surprising result: every strong digraph composition $Q$ in which $n_i \geq 2$ for every $i \in [t]$, has a good pair rooted at every vertex of $Q$. The condition of $n_i \geq 2$ in this result cannot be relaxed. Indeed, the characterization of quasi-transitive digraphs with a good pair [7] provides an infinite family of strong quasi-transitive digraphs which have no good pair rooted at some vertices. In Section 3, we characterize semicomplete compositions with a good pair generalizing the corresponding result in [7]. This allows us to decide in polynomial time whether a given semicomplete composition has a good pair rooted at a given vertex. In Section 4, we discuss some open problems and a recent related result.

Let $p$ and $q$ be integers. Then $[p..q] := \{p, p + 1, \ldots, q\}$ if $p \leq q$ and $\emptyset$, otherwise. In particular, if $p > 0$, $[p]$ will be a shorthand for $[1..p]$.

2 Compositions of digraphs: $T$ arbitrary

A digraph $D = (V, A)$ has a strong arc decomposition if $A$ has two disjoint sets $A_1$ and $A_2$ such that both $(V, A_1)$ and $(V, A_2)$ are strong. Sun et al. [16] obtained sufficient conditions for a digraph composition to have a strong arc decomposition. In particular, they proved the following:

**Theorem 2.1** Let $T$ be a digraph with $t$ vertices ($t \geq 2$) and let $H_1, \ldots, H_t$ be digraphs. Then $Q = T[H_1, \ldots, H_t]$ has a strong arc decomposition if $T$ has a Hamiltonian cycle and one of the following conditions holds:

- $t$ is even and $n_i \geq 2$ for every $i = 1, \ldots, t$;
- $t$ is odd, $n_i \geq 2$ for every $i = 1, \ldots, t$ and at least two distinct subgraphs $H_i$ have arcs;
- $t$ is odd and $n_i \geq 3$ for every $i = 1, \ldots, t$ apart from one $i$ for which $n_i \geq 2$.

**Lemma 2.2** Let $Q = T[H_1, \ldots, H_t]$, where $t \geq 2$. If $T$ has a Hamiltonian cycle and $H_1, \ldots, H_t$ are arbitrary digraphs, each with at least two vertices, then $Q$ has a good pair at any root $r$.

**Proof:** For the case that $t$ is even, by Theorem 2.1, $Q$ has has a pair of arc-disjoint strong spanning subgraphs $Q_1$ and $Q_2$. Observe that in $Q_1$ ($Q_2$, respectively), we can find an out-branching (in-branching, respectively) at $r$ (in polynomial time), as desired.

Now we assume that $t$ is odd. Without loss of generality, let $u_{1,1}$ be the root. Let $T'_1$ be the path $u_{1,1}u_{2,1} \ldots u_{i-1,1}u_{i,1}u_{i+1,1} \ldots u_{t,1}$, and let $T'_2$ be the in-tree rooted at $u_{1,1}$ with arc set $\{u_{i,2}u_{i+1,1} \mid 1 \leq i \leq t - 1\} \cup \{u_{i,1}u_{i+1,2} \mid 2 \leq i \leq t - 1\} \cup \{u_{i,1}u_{i+1,1}, u_{i,2}u_{i+1,1}\}$. By definition, $V(T'_1) = V(T'_2) = \{u_{i,j} \mid 1 \leq i \leq t, 1 \leq j \leq 2\}$. For any vertex $u_{i,j}$ with $1 \leq i \leq t$ and $j \geq 3$, we add the arcs $u_{i-1,1}u_{i,j}$ and $u_{i,j}u_{i+1,1}$ to $T'_1$ and $T'_2$, respectively. Note that here $u_{0,1} = u_{t,1}$ and $u_{t+1,1} = u_{1,1}$. Observe that the resulting two subgraphs
form a pair of out-branching and in-branching rooted at $u_{1,1}$, which are arc-disjoint.

We will use the following decomposition of strong digraphs. An ear decomposition of a digraph $D$ is a sequence $\mathcal{P} = (P_0, P_1, P_2, \ldots, P_t)$, where $P_0$ is a cycle or a vertex and each $P_i$ is a path, or a cycle with the following properties:

(a) $P_i$ and $P_j$ are arc-disjoint when $i \neq j$.

(b) For each $i \in [0, t]$, let $D_i$ denote the digraph with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$. If $P_i$ is a cycle, then it has precisely one vertex in common with $V(D_{i-1})$. Otherwise the end vertices of $P_i$ are distinct vertices of $V(D_{i-1})$ and no other vertex of $P_i$ belongs to $V(D_{i-1})$.

(c) $\bigcup_{i=0}^{t} A(P_j) = A(D)$.

The following result is well-known, see, e.g., [4].

**Theorem 2.3** Let $D$ be a digraph with at least two vertices. Then $D$ is strong if and only if it has an ear decomposition. Furthermore, if $D$ is strong, every cycle can be used as a starting cycle $P_0$ for an ear decomposition of $D$, and there is a linear-time algorithm to find such an ear decomposition.

**Lemma 2.4** Let $Q = T[\bar{K}_2, \ldots, \bar{K}_2]$, where $|V(T)| = t \geq 2$ and $\bar{K}_2$ is the digraph with two vertices and no arcs. If $T$ is strong, then $Q$ has a good pair at any root $r$.

**Proof:** Without loss of generality, let $r = u_{1,1}$. Since $T$ is strong, $u_1$ belongs to some cycle $C$ in $T$. By Theorem 2.3, $T$ has an ear decomposition $\mathcal{P} = (P_0, P_1, P_2, \ldots, P_p)$, such that $P_0 = C$ is the starting cycle. Let $T_i$ denote the subgraph of $T$ with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$.

We will prove the lemma by induction on $i \in \{0, 1, \ldots, p\}$. For the base step, by Lemma 2.2, the subgraph $P_0[\bar{K}_2, \ldots, \bar{K}_2]$ has a good pair rooted at $u_1$. For the inductive step, assume that $T_i[\bar{K}_2, \ldots, \bar{K}_2]$ has a pair of arc-disjoint out-branching $B_r^+$ and in-branching $B_r^-$ rooted at $r$. Without loss of generality, let $P_{i+1} = u_s u_{s+1} \ldots u_\ell$. The following argument will be divided into two cases according to whether $P_{i+1}$ is a cycle.

**Case 1:** $P_{i+1}$ is a cycle. In this case $u_s = u_\ell \in V(T_i)$. By Lemma 2.2, in the subgraph $P_{i+1}[\bar{K}_2, \ldots, \bar{K}_2]$, there is a pair of arc-disjoint out-branching $B_r'^+$ and in-branching $B_r'^-$ rooted at $u_{s,1}$. Let $B_r^+ = B_r'^+ \cup B_r'^-$ and $B_r^- = B_r'^+ \cup B_r'^-$. Observe that $B_r^+$ is an out-branching and $B_r^-$ is an in-branching rooted at $r$ in $T_{i+1}[\bar{K}_2, \ldots, \bar{K}_2]$. Since $P_{i+1}[\bar{K}_2, \ldots, \bar{K}_2]$ and $T_i[\bar{K}_2, \ldots, \bar{K}_2]$ are arc-disjoint, $B_r^+$ and $B_r^-$ are also arc-disjoint.

**Case 2:** $P_{i+1}$ is a path. In this case, $u_s, u_\ell \in V(T_i)$ and $s \neq \ell$. We just consider the case that $\ell - s \geq 2$ since the remaining case is trivial (no need to change the current pair of out- and in-branchings). Let $B_r^+$ be the union of $B_r'^+$ and the two paths $u_{s,i} u_{s+1,i} \ldots u_{\ell-1,i}$ where $1 \leq i \leq 2$. Let $B_r^-$ be the union of $B_r'^-$ and the two paths $u_{s,1} u_{s+1,2} u_{s+2,1} u_{s+3,2} \ldots u_{\ell,1}$ and
Let $Q = T[H_1, \ldots, H_t]$, where $t \geq 2$. If $T$ is strong and $H_1, \ldots, H_t$ are arbitrary digraphs, each with at least two vertices, then $Q$ has a good pair at any root $r$. Furthermore, this pair can be found in polynomial time.

Proof: Without loss of generality, let $r = u_{1,1}$. Let $Q'$ be the subgraph of $Q$ induced by the vertex set $\{u_{i,j} \mid 1 \leq i \leq t, 1 \leq j \leq 2\}$. In $Q'$ delete arcs between vertices $u_{i,1}$ and $u_{i,2}$ for every $i \in [t]$. By Lemma 2.4, $Q'$ contains a pair of arc-disjoint out-tree $T_1'$ and in-tree $T_2'$ rooted at $r$. By definition of out-tree, there is an arc $u_{p_i,q_i}u_{i,2}$ in $T_1'$ for every $i \in [t]$. For every $i \in [t]$ and $j \in [3..n_i]$, add $u_{p_i,q_i}u_{i,j}$ to $T_1'$. This results in an out-branching $T_1$. By definition of in-tree, there is an arc $u_{i,2}u_{q_i,b_i}$ in $T_2'$ for every $i \in [t]$. For every $i \in [t]$ and $j \in [3..n_i]$, add $u_{i,j}u_{q_i,b_i}$ to $T_2'$. This results in an in-branching $T_2$. Observe that $T_1$ and $T_2$ are arc-disjoint since $T_1'$ and $T_2'$ are arc-disjoint and the added arcs have heads and tails from $\{u_{i,j} \mid 1 \leq i \leq t, 3 \leq j \leq n_i\}$, respectively, in the arcs added to $T_1'$ and $T_2'$, respectively. Note that the proofs of Theorem 2.1, Lemmas 2.2 and 2.4, and this theorem are constructive and can be converted into polynomial-time algorithms. This fact and the polynomial-time algorithm of Theorem 2.3 imply that $T_1$ and $T_2$ can be constructed in polynomial time. □

3 Compositions of digraphs: $T$ semicomplete

We use $N^-(v)$ ($N^+(v)$, respectively) to denote the set of all in-neighbours (out-neighbours, respectively) of a vertex $v$ in a digraph $D$.

The next result was obtained by Bang-Jensen and Huang [7].

Theorem 3.1 Let $D$ be a strong digraph and $r$ a vertex of $D$ such that $V(D) = \{r\} \cup N^-(r) \cup N^+(r)$. There is a polynomial-time algorithm to decide whether $D$ has a good pair at $r$.

For a path $P = x_1x_2\ldots x_p$ and $1 \leq i \leq j \leq p$, let $P[x_i, x_j] := x_ix_{i+1}\ldots x_j$.

We now prove the following result on semicomplete compositions which generalizes a similar result for quasi-transitive digraphs by Bang-Jensen and Huang [7].

Theorem 3.2 A strong semicomplete composition $Q$ has a good pair rooted at $r$ if and only if $Q' = Q[\{r\} \cup N^-(r) \cup N^+(r)]$ has a good pair rooted at $r$.

Proof: Let $Q = T[H_1, \ldots, H_t]$ and $A = V(Q) \setminus V(Q')$. Without loss of generality, assume that $r \in V(H_1)$. By definitions of a semicomplete composition and $Q'$, we have $A = V(H_1) \setminus \{r\}$. 

5
Assume that $Q'$ has a good pair rooted at $r$, an out-branching $B_{r}^{+}$ and an in-branching $B_{r}^{-}$. Starting with $B_{r}^{+}$, we can construct an out-branching $B_{r}^{+}$ in $Q$ as follows. Let $v$ be a vertex such that $vr \in B_{r}^{-}$. Then add the arc $vu$ to $B_{r}^{+}$ for each $u \in A$. Similarly, starting with $B_{r}^{-}$, we could construct an in-branching $B_{r}^{-}$ in $Q$ as follows: for each $u \in A$, add the arc $uv'$ to $B_{r}^{-}$, where $vr' \in B_{r}^{+}$. Observe that $B_{r}^{+}$ and $B_{r}^{-}$ are arc-disjoint, as desired.

Now we prove the other direction. Assume that $Q$ has a good pair, an out-branching $B_{r}^{+}$ and an in-branching $B_{r}^{-}$, rooted at $r$. If $B_{r}^{+}[V(Q')]$ and $B_{r}^{-}[V(Q')]$ are branchings, then we are done. Otherwise, we will obtain an in-branching (out-branching, respectively) from $B_{r}^{-}$ ($B_{r}^{+}$, respectively) using the following procedure.

Choose a maximal path $P$ of $B_{r}^{-}$ to $r$, which contains a vertex $w \in A$, and assume that $w$ is furthest from $r$ among vertices in $A \cap V(P)$. If $w$ is the first vertex of $P$, then delete it. Otherwise, the previous vertex $u$ on $P$ has an arc $ur$ to $r$ (the arc $ur$ exists since $A \subseteq V(H_1)$), and we replace $P$ in $B_{r}^{-}$ by two paths: one is $P[p, u|r$, where $p$ is the first vertex of $P$, and the other is $P[w, r]$.

Note that the in-degree $d^{-}(w)$ of $w$ has decreased by one. Thus, after $d^{-}(w)$ such replacements the in-degree of $w$ becomes equal to zero, i.e., $w$ is the first vertex on its maximal path $Q$ to $r$ and therefore $w$ will be deleted when we consider $Q$. This means that after a finite number of replacements, we will delete all vertices of $A$ in $B_{r}^{-}$ and obtain an in-branching $B_{r}^{-}$ of $Q'$ rooted at $r$. Similarly, we can construct an out-branching $B_{r}^{+}$ of $Q'$. Note that to build $B_{r}^{-}$ we add only arcs to $r$ and to build $B_{r}^{+}$ we add only arcs from $r$. This fact and the fact that $B_{r}^{-}$ and $B_{r}^{+}$ are arc-disjoint, imply that $B_{r}^{-}$ and $B_{r}^{+}$ are arc-disjoint, too.

By Theorems 3.1 and 3.2, we immediately have the following:

**Theorem 3.3** Given a semicomplete composition and a vertex $r$, we can decide in polynomial time whether $D$ has a good pair rooted at $r$.

### 4 Open Problems and Related Results

Theorem 3.2 provides a characterization for the following problem for semicomplete compositions: given a digraph $D$ and a vertex $r \in V(D)$ decide whether $D$ has a good pair rooted at $r$. The theorem generalizes a similar characterization by Bang-Jensen and Huang [7] for quasi-transitive digraphs. Strong semicomplete compositions is not the only class of digraphs generalizing strong quasi-transitive digraphs. Other such classes have been studied such as $k$-quasi-transitive digraphs [12] and it would be interesting to see whether a characterization for the problem (or, at least non-trivial sufficient conditions) on $k$-quasi-transitive digraphs can be obtained. As we mentioned above, Bang-Jensen and Huang [9] obtained a characterization for the problem for locally semicomplete digraphs. It would be interesting to see whether a characterization for the problem on in-locally semicomplete digraphs [2,4] can be obtained.
An out-branching and in-branching $B^+_r$ and $B^-_r$ are called $k$-distinct if $B^+_r$ has at least $k$ arcs not present in $B^-_r$. The problem of deciding whether a digraph $D$ has a $k$-distinct pair of out- and in-branchings is NP-complete since it generalizes the good pair problem ($k = |V(D)| - 1$). Bang-Jensen and Yeo [10] asked whether the $k$-distinct problem is fixed-parameter tractable when parameterized by $k$, i.e., whether there is an $O(f(k)|V(D)|^{O(1)})$-time algorithm for solving the problem, where $f(k)$ is an arbitrary computable function in $k$ only. Gutin, Reidl and Wahlström [14] answered this open question in affirmative by designing an $O(2^{O(k \log^2 k)}|V(D)|^{O(1)})$-time algorithm for solving the problem.

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References


