Arc-disjoint in- and out-branchings rooted at the same vertex in compositions of digraphs

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Abstract

A digraph $D=(V,A)$ has a good pair at a vertex r if D has a pair of arc-disjoint in- and out-branchings rooted at r. Let T be a digraph with t vertices $u_1,\dots,u_t$ and let $H_1,\dots H_t$ be digraphs such that H_i has vertices $u_{i,j_i},\ 1\le j_i\le n_i$. Then the composition $Q=T[H_1,\dots,H_t]$ is a digraph with vertex set $\{u_{i,j_i}\mid 1\le i\le t, 1\le j_i\le n_i\}$ and arc set

$$A(Q)=\cup_{i=1}^t A(H_i)\cup\{u_{ij_i}u_{pq_p}\mid u_iu_p\in A(T), 1\le j_i\le n_i, 1\le q_p\le n_p\}.$$

If T is strongly connected, then Q is called a strong composition and if T is semicomplete, i.e., there is at least one arc between every pair of vertices, then Q is called a semicomplete composition.

We obtain the following result: every strong digraph composition Q in which $n_i\ge 2$ for every $1\le i\le t$, has a good pair at every vertex of Q. The condition of $n_i\ge 2$ in this result cannot be relaxed. We characterize semicomplete compositions with a good pair, which generalizes the corresponding characterization by Bang-Jensen and Huang (J. Graph Theory, 1995) for quasi-transitive digraphs. As a result, we can decide in polynomial time whether a given semicomplete composition has a good pair rooted at a given vertex.

Keywords: branching; semicomplete digraph; digraph composition.

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1 Introduction

We use a standard digraph terminology and notation as in [4, 5]. A digraph $D=(V,A)$ is *strongly connected* (or, just *strong*) if there exists a

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path from x to y and a path from y to x in D for every pair of distinct vertices x, y of D. An *out-tree* (*in-tree*, respectively) *rooted at a vertex* r is an orientation of a tree such that the in-degree (out-degree, respectively) of every vertex but r equals one. An *out-branching* (*in-branching*, respectively) in a digraph D is a spanning subgraph of D which is out-tree (in-tree, respectively). It is well-known and easy to show [4,5] that a digraph has an out-branching (in-branching, respectively) rooted at r if and only if D has a unique initial strong connectivity component (terminal strong connectivity component, respectively) and r belongs to this component. Out-branchings and in-branchings when they exist can be found in linear-time using, say, depth-first search from the root.

Edmonds [11] characterized digraphs with k arc-disjoint out-branchings rooted at a specified vertex r. Furthermore, there exists a polynomial algorithm for finding k arc-disjoint out-branchings with a given root r if they exist [4]). However, it is NP-complete to decide whether a digraph D has a pair of arc-disjoint out-branching and in-branching rooted at r, which was proved by Thomassen, see [1]. Following [9] we will call such a pair a *good pair rooted at* r. Note that a good pair forms a strong spanning subgraph of D and thus if D has a good pair, then D is strong. The problem of the existence of a good pair was studied for tournaments and their generalizations, and characterizations (with proofs implying polynomial-time algorithms for finding such a pair) were obtained in [1] for tournaments, [7] for quasi-transitive digraphs and [9] for locally semicomplete digraphs. Also, Bang-Jensen and Huang [7] showed that if r is adjacent to every vertex of D (apart from itself) then D has a good pair rooted at r.

In this paper, we study the existence of good pairs for digraph compositions. Let T be a digraph with t vertices $u_1, \dots, u_t$ and let $H_1, \dots H_t$ be digraphs such that H_i has vertices $u_{i,j_i},\ 1 \leq j_i \leq n_i$. Then the *composition* $Q = T[H_1, \dots, H_t]$ is a digraph with vertex set $\{u_{i,j_i} \mid 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$ and arc set

$$A(Q) = \cup_{i=1}^{t} A(H_i) \cup \{u_{ij_i}u_{pq_p} \mid u_iu_p \in A(T), 1 \leq j_i \leq n_i, 1 \leq q_p \leq n_p\}.$$

If T is strongly connected, then Q is called a *strong composition* and if T is *semicomplete*, i.e., there is at least one arc between every pair of vertices, then Q is called a *semicomplete composition*.

Digraph compositions generalize some families of digraphs. In particular, semicomplete compositions generalize strong quasi-transitive digraphs as every strong quasi-transitive digraph is a strong semicomplete composition in which H_i is either a one-vertex digraph or a non-strong quasi-transitive digraph. To see that strong compositions form a significant generalization of strong quasi-transitive digraphs, observe that the Hamiltonian cycle problem is polynomial-time solvable for quasi-transitive digraphs [13], but NP-complete for strong compositions (see, e.g., [6]). When H_i is the same digraph H for every $i \in [t]$, Q is the lexicographic product of T and H, see, e.g., [15]. While digraph compositions has been used since 1990s to study quasi-transitive digraphs and their generalizations, see, e.g., [3,4,12], the study of digraph decompositions in their own right was initiated only recently in [16].

In the next section, we obtain the following somewhat surprising result: every strong digraph composition Q in which $n_i \geq 2$ for every $i \in [t]$, has a good pair rooted at every vertex of Q. The condition of $n_i \geq 2$ in this result cannot be relaxed. Indeed, the characterization of quasi-transitive digraphs with a good pair [7] provides an infinite family of strong quasi-transitive digraphs which have no good pair rooted at some vertices. In Section 3, we characterize semicomplete compositions with a good pair generalizing the corresponding result in [7]. This allows us to decide in polynomial time whether a given semicomplete composition has a good pair rooted at a given vertex. In Section 4, we discuss some open problems and a recent related result.

Let p and q be integers. Then $[p..q] := \{p, p+1, \ldots, q\}$ if $p \leq q$ and $\emptyset$, otherwise. In particular, if $p > 0$, $[p]$ will be a shorthand for $[1..p]$.

2 Compositions of digraphs: T arbitrary

A digraph $D = (V, A)$ has a *strong arc decomposition* if A has two disjoint sets A_1 and A_2 such that both (V, A_1) and (V, A_2) are strong. Sun et al. [16] obtained sufficient conditions for a digraph composition to have a strong arc decomposition. In particular, they proved the following:

Theorem 2.1 *Let T be a digraph with t vertices ($t \geq 2$) and let $H_1, \ldots, H_t$ be digraphs. Then $Q = T[H_1, \ldots, H_t]$ has a strong arc decomposition if T has a Hamiltonian cycle and one of the following conditions holds:*

- *t is even and $n_i \geq 2$ for every $i = 1, \ldots, t$;*
- *t is odd, $n_i \geq 2$ for every $i = 1, \ldots, t$ and at least two distinct subgraphs H_i have arcs;*
- *t is odd and $n_i \geq 3$ for every $i = 1, \ldots, t$ apart from one i for which $n_i \geq 2$.*

Lemma 2.2 *Let $Q = T[H_1, \ldots, H_t]$, where $t \geq 2$. If T has a Hamiltonian cycle and $H_1, \ldots, H_t$ are arbitrary digraphs, each with at least two vertices, then Q has a good pair at any root r.*

Proof: For the case that t is even, by Theorem 2.1, Q has has a pair of arc-disjoint strong spanning subgraphs Q_1 and Q_2. Observe that in Q_1 (Q_2, respectively), we can find an out-branching (in-branching, respectively) at r (in polynomial time), as desired.

Now we assume that t is odd. Without loss of generality, let $u_{1,1}$ be the root. Let T'_1 be the path $u_{1,1}u_{2,1} \ldots u_{t,1}u_{1,2}u_{2,2} \ldots u_{t,2}$, and let T'_2 be the in-tree rooted at $u_{1,1}$ with arc set $\{u_{i,2}u_{i+1,1} \mid 1 \leq i \leq t-1\} \cup \{u_{i,1}u_{i+1,2} \mid 2 \leq i \leq t-1\} \cup \{u_{t,1}u_{1,1}, u_{t,2}u_{1,1}\}$. By definition, $V(T'_1) = V(T'_2) = \{u_{i,j} \mid 1 \leq i \leq t, 1 \leq j \leq 2\}$. For any vertex $u_{i,j}$ with $1 \leq i \leq t$ and $j \geq 3$, we add the arcs $u_{i-1,1}u_{i,j}$ and $u_{i,j}u_{i+1,1}$ to T'_1 and T'_2, respectively. Note that here $u_{0,1} = u_{t,1}$ and $u_{t+1,1} = u_{1,1}$. Observe that the resulting two subgraphs form a pair of out-branching and in-branching rooted at $u_{1,1}$, which are arc-disjoint. $\square$

We will use the following decomposition of strong digraphs. An *ear decomposition* of a digraph D is a sequence $\mathcal{P} = (P_0, P_1, P_2, \cdots, P_t)$, where P_0 is a cycle or a vertex and each P_i is a path, or a cycle with the following properties:
(a) P_i and P_j are arc-disjoint when $i \neq j$.
(b) For each $i \in [0..t]$, let D_i denote the digraph with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$. If P_i is a cycle, then it has precisely one vertex in common with $V(D_{i-1})$. Otherwise the end vertices of P_i are distinct vertices of $V(D_{i-1})$ and no other vertex of P_i belongs to $V(D_{i-1})$.
(c) $\bigcup_{j=0}^{t} A(P_j) = A(D)$.

The following result is well-known, see, e.g., [4].

Theorem 2.3 *Let D be a digraph with at least two vertices. Then D is strong if and only if it has an ear decomposition. Furthermore, if D is strong, every cycle can be used as a starting cycle P_0 for an ear decomposition of D, and there is a linear-time algorithm to find such an ear decomposition.*

Lemma 2.4 *Let $Q = T[\overline{K_2}, \dots, \overline{K_2}]$, where $|V(T)| = t \geq 2$ and $\overline{K_2}$ is the digraph with two vertices and no arcs. If T is strong, then Q has a good pair at any root r.*

Proof: Without loss of generality, let $r = u_{1,1}$. Since T is strong, u_1 belongs to some cycle C in T. By Theorem 2.3, T has an ear decomposition $\mathcal{P} = (P_0, P_1, P_2, \cdots, P_p)$, such that $P_0 = C$ is the starting cycle. Let T_i denote the subgraph of T with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$.

We will prove the lemma by induction on $i \in \{0, 1, \dots, p\}$. For the base step, by Lemma 2.2, the subgraph $P_0[\overline{K_2}, \dots, \overline{K_2}]$ has a good pair rooted at u_1. For the inductive step, assume that $T_i[\overline{K_2}, \dots, \overline{K_2}]$ has a pair of arc-disjoint out-branching $B_r'^+$ and in-branching $B_r'^-$ rooted at r. Without loss of generality, let $P_{i+1} = u_s u_{s+1} \dots u_\ell$. The following argument will be divided into two cases according to whether P_{i+1} is a cycle.

Case 1: P_{i+1} is a cycle. In this case $u_s = u_\ell \in V(T_i)$. By Lemma 2.2, in the subgraph $P_{i+1}[\overline{K_2}, \dots, \overline{K_2}]$, there is a pair of arc-disjoint out-branching $B_r''^+$ and in-branching $B_r''^-$ rooted at $u_{s,1}$. Let $B_r^+ = B_r'^+ \cup B_r''^+$ and $B_r^- = B_r'^- \cup B_r''^-$. Observe that B_r^+ is an out-branching and B_r^- is an in-branching rooted at r in $T_{i+1}[\overline{K_2}, \dots, \overline{K_2}]$. Since $P_{i+1}[\overline{K_2}, \dots, \overline{K_2}]$ and $T_i[\overline{K_2}, \dots, \overline{K_2}]$ are arc-disjoint, B_r^+ and B_r^- are also arc-disjoint.

Case 2: P_{i+1} is a path. In this case, $u_s, u_\ell \in V(T_i)$ and $s \neq \ell$. We just consider the case that $\ell - s \geq 2$ since the remaining case is trivial (no need to change the current pair of out- and in-branchings). Let B_r^+ be the union of $B_r'^+$ and the two paths $u_{s,i} u_{s+1,i} \dots u_{\ell-1,i}$ where $1 \leq i \leq 2$. Let B_r^- be the union of $B_r'^-$ and the two paths $u_{s,1} u_{s+1,2} u_{s+2,1} u_{s+3,2} \dots u_{\ell,i}$ and

$u_{s,2}u_{s+1,1}u_{s+2,2}u_{s+3,1}\dots u_{\ell,j}$, where $\{i,j\}=\{1,2\}$. Observe that B_r^+ is an out-branching and B_r^- is an in-branching rooted at r in $T_{i+1}[\overline{K_2},\dots,\overline{K_2}]$, moreover, they are arc-disjoint.

Thus, by induction, we are done. □

Theorem 2.5 *Let $Q=T[H_1,\dots,H_t]$, where $t\geq 2$. If T is strong and $H_1,\dots,H_t$ are arbitrary digraphs, each with at least two vertices, then Q has a good pair at any root r. Furthermore, this pair can be found in polynomial time.*

Proof: Without loss of generality, let $r=u_{1,1}$. Let Q' be the subgraph of Q induced by the vertex set $\{u_{i,j}\mid 1\leq i\leq t, 1\leq j\leq 2\}$. In Q' delete arcs between vertices $u_{i,1}$ and $u_{i,2}$ for every $i\in[t]$. By Lemma 2.4, Q' contains a pair of arc-disjoint out-tree T_1' and in-tree T_2' rooted at r. By definition of out-tree, there is an arc $u_{p_i,q_i}u_{i,2}$ in T_1' for every $i\in[t]$. For every $i\in[t]$ and $j\in[3..n_i]$, add $u_{p_i,q_i}u_{i,j}$ to T_1'. This results in an out-branching T_1. By definition of in-tree, there is an arc $u_{i,2}u_{a_i,b_i}$ in T_2' for every $i\in[t]$. For every $i\in[t]$ and $j\in[3..n_i]$, add $u_{i,j}u_{a_i,b_i}$ to T_2'. This results in an in-branching T_2. Observe that T_1 and T_2 are arc-disjoint since T_1' and T_2' are arc-disjoint and the added arcs have heads and tails from $\{u_{i,j}\mid 1\leq i\leq t, 3\leq j\leq n_i\}$, respectively, in the arcs added to T_1' and T_2', respectively. Note that the proofs of Theorem 2.1, Lemmas 2.2 and 2.4, and this theorem are constructive and can be converted into polynomial-time algorithms. This fact and the polynomial-time algorithm of Theorem 2.3 imply that T_1 and T_2 can be constructed in polynomial time. □

3 Compositions of digraphs: T semicomplete

We use $N^-(v)$ ($N^+(v)$, respectively) to denote the set of all in-neighbours (out-neighbours, respectively) of a vertex v in a digraph D.

The next result was obtained by Bang-Jensen and Huang [7].

Theorem 3.1 *Let D be a strong digraph and r a vertex of D such that $V(D)=\{r\}\cup N^-(r)\cup N^+(r)$. There is a polynomial-time algorithm to decide whether D has a good pair at r.*

For a path $P=x_1x_2\dots x_p$ and $1\leq i\leq j\leq p$, let $P[x_i,x_j]:=x_ix_{i+1}\dots x_j$. We now prove the following result on semicomplete compositions which generalizes a similar result for quasi-transitive digraphs by Bang-Jensen and Huang [7].

Theorem 3.2 *A strong semicomplete composition Q has a good pair rooted at r if and only if $Q'=Q[\{r\}\cup N^-(r)\cup N^+(r)]$ has a good pair rooted at r.*

Proof: Let $Q=T[H_1,\dots,H_t]$ and $A=V(Q)\setminus V(Q')$. Without loss of generality, assume that $r\in V(H_1)$. By definitions of a semicomplete composition and Q', we have $A=V(H_1)\setminus\{r\}$.

Assume that Q' has a good pair rooted at r, an out-branching $B_r'^+$ and an in-branching $B_r'^-$. Starting with $B_r'^+$, we can construct an out-branching B_r^+ in Q as follows. Let v be a vertex such that $vr \in B_r'^-$. Then add the arc vu to $B_r'^+$ for each $u \in A$. Similarly, starting with $B_r'^-$, we could construct an in-branching B_r^- in Q as follows: for each $u \in A$, add the arc uv' to $B_r'^-$, where $rv' \in B_r'^+$. Observe that B_r^+ and B_r^- are arc-disjoint, as desired.

Now we prove the other direction. Assume that Q has a good pair, an out-branching B_r^+ and an in-branching B_r^-, rooted at r. If $B_r^+[V(Q')]$ and $B_r^-[V(Q')]$ are branchings, then we are done. Otherwise, we will obtain an in-branching (out-branching, respectively) from B_r^- (B_r^+, respectively) using the following procedure.

Choose a maximal path P of B_r^- to r, which contains a vertex $w \in A$, and assume that w is furthest from r among vertices in $A \cap V(P)$. If w is the first vertex of P, then delete it. Otherwise, the previous vertex u on P has an arc ur to r (the arc ur exists since $A \subseteq V(H_1)$), and we replace P in B_r^- by two paths: one is $P[p,u]r$, where p is the first vertex of P, and the other is $P[w,r]$.

Note that the in-degree $d^-(w)$ of w has decreased by one. Thus, after $d^-(w)$ such replacements the in-degree of w becomes equal to zero, i.e., w is the first vertex on its maximal path Q to r and therefore w will be deleted when we consider Q. This means that after a finite number of replacements, we will delete all vertices of A in B_r^- and obtain an in-branching $B_r'^-$ of Q' rooted at r. Similarly, we can construct an out-branching $B_r'^+$ of Q'. Note that to build $B_r'^-$ we add only arcs to r and to build $B_r'^+$ we add only arcs from r. This fact and the fact that B_r^- and B_r^+ are arc-disjoint, imply that $B_r'^-$ and $B_r'^+$ are arc-disjoint, too.

□

By Theorems 3.1 and 3.2, we immediately have the following:

Theorem 3.3 *Given a semicomplete composition and a vertex r, we can decide in polynomial time whether D has a good pair rooted at r.*

4 Open Problems and Related Results

Theorem 3.2 provides a characterization for the following problem for semicomplete compositions: given a digraph D and a vertex $r \in V(D)$ decide whether D has a good pair rooted at r. The theorem generalizes a similar characterization by Bang-Jensen and Huang [7] for quasi-transitive digraphs. Strong semicomplete compositions is not the only class of digraphs generalizing strong quasi-transitive digraphs. Other such classes have been studied such as k-quasi-transitive digraphs [12] and it would be interesting to see whether a characterization for the problem (or, at least non-trivial sufficient conditions) on k-quasi-transitive digraphs can be obtained. As we mentioned above, Bang-Jensen and Huang [9] obtained a characterization for the problem for locally semicomplete digraphs. It would be interesting to see whether a characterization for the problem on in-locally semicomplete digraphs [2,4] can be obtained.

An out-branching and in-branching B_r^+ and B_r^- are called *k-distinct* if B_r^+ has at least k arcs not present in B_r^-. The problem of deciding whether a digraph D has a k-distinct pair of out- and in-branchings is NP-complete since it generalizes the good pair problem ($k = |V(D)|-1$). Bang-Jensen and Yeo [10] asked whether the k-distinct problem is fixed-parameter tractable when parameterized by k, i.e., whether there is an $O(f(k)|V(D)|^{O(1)})$-time algorithm for solving the problem, where $f(k)$ is an arbitrary computable function in k only. Gutin, Reidl and Wahlström [14] answered this open question in affirmative by designing an $O(2^{O(k\log^2 k)}|V(D)|^{O(1)})$-time algorithm for solving the problem.

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