Intertwined Superfluidity and Density Wave order in a p-Orbital Bose Condensate

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We study a continuum model of the weakly interacting Bose gas in the presence of an external field with minima forming a triangular lattice. The second lowest band of the single-particle spectrum (p-band) has three minima at non-zero momenta. We consider a metastable Bose condensate at these momenta and find that, in the presence of interactions that vary slowly over the lattice spacing, the order parameter space is isomorphic to S^5. We show that the enlarged symmetry leads to the loss of topologically stable vortices, as well as two extra gapless modes with quadratic dispersion. The former feature implies that this non-Abelian condensate is a "failed superfluid" that does not undergo a Berezinskii-Kosterlitz-Thouless (BKT) transition. Order-by-disorder splitting appears suppressed, implying that signatures of the S^5 manifold ought to be observable at low temperatures.

The search for novel quantum states of matter continues to be a central theme in condensed matter physics. The conventional Bose superfluid spontaneously breaks a continuous U(1) symmetry in the global phase of the condensate wavefunction, giving rise to quantized vortices as a signature of the system. Motivated by recent experiments on two completely different platforms, we investigate the possibility of intertwining superfluid order with density wave order as a result of additional degeneracies in the spatial structure of a Bose-Einstein condensate (BEC). We present in this Letter a simple theory of an intertwined state that has a superfluid stiffness, but has a non-Abelian order parameter manifold that lacks the topological protection to support superflow at non-zero temperatures.

Our first experimental motivation comes from torsional oscillator experiments on ^{4}He bilayers on graphite suggesting that a 2D superfluid in a periodic potential may exhibit an unconventional quantum phase [1], notably characterized by the lack of a BKT transition [2] and a linear temperature dependence of the normal fraction (*c.f.* cubic behavior in the conventional superfluid). The system is close to an incommensurate solid phase. An order parameter was postulated [1] with intertwined superfluidity and crystallinity such that the global phase and translational degrees of freedom are no longer independent. This order parameter exists on an enlarged symmetry manifold. The BKT transition would be eliminated since global phase vortices would not be topologically protected, while unconventional Goldstone modes could be the source of an enhanced normal fraction.

Secondly, there has also been a series of remarkable experiments [3] that succeeded in creating a spatially modulated atomic BEC in an optical lattice that arose from the spontaneous occupation of photon cavity modes in two intersecting cavities. A new U(1) Goldstone mode was observed associated with a degenerate set of density wave patterns. This again raises the tantalizing possibility of intertwined density wave order and superfluidity.

In this Letter, we study a simple bosonic Hamiltonian to understand such an intertwined quantum state. We will see that two ingredients are required. We need degenerate single-particle states with different spatial structures to form an enlarged order parameter space. To preserve this degeneracy, the interaction between particles has to be smooth and long-ranged. We consider bosons in 2D with mass m in an external potential: $U(\mathbf{r}) = 2U\sum_{j=1}^{3}\sin^2(\mathbf{G}_j\cdot\mathbf{r}/2)$ where $\mathbf{G}_j = G(\cos\theta_j, \sin\theta_j)$ with $\theta_j = \pi(j-1)/3$ $(j = 1, ..., 6)$ are the reciprocal lattice vectors. We take $U > 0$ which gives potential minima on a triangular lattice with lattice constant $a = 4\pi/\sqrt{3}G$. The Hamiltonian is

$$H = \sum_{\mathbf{k}}(\epsilon_{\mathbf{k}} - \mu)\, b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - \frac{U}{2}\sum_{j,\mathbf{k}}\left(b^\dagger_{\mathbf{k}+\mathbf{G}_j} b_{\mathbf{k}} + \text{h.c.}\right) + \frac{1}{2L^2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V_{\mathbf{q}} b^\dagger_{\mathbf{k}-\mathbf{q}} b^\dagger_{\mathbf{k}'+\mathbf{q}} b_{\mathbf{k}'} b_{\mathbf{k}} \quad (1)$$

where $\epsilon_{\mathbf{k}} = \hbar^2k^2/2m$, μ is the chemical potential, $V_{\mathbf{q}}$ is the Fourier transform of the interaction potential, L is the linear size of the system and $b_{\mathbf{k}}$ is the bosonic annihilation operator.

For the degeneracy requirement, we exploit the p-band of this system which has degenerate minima in the first Brillouin zone (see below). We note that p-band BECs can be realized experimentally with ultracold atoms [4, 5] with novel features due to symmetries from "internal" degrees of freedom, *e.g.* orbitals or spins [6, 7]. Our proposal is different in that we make use of degeneracies in spatial structure rather than internal symmetries.

We will now discuss the requirement on the form of the interactions. Theoretical studies of ultracold atoms typically employ a zero-range contact interaction to model s-wave scattering because the range of the interaction is short compared to the wavelength of the condensate. In this work, we are interested in a condensate with spatial modulation commensurate with a wavevector of magnitude $G/2$. We will study interactions $V_{\mathbf{q}}$ that are smooth over a length scale R that is long compared to $1/G$, such that there are momentum transfers of the order of G are suppressed. For our calculations below, we use the simple mathematical form: $V_{\mathbf{q}} = V_0 \exp(-q^2R^2)$ with $GR \gg 1$. We note that smooth interactions can be realized in dipolar atoms/molecules [8] by tuning a Feshbach resonance so that the contact repulsion is cancelled by the dipolar attraction at short range. The cooling of dipolar atoms is challenging but an Er BEC in an optical lattice has been achieved recently [9].

Let us examine now the single-particle band structure ($V_{\mathbf{q}} = 0$) of this system. The lowest band has the lowest energy at the Γ point. We will focus on the next lowest band which corresponds in the tight-binding limit to Bloch states formed by p_x and p_y orbitals in each well of the external potential. This p-band has three degenerate local minima (due to rotational symmetry) with energy ϵ_M and crystal momenta $\mathbf{Q}_i \equiv \mathbf{G}_i/2$ $(i = 1, 2, 3)$ at the three M points ($M_{1,2,3}$) of the first Brillouin zone of the triangular lattice (Fig. 1c). The annihilation operators for these Bloch states, $B_{1,2,3}$, can be written as a superposition of plane-wave states connected by reciprocal lattice vectors $\mathbf{G} = p_1\mathbf{G}_1 + p_2\mathbf{G}_2$ for integer $p_{1,2}$: $B_i = \sum_{\mathbf{G}} d_{\mathbf{G},i} b_{\mathbf{Q}_i+\mathbf{G}}$. From these three states, we can construct a degenerate set of single-particle states $(c_1B_1^\dagger + c_2B_2^\dagger + c_3B_3^\dagger)\,|\mathrm{vac}\rangle$ with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$, each of which can be represented as a point on the S^5 surface in $\mathbb{R}^6$. Depending on the magnitude and phases of the c_i's, these degenerate states have quite different density profiles, generically breaking completely the point-group symmetries of the external potential. We will study the condensation of bosons into these states. Two of the more symmetric cases are plotted in Fig. 1.

We now turn to the effect of interactions. One would expect short-ranged interactions to select Bose condensation into a unique member of this manifold [10]. In this work, we aim to explore the novel properties of a Bose-condensed system where the single-particle degeneracy is preserved even in the presence of interactions, by using an interaction that is smooth on the scale of the lattice spacing a. To be more precise, we consider the system in a coherent state [11] of the form [1, 12–14]:

$$|\Psi\rangle = e^{-N/2} \exp\left(\sqrt{N}\sum_{i=1}^{3} c_i B_i^\dagger\right)|\mathrm{vac}\rangle \qquad (2)$$

where N is the total number of particles in the system set by the chemical potential μ. The mean field energy for this ansatz is of the form $\langle\Psi|H|\Psi\rangle = Nu_{\mathrm{MF}}$:

$$\begin{aligned} u_{\mathrm{MF}} =& \frac{\tilde{V}_0\bar{n}}{2} + \frac{\tilde{V}_2\bar{n}}{4}\sum_{i=1}^{3}|c_i|^4 \\ &+ \frac{\tilde{V}_1\bar{n}}{4}\sum_{i=1}^{3}\left[2|c_i|^2|c_{i+1}|^2 + (c_i^{*2}c_{i+1}^2 + \mathrm{c.c.})\right] \end{aligned} \qquad (3)$$

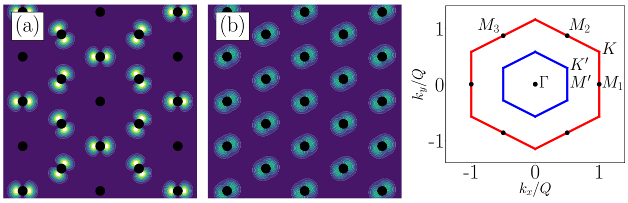

FIG. 1. (a) Density profile for $(c_1, c_2, c_3) = (1, 1, 1)/\sqrt{3}$ has point-group symmetry of honeycomb lattice. States with equivalent densities (up to a translation) can be constructed by changing the sign of one of the amplitudes, leading to a $\mathbb{Z}_4$ symmetry; (b) Density profile of $(c_1, c_2, c_3) = (1, i, 0)/\sqrt{2}$ has a C_{2v} point group. Black dots: potential minima. (c) Brillouin zone of the external potential (red) with reciprocal lattice vectors $\mathbf{G}_i$. The p-band minima occur at three inequivalent M points at $\mathbf{Q}_i = \mathbf{G}_i/2$. Upon condensation at the M points, the spatial periodicity of the system doubles, and excitations have a reduced Brillouin zone (blue).

where the addition in the i-index is modulo 3, $\bar{n} = N/L^2$ is the average boson number and $\tilde{V}_0 \simeq V_0$, $\tilde{V}_1 \simeq V_Q + V_{\sqrt{3}Q}$ and $\tilde{V}_2 \simeq V_{2Q}$ for $U \ll \epsilon_Q$ or $QR \gg 1$. $\tilde{V}_1$ controls intervalley scattering of particles from one M point to another (Fig. 2 left) and is the only term allowed by the conservation of crystal momentum. The key issue arising from interactions is this: generically the interactions, $\tilde{V}_{1,2}$ will fix the relative weights and phases for the amplitudes c_i in the coherent state (2), leading to a reduction of the degenerate manifold from S^5 to the conventional U(1) manifold. Indeed, for $\tilde{V}_1 \geq \tilde{V}_2 > 0$, Liu and Wu [7, 10] showed that the system favors $c_1 = \pm ic_2$, $c_3 = 0$ (and permutations) which has a U(1) symmetry in the global phase as well as discrete symmetries: $\mathbb{Z}_2$ for time reversal and $\mathbb{Z}_3$ for the choice of the empty state (Fig. 1b).

In the spirit of preserving the degeneracy of the S^5 manifold (2), we specialize to a spatially smooth interaction such that $V_{q\geq Q} = 0$, *i.e.* $\tilde{V}_1 = 0$ and $\tilde{V}_2 = 0$. Then intervalley processes are absent and the degeneracy on the S^5 manifold is not lifted by interactions at the mean field level. The mean field energy does not depend on the relative phases of the amplitudes c_i due to separate number conservation at each M point at this level. We will from now on focus on this degenerate scenario. (A similar situation applies to the spontaneous optical lattice experiments [3] where the additional U(1) symmetry is only approximate in the presence of s-wave scattering.)

We discuss now the emergent SU(3) symmetry arising from the degeneracy of the S^5 manifold (2). The state

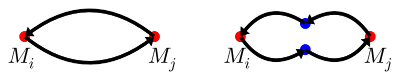

FIG. 2. Direct exchange (left) and superexchange (right) between condensed particles at momenta M_i and M_j. Direct exchange is absent if $V_Q = 0$.

$(c_1 B_1^\dagger + c_2 B_2^\dagger + c_3 B_3^\dagger)\,|\text{vac}\rangle$ is a Schwinger boson formulation of the fundamental representation of SU(3). The unitary transformation that connects any two states in the manifold can be written as an SU(3) transformation with the generators given by $\Lambda^a = \sum_{i,j=1,2,3} B_i^\dagger \lambda^a_{i,j} B_j/2$, where λ^a are the eight Gell-Mann matrices as defined in [15]. Any state in (2) can be generated by an SU(3) rotation on the highest-weight state $(c_1, c_2, c_3) = (0, 0, 1)$. The manifold is isomorphic to $\mathrm{SU}(3)/\mathrm{SU}(2) \sim S^5$ [16].

A non-Abelian symmetry manifold is interesting in the context of superfluid BECs. Conventional 2D superfluids exhibit a topological (BKT) phase transition due to the unbinding of U(1) phase vortices, separating a normal phase at high temperature and a low-temperature phase with quasi-long-range order. However, the first homotopy group for our S^5 manifold is trivial: $\pi_1(S^5) = 0$, meaning that any closed loop in the manifold of S^5 may be continuously shrunk to a point. This means that phase vortices are not topologically stable. Explicitly, we can destroy the phase of the amplitude at M_i by a trajectory on the S^5 manifold that takes the coherent state through a region where $c_i = 0$. In addition, $\pi_2(S^5)$ is also trivial and so there are no topologically stable defects for our coherent states in two dimensions. In the absence of topological protected phases, the Mermin-Wagner theorem [17] implies that a 2D S^5-degenerate condensate is a rare example of an interacting Bose system without superfluidity in the thermodynamic limit at any nonzero temperature. In this sense, a condensate with non-Abelian symmetry generators may be viewed as a "failed superfluid" in two dimensions.

To be more quantitative, we will now compute the excitation spectrum of the condensate and investigate the implications for the system. We focus on long-wavelength fluctuations ($|\mathbf{k}| \ll Q$) around a coherent state on the manifold (2). This corresponds to introducing spatial variations, $c_i \to c_i(\mathbf{r}) = \sqrt{n_i(\mathbf{r})}\exp[i\theta_i(\mathbf{r})]$, that are smooth on the scale of the lattice spacing. The Lagrangian density for these fluctuations is given by [18]

$$L = \bar{n}\sum_{j=1}^{3} c_i^* \left(i\hbar\partial_t - \epsilon_{j\hat{\mathbf{k}}}\right) c_j - \bar{n} u_{\mathrm{MF}}[\{c_j(\mathbf{r})\}] \qquad (4)$$

where $\epsilon_{j\mathbf{k}}$ ($j = 1, 2, 3$) is the energy of p-band Bloch state near the M_j point with crystal momentum $\mathbf{Q}_j + \mathbf{k}$, and $\epsilon_{j\hat{\mathbf{k}}}$ is obtained by replacing $\mathbf{k}$ with $\hat{\mathbf{k}} = -i\hbar\nabla$. We study small fluctuations and retain only terms quadratic in the fluctuations. If we denote $B^\dagger_{j\mathbf{k}}$ as the creation operator for the Bloch state, this is equivalent to Bogoliubov theory which approximates $B_{j,\mathbf{k}=0} \to \sqrt{N} c_j$ in the Hamiltonian and keeps only terms quadratic in $B_{j\mathbf{k}}$.

In this p-band model, we obtain three bands of Bogoliubov quasiparticles with energy $E_{\mu\mathbf{k}}$ and crystal momentum $\hbar\mathbf{k}$ ($\mu = 1, 2, 3$). We have also verified this long-wavelength model by a numerical Bogoliubov calculation not restricted to the p-band. Unlike the mean field energy, the Bogoliubov spectrum does depend on the relative phases of the amplitudes c_i even if $V_Q = 0$. This is because particles at different M points can be coupled by intervalley 'superexchange' involving an intermediate state with two uncondensed particles (Fig. 2 right). This process rules out a fragmented condensate [11].

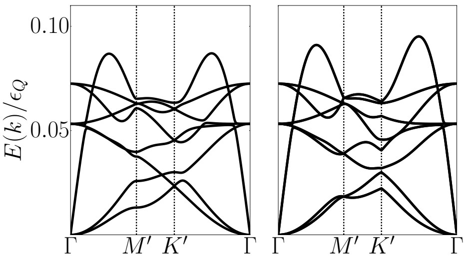

FIG. 3. The eight lowest, positive-energy modes in the Bogoliubov spectrum for the state $(c_1, c_2, c_3) = (1, 1, 1)/\sqrt{3}$ (left) $(1, i, 0)/\sqrt{2}$ (right) with $1/QR = 0.3$, $\bar{n}V_0 = \epsilon_Q$, $U = 6\epsilon_Q$. The spectrum converges using a $m_c = 271$ plane-wave basis.

Figure 3 shows the lowest, positive excitation energies for fluctuations around two coherent states: the symmetric state $(c_1, c_2, c_3) = (1, 1, 1)/\sqrt{3}$ and the Liu-Wu state $(1, i, 0)/\sqrt{2}$. For both states, we find a gapless mode with a linear dispersion. If we approximate the single-particle dispersion around the M points as isotropic with band mass m^*, $E_{\mu=1,\mathbf{k}} \simeq (\tilde{V}_0\bar{n}/m^*)^{1/2}\,\hbar k$ for $k \ll Q$ and $\tilde{V}_0 \gg \tilde{V}_{1,2}$ [18]. This corresponds to fluctuations in the overall density, analogous to the usual U(1) superfluid Goldstone mode arising from global phase invariance. The corresponding phase variable is $\delta\Phi = \sqrt{\bar{n}}\sum_i |c_i|^2\delta\theta_i$ where $\delta\theta_i$ is the fluctuations in the phase of c_i. This variable is not quantized because $0 \le |c_i| \le 1$, consistent with the lack of topological defects in the system.

For our S^5 degenerate scenario ($\tilde{V}_1 = \tilde{V}_2 = 0$), there are also two quadratic modes, $E_{\mu=2,3,\mathbf{k}}$. Their $k \to 0$ eigenvectors correspond to excitations to p-band states orthogonal to the condensed single-particle state. Such excitations are analogous to "phasons": a continuous internal rearrangement of a crystal [19] sampling various density configurations (Fig. 1). We note that spin waves in ferromagnets are also quadratic modes and this dispersion is associated with the order parameter being a good quantum number of the Hamiltonian. The analogous conservation law in our system (4) is the conservation of the number of particles in the two single-particle states orthogonal to the condensed state. More mathematically, by adapting the analysis for the Watanabe-Brauner counting rules [20, 21], we can show that quadratic modes exist if the expectation values of the commutators of the non-Abelian generators, $\{\Lambda^a\}$, do not vanish for states within the manifold. This connection with the enlarged symmetry of the manifold is consistent with these modes acquiring energy gaps of $(2\tilde{V}_1 - \tilde{V}_2)\bar{n}/4$ and $(\tilde{V}_1\tilde{V}_2/2)^{1/2}\bar{n}$

if $\tilde{V}_{1,2} \neq 0$ (for the $(1, i, 0)$ state). (We address the influence of a small gap later.)

We also find Bogoliubov eigenstates with negative energies corresponding to s-band states. Metastability against scattering into the s-band has been addressed [6, 7] and a metastable p-band atomic BEC has been achieved [4, 5]. This is however is difficult to realize in helium films due to inelastic scattering with the substrate.

The S^5 symmetry is emergent meaning that the generators described above do not commute with the Hamiltonian but only do so in the expectation value of the macroscopic coherent state. Such emergent symmetries are typically broken by the "order-by-disorder" mechanism [22–24] which reduces the symmetry by picking the state that minimizes the quantum zero-point energy $E_{\rm zp} = \frac{1}{2}\sum_{\mu\mathbf{k}}(E_{\mu\mathbf{k}} - E^{(0)}_{\mu\mathbf{k}})$ where $E^{(0)}_{\mu\mathbf{k}}$ are the non-interacting band energies. In systems where order-by-disorder is typically important, this quantity is on the order of ϵ_Q [24]. We have evaluated the zero-point energy for the parameters considered in Fig. 3. We find that the order-by-disorder mechanism favors the symmetrically condensed state, $(c_1, c_2, c_3) \propto (1, 1, 1)$ and its three other degenerate counterparts: $(-1, 1, 1)$ and permutations. In other words, this reduces the degeneracy on the S^5 manifold to a $\mathrm{U}(1)\otimes\mathbb{Z}_4$ symmetry. These states have a zero-point energy per particle of $E_{\rm zp}/N \simeq 4 \times 10^{-2}\epsilon_Q/n_{\rm cell}$ where $n_{\rm cell} = \sqrt{3}\bar{n}a^2/2$ is the number of particles per unit cell. (The p-band has a bandwidth of $\sim 10^{-1}\epsilon_Q$ for these parameters.) However, the range of zero-point energies over the whole manifold is only 1% of this quantity: $\Delta E_{\rm zp}/N \sim 10^{-4}\epsilon_Q/n_{\rm cell}$, with the $(1, i, 0)$ state having the highest zero-point energy. We believe that this surprisingly small range can be related to the small matrix elements for the intervalley superexchange contribution to the zero-point energy (Fig. 2 right). This involves intermediate states produced by momentum transfers of the order of $Q/2$ from a condensed wavevector M, and we see numerically that the splitting scales approximately as $V^2_{Q/2}/V_0$ for $V_0 \gg V_{Q/2}$ $(= 6 \times 10^{-2}V_0$ in Fig. 3). Such small energy differences between the coherent states in the S^5 manifold means that they should all be accessible at low temperatures.

To establish the stability of the S^5-symmetric condensate, we have calculated the condensate depletion, defined as the fraction of bosons not in the condensate. This diverges with the system size L as $T\log[L/l(T)]$ where $l(T) \sim 1/T$ is a lengthscale beyond which fluctuations destroy the condensate, similar to a conventional 2D U(1) condensate. We also calculated the normal fluid density ρ_n which exhibits viscosity due to thermally excited quasiparticles [25]. This quantity exhibits anomalous behaviour — a simple calculation (ignoring interactions between the Bogoliubov quasiparticles) shows that $\rho_n \sim T\log[L/l(T)]$. This is quite different from the T^3 behaviour for a conventional 2D superfluid but is intriguingly reminiscent of helium on graphite [1]. This new behavior can be traced to the quadratic modes that are absent from U(1) superfluid. (See [18] for details.)

We return now to the issue of the small energy gap due to a small V_Q: $E_{\rm g} \simeq 10^{-5}\epsilon_Q$ for parameters in Fig. 3. This reflects the anisotropy on the S^5 manifold of coherent states, reducing the symmetry to U(1). Scaling theory [26] suggests that this restores the BKT phase at a critical temperature suppressed from a simple U(1) condensate by an order of magnitude [27] $\sim 1/\log(\epsilon_Q/E_{\rm g}) \sim 10^{-1}$. Moreover, the U(1) vortex size $\sim (\epsilon_Q/E_{\rm g})^{-1/2}a \sim 300a$ is large so that optical lattices smaller than this size will not see the vortex-unbinding transition. This leaves us scope to explore the non-Abelian condensate as a failed superfluid.

In this work, we have proposed a non-Abelian condensate with spatial density modulations. Can this be a candidate for a "supersolid" phase that spontaneously breaks both translational and global gauge symmetries? The condensation at non-zero momenta may be induced by certain two-body interaction potentials with negative Fourier components at the ordering wavevector. Such condensation may occur even in the absence of an applied field by creating a roton instability [12, 13]. Such a system is generically an Abelian condensate with decoupled Bogoliubov and phonon modes. (In the context of our S^5 manifold, the interactions creating the roton instability will generically determine all the relative weights and phases of the amplitudes c_i, other than the ones responsible for these U(1) modes.) Therefore, one expects to see a BKT transition in contrast to [1]. Nevertheless, we can show [28] that non-Abelian condensates can be local minima in mean-field theory for special fine-tuned Hamiltonians. These states, however, appear to be dynamically unstable in general, as evidenced by imaginary eigenvalues in their Bogoliubov spectra. A notable exception arises if the single-particle spectrum deviates from the typical quadratic kinetic energy dispersion (*e.g.* due to band structure, or internal degrees of freedom) where we have found a condensate with SU(2) symmetry in addition to the U(1) translational symmetries. Such a setup will be the focus of future studies.

In summary, we have proposed a scenario for a condensate with non-Abelian features, such as a lack of a BKT transition and additional gapless "phason" modes. We believe this is the first example of such a condensate that exploits spatial structure instead of additional internal degrees of freedom. By leveraging the single-particle degeneracy of the p-band, we study an interaction that does not spoil the SU(3) symmetry of the system at the mean-field level. We find a "failed superfluid" in two dimensions. Our scenario is not confined to a triangular lattice and is anticipated to generalize to degenerate higher band condensates in *e.g.* square, hexagonal lattices and in three dimensions. Intriguingly, our failed superfluid shares similar low-temperature behavior and a lack of a BKT transition with the ^{4}He bilayer on graphite [1].

Nevertheless, a complete theory for this helium system that motivated our story remains elusive.

SL is supported by the Imperial College President's Scholarship. PC is supported by the US Department of Energy, Office of Basic Energy Sciences grant DE-FG02-99ER45790. Since the submission of this work, an SO(3) generalization of the spontaneous optical lattice experiment has been proposed [29].

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Supplemental Material 1: Dipolar interactions

In this section, we review the work of Fischer [8] which demonstrated how a finite-range interaction can be realized for dipolar bosons confined to a cloud in the xy-plane.

Consider dipolar bosons of mass m with dipole moment d_e, polarized by a strong electric field in the z-direction. The interaction between two bosons at a (three-dimensional) displacement of $\mathbf{r}$ consists of two components. Firstly, there is a contact interaction parametrized by an s-wave scattering length a_s or an interaction strength $g_{3\mathrm{D}} = 4\pi\hbar^2 a_s/m$. There is also a dipole-dipole interaction of the form $V_{dd}(\mathbf{r}) = (3g_d/4\pi r^3)(1-3z^2/r^2)$ with $g_d = d_e^2/3\epsilon_0$. This is repulsive when $x^2+y^2 \gg z^2$ and attractive when $x^2+y^2 \ll z^2$ (when the dipoles are nearly collinear in the z-direction).

When these bosons are confined by harmonic trap to a Gaussian wavepacket of width d_z in the z-direction, the Fourier transform of the effective interaction in the 2D plane can be written as

$$V_{\mathbf{q}} = \frac{g_d}{d_z}\left[\frac{1+g_{3\mathrm{D}}/2g_d}{\sqrt{\pi/2}} - \frac{3}{2}qd_z w\left(\frac{qd_z}{\sqrt{2}}\right)\right] , \qquad \text{with} \qquad w(x) = e^{x^2}\mathrm{erfc}(x) , \tag{S1}$$

where $\mathbf{q}$ is the 2D wavevector of the Fourier transform. Fischer [8] proposed that the contact interaction strength $g_{3\mathrm{D}}$ can be tuned to be equal to g_d so that $V_{\mathbf{q}} \to 0$ as $q \to \infty$. Thus, the short-range contributions from the dipolar interaction and contact interaction cancel each other, producing an interaction with lengthscale d_z. For this study, we want this lengthscale to be large compared to the wavelength $\sim 1/Q$ of the density modulations of our coherent state (2). This corresponds to the condition that confinement in the z-direction must be larger than $\sqrt{3}a$ where a is the length of the triangular lattice vector. This suppresses intervalley processes that break the S^5 symmetry. In summary, we impose two conditions on the interaction to observe the S^5 symmetry

$$g_d = g_{3\mathrm{D}} , \qquad d_z > \sqrt{3}a , \tag{S2}$$

which can be achieved by using a Feshbach resonance and by adjusting the out-of-plane confinement of the trap. We plot the resulting Bogoliubov spectrum in Fig. S1.

We should recall that the bosons are regarded as quasi-2D because they are condensed in the ground state of the confinement potential in the z-direction. The next lowest state is higher in energy by $\hbar^2/md_z^2$. We should not allow d_z to be so large that $\hbar^2/md_z^2$ becomes smaller than the bandwidth of the p-band for motion in the xy-plane. This can be achieved without violating the condition (S2) in a deep optical lattice where the bandwidth is only a fraction of ϵ_Q. For instance, $U = 6\epsilon_Q$ (used in Fig. 3), gives a bandwidth of the order of $0.05\epsilon_Q$. Thus, there is a window of d_z where the system remains quasi-two-dimensional while the interparticle interaction is smooth over the lattice spacing.

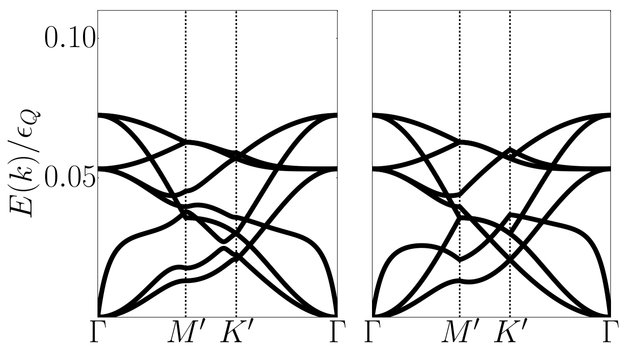

FIG. S1. The eight lowest, positive-energy modes in the Bogoliubov spectrum for the state $c_1 = c_2 = c_3$ (left) and $c_1 = 1, c_2 = c_3 = 0$ (right). Parameters: $g_d = g_{3\mathrm{D}}$, $d_z = 7.5/Q$, $3\sqrt{2/\pi}\, g_d\bar{n}/2d_z = \epsilon_Q$, $U = 6\epsilon_Q$.

Supplemental Material 2: Symmetry generators and counting rule

In the main text, we found that as long as the condition $V_{k\geq Q} = 0$ is obeyed, the coherent state is degenerate in mean field theory on the manifold

$$|\Psi\rangle \sim \exp\left[\sqrt{N}\left(c_1 B_1^\dagger + c_2 B_2^\dagger + c_3 B_3^\dagger\right)\right]|\mathrm{vac}\rangle , \qquad |c_1|^2 + |c_2|^2 + |c_3|^2 = 1 \tag{S3}$$

where $B^\dagger_{1,2,3}$ are the creation operators of the three Bloch states atat the M points of the first Brillouin zone. If we project the Hamiltonian onto this set of states, we can write

$$PHP = \epsilon_M \sum_i B_i^\dagger B_i + \frac{V_0}{2}\left(\sum_i B_i^\dagger B_i\right)^2 \tag{S4}$$

The degeneracy arises from the fact that

$$\langle\Psi|[B_i^\dagger B_j, H]|\Psi\rangle = 0 \tag{S5}$$

for all the coherent states defined above. Physically, this corresponds to the fact the mean-field energy is the same for any arbitrary occupation of the three Bloch states with the same total boson number. We can take linear combinations of these commutators to show that $\langle\Psi|[\Lambda^a, H]|\Psi\rangle = 0$ where $\Lambda^a = \sum_{i,j=1,2,3} B_i^\dagger \lambda_{ij}^{(a)} B_j/2$ and $\lambda^{(a)}$ are the Gell-Mann matrices as

$$\lambda^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^{(2)} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^{(4)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda^{(5)} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^{(6)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^{(7)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^{(8)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{S6}$$

There is an SU(3) symmetry in the sense that any two coherent states on this manifold can be connected by a SU(3) unitary transformation which can be parametrized by $\exp\left[i\sum_{j=1}^{8}\theta_a\Lambda^a\right]$. This manifold may be parameterized via an SU(3) rotation on the highest-weight state using the generators according to:

$$|\Psi\rangle = \exp\left[i\sum_{a=4}^{8}\theta_a\Lambda^a\right]\exp\left[\sqrt{N}B_3^\dagger\right]|\text{vac}\rangle \tag{S7}$$

Note that rotations involving the generators $\Lambda^{1,2,3}$ on the highest-weight state leaves it invariant and therefore the entire manifold may be parameterized parsimoniously using only five generators: $\Lambda^{4,\cdots,8}$, *i.e.* rotations involving the generators which form an $SU(2)$ subgroup do not alter the highest-weight state.

We discuss the Watanabe-Brauner counting rule [20] for the Nambu-Goldstone modes, which was generalized by Takahaski and Nitta to apply to emergent symmetries in the Bogoliubov framework [21]. The Bogoliubov Hamiltonian which we consider has a term which is linear in k and thus deviates from the conditions under which the counting rule was derived by these authors; nevertheless we can show that the rule is respected. We summarize this algorithm. Suppose we have n broken symmetry generators $P_1, \ldots, P_n$. Define a matrix: $\Gamma_{ij} = \langle\Psi|[P_i, P_j]|\Psi\rangle$. The number of modes with even dispersion in k are given by $n_{\text{even}} = \frac{1}{2}\text{rank}(\Gamma)$ and the number of odd modes are $n_{\text{odd}} = n - 2n_{\text{even}}$. Using the operators $\Lambda^{4,\cdots,8}$ as the symmetry generators, we find that $n_{\text{odd}} = 1, n_{\text{even}} = 2$, agreeing with our perturbative analysis and numerical simulations.

Supplemental Material 3: Excitations

In this section, we outline two methods to study the excitation spectrum. The first method applies the Bogoliubov approximation to the full Hamiltonian and obtains numerical results for the excitations for multiple bands in the whole Brillouin zone. The second method describes an effective theory for long-wavelength fluctuations in the p-band of the system. This is useful in understanding the Goldstone modes of the system.

Numerical Bogoliubov Calculation

Consider first non-interacting bosons in the triangular potential $U(\mathbf{r})$ [see (1)] consisting of Fourier components at the reciprocal lattice vectors $\pm\mathbf{G}_{1,2,3}$. For our numerical work, we work in the plane-wave (Fourier) basis. The eigenstates are Bloch states. A Bloch state in the band γ is created by the creation operator

$$(B_{\mathbf{q}}^{(\gamma)})^\dagger = \sum_{\mathbf{G}} d_{\mathbf{G},\mathbf{q}}^{(\gamma)} b_{\mathbf{G}+\mathbf{q}}^\dagger\,. \tag{S8}$$

where $\mathbf{G} = p_1\mathbf{G}_1 + p_2\mathbf{G}_2$ with integer $p_{1,2}$ are the reciprocal lattice vectors, $b^\dagger_{\mathbf{p}}$ creates a plane-wave state at wavevector $\mathbf{p}$, and $\mathbf{q}$ is restricted to the first Brillouin zone of the non-interacting problem (red hexagon in Fig. S2). In other words, it is a superposition of plane waves with wavevectors separated by reciprocal lattice vectors $\mathbf{G}$. There are two p-bands, the lower one of which has energy minima at the $M_{1,2,3}$ points of Brillouin zone corresponding to $\mathbf{q} = \mathbf{Q}_{1,2,3} = \mathbf{G}_{1,2,3}/2$. We will from now on refer to this lower band as "the p-band".

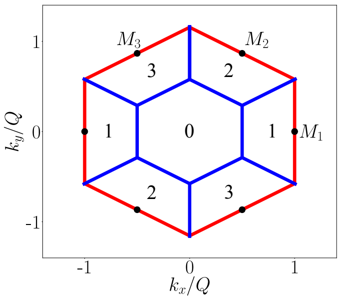

FIG. S2. The Brillouin zone of the external potential (red). The p-band minima occur at the three inequivalent M points. Upon condensation at the M points, the periodicity of the system doubles in real space, and excitations have a reduced Brillouin zone (blue). The larger (red) Brillouin zone is now divided into four smaller Brillouin zones. They can be labelled $m = 0$ (centered at the Γ point), and $m = 1, 2, 3$ centered at $M_{1,2,3}$ (up to translation a reciprocal lattice vector $\mathbf{G}_m$).

We are concerned with the excitation spectrum after the particles have Bose-condensed into the Bloch states at the three M points. The spatial modulation of the condensate has Fourier components at integer multiples of $\mathbf{Q}_{1,2,3}$. This means that the Brillouin zone for the excitations is halved in each direction (Fig. S2). So, it is more convenient to label states in four reduced Brillouin zones ($m = 0, 1, 2, 3$, shown in Fig. S2). Let us also divide all the plane-wave states into reduced Brillouin zones centered at $\mathbf{Q}_m = p_1\mathbf{Q}_1 + p_2\mathbf{Q}_2$ for some integers $p_{1,2}$. (The numerical calculation cuts off the basis at $m = m_c$.) Let us denote the creation operator for a free particle with wavevector $\mathbf{Q}_m + \mathbf{k}$ and energy $\hbar^2(\mathbf{Q}_m + \mathbf{k})^2/2m$ as $b^\dagger_{m,\mathbf{k}} \equiv b^\dagger_{\mathbf{Q}_m+\mathbf{k}}$ where $\mathbf{k}$ is restricted to the first Brillouin zone (central $m = 0$ blue hexagon in Fig. S2) of the reduced Brillouin zones. In this work, we will focus on condensation into the p-band which has energy minima at the $M_{1,2,3}$ points (Fig. S2). The condensate creation operator can be written as:

$$c_1 B^\dagger_{1,\mathbf{k}=0} + c_2 B^\dagger_{2,\mathbf{k}=0} + c_3 B^\dagger_{3,\mathbf{k}=0} = \sum_m \alpha_m b^\dagger_{m,\mathbf{k}=0} \tag{S9}$$

where $B^\dagger_{j,\mathbf{k}}$ creates a Bloch state with crystal momentum $\mathbf{k}$ near the M_j point. (For each j, $B^\dagger_{j,\mathbf{k}=0}$ superposes a set of plane waves at wavevectors $\mathbf{G}_m + \mathbf{Q}_j = \mathbf{G}_m + \mathbf{G}_j/2$. The three sets of plane waves for the three different j's are disjoint and they span the set of plane waves at all the reciprocal lattice vectors of the reduced Brillouin zone, $\mathbf{Q}_m$.)

To construct the Bogoliubov Hamiltonian, H_{Bog}, we make the shift in the microscopic Hamiltonian (1) using $b_{m,0} \to \sqrt{N}\alpha_m$. The Bogoliubov approximation keeps only terms quadratic in α_m. These terms are quadratic in the boson operators and can be written in the Nambu form:

$$H_{\mathrm{Bog}} = \frac{1}{2}\sum_{\mathbf{k}\in\mathrm{BZ}} \mathbf{b}^\dagger_{\mathbf{k}} H_{\mathbf{k}} \mathbf{b}_{\mathbf{k}} - \frac{1}{2}\sum_{m\mathbf{k}} \epsilon_{\mathbf{k}+\mathbf{Q}_m} \tag{S10}$$

where $\mathbf{b}_{\mathbf{k}} = (b_{0,\mathbf{k}}, ..., b_{m_c,\mathbf{k}}, b^\dagger_{0,-\mathbf{k}}, ..., b^\dagger_{m_c,-\mathbf{k}})^T$. The Nambu form contains kinetic energy terms of the form $bb^\dagger$ but the original Hamiltonian only has terms of the normal ordered form $b^\dagger b$. The constant term above has been inserted to subtract out an unwanted constant from this rearrangement.

The eigenenergies and eigenvectors of the Bogoliubov quasiparticles are obtained by solving the equation $H_{\mathbf{k}}\mathbf{b}_{\mathbf{k}} = E_{\mathbf{k}}\sigma_3\mathbf{b}_{\mathbf{k}}$ where $\sigma_3 = \mathrm{diag}(1, 1, 1, \ldots, -1, -1, -1, \ldots)$ is a block-diagonal $2m_c \times 2m_c$ matrix. This is equivalent to a diagonalization using the Bogoliubov transformation $\mathbf{b}_{\mathbf{k}} = T_{\mathbf{k}}\boldsymbol{\beta}_{\mathbf{k}}$ with

$$T_{\mathbf{k}} = \begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}} \\ v^*_{-\mathbf{k}} & u^*_{-\mathbf{k}} \end{pmatrix}, \quad u_{\mathbf{k}}u^\dagger_{\mathbf{k}} - v_{\mathbf{k}}v^\dagger_{\mathbf{k}} = \mathbf{1} \tag{S11}$$

giving us a diagonal form of the quadratic Hamiltonian

$$H_{\mathrm{Bog}} = \sum_{\mu\mathbf{k}} E_{\mu\mathbf{k}}\beta^\dagger_{\mu\mathbf{k}}\beta_{\mu\mathbf{k}} + \frac{1}{2}\sum_{\mu\mathbf{k}}\left(E_{\mu\mathbf{k}} - \epsilon_{\mathbf{Q}_\mu+\mathbf{k}}\right). \tag{S12}$$

Single Band Effective Theory

Here, we outline a minimal effective theory to describe the long-wavelength excitations of the p-band condensate. We will use a number-phase representation of the condensate which reveals the physical content of these excitations. (For this section, we have set $\hbar = k_B = 1$.)

Let $B^\dagger_{i,\mathbf{k}}$ be the creation operator for a Bloch state with crystal momentum $\mathbf{Q}_i + \mathbf{k}$ near the M_i point with energy $\epsilon_{i\mathbf{k}}$ ($i = 1, 2, 3$). We will provide analytic results for the simplified case when the dispersion relation is isotropic $\epsilon_{i\mathbf{k}} = \epsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/2m^*$. This calculation is easily generalized to the actual anisotropic dispersion but the analytic results are cumbersome.

We study fluctuations around the coherent state (2): $|\Psi\rangle = e^{-N/2}\exp[\sqrt{N}(c_1 B^\dagger_{1,\mathbf{k}=0} + c_2 B^\dagger_{2,\mathbf{k}=0} + c_3 B^\dagger_{3,\mathbf{k}=0})]\,|\mathrm{vac}\rangle$. In the number-phase representation,

$$c_j(\mathbf{r}) = \sqrt{n_j(\mathbf{r})}e^{i\theta_j(\mathbf{r})}\,, \tag{S13}$$

this coherent state has a mean-field energy (3) density per unit area of

$$\bar{n}u_{\mathrm{MF}} = \frac{V_0}{2}(n_1+n_2+n_3)^2 + \frac{\tilde{V}_1}{2}\sum_{j=1}^{3} n_j n_{j+1}\left[1+\cos(2\theta_j - 2\theta_{j+1})\right] + \frac{\tilde{V}_2}{4}\sum_{j=1}^{3} n_j^2\,. \tag{S14}$$

where $\bar{n}$ is the mean-field number density, and the addition in the j-index is modulo 3. We concentrate on long-wavelength fluctuations ($|\mathbf{k}| \ll Q$) in the amplitudes

$$\sqrt{N}c_j \to \sqrt{N}c_j(\mathbf{r}) = \sqrt{\frac{N}{L^2}}\sum_{\mathbf{k}}^{k\ll Q} c_{j,\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{S15}$$

where L^2 is the area of the system. To be more precise, we consider states of the form

$$\begin{gathered}|\{c_j(\mathbf{r})\}\rangle = e^{-N/2}\exp\left[\sqrt{N}\int c_j(\mathbf{r})\psi_j^\dagger(\mathbf{r})\,d^2\mathbf{r}\right]|\mathrm{vac}\rangle = \exp\left[\sqrt{N}\sum_{j,\mathbf{k}} c_{j\mathbf{k}}B^\dagger_{j,\mathbf{k}}\right]|\mathrm{vac}\rangle\ , \\ c_j(\mathbf{r}) = \frac{1}{L}\sum_{j,\mathbf{k}} c_{j,\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}\,,\quad \psi_j^\dagger(\mathbf{r}) = \frac{1}{L}\sum_{\mathbf{k}} B^\dagger_{j,\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}\end{gathered} \tag{S16}$$

where $c_{j,\mathbf{k}}$ is not small only for $k \ll Q$ and $\psi_j(\mathbf{r})$ is the field operator projected onto the Bloch states in the p-band around the M_j point. The Lagrangian density $\mathcal{L}$ for the long-wavelength fluctuations can be written as

$$\mathcal{L} = \bar{n}\sum_{j=1}^{3} c_j^*\left(i\hbar\partial_t - \epsilon_{j\hat{\mathbf{k}}}\right)c_j - \bar{n}u_{\mathrm{MF}}[\{c_j(\mathbf{r})\}] \tag{S17}$$

where $\epsilon_{j\hat{\mathbf{k}}}$ is obtained from the single-particle band energies $\epsilon_{j\mathbf{k}}$ by replacing $\mathbf{k} \to \hat{\mathbf{k}} = -i\hbar\nabla$.

For small fluctuations in the density and phase, $n_j(\mathbf{r}) = \bar{n}_j + \delta n_j$ and $\theta_j = \bar{\theta}_j + \delta\theta_j$, we write $c_j \simeq \sqrt{\bar{n}_j}\exp(i\bar{\theta}_j)(1 + \delta n_j/2\bar{n}_j + i\delta\theta_j)$ where $\sqrt{\bar{n}_j}\exp(i\bar{\theta}_j)$ is the mean field value of c_j that minimizes u_{MF}. Then, we expand $\mathcal{L}$ and collect the terms quadratic in $\delta\theta_j$ and δn_j. Consider first the SU(3) symmetric Hamiltonian with $\tilde{V}_{1,2} = 0$ with the S^5 manifold of degenerate coherent states described by any (c_1, c_2, c_3) with $\bar{n} = \bar{n}_1 + \bar{n}_2 + \bar{n}_3$ fixed. The quadratic fluctuations are described by the Lagrangian density:

$$\delta\mathcal{L}_{S^5} = \sum_{i=1}^{3}\left[-\delta n_i\partial_t\delta\theta_i - \frac{1}{2m^*}\left(\bar{n}_i|\nabla\delta\theta_i|^2 + \frac{1}{4\bar{n}_i}|\nabla\delta n_i|^2\right)\right] - \frac{V_0}{2}(\delta n_1 + \delta n_2 + \delta n_3)^2\,. \tag{S18}$$

Note that δn_i and $\delta\theta_i$ are canonically conjugate variables. Three pairs of natural canonical conjugates for this problem are

$$\nu_i = \sum_j P_{ij}\frac{\delta n_j}{\sqrt{\bar{n}_j}}\,,\quad \phi_i = \sum_j P_{ij}\sqrt{\bar{n}_j}\delta\theta_j\,,\quad P = \begin{pmatrix} \sqrt{\bar{n}_1/\bar{n}} & \sqrt{\bar{n}_2/\bar{n}} & \sqrt{\bar{n}_3/\bar{n}} \\ -\sqrt{\bar{n}_2/(\bar{n}-\bar{n}_3)} & \sqrt{\bar{n}_1/(\bar{n}-\bar{n}_3)} & 0 \\ -\sqrt{\bar{n}_1\bar{n}_3/\bar{n}(\bar{n}-\bar{n}_3)} & -\sqrt{\bar{n}_2\bar{n}_3/\bar{n}(\bar{n}-\bar{n}_3)} & \sqrt{(\bar{n}-\bar{n}_3)/\bar{n}} \end{pmatrix} \tag{S19}$$

Using these canonical variables, we can write

$$\delta L_{S^5} = \sum_{i=1}^{3} \left[-\nu_i \partial_t \phi_i - \frac{1}{2m^*} \left(|\nabla \phi_i|^2 + \frac{1}{4} |\nabla \nu_i|^2 \right) \right] - \frac{V_0 \bar{n}}{2} \nu_1^2 \tag{S20}$$

The spectrum for this system can be easily extracted by comparing this with the Langrangian for a simple harmonic oscillator with frequency ω: $L_{\text{SHO}} = p\dot{q} - p^2/2m - m\omega^2 q^2/2$. We find three gapless modes. Mode 1 has the dispersion relation $E_{1\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2V_0\bar{n})}$ which is linear in the wavevector k for small k. This corresponds to overall density fluctuations $\delta n = \sqrt{\bar{n}}\nu_1 = \delta n_1 + \delta n_2 + \delta n_3$. The conjugate phase variable is $\delta\Phi = \phi_1/\sqrt{\bar{n}} = \bar{n}_1\delta\theta_1 + \bar{n}_2\delta\theta_2 + \bar{n}_3\delta\theta_3$. The two other modes are degenerate and simply have the non-interacting dispersion $E_{2\mathbf{k}} = E_{3\mathbf{k}} = \epsilon_{\mathbf{k}}$. In second-quantized form, the annihilation operators for the three modes can be written as

$$a_{j\mathbf{k}} = l_{j\mathbf{k}} \phi_{j\mathbf{k}} - \frac{i}{2 l_{j\mathbf{k}}} \nu_{j\mathbf{k}} \,, \quad l_{1\mathbf{k}}^2 = \left(\frac{\epsilon_{\mathbf{k}}}{2V_0\bar{n} + \epsilon_{\mathbf{k}}} \right)^{1/2} = \frac{\epsilon_{\mathbf{k}}}{E_1(\mathbf{k})} \,, \qquad l_{2/3,\mathbf{k}} = 1 \,. \tag{S21}$$

We can show that the existence of a linear mode and two quadratic modes is robust when we restore the anisotropy in the band energies around the three M points. The quadratic modes become non-degenerate and their energies do depend on the interaction strength V_0. Moreover, the spectrum becomes dependent on the choice of the mean-field coherent state.

The $\tilde{V}_{1,2}$ interaction terms break the S^5 symmetry. The ground state is $(c_1, c_2, c_3) = (1, \pm i, 0)/\sqrt{2}$ with a U(1) symmetry for the overall phase. Fluctuations around this state can be described by the number and phase fluctuations at the two condensed amplitudes $c_{1,2}$ and a decoupled single-particle Hamiltonian for fluctuations around $c_3 = 0$.

$$\begin{aligned} \delta\mathcal{L}_2 = &- \left[\nu_1 \partial_t \phi_1 - \phi_1 \epsilon_{\hat{\mathbf{k}}} \phi_1 + \frac{1}{8} \nu_1 \left(4V_0\bar{n} + \tilde{V}_2 \bar{n} + 2\epsilon_{\hat{\mathbf{k}}} \right) \nu_1 \right] \\ &- \left[\nu_2 \partial_t \phi_2 + \phi_2 \left(\tilde{V}_1 \bar{n} + \epsilon_{\hat{\mathbf{k}}} \right) \phi_2 + \frac{1}{8} \nu_2 \left(\tilde{V}_2 \bar{n} + 2\epsilon_{\hat{\mathbf{k}}} \right) \nu_2 \right] , \end{aligned} \tag{S22}$$

$$\delta\mathcal{L}_3 = \bar{n} c_3^* \left(i\partial_t - \epsilon_{\hat{\mathbf{k}}} - \frac{\tilde{V}_1 \bar{n}}{2} + \mu - V_0 \bar{n} \right) c_3 \tag{S23}$$

with the chemical potential $\mu = 2\bar{n}u_{\text{MF}} = (V_0 - \tilde{V}_2/4)\bar{n}$. Mode 1 for overall density fluctuations remains linear. The quadratic modes from the SU(3)-symmetric case now have energy gaps of $(\tilde{V}_1\tilde{V}_2/2)^{1/2}\bar{n}$ and $(2\tilde{V}_1 - \tilde{V}_2)\bar{n}/4$.

Condensate depletion

The condensate depletion Δ is defined as the fraction of particles with momenta different from the ones in the coherent state (2).

$$\Delta = \frac{1}{N} \sum_{j,\mathbf{k}\neq 0} \langle c_{j,\mathbf{k}}^\dagger c_{j,\mathbf{k}} \rangle \tag{S24}$$

At the level of our approximation of small fluctuations

$$\Delta \simeq \frac{1}{N} \sum_{j,\mathbf{k}\neq 0} \left\langle \left(\frac{\nu_{j,\mathbf{k}}}{2} - i\phi_{j,\mathbf{k}} \right) \left(\frac{\nu_{j,-\mathbf{k}}}{2} + i\phi_{j,-\mathbf{k}} \right) \right\rangle = \frac{1}{2\bar{n}} \sum_j \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left[\frac{1}{2} \left(l_{j\mathbf{k}}^2 + l_{j\mathbf{k}}^{-2} - 2 \right) + \frac{l_{j\mathbf{k}}^2 + l_{j\mathbf{k}}^{-2}}{e^{E_{j\mathbf{k}}/T} - 1} \right] . \tag{S25}$$

The first term is the depletion at zero temperature. It is finite in 2D. The second term arises from the thermal excitation of quasiparticles. Both linear and quadratic modes contribute terms that scale as $T\log(LT)$.

Superfluid density

The local current density is given by $\mathbf{J} = N\sum_i c_i^* \nabla_{\hat{\mathbf{k}}} \epsilon_{\hat{\mathbf{k}}} c_i$. For an isotropic quadratic dispersion around the M points, this gives $\mathbf{J} = -iN\sum_i (c_i^* \nabla_{\mathbf{r}} c_i - c_i \nabla_{\mathbf{r}} c_i^*)/2m^*$. If we confine our attention to slow spatial variations only, the current is given in the number-phase representation by

$$\mathbf{J} \simeq \frac{1}{m} \sum_i n_i \nabla\theta_i \simeq \frac{1}{m^*} \sum_i \left(\bar{n}_i \nabla\delta\theta_i + \delta n_i \nabla\delta\theta_i \right) \tag{S26}$$

The first term involves excitations of a single quasiparticle while the latter involves two quasiparticles. The first is longitudinal and therefore does not contribute to the normal fluid response. The second term is diagonal in the index i and remains so after the orthogonal basis transformation (S19). Its Fourier transform is

$$\mathbf{J}_{\perp\mathbf{q}} \simeq \frac{1}{m^*}\sum_i \int \nu_i(\nabla\phi_i)_{\perp\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{d^2\mathbf{r}}{L^2} = \frac{i}{m^*}\sum_{i\mathbf{k}} \mathbf{k}_{\perp\mathbf{q}}\,\nu_{i,\mathbf{q}-\mathbf{k}}\phi_{i\mathbf{k}} = \frac{1}{2m^*}\sum_{i\mathbf{k}} \frac{l_{i,\mathbf{q}-\mathbf{k}}}{l_{i\mathbf{k}}}\mathbf{k}_{\perp\mathbf{q}}\,(a_{i\mathbf{q}-\mathbf{k}} - a^\dagger_{i\mathbf{k}-\mathbf{q}})(a_{i\mathbf{k}} + a^\dagger_{i-\mathbf{k}})\,. \tag{S27}$$

The normal fluid density is given by [25]

$$\rho_n = \frac{2}{L^2 Z}\lim_{\mathbf{q}\to 0}\sum_{\nu\nu'} \frac{e^{-\mathcal{E}_\nu/T}|\langle\nu'|J_{\perp\mathbf{q}}|\nu\rangle|^2}{\mathcal{E}_{\nu'}-\mathcal{E}_\nu}\,, \tag{S28}$$

where ν labels eigenstates with energies $\mathcal{E}_\nu$ of the gas of Bogoliubov excitations with partition function Z and $J_{\perp\mathbf{q}}$ is the component of the current operator transverse to $\mathbf{q}$. Inserting the quasiparticle spectrum gives

$$\rho_n = \frac{2}{m^{*2}L^2}\lim_{\mathbf{q}\to 0}\sum_{i\mathbf{k}}\left(\frac{\mathbf{k}_{\perp\mathbf{q}} l_{i,\mathbf{q}-\mathbf{k}}}{l_{i\mathbf{k}}}\right)^2\left[\frac{N_{i,\mathbf{q}-\mathbf{k}}+N_{i\mathbf{k}}}{E_{i,\mathbf{q}-\mathbf{k}}+E_{i\mathbf{k}}} - \frac{N_{i,\mathbf{q}-\mathbf{k}}-N_{i\mathbf{k}}}{E_{i,\mathbf{q}-\mathbf{k}}-E_{i\mathbf{k}}}\right] \tag{S29}$$

where $N_{i\mathbf{k}}$ is the Bose occupation number of the eigenstate with energy $E_{i,\mathbf{k}}$. The contribution from long-wavelength fluctuations at a fixed low temperature T can be obtained by noting that $N_{i\mathbf{k}} \simeq T/E_{i\mathbf{k}}$ (equipartition) and summing only up to $E \sim T$.

$$\rho_n \simeq \frac{4T}{m^{*2}L^2}\lim_{\mathbf{q}\to 0}\sum_{i\mathbf{k}}^{E_{i\mathbf{k}}<T}\left(\frac{\mathbf{k}_{\perp\mathbf{q}}}{E_{i\mathbf{k}}}\right)^2 \tag{S30}$$

The contribution from the linear mode gives a dependence of T^3 while the quadratic mode gives $T\log(LT)$.

When the anisotropy of the dispersion around the M points is included in the calculation, these temperature dependences are robust. There is also a reduction of the superfluid fraction from unity at zero temperature, as expected on general grounds due to the loss of Galilean invariance.

Supplemental Material 4: Suppression of the BKT Transition

In this section, we estimate the temperature scale at which the breaking of the symmetry of the S^5 degenerate manifold due to a small non-zero interaction V_Q which couples bosons at two M points by a momentum transfer of Q. This reduces the symmetry of the degenerate manifold to U(1).

We borrow from Nelson and Pelcovits [26] and Fellows *et al* [26] and consider the O(M+2) non-linear sigma model with a small anisotropic term, defined by the $(M+2)\times(M+2)$ matrix D, that breaks the symmetry to an O(2) model. This is described by the energy density:

$$\mathcal{H} = \frac{J}{2}(\nabla\mathbf{n})^2 + \frac{J_\perp}{2a^2}\mathbf{n}^{\mathrm{T}}D\mathbf{n} \tag{S31}$$

where $\mathbf{n}$ is a unit vector on the S^{M+1}-sphere, $J_\perp$ is a dimensionless measure of the anisotropy and $a = 2\pi/\sqrt{3}Q$ is the lattice spacing. In the absence of the anistropy, the Mermin-Wagner theorem states that the system is disordered in the thermodynamic limit in two dimensions at any non-zero temperature. For a non-zero $J_\perp \ll 1$, a BKT transition occurs at a critical temperature $T_c \simeq J/\ln(J/J_\perp)$. In the opposite limit of large $J_\perp$, this is equivalent to the O(2) model which has a critical temperature of $T_{\mathrm{BKT}} \simeq \pi J/2$.

We should also note that the anisotropy gives rise to topologically stable vortices. The size of these vortices diverge as $\xi \sim a/\sqrt{J/J_\perp}$ as $J_\perp \to 0$.

Our system cannot be mapped directly onto an O(M+2) model. However, we believe that we can use these results to estimate the effect of anistropy. We estimate that $J \sim \hbar^2\bar{n}/2m^*$ where m^* is the effective mass of the single-particle dispersion relation around the M points. Since the anisotropy arises from intervalley exchange, the anistropy energy per unit area is controlled by $\bar{n}^2\tilde{V}_1 \sim \bar{n}E_{\mathrm{g}}$. We estimate $J_\perp/a^2 \sim \bar{n}E_{\mathrm{g}}$. This gives $J_\perp/J \sim 8\pi^2 m^* E_{\mathrm{g}}/\hbar^2 = (4\pi^2/3)(m^*/m)(E_{\mathrm{g}}/\epsilon_Q)$.

For Fig. 3 where $U = 6\epsilon_Q$, $V_0 = \bar{n}\epsilon_Q$ and $\tilde{V}_1 \simeq 10^{-5}V_0$, we find $m^*/m \sim 0.2$ this gives $J_\perp/J \sim 10^{-5}$. So, the BKT transition temperature is suppressed by a factor of $1/\ln(J_\perp/J) \sim 10^{-1}$ for an infinite system. This will be observable for systems larger than the vortex size $\xi \sim 300a$.