Random Serial Dictatorship: The One and Only.

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Abstract

Fix a Pareto optimal, strategy proof, and non-bossy deterministic matching mechanism and define a random matching mechanism by assigning agents to the roles in the mechanism via a uniform lottery. Given a profile of preferences, the lottery over outcomes that arises under the random matching mechanism is identical to the lottery that arises under random serial dictatorship, where the order of dictators is uniformly distributed. This result extends the celebrated equivalence between the core from random endowments and random serial dictatorship to the grand set of all Pareto optimal, strategy proof and non-bossy matching mechanisms.

1 Introduction

A matching problem consists of a finite set of agents, a finite set of indivisible objects, henceforth called houses, and a profile of all agents' preferences

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over all houses. A matching is a maximal set of agent-house pairs (with no agent or house taking part in more than one pair). Each agent's preference over matchings only depends on the houses he is matched with. Each agent strictly ranks any two houses and considers any house better than being homeless. Mechanisms map preference profiles to matchings, with serial dictatorship and Gale's top trading cycles (GTTC) the two most prominent examples. Serial dictatorship matches one agent - the first dictator - to his most preferred house among all houses. A next agent - the second dictator - is matched with his most preferred remaining house, and so forth. A GTTC starts with an initial matching called the endowment and requires that there are equally many agents and houses. In a first round each agent points to his most preferred house and each house points to the agent it was endowed to. At least one pointing cycle forms. Each agent in such a cycle is matched with the house he points to. As long as some agents remain unmatched this procedure is repeated with all unmatched agents and houses.

Serial dictatorship and GTTC satisfy three central properties. Both are *Pareto optimal*: they map each profile of preferences to a matching that is Pareto optimal at that profile. Both are *strategy proof*: at no profile of preferences does any agent benefit from reporting a false preference, keeping the reports of all other agents fixed. Both are *non-bossy*: if an agent at some profile of preference changes his report in a way to change someone else's match, then his own match must also change. Call any mechanism that satisfies these three properties *good*.

No good mechanism treats equals equally: when two different agents submit the same preference they end up with different houses. Randomization fixes this flaw. Consider drawing the sequence in which agents become dictators in a serial dictatorship from a uniform distribution on all such sequences. The resulting random matching mechanism is known as *random serial dictatorship* or *random priority*. Any two agents who submit the same preferences in random serial dictatorship face the same lottery over houses. So random serial dictatorship does treat equals equally.

The same method can be used to symmetrize any good mechanism. Instead of assigning one agent, say Anton, to assume the role of agent 1 in a mechanism, and assigning Betty to the role of agent 2, and so forth, the *sym-*

metrization of the mechanism uses a uniform lottery over all possible such assignments. To better understand such symmetrizations, fix a mechanism, a profile of preferences and a matching. The probability of the matching under the symmetrization of the mechanism at the given profile of preferences, equals the probability of the assignments (of agents to roles) with which the mechanism maps the profile to the matching.

The set of all good mechanisms is large. So this method would seem to generate many different random matching mechanisms, viewed as mappings from profiles of preferences to lotteries over matchings. This is not the case: Theorem 1 shows that the symmetrization of any good mechanism coincides with random serial dictatorship.

The first observations of this sort relate to GTTC: Abdulkadiroglu and Sönmez [1] and Knuth [9] independently proved the identity of random serial dictatorship and the symmetrization of GTTC, known as the *core from random endowments*. Both their proofs start by fixing an arbitrary profile of preferences. They then construct a bijection between the set of all sequences of agents as dictators and the set of initial endowments such that the outcome of the serial dictatorship with a given sequence equals the outcome of GTTC with the image of this given sequence. The bijection ensures that the number of sequences with which serial dictatorship yields some fixed matching equals the number of initial endowments with which GTTC yields the same matching. To see that this equality implies the result fix an arbitrary matching. Under random serial dictatorship and the core from random endowments the probability of this matching respectively equals proportion of sequences with which serial dictatorship yields this matching and the proportion of initial endowments with which GTTC yields the matching. This result has been extended to increasingly larger sets of good mechanisms by Pathak and Sethuraman [12], Caroll [5], and Lee and Sethuraman [10].

There are three differences between Theorem 1 and the preceding results. First, I show the equivalence for *all* good mechanisms. Second, Theorem 1 holds whether there are equally many agents and houses or not; in fact my proof does not distinguish between these cases. These two differences are made possible by the third innovation: a new simple strategy of proof.

This strategy relies on the construction of a sequence of good mechanisms $M^0, M^1, \cdots, M^K$ from an arbitrary good mechanism M^0 to a serial dictatorship M^K, such that any two consecutive mechanisms M^k, M^{k+1} differ minimally and have identical symmetrizations. I follow the bijective strategy pioneered by Abdulkadiroglu and Sönmez [1] and Knuth [9] to establish the identity of the symmetrizations of any two consecutive mechanisms M^k, M^{k+1}. The minimal difference between any two consecutive mechanisms makes this step as straightforward as possible.

To prove Theorem 1 I characterize the set of all good mechanisms as trading and braiding mechanisms in Theorem 2. Like GTTC, trading and braiding mechanisms use rounds of trade in pointing cycles to determine matchings. In any trading round, houses point to their owners and owners point to their most preferred houses. Matchings are obtained through the consecutive elimination of trading cycles. Trading and braiding mechanisms generalize GTTC in three dimensions. One agent might own multiple houses, a feature introduced by Papai [11]. There is a second form of control called brokerage, introduced by Pycia and Unver [13]. Finally, if there are only three houses left a trading and braiding mechanism might terminate in a *braid*. Theorem 2 and its proof owe a large debt to Pycia and Unver [13], which, in light of corrections implied by the present proof, provided a characterization of all good mechanisms when there are strictly more houses than agents.

My paper is concerned with a fully symmetric treatment of all agents. It does not speak to existing results on random matching mechanisms that treat different agents differently. Ekici [6], for example, considers two different matching mechanisms that both respect initial (private) allocations. He shows that a uniform randomization of any additional "social" endowment of houses in the two mechanisms yields the same random matching mechanism. Similarly Carroll [5] and Lee and Sethuraman [10] provide equivalence results for the case in which agents are treated symmetrically only within some fixed groups.

2 A sketch of the proof: GTTC

To illustrate the new strategy of proof, let M^0 be the GTTC where each agent i is endowed with house h_i. Construct a sequence of mechanisms $M^1, \ldots, M^K$

such that each mechanism is derived from its predecessor by consolidating the ownership of exactly two agents. The sequence terminates with a serial dictatorship M^K when all ownership has been maximally consolidated. M^1 is identical to GTTC except that agent 1 owns h_1 and h_2. Once agent 1 is matched with a house, he exits and agent 2 inherits the unmatched house in $\{h_1, h_2\}$.

To show that the symmetrizations of M^0 (GTTC) and M^1 are identical I fix an arbitrary profile of preferences and construct a bijection on the set of all assignments (of agents to roles). This bijection maps any assignment to a new assignment such that (at the given profile of preferences) the outcome of M^0 for the original assignment equals the outcome of M^1 for the new assignment. The existence of such a bijection implies that the proportions of assignments for which M^0 and M^1 (at the given profile of preferences) yield some particular fixed outcome are identical, implying in turn that their symmetrizations are identical.

To gain some insight into the construction of this bijection, consider the assignment of agents to roles where i owns h_i (so each agent assumes the role he is assigned to in the endowment). Consider two different profiles of preferences for which exactly one cycle forms at the first round of M^0. If agent 1 but not 2 takes part in this cycle, then the same cycle forms at the start of M^1. Once this cycle is matched, M^0 and M^1 continue identically as agent 2 inherits house h_2 under M^1. So M^0 and M^1 yield the same outcome at the given profile of preferences and assignment of agents to roles.

Now consider a profile of preferences such that M^0 starts with a single cycle involving agent 2 but not 1. Say this cycle is $2 \to h_3 \to 3 \to h_2 \to 2$ and assume that agent 1 prefers h_3 to all other houses. Under M^1, 1 owns h_2, the cycle $1 \to h_3 \to 3 \to h_2 \to 1$ forms in the first round. So M^0 and M^1 match h_3 with two different agents. If we switch the roles of agents 1 and 2 in M^1, agent 2 owns h_1 and h_2 at the start of M^1. The cycle $2 \to h_3 \to 3 \to h_2 \to 2$ which forms in the first round of M^0 also forms in the first round of M^1 with this new assignment; the fact that 2 additionally owns h_1 is irrelevant. Once this cycle is matched, M^1 with the new assignment continues identically to M^0 with the original assignment (given that 1 inherits house h_1). So M^0 with the original assignment and M^1 with the new assignment yield the same outcome at the given profile of preferences.

To generalize the above observations, fix any assignment of agents to roles and define its *permuted assignment* such that the agent who originally plays the role of agent 2 gets the role of agent 1 and vice versa, keeping all else equal. Now fix any profile of preferences. If M^0 and M^1 with the given assignment map the given profile to two different matchings, then M^1 with the permuted assignment maps the profile to the same matching as does M^0 with the original assignment. So the required bijection is easy to construct: any assignment that is not mapped onto itself is mapped to its permuted assignment.

The remaining mechanisms in the sequence $M^2, \cdots, M^K$ are constructed via the further consolidation of ownership. At the start of M^2 agent 1 owns h_1, h_2 and h_3, at the start of M^3 he also owns h_4. The process of consolidation terminates once M^0 has been transformed into a serial dictatorship M^K. The proof that M^0 and M^1 have identical symmetrizations is easily amended to any pair M^k, M^{k+1} in this sequence.

To apply the consolidation strategy to any good mechanism M^0 the above arguments have to be extended to the case where agents may own multiple houses. A somewhat different approach is needed to absorb brokers and to replace braids with serial dictatorships. One feature that all these cases share with the simple case of M^0 and M^1 is that the bijections between assignments of roles which show that the symmetrization of two consecutive mechanisms M^k and M^{k+1} are identical, switch the roles of at most three agents.

3 Definitions

A housing problem consists of a set of agents $N\colon = \{1, \cdots, n\}$, a finite set of houses H and a profile of preferences $R = (R_i)_{i=1}^n$. The option to stay homeless $\emptyset$ is always available: $\emptyset \in H$. Preferences R_i are linear orders[1] on H and each agent prefers any house to homelessness, so $hR_i\emptyset$ holds for all

[1] So hR_ih' and $h'R_ih$ together imply $h = h'$.

$i \in N, h \in H$.[2] The notation hR_iH' means that agent i prefers h to each house in H'. If hR_iH holds I also write $R_i : h$ and similarly $R_i : h, e, g$ means that R_i ranks h, e, and g respectively in first, second and third place. The set of all profiles R is denoted by $\mathcal{R}$. The **restriction** of R to some $N' \subset N$ and $H' \subset H$ is a profile of preferences $\overline{R}$ (defined for N' and H') with $h\overline{R}_ig \Leftrightarrow hR_ig$ for all $h, g \in H'$ and $i \in N'$.

A **match** is a pair (i, h) of one agent i and a house h. A **submatching** ν is a set of such matches where no agent or house takes part in more than one match, so $(i, h) \in \nu$, $i' \neq i$, and $h' \neq h$ imply $(i, h'), (i', h) \notin \nu$. If $(i, h) \in \nu$ then I write $\nu(i)$ for h, if agent i is not matched under ν, (so $(i, h) \in \nu$ for no $h \in H$), then I write $\nu(i) = \emptyset$. The sets of agents and houses matched under ν are $N_\nu :=\{i : (i, h) \in \nu \text{ for some } h \in H\}$ and $H_\nu :=\{h : (i, h) \in \nu \text{ for some } i \in N\}$; the complementary sets of unmatched agents and houses are $\overline{N}_\nu := N \setminus N_\nu$ and $\overline{H}_\nu := H \setminus H_\nu$. If $N_\nu \cap N_{\nu'} = \emptyset = H_\nu \cap H_{\nu'}$ holds for two submatchings ν and ν',then $\nu \cup \nu'$ is a well-defined submatching. A submatching ν is considered **maximal** (**minimal**) in a set of submatchings if there exists no ν' in the set such that $\nu \subsetneq \nu'$ ($\nu' \subsetneq \nu$). Any maximal submatching (in the set of all submatchings) is a **matching**, so μ is a matching if and only if $H_\mu = H$ or $N_\mu = N$ (or both) hold. The sets of all matchings and of all lotteries over matchings are denoted $\mathcal{M}$ and $\Delta\mathcal{M}$ respectively.

A (deterministic) **mechanism** is a function $M : \mathcal{R} \to \mathcal{M}$ where i is matched with $M(R)(i)$ under M at R. A mechanism M is **Pareto optimal** if for no R there exists a matching $\mu \neq M(R)$ such that $\mu(i)R_iM(R)(i)$ for all i.[3] A mechanism M is **strategy proof** if $M(R)(i)R_iM(R'_i, R_{-i})(i)$ holds for all triples R, R'_i, i: declaring one's true preference is a weakly dominant strategy. A mechanism M is **non-bossy** if $M(R)(i) = M(R'_i, R_{-i})(i)$ implies

[2]The assumption that agents never rank homelessness above a house is not without loss of generality: Erdil [7], for example, shows that random serial dictatorship is ex-ante Pareto dominated by other strategyproof, non-bossy and fair mechanisms if agents may opt to stay unmatched. Similarly Kesten and Kurino [8] argue that the trade-off between the welfare and incentive properties of optimal school matching mechanisms depends on the existence of outside options.

[3]Since all R_i are linear some i^* must strictly prefer $\mu(i^*)$ to $M(R)(i^*)$ for μ to differ from $M(R)$ and for $\mu(i)R_iM(R)(i)$ to hold for all i.

$M(R) = M(R'_i, R_{-i})$ for all triples R, R'_i, i, so an agent can only change someone else's match if he also changes his own match. A mechanism M is **good** if it is Pareto optimal, strategy proof and non-bossy.

The set of all permutations $p : N \to N$ is denoted P, and $p_{i,j}$ is the permutation involving only agents i and j, so $p_{i,j}(i) = j$, $p_{i,j}(j) = i$, and $p_{i,j}(i') = i'$, for $i' \in N \setminus \{i, j\}$. For any mechanism M and any permutation p define the **permuted mechanism** $p \odot M : \mathcal{R} \to \mathcal{M}$ via $(p(i), h) \in (p \odot M)(R)$ if and only if $(i, h) \in M(R_{p(1)}, \cdots, R_{p(n)})$ for all $i \in N$.[4] The permutation p assigns each agent in N to a "role" in the mechanism, such that agent $p(i)$ under $p \odot M$ assumes the role that agent i plays under M. If $S : \mathcal{R} \to \mathcal{M}$ is the serial dictatorship with agent i as the ith dictator, then $p(i)$ is the i-th dictator under $p \odot S$. To calculate $(p \odot S)(R)$ we need to substitute $p(i)$'s preference for agent i's preference to obtain the new profile of preferences $(R_{p(1)}, \cdots, R_{p(n)})$. Under $S(R_{p(1)}, \cdots, R_{p(n)})$ agent 1 is matched $p(1)$'s most preferred house. Under $(p \odot S)(R)$ this house is matched with $p(1)$.

A **(random matching) mechanism** is a function that maps the set of preference profiles $\mathcal{R}$ to the set of all lotteries over matchings $\Delta\mathcal{M}$: The **symmetrization** of a mechanism $M : \mathcal{R} \to \mathcal{M}$ is a random matching mechanism $\Delta M : \mathcal{R} \to \Delta\mathcal{M}$ that calculates the probability of matching μ at the profile R as the probability of a permutation p with $\mu = (p \odot M)(R)$ under the uniform distribution on P. So we have

$$\Delta M(R)(\mu) := \frac{|\{p \in P : (p \odot M)(R) = \mu\}|}{n!}.$$

Abdulkadiroglu and Sönmez [1] call ΔM a **random serial dictatorship** if M is a serial dictatorship and the **core from random endowments** if M is GTTC.

Definition 1 *Two (deterministic) mechanisms M and M' are **s-equivalent**[5] if $\Delta M = \Delta M'$.*

[4] The symbol $\odot$ is chosen as a reminder that $p \odot M$ arises out of a non-standard composition of the permutation p and the mapping M: $\odot$ is similar to but different from $\circ$, the standard operator for compositions. Any such non-standard composition of a permutation with another object is denoted by the operator $\odot$.

[5] The letter "s" is a reminder that symmetrizations are the base of s-equivalence.

4 The Result

Theorem 1 *Any good mechanism is s-equivalent to serial dictatorship.*

For the proof fix an arbitrary good mechanism M^0 and construct a sequence of mechanisms $M^1, \cdots, M^K$, with M^K a serial dictatorship and such that $\Delta M^k = \Delta M^{k+1}$ holds for $0 \leq k < K$. The s-equivalence of any two adjacent M^k, M^{k+1} implies the s-equivalence of M^0 and M^K. The minimal difference between M^k and M^{k+1} is crucial. It simplifies the task to find a bijection $f : P \to P$ with $(p \odot M^k)(R) = (f(p) \odot M^{k+1})(R)$ for all $p \in P$. Such a bijection exists if and only if $|\{p : (p \odot M^k)(R) = \mu\}| = |\{p : (p \odot M^{k+1})(R) = \mu\}|$ holds for all R and all matchings μ. The equality $|\{p : (p \odot M^k)(R) = \mu\}| = |\{p : (p \odot M^{k+1})(R) = \mu\}|$ is, in turn, equivalent to $\Delta M^k(R)(\mu) = \Delta M^{k+1}(R)(\mu)$, implying that a bijection f exists if and only if M^k and M^{k+1} are s-equivalent.

5 Trading and braiding mechanisms

My construction of marginally different good mechanisms M^k, M^{k+1} relies on the characterization of the set of all good mechanisms as trading and braiding mechanisms. Just like GTTC, trading and braiding mechanisms use sequential trading rounds to determine matchings. In such trading rounds owned houses point to their owners, and owners point to their most preferred houses. Agents in cycles are matched with the houses they point to. If a matching results the mechanism terminates, if not a new round ensues. There are three differences between GTTC and trading and braiding mechanisms. Following Papai's [11] hierarchical exchange mechanisms trading and braiding mechanisms permit the ownership of multiple houses. Following Pycia and Unver's [13] trading cycles mechanisms there is a second form of control in trading and braiding mechanisms: houses might be "brokered". At any given trading round there is at most one broker and he brokers exactly one house. This house points to the broker and the broker points to his most preferred house among the owned ones. Finally when there are only three houses left a trading and braiding mechanism might terminate as a braid. Braids are good mechanisms that match three houses to three agents with the aim to obtain a matching that maximally differs from some fixed “avoidance matching”. Braids can be thought of as mechanisms with three houses and three brokers: the definition of brokerage makes it unlikely for any broker to be matched with the house he brokers. When there are more houses, it turns out to be impossible to align Pareto optimality, strategyproofness and non-bossiness with the goal to avoid one particular matching.

The **braid** $B^\omega : \mathcal{R} \to \mathcal{M}$ is a mechanism for a problem with exactly three houses and at least as many agents. It is fully defined by the **avoidance matching** ω. Outcomes $B^\omega(R)$ are chosen to avoid any match $(i, \omega(i))$, while matching the same agents as does ω. For any R let $PO(R)$ be the set of Pareto optima μ with $N_\omega = N_\mu$. Let $Mini(R)$ be the subset of $PO(R)$ of matchings that minimally coincide with ω, so $\mu \in Mini(R)$ if $\mu \in PO(R)$ and there does not exist any $\mu' \in PO(R)$ with $\mu' \cap \omega \subsetneq \mu \cap \omega$. If $Mini(R)$ contains a single matching μ let $B^\omega(R) := \mu$. If not, at least two agents in N_ω must rank some house $\omega(i)$ at the top. If only one agent $j \neq i$ ranks $\omega(i)$ at the top then $B(R)$ is the unique element in $Mini(R)$ that matches j to $\omega(i)$. If both agents $j \neq i$ rank $\omega(i)$ at the top, then $B^\omega(R)$ is i's preferred matching in $Mini(R)$.

Example 1 Let $H = \{e, f, g\}$ and $N = \{1, 2, 3\}$. Since any matching μ satisfies the condition $N_\omega = N_\mu$ it can be ignored. For convenience denote matchings as vectors with the understanding that the i-th component represents agent i's match. Let $\omega := (e, f, g)$. There are exactly two matchings that maximally avoid ω: $\omega' \cap \omega = \emptyset = \omega'' \cap \omega$ holds for $\omega' := (g, e, f)$, and $\omega'' := (f, g, e)$. If all three agents rank e at the top and g at the bottom under R then $Mini(R) = \{\omega', \omega''\}$. Since 2 and 3 both rank $\omega(1) = e$ at the top, agent 1's preference of $f = \omega''(1)$ over $g = \omega'(1)$ implies $B^\omega(R) = \omega''$. Under (R'_2, R_{-2}) where R'_2 ranks f at the top and e at the bottom there are four Pareto optima: $\omega, (e, g, f), (g, f, e)$ and ω''. So $Mini(R) = \{\omega''\}$ and $B^\omega(R)$ equals ω''.

To formally define trading and braiding mechanisms I use Pycia and Unver's [13] notational system for trading algorithms. In this parsimonious system mechanisms are defined via sets of control rights functions. A **control rights function** $c_\nu : \overline{H}_\nu \to \overline{N}_\nu \times \{o, b\}$ at some submatching ν assigns control rights over any unmatched house to some unmatched agent and specifies a type of control. If $c_\nu(h) = (i, x)$, then agent i **controls** house h at ν. If $x = o$, then i **owns** h; if $x = b$ he **brokers** h. Control rights functions must satisfy the following three criteria:

(C1) If more than one house is brokered, then there are exactly three houses and they are brokered by three different agents.
(C2) If exactly one house is brokered then there is at least one owner.
(C3) No broker owns a house.

In order to obtain a unique representation of good mechanisms I also consider control rights functions that satisfy the stronger condition (C2') instead of (C2):

(C2') If exactly one house is brokered then there are at least two owners.

The following algorithm uses a set of control rights functions c to map any profile of preferences R to a matching.[6] For now assume that the set c contains all control rights functions c_ν used in the upcoming algorithm.

Initialize with $r = 1$, $\nu_1 = \emptyset$

Round r: only consider the remaining houses and agents $\overline{H}_{\nu_r}$ and $\overline{N}_{\nu_r}$.

Braiding: If more than one house is brokered under c_{ν_r}, terminate the process with $M(R) = \nu_r \cup B^\omega(\overline{R})$ where the avoidance matching ω is defined via $c_{\nu_r}(\omega(i)) = (i, b)$ and $\overline{R}$ is the restriction of R to $\overline{H}_{\nu_r}$ and $\overline{N}_{\nu_r}$. If not, go on to the next step.

Pointing: Each house points to the agent who controls it, so $h \in \overline{H}_{\nu_r}$ points to $i \in \overline{N}_{\nu_r}$ with $c_{\nu_r}(h) = (i, \cdot)$. Each owner points to his most preferred house,

[6]Without any further conditions, the algorithm may map some R to multiple matchings. Consider the problem $H = \{e, g\}$, $N = \{1, 2, 3\}$, $c_\emptyset(e) = (1, o)$, $c_\emptyset(g) = (2, o)$, $c_{\{(1,e)\}}(g) = c_{\{(2,g)\}}(e) = (3, o)$ and R such that 1 and 2 respectively rank e and g highest. In the first round two cycles form. Removing both at once the matching $\{(1, e), (2, g)\}$ results. Removing only $\{(1, e)\}$, the cycle involving g and 3 forms in the next round yielding the matching $\{(1, e), (3, g)\}$. Conditions (C4), (C5), and (C6) below ensure that the set c together with the algorithm define a mechanism. Section 4 of the online appendix proves this claim.

so owner $i \in \overline{N}_{\nu_r}$ points to house $h \in \overline{H}_{\nu_r}$ if $hR_i\overline{H}_{\nu_r}$. Each broker points to his most preferred owned house, so broker $i_b \in \overline{N}_{\nu_r}$ with $c_{\nu_r}(h_b) = (i_b, b)$ points to house $h \in \overline{H}_{\nu_r} \setminus \{h_b\}$ if $hR_{i_b}\overline{H}_{\nu_r} \setminus \{h_b\}$.

Cycles: Select at least one cycle. Define ν such that $(i,h) \in \nu$ if i points to h in one of the selected cycles.

Continuation: Let $\nu_{r+1} := \nu_r \cup \nu$. If ν_{r+1} is a matching terminate the process with $M(R) = \nu_{r+1}$. If not, continue with round $r+1$.

A submatching ν is **reachable** under c at R if one can choose to match cycles in the above algorithm (for the given c and R) such that some round r starts with $\nu = \nu_r$. A submatching ν is **c-relevant** if it is reachable under c at some R.[7] A submatching ν is a **direct c-successor** of some c-relevant ν° if there exists a profile of preferences R such that ν° is reachable under c at R and ν arises out of matching a single cycle at ν°. The following requirements (C4), (C5), and (C6) link control rights functions for different submatchings.

Fix a c-relevant ν° and a direct c-successor ν to ν°.

(C4) If $i \notin N_\nu$ owns h at ν° then i owns h at ν.
(C5) If at least two owners at ν° remain unmatched at ν and if $i_b \notin N_\nu$ brokers h_b at ν° then i_b brokers h_b at ν.
(C6) If i owns h at ν° and ν and if $i_b \notin N_\nu$ brokers h_b at ν° but not at ν, then i owns h_b at ν and i_b owns h at $\nu \cup \{(i, h_b)\}$.

A set of control rights functions c is a **control rights structure** if it satisfies (C1), (C2), (C3), (C4), (C5), and (C6). If c additionally satisfies (C2'), then c is a **tight (control rights) structure**. Any tight structure defines a **trading and braiding mechanism** $M^c : \mathcal{R} \to \mathcal{M}$, where $M^c(R)$ is calculated via the above algorithm. Using the same algorithm a control

[7] Consider a control rights structure c with three agents $\{1,2,3\}$ and 4 houses $\{e,f,g,h\}$, where agent 1 starts out owning house e and agent 2 starts out owning the remainder. Let R be such that 1 and 2 respectively rank e, and f at the top. The submatchings $\{(1,e)\}$, $\{(2,f)\}$ and $\{(1,e),(2,f)\}$ are reachable under c at R; $\{(2,g)\}$ is c-relevant since 2 could appropriate house g, but it is not reachable under c at R given that 2 prefers f to g; $\{(3,h)\}$ is not c-relevant since 3 does not own h at $\emptyset$. Note that $\emptyset$ is c-relevant for any control rights structure c.

rights structure c defines a **lax (trading and braiding) mechanism**. Since (C2') is strictly stronger than (C2) any trading and braiding mechanism is also a lax mechanism, and any tight structure is a control rights structure.

Example 2 Fix a matching problem with 7 agents $\{1,\ldots,7\}$ and 6 houses $H:=\{h_1,\ldots,h_6\}$. Define the following control rights functions c_ν for the submatchings $\emptyset$, $\{(6,h)\}$ for all $h\in H$, $\{(6,h_5),(1,h_1)\}$, $\{(6,h_6)(5,h_5)\}$ and $\{(6,h_5),(5,h_6),(4,h_4)\}$

$$c_\emptyset(h)=(6,o) \text{ if } h\in H$$

$$c_{\{(6,h)\}}(h')=(5,o) \text{ if } h\in H\setminus\{h_5\} \text{ and } h'\in H\setminus\{h\}$$

$$c_{\{(6,h_5)\}}(h_i)=(i,o) \text{ if } i\in\{1,2,4\}$$
$$c_{\{(6,h_5)\}}(h_3)=(1,o),\ c_{\{(6,h_5)\}}(h_6)=(7,b)$$

$$c_{\{(6,h_5),(1,h_1)\}}(h_i)=(i,o) \text{ if } i\in\{2,4\}$$
$$c_{\{(6,h_5),(1,h_1)\}}(h_3)=(5,o),\ c_{\{(6,h_5),(1,h_1)\}}(h_6)=(7,b)$$

$$c_{\{(6,h_6),(5,h_5)\}}(h)=(4,o) \text{ if } h\in H\setminus\{h_5,h_6\}$$

$$c_{\{(6,h_6),(5,h_5),(4,h_4)\}}(h_i)=(i,b) \text{ if } i\in\{1,2,3\}.$$

To get a sense for conditions (C1)-(C6), note that none of the above control rights functions assigns brokerage rights to more than three agents. At $\{(6,h_6),(5,h_5),(4,h_4)\}$ three different agents broker three different houses as required by (C1) and (C3). If this submatching is reached, then a braid matches the remaining agents and houses. At $\{(6,h_5)\}$ exactly one house, namely h_6, is brokered and there are at least two owners as required by (C2'). Consistently with (C3), agent 7, the broker of h_6, does not own a house at $\{(6,h_5)\}$.

The conditions (C4), (C5), and (C6) are satisfied by $\nu^\circ=\{(6,h_5)\}$ and $\nu=\{(6,h_5),(1,h_1)\}$: At $\nu^\circ=\{(6,h_5)\}$ agents 2 and 4 respectively own houses h_2 and h_4. Since they remain unmatched at $\nu = \{(6, h_5), (1, h_1)\}$ they must by (C4) continue to own these houses. Since two agents, namely 2 and 4, are owners at ν° and ν and since agent 7 the broker of h_6 at ν°, remains unmatched at ν, agent 7 must by (C5) continue to broker h_6 at ν. Since no broker looses control from ν° to ν, (C6) is trivially satisfied.

Now assume that c is a tight structure that is consistent with the above control rights functions. Conditions (C4), (C5), and (C6) restrict the freedom with which c_ν can be defined on c-relevant submatchings not mentioned above. The submatching $\{(6, h_5), (1, h_2), (2, h_1)\}$, for example, is a direct c-successor to $\{(6, h_5)\}$: It can be reached since agents 1 and 2 respectively own houses h_1 and h_2 at $\{(6, h_5)\}$. By (C4) agent 4, the owner of h_4 at $\{(6, h_5)\}$ must continue to own h_4 at $\{(6, h_5), (1, h_2), (2, h_1)\}$. However, since agent 4 is the only owner at both $\{(6, h_5)\}$ and $\{(6, h_5), (1, h_2), (2, h_1)\}$, (C5) does not require agent 7's brokerage to continue. If agent 4 owns h_6 at $\{(6, h_5), (1, h_2), (2, h_1)\}$, then agent 7 must by (C6) own h_4 at $\{(6, h_5), (1, h_2), (2, h_1), (4, h_6)\}$.

To see that an unmatched broker may have to lose control, consider $\nu' := \{(6, h_5), (2, h_4), (4, h_2)\}$. The submatching ν' is reached via agent 6 first appropriating house h_5 and then agents 2 and 4 swapping houses h_4 and h_2. By (C4) agent 1, the owner of h_1 and h_3 at $\{(6, h_5)\}$, must continue to own these houses at ν'. Since agent 1 owns all houses but h_6, (C2') implies that agent 7 cannot continue to broker h_6 at ν'. If we only require (C2) instead of (C2'), then agent 7 may continue to broker h_6 at ν'.

To see the algorithm at work consider a profile R^* with

$$R_1^* : h_1, h_2, \quad R_2^* : h_1, h_3, \quad R_3^* : h_5, h_6, \quad R_4^* : h_2, h_1, h_4,$$
$$R_5^* : h_5, h_6, h_1, h_2, \quad R_6^* : h_5, h_1, \quad R_7^* : h_1, h_2, h_3.$$

According to the algorithm, agent 6 appropriates house h_5, his most preferred house at $\emptyset$. At $\{(6, h_5)\}$ agent 1 owns h_1, his top ranked house, so that $\{(6, h_5), (1, h_1)\}$ is reached next. At $\{(6, h_5), (1, h_1)\}$ agents 2, 5, and 7 form a cycle and the submatching $\{(6, h_5), (1, h_1), (2, h_3), (5, h_6), (7, h_2)\}$ is reached. Finally agent 4 is matched with house h_4 and we obtain $M^c(R^*) = \{(6, h_5), (1, h_1), (2, h_3), (5, h_6), (7, h_2), (4, h_4)\}$.

If one strengthens (C1) to require that at most one house is brokered, one obtains the set of Pycia and Unver [13] trading cycles mechanisms. Following on the characterization in the present paper, Pycia and Unver [14] weakened (C1) to cover all good mechanisms. The mechanism in Example 2 is trading cycles mechanism according to Pycia and Unver [14] but not according to Pycia and Unver [13].

Theorem 2 *Any lax trading and braiding mechanism is good. Any good mechanism has a unique representation as a trading and braiding mechanism.*

The proof of Theorem 2 is in the online appendix. It starts by showing that braids are good. The remainder is proved by induction over the number of agents n. With only one agent Theorem 2 obviously holds. The next step is to show that lax mechanisms are well-defined: the order of the elimination of trading cycles does not matter. This flexibility together with the inductive hypothesis that any submechanism with fewer than n agents is good shortens the proof that any lax mechanism is good to a page. Pycia and Unver [13] broke the path for converse direction of the proof (any good mechanism can be represented as a trading and braiding mechanism). While my proof builds on the groundwork laid in Pycia and Unver [13], it ultimately deviates to show that more than one house might be brokered at some round of the mechanism and that any such round is a braid.

6 Tools for Trading and Braiding Mechanisms

Fix a control rights structure c and ν either a matching or a c-relevant submatching. A c-**path** (towards ν) is a set of c-relevant submatchings $\nu^0, \ldots, \nu^{L-1}$ such that $\nu^0 = \emptyset$, $\nu^L = \nu$ and ν^{l+1} is for each $l < L$ a direct c-successor of ν^l. If a single agent i owns all remaining houses $\overline{H}_\nu$ at some relevant submatching ν, so $i = c_\nu(\overline{H}_\nu)$, then ν as well as c_ν are c-**dictatorial** and i is the c-**dictator** at ν. A c-path $\nu^0, \ldots, \nu^{L-1}$ towards $\nu = \nu^L$ is a c-**dictatorial path towards** ν if each ν^l on the path is c-dictatorial. A submatching ν is c-**dictatorially chosen** if either $\nu = \emptyset$ or if there exists a c-dictatorial path towards ν. The control rights structure c **starts with a dictator** if the submatching $\emptyset$ is c-dictatorial.

Lemma 1 *Fix a control rights structure structure c and a c-dictatorially chosen submatching ν. Then there is exactly one c-path towards ν.*

Proof If $\nu = \emptyset$ we are done. If not, then $\emptyset = \nu^0$ must be on any path towards ν. Since some c-dictatorial path towards ν exists, ν^0 must be c-dictatorial. So the next submatching on any c-path towards ν must be of the form $\{(i^0, h)\}$ where i^0 is the dictator at ν^0. Since $\{(i^0, h)\}$ is on a c-path towards ν, h must equal $\nu(i^0)$. So $\nu^1 = \{(i^0, \nu(i^0))\}$ is on any c-path towards ν. If $\nu^1 = \nu$ we are done. If not repeat the above argument replacing ν^0 and i^0 with ν^1 and i^1. Inductively applying this argument to all submatchings ν^l on any c-path towards ν, we see that there is a unique c-path $(\nu^0, \ldots, \nu^{L-1})$ towards $\nu = \nu^L$. □

Example 3 To illustrate Lemma 1 consider the tight structure c defined in Example 2 together with the c-dictatorially chosen submatching $\nu\colon = \{(6, h_6), (5, h_5), (4, h_4)\}$. To see that $\big(\emptyset, \{(6, h_6)\}, \{(6, h_6), (5, h_5)\}\big)$ is the unique c-path towards ν, note that agent 6, the dictator at $\emptyset$, must appropriate house $\nu(6) = h_6$ at $\emptyset$ under any c-path towards ν. If not, ν cannot be reached. By the same token agent 5 the dictator at $\{(6, h_6)\}$ must appropriate $\nu(5) = h_5$ for ν to be reached. Submatchings that are not c-dictatorially chosen may be reached via multiple c-paths: $\nu'\colon = \{(6, h_5), (1, h_1), (2, h_2)\}$, for example is reached via the c-paths $\big(\emptyset, \{(6, h_5)\}, \{(6, h_5), (1, h_1)\}\big)$ and $\big(\emptyset, \{(6, h_5)\}, \{(6, h_5), (2, h_2)\}\big)$.

A control rights structure c defines a **path dependent dictatorship** if any c-relevant submatching is c-dictatorial. A path dependent dictatorship M^c is a serial dictatorship if the dictator at any c-relevant ν depends only on the size of ν.

Any control rights structure c together with a c-relevant submatching ν^* define a **submechanism** $M^{c[\nu^*]}$ that maps restrictions $\overline{R}$ (of $R \in \mathcal{R}$ to $\overline{N}_{\nu^*}$ and $\overline{H}_{\nu^*}$) to submatchings $\overline{\nu}$ with the feature that $\nu^* \cup \overline{\nu}$ is a matching in the original problem. The control rights structure $c[\nu^*]$ is such that $\nu = \nu^* \cup \nu'$ is c-relevant if and only if ν' is $c[\nu^*]$-relevant. For any such pair ν, ν' we have $c[\nu^*]_{\nu'} = c_\nu$. Fixing R such that ν^* is reachable under c at R, the definition of the trading-cycles algorithm implies $M^c(R) = \nu^* \cup M^{c[\nu^*]}(\overline{R})$ where $\overline{R}$ is the restriction of R to $\overline{N}_{\nu^*}$ and $\overline{H}_{\nu^*}$. If c is tight, then so is $c[\nu^*]$, if c defines a path dependent or a serial dictatorship then $c[\nu^*]$ does too.

Example 4 Reconsidering Example 2 let $\nu^* := \{(6, h_5), (1, h_1)\}$. The submechanism $M^{c[\nu^*]}$ maps the preferences of agents 2, 3, 4, 5, 7 over houses h_2, h_3, h_4, h_6 to matchings for this restricted problem. Since $c[\nu^*]_\emptyset(h_i)$ equals $c_{\nu^*}(h_i)$ for all h_i that remain unmatched at ν^* we obtain $c[\nu^*]_\emptyset(h_2) = (2, o)$, $c[\nu^*]_\emptyset(h_3) = (5, o)$, $c[\nu^*]_\emptyset(h_4) = (4, o)$, and $c[\nu^*]_\emptyset(h_6) = (7, b)$. The submatching ν^* is reached under $M^c(R^*)$, where R^* is the profile defined in Example 2. So $M^c(R^*)$ equals $\nu^* \cup M^{c[\nu^*]}(\overline{R}^*)$, with

$$\overline{R}_2^* : h_3 \quad \overline{R}_3^* : h_6 \quad \overline{R}_4^* : h_2, h_4 \quad \overline{R}_5^* : h_6, h_2 \quad \overline{R}_7^* : h_2, h_3.$$

Agents 2, 5, and 7 form a cycle at the start of $M^{c[\nu^*]}(\overline{R}^*)$ and $\{(2, h_3), (5, h_6), (7, h_2)\}$ is reached. Agent 4 continues to own h_4 at $\{(2, h_3), (5, h_6), (7, h_2)\}$. Since agent 4 prefers to be matched with any house, to not being matched we obtain $M^{c[\nu^*]}(\overline{R}) = \{(2, h_3), (5, h_6), (7, h_2), (4, h_4)\}$. Consistently with Example 2 we obtain $M^c(R^*) = \nu^* \cup M^{c[\nu^*]}(\overline{R})$.

The next Lemma applies to any R and any two tight structures c and c' that only differ following some submatching ν^*. If at least one match $(i, \nu^*(i))$ does not form under M^c at some R, then the mechanisms induced by c and c' have identical outcomes at R.

Lemma 2 *Fix two tight structures c and c', a submatching ν^*, and a profile of preferences R. If $c_\nu = c'_\nu$ holds for all c-relevant ν with $\nu^* \not\subset \nu$ and if $\nu^* \not\subset M^c(R)$, then $M^{c'}(R) = M^c(R)$.*

Proof Since $\nu^* \not\subset M^c(R)$ and since $c_\nu = c'_\nu$ holds for all c-relevant ν with $\nu^* \not\subset \nu$, c_ν equals c'_ν for all submatchings ν that are reachable under $M^c(R)$. This in turn implies that the same cycles form under $M^{c'}(R)$ and $M^c(R)$ at each ν that is reachable under $M^{c'}(R)$ and $M^{c'}(R) = M^c(R)$ holds. □

Example 5 Reconsidering c and R^* defined in Example 2 define a tight structure c' with $c_\nu = c'_\nu$ for all ν with $\{(6, h_5), (1, h_1), (2, h_2)\} \not\subset \nu$. By Example 2 we have $\{(6, h_5), (1, h_1), (2, h_2)\} \notin M^c(R^*)$. So $M^{c'}(R^*)$ equals

$M^c(R^*)$ no matter how we define c' on c'-relevant submatchings ν with $\{(6,h_5),(1,h_1),(2,h_2)\} \subset \nu$.

By Theorem 2 any lax trading and braiding mechanism is good and any good mechanism has a unique representation as a trading and braiding mechanism. So for any control rights structure $\overline{c}$ there must exist a tight structure c such that $M^{\overline{c}}$ and M^c define the same mechanism viewed as a mapping from preference profiles to matchings. The next lemma explains how to construct such alternative representations.

Lemma 3 *Fix any control rights structure $\overline{c}$. For any $\overline{c}$-relevant ν that satisfies (C2') let $c_\nu = \overline{c}_\nu$. For any $\overline{c}$-relevant ν with $\overline{c}_\nu(h_b) = (i_b, b)$ and $\overline{c}_\nu(h) = (i^*, o)$ for some $i_b, i^* \in \overline{N}_\nu$, $h_b \in \overline{H}_\nu$ and all $h \in \overline{H}_\nu \setminus \{h_b\}$ let $c_\nu(h) = (i^*, o)$ for all $h \in \overline{H}_\nu$ and $c_{\nu \cup \{(i^*, h_b)\}}(h) = (i_b, o)$ for all $h \in \overline{H}_\nu \setminus \{h_b\}$. Then c is a tight structure and $M^c(R) = M^{\overline{c}}(R)$ holds for all $R \in \mathcal{R}$.*

The proof of Lemma 3 is in the online appendix.

Example 6 To illustrate Lemma 3 recall the tight structure c defined in Example 2. Say that $\overline{c}$ is a control rights structure, such that c and $\overline{c}$ coincide on all c-relevant submatchings with $\nu^* \not\subset \nu$ for $\nu^* := \{(6,h_5),(2,h_4),(4,h_2)\}$ consistently with Example 2 define control rights for the c- and $\overline{c}$-successors to ν^* as follows:

$$
\begin{aligned}
&c_{\nu^*}(h_i) = \overline{c}_{\nu^*}(h_i) = (1,o) \text{ for } h_i \in \{h_1,h_3\} \\
&c_{\nu^*}(h_6) = (1,o) \neq (7,b) = \overline{c}_{\nu^*}(h_6) \\
&c_{\nu^* \cup \{(1,h_6)\}}(h_i) = (7,o) \text{ for } h_i \in \{h_1,h_3\} \\
&\overline{c}_\nu = c_\nu \text{ for all } \overline{c} - \text{relevant } \nu \text{ with } \nu^* \subsetneq \nu.
\end{aligned}
$$

The application of the Lemma 3 to the control rights structure $\overline{c}$ yields the tight structure c: $\nu^* = \{(6,h_5),(2,h_4),(4,h_2)\}$ is the only $\overline{c}$-relevant submatching that does not satisfy (C2'). Letting $i_b := 7$, $h_b := h_6$, $i^* := 1$ we obtain the control rights prescribed by c for ν^* and its direct c-successors.

To see that $M^c(R) = M^{\overline{c}}(R)$ holds for all $R \in \mathcal{R}$ fix an arbitrary R. If $\nu^* \not\subset M^c(R)$ then $M^c(R) = M^{\overline{c}}(R)$ holds by Lemma 2. So suppose that

$\nu^* \subset M^c(R)$. For concreteness consider $R_6 : h_5$, $R_2 : h_2$, and $R_4 : h_4$ where agents 2 and 4 swap houses after agent 6 has appropriated h_5.

At ν^* agent 1 owns h_1 and h_3 according to both control rights structures. If he appropriates either one, both control rights structures continue identically and we have $M^c(R) = M^{\overline{c}}(R)$. Only one case remains to be considered: if R_1 ranks h_6 above h_1 and h_3 then agent 1 appropriates h_6 at ν^* if he owns it. So $\nu^* \cup \{(1, h_6)\}$ is reached under $M^c(R)$. Agent 7 then becomes the owner of h_1 and h_3. Assuming that h^* is the R_7-preferred house among h_1 and h_3 the submatching $\nu^* \cup \{(1, h_6), (7, h^*)\}$ is reached next under $M^c(R)$. Under $M^{\overline{c}}(R)$ agent 1 and 7 respectively point to h_6 and h^* at ν^* and $\nu^* \cup \{(1, h_6), (7, h^*)\}$ is reached immediately. Both c and $\overline{c}$ then prescribe the same control rights for the sole house that remains unmatched at $\nu^* \cup \{(1, h_6), (7, h^*)\}$ and $M^c(R)$ equals $M^{\overline{c}}(R)$ in all possible cases.

The permutation of any lax mechanism $p \odot M^c$ (and therefore and trading and braiding mechanism) can now be defined by applying the permutation p directly to the control rights structure c that defines the mechanism. To do so, permutations of submatchings have to be defined first: For any submatching ν and permutation p on the set of agents N define $p \odot \nu$ so that $(p(i), h) \in p \odot \nu$ holds if and only if $(i, h) \in \nu$. So $p \odot \nu$ matches agent $p(i)$ with house h wherever ν matches agent i with this house h. The control rights structure $p \odot c$ is defined by $(p \odot c)_{p \odot \nu}(h) = (p(i), x)$ if and only if $c_\nu(h) = (i, x)$ for any c-relevant ν, $i \in \overline{N}_\nu$, $h \in \overline{H}_\nu$ and $x \in \{b, o\}$. Since a submatching $p \odot \nu$ is $(p \odot c)$-relevant if and only if ν is c-relevant properties (C1), (C2), (C2'), (C3), (C4), (C5), and (C6) hold for $p \odot c$ if and only if they hold for c. So $p \odot c$ is a (tight) control rights structure if and only if c is a (tight) control rights structure. The permuted control right structure $p \odot c$ defines the mechanism $p \odot M^c$ in the sense that $p \odot M^c(R) = M^{p \odot c}(R)$ for all R.

Example 7 Reconsidering the tight structure c in Example 2, define a new tight structure $p \odot c$ where p is the permutation on N for which $p(1) = 6$, $p(2) = 1$, $p(6) = 2$, and $p(i) = i$ otherwise. So agents 1, 2, and 6 exchange roles when applying p to the mechanism M^c; the others keep their roles. Since agent $2 = p(6)$ assumes the role of agent 6 who in turn is the dictator at $\emptyset$ according to c, we have $(p \odot c)_\emptyset(h) = (2, o)$ for all $h \in H$. If agent 2 appropriates house h_5 the submatching $\{(2,h_5)\} = \{(p(6),h_5)\} = p \odot \{(6,h_5)\}$ is reached. Since $c_{\{(6,h_5)\}}(h_1) = (1,o)$ and $c_{\{(6,h_5)\}}(h_2) = (2,o)$ we obtain $(p \odot c)_{\{(2,h_5)\}}(h_1) = (p(1),o) = (6,o)$ as well as $(p \odot c)_{\{(2,h_5)\}}(h_2) = (p(2),o) = (1,o)$. Since $p(i) = i$ holds for each agent $i \in \{3,4,5,7\}$, $(p \odot c)_{\{(2,h_5)\}}(h)$ equals $c_{\{(6,h_5)\}}(h)$ whenever $c_{\{(6,h_5)\}}(h) = (i,\cdot)$ for $i \in \{3,4,5,7\}$. Agent 7, for example, brokers house h_6 at $\{(6,h_5)\}$ given c and at $p \odot \{(6,h_5)\} = \{(2,h_5)\}$ given $p \odot c$.

If two control rights structures c and c' define s-equivalent mechanisms M^c and $M^{c'}$ then I say that c and c' are **s-equivalent**.

7 Proof of Theorem 1: Overview

Using the characterization of all good mechanisms the outline of the proof of Theorem 1 can be fleshed out further. To do so, fix an arbitrary good mechanism represented by some tight structure c^0. This arbitrary mechanism is s-equivalent to serial dictatorship if there is a sequence of s-equivalent control rights structures $c^0, c^1, \ldots, c^K$ where c^K defines a serial dictatorship.

To construct such a sequence assume that c^0 does not define a path dependent dictatorship, so that some c^0-relevant submatchings are not c^0-dictatorial. Fix a minimal such submatching ν^*. I first show that $c^0[\nu^*]$ has a s-equivalent tight structure $c^1[\nu^*]$ that starts with a dictator. In Section 8 I show that the ownership in any tight structure that starts with m owners can be consolidated to find a s-equivalent control rights structure that starts with $m-1$ owners. By Lemma 3 such a control rights structure can w.l.o.g be replaced by a tight structure that starts with $m-1$ owners.[8] The inductive application of these arguments shows that any tight structure that starts with m owners is s-equivalent to a tight structure that starts with a

[8] Instead of defining both control rights structures and tight structures, I could have directly transformed tight structures starting with m owners into s-equivalent tight structures with $m-1$ owners. The simple approach to consolidate ownership in Section 8 would then have to be replaced with a more involved approach, that not only consolidates ownership but also guarantees that there newly constructed control rights allow for brokers only when there are at least two owners. The definition of both types of control rights is useful to disentangle the consolidation of ownership from the absorption of brokers.

dictator. If the initial tight structure starts with a broker then the last step of the above process transforms a control rights structure that starts with one owner and a broker to a tight structure that starts with a dictator and represents the same mechanism. Section 9 covers the case when c^0 does not start with any owner: If $c^0[\nu^*]$ defines a braid, then $c^0[\nu^*]$ is s-equivalent to some $c^1[\nu^*]$ that defines a serial dictatorship.

Section 10 then defines a new tight structure c^1 by using $c^1[\nu^*]$ following on ν^* and setting c^1_ν to c^0_ν on all c^0-relevant ν with $\nu^* \not\subset \nu$. This new tight structure c^1 is s-equivalent to c^0. While any direct c^1-successor to ν^* is c^1-dictatorially chosen, no direct c^0-successor to ν^* is c^0-dictatorially chosen. So the set of c^0-dictatorially chosen submatchings is a strict subset of the set of c^1-dictatorially chosen submatchings.

Inductively repeating the step outlined in the above paragraph we obtain a sequence $c^0, c^1, \ldots, c^{K'}$ of s-equivalent tight structures where the set of c^k-dictatorially chosen submatchings increases with each step. Since there are only finitely many submatchings this sequence must reach a tight structure $c^{K'}$ where each $c^{K'}$-relevant submatching if $c^{K'}$-dictatorial, so that $c^{K'}$ defines a path dependent dictatorship. The tools developed in Section 10 suffice to then show that dictators can be reordered to construct some further s-equivalent tight structures $c^{K'}, c^{K'+1}, \ldots, c^K$ where c^K defines a serial dictatorship, completing the proof.

8 Consolidation of Ownership

In this Section I show that any tight structure that starts with at least two owners is s-equivalent to a tight structure that starts with a dictator.

Lemma 4 *Fix a tight structure c with $m \geq 2$ owners at $\emptyset$. Then there exists a s-equivalent tight structure $\tilde{c}$ that starts with a dictator.*

The proof of this Lemma is broken into three steps. For any fixed arbitrary tight structure c that starts with $m \geq 2$ owners Lemma 5 constructs a similar control rights structure $\bar{c}$ that starts with $m-1$ owners. Lemma 6 then shows that this similar control rights structure is s-equivalent to the control rights structure it was derived from. By Lemma 3 the mechanism represented by the similar control right structure can be represented by a tight structure that starts with equally many owners. So any tight structure that starts with at least two owners has a s-equivalent tight structure that starts with one fewer owner. Iteratively applying this last observation to the original tight structure we obtain a sequence of s-equivalent tight structures that start with fewer and fewer owners. This sequence terminates with a tight structure that starts with a dictator and is s-equivalent to the original tight structure c, proving Lemma 4.

To state Lemma 5 say that a c-path $\{\nu^0, \nu^1, \cdots, \nu^{L-1}\}$ towards $\nu = \nu^L$ matches agent i before agent j if some submatching ν_l with $l \in \{1, \ldots, L\}$ matches i but not j so $i \in N_{\nu^l}$ and $j \notin N_{\nu^l}$.

Lemma 5 *Fix a tight structure c such that agents 1 and 2 are owners at $\emptyset$. Then $\bar{c}$, defined as follows, is a control rights structure.*

If ν is c-relevant, $1, 2 \notin N_\nu$ and $h \in \overline{H}_\nu$, let

$$\bar{c}_\nu(h) = \begin{cases} (1, o) & \text{if } c_\nu(h) = (1, o) \text{ or } c_\nu(h) = (2, o) \\ c_\nu(h) & \text{otherwise.} \end{cases}$$

If $1 \in N_\nu$, let

$$\bar{c}_\nu = \begin{cases} c_\nu & \text{if a } c\text{-path towards } \nu \text{ matches } 1 \text{ before } 2 \\ (p_{1,2} \odot c)_\nu & \text{if a } p_{1,2} \odot c\text{-path towards } \nu \text{ matches } 1 \text{ before } 2 \end{cases}$$

Proof Fix an arbitrary tight structure c. To see that the Lemma defines control rights $\bar{c}_\nu$ for each $\bar{c}$-relevant ν, firstly note that $\bar{c}_\emptyset$ is uniquely defined since $1, 2 \notin N_\emptyset = \emptyset$. So suppose there exists some $\bar{c}$-relevant ν° such that $\bar{c}_{\nu'}$ is well-defined for all $\bar{c}$-relevant $\nu' \subset \nu^\circ$ and suppose that ν is a direct $\bar{c}$-successor to ν°. If $1, 2 \notin N_\nu$ then ν is $\bar{c}$-relevant if and only if it is c-relevant and $\bar{c}_\nu$ is uniquely defined. If $1, 2 \notin N_{\nu^\circ}$ but $\{1, 2\} \cap N_\nu \neq \emptyset$, say σ is such that $\nu^\circ \cup \sigma = \nu$. Since agent 2 does not control any houses according to $\bar{c}_{\nu^\circ}$, agent 1 must be matched by σ. Say h is the unique house in H_σ owned by agent 1 according to $\bar{c}_{\nu^\circ}$. If $c_{\nu^\circ}(h) = (1, o)$, then ν is c-relevant so that $\bar{c}_\nu$ equals c_ν. If not, then the definition of $\bar{c}$ implies $c_{\nu^\circ}(h) = (2, o)$ and ν is $p_{1,2} \odot c$-relevant, so that $\bar{c}_\nu$ equals $(p_{1,2} \odot c)_\nu$. Finally consider the case that $\{1, 2\} \cap N_{\nu^\circ} \neq \emptyset$: If ν° is c-relevant then ν is too and $\bar{c}_\nu$ equals c_ν. If ν° is not

c-relevant, then ν° and ν are both $p_{1,2} \odot c$-relevant and $\bar{c}_\nu$ equals $(p_{1,2} \odot c)_\nu$.[9]

To see that $\bar{c}$ is a control rights structure, first fix an arbitrary $\bar{c}$-relevant ν with $1 \in N_\nu$. For any such ν $\bar{c}[\nu]$ either equals $c[\nu]$ or $(p_{1,2} \odot c)[\nu]$, implying that $\bar{c}[\nu]$ is a tight structure. So it only remains to be seen that (C1)- (C6) hold for $\bar{c}$-relevant ν with $1 \notin N_\nu$. Since $\bar{c}_\nu$ for any such ν is defined via the consolidation of ownership of two owners it satisfies (C1), (C2), and (C3). To see (C4), (C5), and (C6) fix a pair ν°, ν such that ν° is $\bar{c}$-relevant, ν is a direct $\bar{c}$-successor to ν° and $1 \notin N_{\nu^\circ}$. Fix any $h \in \overline{H}_\nu$ and $i \in \overline{N}_\nu$.

Case 1: $1 \notin N_\nu$, so that ν is c-relevant. If $\bar{c}_{\nu^\circ}(h) = (i, o)$ and $i \neq 1$, then $\bar{c}_{\nu^\circ}(h) = c_{\nu^\circ}(h) = c_\nu(h) = \bar{c}_\nu(h)$ holds by the definition of $\bar{c}$ and the fact that c satisfies (C4). If $i = 1$, the same arguments yield

$$\begin{aligned} \bar{c}_{\nu^\circ}(h) = (1, o) &\Rightarrow c_{\nu^\circ}(h) \in \{(1, o), (2, o)\} \\ \Rightarrow c_\nu(h) \in \{(1, o), (2, o)\} &\Rightarrow \bar{c}_\nu(h) = (1, o). \end{aligned}$$

Since agents 1 and 2 are owners according to c_{ν° and c_ν and since c satisfies (C5) and (C6), $\bar{c}_{\nu^\circ}(h) = c_{\nu^\circ}(h) = c_\nu(h) = \bar{c}_\nu(h)$ also holds if $\bar{c}_{\nu^\circ}(h) = (i, b)$. So $\bar{c}_{\nu^\circ}(h) = \bar{c}_\nu(h)$ holds for any $h \in \overline{H}_\nu$ and $\bar{c}$ satisfies (C4), (C5), and (C6) at ν°, ν.

Case 2: $1 \in N_\nu$ and ν is c-relevant. To see that $\bar{c}$ satisfies (C4) at ν°, ν assume $\bar{c}_{\nu^\circ}(h) = (i, o)$. Since 1 is matched under ν and since 2 is not an

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[9] To see that $\bar{c}_\nu$ is uniquely defined when $1 \in N_{\nu^\circ}$, suppose a c-path and a $p_{1,2} \odot c$-path both, of which match 1 before 2 reach ν. As long as agents 1 and 2 remain unmatched, c and $p_{1,2} \odot c$ prescribe the same control rights for all agents other than 1 and 2. The unique maximal c-relevant submatching ν^* in the set of all $\nu' \subset \nu$ with $1, 2 \notin N_{\nu'}$ is therefore also the unique maximal $p_{1,2} \odot c$-relevant submatching in this set. Since trading cycles may be matched in any order we can without loss of generality assume that ν^* is on both paths. Say that $\nu^* \cup \sigma'$ and $\nu^* \cup \sigma''$ directly succeed ν^* on the two paths, so we have $1 \in N_{\sigma'} \cap N_{\sigma''}$, $2 \notin N_{\sigma'} \cup N_{\sigma''}$, $\nu^* \cup \sigma' \subset \nu$ and $\nu^* \cup \sigma'' \subset \nu$. Since $\sigma' \subset \nu$ and $\sigma'' \subset \nu$, $\sigma'(1)$ and $\sigma''(1)$ must both equal $\nu(1)$. Since there exists no house h with $c_{\nu^*}(h) = (1, \cdot)$ and $(p_{1,2} \odot c)_{\nu^*}(h) = (1, \cdot)$, an agent $i \in N_{\sigma'} \setminus \{1\}$ must control the house $\nu(1)$ at c_{ν^*}. Since $2 \notin N_{\sigma'}$ we have $i \neq 2$. Since c_{ν^*} and $p_{1,2} \odot c$ prescribe the same control rights for all agents $(N_{\sigma'} \cup N_{\sigma''}) \setminus \{1, 2\}$ house $\nu(1)$ must be controlled by the same agent i at ν^* under $p_{1,2} \odot c$. Since $\nu^* \cup \sigma' \subset \nu$ and $\nu^* \cup \sigma'' \subset \nu$ we must have $\sigma'(i) = \sigma''(i) = \nu(i)$. By the same argument as above, this house $\nu(i)$ must be controlled by an agent other than agent 1. Proceeding inductively we see that no agent in either $N_{\sigma'}$ and $N_{\sigma''}$ is matched with a house controlled by agent 1 at ν^*, a contradiction to $1 \in N_{\sigma'} \cap N_{\sigma''}$.

owner under $\overline{c}_{\nu^\circ}$, i neither equals 1 nor 2. Since c satisfies (C4) the definition of $\overline{c}$ then implies $\overline{c}_{\nu^\circ}(h) = c_{\nu^\circ}(h) = c_\nu(h) = \overline{c}_\nu(h)$ and $\overline{c}$ satisfies (C4) at ν°, ν. To see that $\overline{c}$ satisfies (C5) and (C6) at ν°, ν assume $\overline{c}_{\nu^\circ}(h) = (i, b)$ and $\overline{c}_\nu(h) \neq (i, b)$. By the definition of $\overline{c}$, $c_{\nu^\circ}(h) = (i, b)$ and $c_\nu(h) \neq (i, b)$ must hold. Since c satisfies (C5) and since 2 is an owner according to c_{ν° and c_ν, no agent $j \neq 2$ is an owner according to both c_{ν° and c_ν. The definition of $\overline{c}$ then implies that no agent is an owner according to both $\overline{c}_{\nu^\circ}$ and $\overline{c}_\nu$, implying that $\overline{c}$ satisfies (C5) and (C6) at ν°, ν.

Case 3: $1 \in N_\nu$ and ν is not c-relevant. Switch agents 1 and 2 in the arguments covering Case 2, including the replacement of ν by $p_{1,2} \odot \nu$. $\square$

Example 8 To see Lemma 5 at work consider tight structure $c[\{(6, h_5)\}]$ for agents $\{1, 2, 3, 4, 5, 7\}$ and houses $\{h_1, h_2, h_3, h_4, h_6\}$ derived from c defined in Example 2. At the start agent 1 owns h_1 and h_3 while agent 2 owns h_2. Use the construction in Lemma 5 to define a control rights structure $\overline{c}$ that consolidates the ownership of agents 1 and 2: According to $\overline{c}_\emptyset$ agent 1 owns all houses owned by 1 and 2 according to $c[\{(6, h_5)\}]_\emptyset$, so we have $\overline{c}_\emptyset(h) = (1, o)$ for $h \in \{h_1, h_2, h_3\}$. All other control rights are chosen to minimize the difference between $\overline{c}$ and $c[\{(6, h_5)\}]$ subject to the constraint that $\overline{c}$ is a tight structure.

There are two reasons why $\overline{c}_\nu$ cannot be set to $c_{\{(6,h_5)\}\cup\nu}$ for all $\overline{c}$-relevant $\nu \neq \emptyset$: Agent 1's ownership must by (C4) continue and $\overline{c}_\nu$ must be defined if ν is $\overline{c}$- but not $c[\{(6, h_5)\}]$-relevant. The continuity of ownership (C4) together with $\overline{c}_\emptyset(h_2) = (1, o)$, for example, forces $\overline{c}_{\{(4,h_4)\}}(h_2) = (1, o)$ even though $c[\{(6, h_5)\}]_{\{(4,h_4)\}}(h_2) = (2, o)$. To avoid this problem, the ownership of agents 1 and 2 is "fully" consolidated in the sense that, agent 1 - as long as he remains unmatched - owns all houses that would otherwise be owned by agents 1 or 2. To understand the second problem consider the submatching $\nu = \{(1, h_6), (7, h_2)\}$. Since this submatching is is $\overline{c}$- but not $c[\{(6, h_5)\}]$-relevant, we cannot equate $\overline{c}_\nu$ with $c[\{(6, h_5)\}]_\nu$ which is not defined. To keep the distance between $c[\{(6, h_5)\}]$ and $\overline{c}$ minimal, $\overline{c}_\nu$ is set to $(p_{1,2} \odot c[\{(6, h_5)\}])_\nu$, which is possible since ν is $p_{1,2} \odot c[\{(6, h_5)\}]$-relevant. Once ν is reached $\overline{c}$ continues as if agent 1 had always played the role of agent 2 in $c[\{(6, h_5)\}]$.

The control rights structure $\bar{c}$ need not be tight: at some $\bar{c}$-relevant ν, there may be exactly one broker and one owner, a violation of (C2'). In the present case agents 1 and 2 are owners while agent 7 is a broker according to $c[\{(6, h_5)\}]_{\{(4,h_4)\}}$. Since $\bar{c}$ consolidates the ownership of agents 1 and 2 into the hands of agent 1, there is a broker (agent 7) and exactly one owner (agent 1) according to $\bar{c}_{\{(4,h_4)\}}$.

Lemma 6 *Fix a tight structure c with $m \geq 2$ owners at $\emptyset$. Then there exists a s-equivalent control rights structure $\bar{c}$ with $m - 1$ owners at $\emptyset$.*

Proof Assume without loss of generality that agents 1 and 2 are owners according to $c_\emptyset$. Use Lemma 5 to define a control rights structure $\bar{c}$. To see that c and $\bar{c}$ are s-equivalent fix a profile of preferences R. In the next paragraph I define a function $f : P \to P$. Reasoning by cases I then show that $M^{p\odot c}(R) = M^{f(p)\odot\bar{c}}(R)$ holds for all $p \in P$. Finally in the last paragraph of the proof I show that f is a bijection so that c and $\bar{c}$ are indeed s-equivalent.

To define the function $f : P \to P$ fix a permutation $p \in P$. Say that $p\odot\nu$ is the maximal reachable submatching under $M^{p\odot c}(R)$ matching neither $p(1)$ nor $p(2)$.[10] Consequently any cycle that forms at $p \odot \nu$ under $M^{p\odot c}(R)$ involves agent $p(1)$ or $p(2)$ (or both). If there are two such cycles, choose the one involving $p(1)$, otherwise choose the unique such cycle. Say the chosen cycle results in the submatching $p\odot\sigma$. Index the houses in H_σ such that agent $p(i)$ controls house h^i at $p \odot \nu$, so $c_\nu(h^i) = (i, \cdot)$ holds for each $i \in N_\sigma$ and $h^i \in H_\sigma$. If $1 \in N_\sigma$ let $f(p) = p$, otherwise let $f(p)(1) = p(2)$, $f(p)(2) = p(1)$ and $f(p)(i) = p(i)$ for all $i \neq 1, 2$.

Since $1, 2 \notin N_\nu$ the submatching $p\odot\nu$ is also reachable under $M^{f(p)\odot\bar{c}}(R)$. At $p\odot\nu$ all agents and all houses in the cycle $p\odot\sigma$ except possibly h^1 and h^2 point into the same direction under $M^{p\odot c}(R)$ and $M^{f(p)\odot\bar{c}}(R)$: The pointing of agents remains constant, as it only depends on their preferences and the set of unmatched houses. The pointing of any house h^i with $i \notin \{1, 2\}$ remains constant, since any such h^i is controlled by the same agent according to $p \odot c$ and $f(p) \odot \bar{c}$. The houses h^1 and h^2 point to either $p(2)$ or $p(1)$ under

[10] The fact that the order in which trading cycles are matched is irrelevant ensures the uniqueness of $p \odot \nu$.

$M^{f(p)\odot\overline{c}}(R)$. To show that $p \odot M^c(R)$ equals $f(p) \odot M^{\overline{c}}(R)$ for all $p \in P$, I reason by cases.

Case 1: $1 \in N_\sigma$, $2 \notin N_\sigma$. By the definition of $\overline{c}$ house h^1 points to $p(1)$ at $p \odot \nu$ under $(p \odot M^{\overline{c}})(R)$. So the cycle yielding $p \odot \sigma$ also forms under $(p \odot M^{\overline{c}})(R)$ at $p \odot \nu$ and $p \odot (\nu \cup \sigma)$ is reached. Since $1 \in N_{\nu\cup\sigma}$, the definition of $\overline{c}$ implies $M^{\overline{c}}[\nu \cup \sigma] = M^c[\nu \cup \sigma]$. Since $f(p) = p$ holds in Case 1, we have $(f(p) \odot M^{\overline{c}})(R) = (p \odot M^{\overline{c}})(R) = (p \odot M^c)(R)$.

Case 2: $2 \in N_\sigma$, $1 \notin N_\sigma$. By the definition of $\overline{c}$ and $f(p)$ house h^2 points to $f(p)(1) = p(2)$ at $p \odot \nu$ under $(f(p) \odot M^{\overline{c}})(R)$. So the cycle yielding $p \odot \sigma$ also forms under $(f(p) \odot M^{\overline{c}})(R)$ at $p \odot \nu$, so that $p \odot (\nu \cup \sigma)$ is reached under $(f(p) \odot M^{\overline{c}})(R)$. Since $2 \in N_{\nu\cup\sigma}$ and $1 \notin N_{\nu\cup\sigma}$, $\overline{c}[\nu \cup \sigma]$ equals $p_{1,2} \circ c[\nu \cup \sigma]$. The application of f to the permutation p then yields $f(p) \odot \overline{c}[\nu \cup \sigma] = f(p) \odot p_{1,2} \odot c[\nu \cup \sigma]$. Since $f(p)(1) = p(2)$ and $f(p)(2) = p(1)$ the two swaps via $p_{1,2}$ and $f(p)$ neutralize each other and we obtain $f(p) \odot \overline{c}[\nu \cup \sigma] = p \odot c[\nu \cup \sigma]$. In sum we have $(p \odot M^c)(R) = (f(p) \odot M^{\overline{c}})(R)$.

Case 3: $1, 2 \in N_\sigma$. By the definition of $\overline{c}$ houses h^1 and h^2 point to $p(1)$ at $p \odot \nu$ under $(p \odot M^{\overline{c}})(R)$ so that the cycle $p(1) \to \sigma(1) \to \dots h^2 \to p(1)$ forms. This cycle yields a submatching $p \odot \sigma' \subset p \odot \sigma$. Since $\nu \cup \sigma'$ is not c-relevant (as it matches agent 1 but no house controlled by agent 1 according to c_ν), the definition of $\overline{c}$ implies $\overline{c}_{\nu\cup\sigma'}(h^1) = p_{1,2} \odot c_{p_{1,2}\odot(\nu\cup\sigma')}(h^1) = (p_{1,2}(1), o) = (2, o)$. So $p(2)$ owns h^1 at $p \odot (\nu \cup \sigma')$ according to $p \odot \overline{c}$.

Case 3a) Each house in $H_\sigma \setminus (H_{\sigma'} \cup \{h^1\})$ points to the same agent at $p \odot (\nu \cup \sigma')$ and at $p \odot \nu$. So the cycle $p(2) \to \sigma(2) \to \dots \to h^1 \to p(2)$ yielding $p \odot \sigma''$ with $\sigma'' = \sigma \setminus \sigma'$ forms at $p \odot (\nu \cup \sigma')$ and $p \odot (\nu \cup \sigma' \cup \sigma'') = p \odot (\nu \cup \sigma)$ is reached under $p \odot M^{\overline{c}}(R)$.

Case 3b) at least one house $h^{i_b} \in H_\sigma \setminus (H_{\sigma'} \cup \{h^1\})$ points to different agents at $p \odot \nu$ and $p \odot (\nu \cup \sigma')$ according to $p \odot \overline{c}$. By (C4) all ownership continues and agent $p(i_b)$ must broker house h^{i_b} given $(p \odot \overline{c})_{p\odot\nu}$. By (C5) then, at most one agent is an owner given c_ν and $c_{\nu\cup\sigma'}$. Since agent 2 does own houses given c_ν and $c_{\nu\cup\sigma'}$ and since there can never be more than one broker at any submatching, agents 2 and agent i_b are the only two agents matched by σ but not by σ'. So the cycle yielding $p \odot \sigma$ at $p \odot \nu$ under $p \odot M^c(R)$ is such that $p(1) \to \sigma(1) \to \dots p(2) \to h^{i_b} \to p(i_b) \to h^1$. By (C6) agent 2 owns h^{i_b} at $\nu \cup \sigma'$. Since $p(2)$ prefers h^{i_b} to all houses $\overline{H}_\nu$, he points to h^{i_b} at $p \odot (\nu \cup \sigma')$ and the cycle $p(2) \to h^{i_b} \to p(2)$ forms. By (C6) agent i_b must inherit house h^2 at $\nu \cup \sigma' \cup \{(2, h^{i_b})\}$ given $\bar{c}$. Another short cycle $p(i_b) \to h^2 \to p(i_b)$ forms at $p \odot (\nu \cup \sigma' \cup \{(2, h^{i_b})\})$ under $p \odot M^{\bar{c}}(R)$ and $p \odot (\nu \cup \sigma)$ is reached. The following figure shows the cycles yielding $p \odot \sigma$ under $p \odot M^c(R)$ and under $p \odot M^{\bar{c}}(R)$ in Cases 3a) and 3b).

Original cycle in Case 3

$p(1) \to \sigma(1) \to \cdots \to h^2 \to \ p(2) \to \sigma(2) \to \qquad \ldots \to h^1 \to \ p(1)$

Cycles in Case 3a)

$p(1) \to \sigma(1) \to \cdots \to h^2 \to \ p(1)$

$p(2) \to \sigma(2) \to \qquad \ldots \to h^1 \to \ p(2)$

Cycles in Case 3b)

$p(1) \to \sigma(1) \to \cdots \to h^2 \to \ p(1)$

$p(2) \to \sigma(2) = h^{i_b} \to \ p(2)$

$p(i_b) \to h^2 \to \ p(i_b)$

In either case, the submechanisms defined by $c[\nu \cup \sigma]$ and $\bar{c}[\nu \cup \sigma]$ are identical, since $1, 2 \in N_{\nu \cup \sigma}$ and since $p_{1,2}(i) = i$ holds for all $i \notin N_{\nu \cup \sigma}$. In sum we have $(p \odot M^c)(R) = (p \odot M^{\bar{c}})(R) = (f(p) \odot M^{\bar{c}})(R)$.

To see that f is a bijection define $P^0 \colon = \{p \in P : f(p) = p\}$. Since f restricted to P^0 is a bijection and since $f(p) \neq f(p')$ holds for any two different $p, p' \in P \backslash P^0$, it suffices to show that $f(p) \notin P^0$ holds for any $p \notin P^0$. Fix any $p \notin P^0$ and define ν and σ as above. Noting that $f(p)(1) = p(2)$ and $f(p)(2) = p(1)$ and $f(p)(i) = p(i)$ for all $i \notin \{1, 2\}$, we see that $M^{p \odot c}(R)$ and $M^{f(p) \odot c}(R)$ proceed identically as long as $p(1)$ and $p(2)$ remain unmatched. So $p \odot \nu = f(p) \odot \nu$ is also the maximal submatching with $f(p)(1), f(p)(2) \notin p \odot \nu$ that is reachable under $M^{f(p) \odot c}(R)$. Since $p \notin P^0$ we have that $2 \in N_\sigma$ and $1 \notin N_\sigma$, so that there is a pointing cycle $p(2) \to \sigma(2) \to \ldots h^2 \to p(2)$ (yielding $p \odot \sigma$) and a pointing chain $p(1) \to h^* \to \cdots \to p(2)$ under $M^{p \odot c}(R)$ at $p \odot \nu$. Conversely under $M^{f(p) \odot c}(R)$ at $p \odot \nu$, the pointing cycle $p(1) = f(p)(2) \to h^* \to \cdots \to h^2 \to f(p)(2) = p(1)$ forms since h^2 is controlled by $f(p)(2) = p(1)$ given $(f(p) \odot c)_{p \odot \nu}$. By the same token there is a pointing chain $f(p)(1) = p(2) \to \sigma(2) \to \dots h^2 \to p(1) = f(p)(2)$ under $M^{f(p)\odot c}(R)$ at $p \odot \nu$. Since $p(1) = f(p)(2) \to h^* \to \cdots \to h^2 \to f(p)(2) = p(1)$ is the only cycle at $p \odot \nu$ under $f(p) \odot c$ at R and since this cycle involves agent $f(p)(2)$ but not agent $f(p)(1)$ we have $f(p) \notin P^0$ as required. $\square$

Example 9 To see that the tight structure $c[\{(6, h_5)\}]$ and the control rights structure $\overline{c}$, as constructed in Example 8 are s-equivalent fix a profile or preferences R for agents $\{1, 2, 3, 4, 5, 7\}$ and houses $\{h_1, h_2, h_3, h_4, h_6\}$ with

$$R_1 : h_2, h_3 \quad R_2 : h_4, h_2, h_6 \quad R_3 : h_2, h_4, h_1 \quad R_i : h_1 \text{ for } i = 4, 5, 7.$$

To understand the definition of f for all permutations first consider id, the identity. Since the only cycle that forms at $\emptyset$ given $M^{c[\{(6,h_5)\}]}(R)$ yields the submatching $\{(1, h_2), (2, h_4), (4, h_1)\}$, f maps id onto itself. Under $M^{\overline{c}}(R)$ a cycle yielding $\{(1, h_2)\}$ forms at $\emptyset$. At $\{(1, h_2)\}$ under $M^{\overline{c}}(R)$ agent 2 owns h_1 and a cycle yielding $\{(2, h_4), (4, h_1)\}$ forms. Once $\{(1, h_2), (2, h_4), (4, h_1)\}$ is reached $M^{c[\{(6,h_5)\}]}(R)$ and $M^{\overline{c}}(R)$ continue identically. So $M^{c[\{(6,h_5)\}]}(R)$ equals $M^{\overline{c}}(R)$ and we obtain $M^{c[\{(6,h_5)\}]}(R) = (id \odot M^{c[\{(6,h_5)\}]})(R) = (f(id) \odot M^{\overline{c}})(R) = M^{\overline{c}}(R)$ as required.

Next consider the permutation $p_{2,3}$ for which only agents 2 and 3 swap roles. The only cycle that forms at $\emptyset$ given $(p_{2,3} \odot M)^{c[\{(6,h_5)\}]}(R)$ yields the submatching $\{(p_{2,3}(2), h_2)\} = \{(3, h_2)\}$ which involves agent $p_{2,3}(2) = 3$ but not agent $p_{2,3}(1) = 1$. We therefore have $f(p_{2,3})(1) = p_{2,3}(2) = 3$, $f(p_{2,3})(2) = p_{2,3}(1) = 1$, and $f(p_{2,3})(i) = p_{2,3}(i)$ for all $i \neq 1, 2$. Under $(f(p_{2,3}) \odot \overline{c})_\emptyset$ agent $p_{2,3}(2) = 3$ owns all houses owned by agents 1 and 2 under $c[(6, h_5)]_\emptyset$, in particular house h_2. So the cycle yielding the submatching $\{f(p_{2,3}(1), h_2)\} = \{(p_{2,3}(2), h_2)\} = \{(3, h_2)\}$ forms at $\emptyset$ under $(f(p_{2,3}) \odot M^{\overline{c}})(R)$. Once this submatching is reached $(p_{2,3} \odot M^{c[\{(6,h_5)\}]})(R)$ and $(f(p_{2,3}) \odot M^{\overline{c}})(R)$ continue identically and we obtain $(p_{2,3} \odot M^{c[\{(6,h_5)\}]})(R) = (f(p_{2,3}) \odot M^{\overline{c}})(R)$ as required.

With Lemmas 5 and 6 in place we are now ready to prove Lemma 4.

Proof of Lemma 4. Say there are $m \geq 2$ owners at the start of c. By Lemma 6 there exists an s-equivalent control rights structure $\overline{c}$ with $m - 1$ owners. By Lemma 3 the lax mechanism $M^{\overline{c}}$ can alternatively be represented as a trading and braiding mechanism M^{c^1} such that the tight structure c^1 and the control right structure $\overline{c}$ start with equally many owners. So c has an s-equivalent structure c^1 starting with $m-1$ owners. Proceeding inductively construct a sequence of s-equivalent tight structures $c = c^0, c^1, c^2 ... c^{m-1}$ such that there is one fewer owner at $c^{k+1}_{\emptyset}$ than at $c^k_{\emptyset}$ for all $0 \leq k$ in the sequence. Since c starts out with m owners, there is exactly one owner at the start of c^{m-1}. By (C2') there is no broker at $c^{m-1}_{\emptyset}$, so $c^{m-1}_{\emptyset}$ is dictatorial. Since s-equivalence is transitive c is s-equivalent to c^{m-1}. □

Example 10 Once again consider the tight structure the $c[\{(6, h_5)\}]$ for agents $\{1, 2, 3, 4, 5, 7\}$ and houses $\{h_1, h_2, h_3, h_4, h_6\}$. Note that agents 1, 2, and 4 are owners according to $c[\{6, h_5\}]$. In Example 8 I applied Lemma 6 to construct a s-equivalent control rights structure $\overline{c}$ which is, as I noted, not tight. By Lemma 3 the mechanism represented by $\overline{c}$ can alternatively be represented by a tight structure c^1 that has the same number of owners as $\overline{c}$ at any $\overline{c}$-relevant submatching. The only owner according to $\overline{c}_{\{(4,h_4)\}}$, for example, is the c^1-dictator at $\{(4, h_4)\}$.

There is one fewer owner at the start of c^1 than at the start of $c[\{(6, h_5)\}]$: only 1 and 4 are owners according to $c^1_{\emptyset}$. Now consolidate the ownership of agents 1 and 4 in c^1 along the lines of Lemma 6 to obtain a s-equivalent control rights structure c'. This control rights structure violates (C2') as agent 7 brokers house h_6 at $\emptyset$ even though there is only one owner. By Lemma 3 there exists a tight structure c^2 with 1 as the dictator at $\emptyset$ that represents the same mechanism as the control rights structure c'. So the tight structure $c[\{(6, h_5)\}]$ that starts with 3 owners is s-equivalent to the tight structure c^2 that starts with a dictator.

9 Transformation of Braids

The strategy of proof requires that any tight structure c has a s-equivalent tight structure $\tilde{c}$ that starts with a dictator. Lemma 4 shows that such s-equivalent tight structures exist if c starts with two or more owners. But if c is a braid then there are no owners to begin with: all houses are brokered.

Lemma 8 below, shows that even tight structures that start with no owners are s-equivalent to tight structures that start with a dictator.

Some more concepts are needed to prove Lemma 8. A random matching mechanism $\mathfrak{M} : \mathcal{R} \to \Delta M$ is **ex post Pareto optimal** if $\mathfrak{M}(R)$ can for each R be represented as a lottery over matchings that are Pareto optimal at R. The same mechanism $\mathfrak{M}$ **treats equals equally** if two agents who declare the same preference face the same lottery over houses.

A random matching mechanism $\mathfrak{M} : \mathcal{R} \to \Delta M$ for three agents and three houses is **ordinally strategyproof** if for any agent i, any profile R with $R_i : h, h', h''$ and deviation R'_i, $\mathfrak{M}(R)$ assigns (weakly) higher respectively lower probability to the matches (i, h) and (i, h'') than $\mathfrak{M}(R'_i, R_{-i})$. So there is no profile R at which any agent can distort his preference to either increase the probability of his most preferred house or decrease the probability of his least preferred house. [11]

The proof of Lemma 8 starts with Lemma 7 on random matching mechanism for small matching problems: With only three agents and three houses, random serial dictatorship is the unique ex post Pareto optimal, ordinally strategyproof random matching mechanism that treats equals equally. Since braids involve only three houses, the result applies to braids with three agents. Braids, in turn are the only trading and braiding mechanisms that start with no owners.

Lemma 7 *Let $H = \{a, b, c\}$ and $N = \{1, 2, 3\}$. Any ex post Pareto optimal and ordinally strategy proof $\mathfrak{M} : \mathcal{R} \to \Delta\mathcal{M}$ that satisfies equal treatment of equals, is a random serial dictatorship.*

The proof of Lemma 7 directly follows from Lemma 2 and Proposition 2 in Bogomolnaia and Moulin [4]. To keep the present treatment as self-contained as possible I state an alternative proof in the online appendix. The main difference between the proofs in Bogomolnaia and Moulin [4] and the online appendix, is that my proof does not use the concept of ordinal efficiency. Lemma 8 applies Lemma 7 to show that the symmetrization of

[11] A random matching mechanism is non-bossy if not agent can alter someone else's lottery over matches without altering his own. For a proof that non-bossiness is robust to randomization when the base mechanism is strategyproof see Bade [3].

any braid is identical to random serial dictatorship.

Lemma 8 *Let* $|H| = 3$, $|N| \geq 3$. *Any braid is s-equivalent to any serial dictatorship.*

Proof Say c defines a braid where $c_\emptyset$ assigns agents 1, 2, and 3 brokerage rights. Say c' defines a serial dictatorship with agents 1, 2, and 3 as the first three dictators. Fix an arbitrary matching μ and profile of preferences R. The agents N_μ are matched under $(p \circ M^c)(R)$ if and only if $N_\mu = p(\{1, 2, 3\})$. The same holds for c'. So the probability that the agents N_μ are matched under $\Delta M^c(R)$ (or respectively $\Delta M^{c'}(R)$) equals $\frac{1}{|\mathcal{S}|}$, where $\mathcal{S}$ is the set of all three agent subsets of N.

Define $\overline{R}$ as the restriction of R to N_μ and say that $\overline{c}$ and $\overline{c}'$ respectively define a braid and a serial dictatorship for N_μ and H. Note that μ is also a matching in the restricted problem with N_μ and H the sets of agents and houses. The probabilities of μ under the symmetrizations of $M^{\overline{c}}$ and $M^{\overline{c}'}$ are $\Delta M^{\overline{c}}(\overline{R})(\mu)$ and $\Delta M^{\overline{c}'}(\overline{R})(\mu)$. Since the braid defined by $\overline{c}$ is Pareto optimal and strategyproof, its symmetrization $\Delta M^{\overline{c}}$ is ex post Pareto optimal, ordinally strategyproof and satisfies equal treatment of equals. Since $\overline{c}$ defines a mechanism for three agents and three houses Lemma 7 applies, and $\Delta M^{\overline{c}}$ equals random serial dictatorship $\mathfrak{M}$, which, by definition, equals $\Delta M^{\overline{c}'}$. So we obtain

$$\Delta M^c(R)(\mu) = \frac{\Delta M^{\overline{c}}(\overline{R})(\mu)}{|\mathcal{S}|} = \frac{\Delta M^{\overline{c}'}(\overline{R})(\mu)}{|\mathcal{S}|} = \Delta M^{c'}(R)(\mu).$$

Since R and μ were chosen arbitrarily we obtain $\Delta M^c(R)(\mu) = \Delta M^{c'}(R)(\mu)$ for all R and μ and c is s-equivalent to c'. □

10 Abstract Transformations

The main result of the present section, Lemma 9 applies to tight structures that are identical except for their submechanisms following on some dictatorially chosen submatching ν^*. Such tight structures are s-equivalent if their submechanisms following on ν^* are s-equivalent. Using Lemmas 4 and 8 to define s-equivalent submechanisms, Lemma 9 allows me to define a sequence of s-equivalent tight structures with more and more dictatorially chosen submatchings as is required for the proof of Theorem 1.

Lemma 9 *Fix a tight structure c and a c-relevant submatching ν^* that is c-dictatorially chosen. Define a set of control rights functions c' such that $c_\nu = c'_\nu$ holds for all c-relevant ν with $\nu^* \not\subset \nu$ and $c'[\nu^*]$ a tight structure. If $c[\nu^*]$ and $c'[\nu^*]$ are s-equivalent, then c and c' are s-equivalent tight structures.*

The proof of Lemma 9 builds on the preliminary Lemma 10, which shows that the set of control rights c' defined in Lemma 9 indeed constitute a tight structure.

Lemma 10 *Fix a tight structure c and a c-relevant submatching ν^* that is c-dictatorially chosen. Define a set of control rights functions c' such that $c_\nu = c'_\nu$ holds for all c-relevant ν with $\nu^* \not\subset \nu$ and $c'[\nu^*]$ a tight structure. Then c' is a tight structure.*

Proof Fix an arbitrary submatching ν. If $\nu^* \not\subset \nu$ then ν is c'-relevant if and only if it is c-relevant and $c'_\nu : = c_\nu$ is welldefined for any c'-relevant ν with $\nu^* \not\subset \nu$. If ν with $\nu^* \subset \nu$ is c'-relevant then Lemma 1 (on unique c-paths towards c-dictatorially chosen matchings) implies that there exists a $c'[\nu^*]$-relevant ν' such that $\nu = \nu^* \cup \nu'$. In this case c'_ν is uniquely defined as $c'[\nu^*]_{\nu'}$. So c' defines control rights c'_ν for each c'-relevant submatching ν.

Since c and $c'[\nu^*]$ are both tight structures c'_ν satisfies (C1), (C2') and (C3) for each c'-relevant ν. Now fix a c'-relevant ν° and a direct c'-successor ν to ν°. If $\nu^* \subset \nu^\circ$ or $\nu^* \not\subset \nu$ then (C4), (C5), and (C6) hold for the pair of ν° and ν since $c'[\nu^*]$ and c are tight structures. So we only need to consider the case that $\nu^* \not\subset \nu^\circ$ and $\nu^* \subset \nu$. By Lemma 1 there is a unique c-dictatorial path towards ν^*. So ν equals ν^* and ν° is the unique c-dictatorial submatching with $\nu^\circ \cup \{(i, \nu^*(i))\} = \nu^*$ where i is the c-dictator at ν°. Since $\nu^* \not\subset \nu^\circ$ we have $c_{\nu^\circ} = c'_{\nu^\circ}$, so that ν° is c' dictatorial. Since ν° is c'-dictatorial, c' trivially satisfies (C4), (C5), and (C6) at the pair ν°, ν^*. □

Example 11 Reconsider the tight structure c in Example 2 together with the submatching $\nu^* :\ = \{(6, h_6), (5, h_5), (4, h_4)\}$, which is by Example 3 c-dictatorially chosen. Use Lemma 10 to generate two tight structures c^1 and c^G by respectively replacing the braid following the submatching $\{(6, h_6), (5, h_5), (4, h_4)\}$ with a serial dictatorship and Gale's top trading cycles. Concretely define c^1 and c^G such that $c^1[\nu^*]$ defines a serial dictatorship (with any order of agents as dictators), $c^G[\nu^*]$ defines Gale's top trading cycles with $c^G[\nu^*](h_i) = i$ for $i \in \{1, 2, 3\}$, and such that $c^1_\nu = c^G_\nu :\ = c_\nu$ holds for each c-relevant ν with $\nu^* \not\subset \nu$.

We are now ready to show that the tight structure c' constructed in Lemma 9 is indeed s-equivalent to c.

Proof of Lemma 9. By Lemma 10 c' is a tight structure. Fix an arbitrary profile of preferences R. To prove the s-equivalence of c and c' construct a bijection $f : P \to P$ such that $(p \odot M^c)(R) = (f(p) \odot M^{c'})(R)$ holds for all $p \in P$. To do so partition P and, for each cell of this partition, define a bijection satisfying the preceding equality.

Firstly define $P^0 :\ = \{p \in P \mid p \odot \nu^* \not\subset (p \odot M^c)(R)\}$. By Lemma 2 $(p \odot M^{c'})(R) = (p \odot M^c)(R)$ holds for all $p \in P^0$. So $f^0 : P^0 \to P^0$ with $f^0(p) = p$ for all $p \in P^0$ is a bijection on P^0 with $(p \odot M^c)(R) = (f^0(p) \odot M^c)(R)$ for all $p \in P^0$. Partition the remainder $P \setminus P^0$ into sets $P^1, \dots P^K$, so that all permutations p in the same cell P^k assign each role $i \in N_{\nu^*}$ to the same agent. Formally, any two permutations $p, p' \notin P^0$ belong to the same cell P^k for some $k \in \{1, \dots, K\}$ if and only if $p(i) = p'(i)$ holds for all $i \in N_{\nu^*}$.

To define bijections $f^k : P^k \to P^k$ fix an arbitrary $k \in \{1, \dots, K\}$ and $p^k \in P^k$. Say $\overline{R}$ is the restriction of R to $\overline{N}_{p^k \odot \nu^*}$ and $\overline{H}_{p^k \odot \nu^*}$, the sets of agents and houses not matched by $p^k \odot \nu^*$, and $\overline{P}$ is the set of all permutations on $\overline{N}_{p^k \odot \nu^*}$. There exists a bijection $g : P^k \to \overline{P}$ such that $g(p)(i) = p(i)$ for all $i \in \overline{N}_{\nu^*}$: Each agent i not matched by $p^k \odot \nu^*$ is assigned the same role by p and $g(p)$. By assumption $c[\nu^*]$ and $c'[\nu^*]$ are s-equivalent. In light of the arguments laid out in Section 4, this s-equivalence implies that there exists a bijection $\overline{f} : \overline{P} \to \overline{P}$ with $(q \odot M^{c[\nu^*]})(\overline{R}) = (\overline{f}(q) \odot M^{c'[\nu^*]})(\overline{R})$ for all $q \in \overline{P}$. To define $f^k : P^k \to P^k$ let $f^k(p)(i) = p^k(i)$ if $i \in N_{\nu^*}$ and

$f^k(p)(i) = \overline{f}(g(p))(i)$ otherwise.

To see that $(p \odot M^c)(R) = (f^k(p) \odot M^{c'})(R)$ holds first note that p as well as $f^k(p)$ are elements of P^k implying that p as well as $f^k(p)$ assign each agent $i \in N_{\nu^*}$ to the role of $p^k(i)$. The definition of c, c' then implies that $p^k \odot \nu^*$ is reached under $(p \odot M^c)(R)$ as well as under $(f^k(p) \odot M^{c'})(R)$. By the definition of f^k and g we have $(f^k(p) \odot M^{c'})(R) = (p^k \odot \nu^*) \cup \big(\overline{f}(g(p)) \odot M^{c'[\nu^*]}\big)(\overline{R})$ as well as $(p \odot M^c)(R) = (p^k \odot \nu^*) \cup \big(g(p) \odot M^{c[\nu^*]}\big)(\overline{R})$. But $\overline{f}$ was defined such that $(q \odot M^{c[\nu^*]})(\overline{R}) = (\overline{f}(q) \odot M^{c'[\nu^*]})(\overline{R})$ holds for all $q \in \overline{P}$, in particular for $q = g(p)$. We in sum obtain:

$$(f^k(p) \odot M^{c'})(R) = (p^k \odot \nu^*) \cup \big(\overline{f}(g(p)) \odot M^{c'[\nu^*]}\big)(\overline{R}) = \\ (p^k \odot \nu^*) \cup \big(g(p) \odot M^{c[\nu^*]})(\overline{R}\big) = (p \odot M^{c'})(R).$$

□

Example 12 Reconsider the tight structures c, c^1 and c^G defined in Examples 2 and 11 together with $\nu^* = \{(6, h_6), (5, h_5), (4, h_4)\}$. Since $c^1[\nu^*]$ defines a serial dictatorship it is by Lemma 8 s-equivalent to $c[\nu^*]$ which defines a braid. Since $c^G[\nu^*]$ defines Gale's top trading cycles it is by Abdulkadiroglu and Sonmez [1] as well as by Knuth [9][12] s-equivalent to $c^1[\nu^*]$ which defines a serial dictatorship. So c, c^1 and c^G are by Lemma 9 s-equivalent.

11 Proof of Theorem 1

Following the overview on the proof of Theorem 1 given in Section 7 any arbitrary tight structure c^0 has to be linked via a sequence of s-equivalent tight structures $c^1, \ldots, c^K$ to a tight structure c^K that defines a serial dictatorship. To construct such a sequence for any tight structure c^0 use the following algorithm. To start go to Step $(\alpha, 1)$.

[12] Some additional arguments similar to the ones given in Lemma 8 are required to extend these equivalence results to the present case of four owners and three houses

Step (α, k): If c^{k-1} defines a serial dictatorship end with $K\colon = k-1$. If c^{k-1} does not define a path dependent dictatorship, go to Step (β, k). In all other cases, go to Step (γ, k).

Step (β, k): Fix a minimal c^{k-1}-relevant but not c^{k-1}-dictatorial submatching ν^*. Define c^k such that $c^k_\nu\colon = c^{k-1}_\nu$ for any c^{k-1}-relevant ν with $\nu^* \not\subset \nu$. Define $c^k[\nu^*]$ as a tight structure that is s-equivalent to $c^{k-1}[\nu^*]$ and starts with a dictator. Go to Step $(\alpha, k+1)$.

Step (γ, k): Fix three c^{k-1}-relevant submatchings $\nu^* \subset \nu' \subset \nu''$, such that $c^{k-1}[\nu^*]$ is a serial dictatorship with i the dictator at ν' and $j < i$ the dictator at ν''. Define $c^k_\nu\colon = c^{k-1}_\nu$ for any c^{k-1}-relevant ν with $\nu^* \not\subset \nu$. Define $c^k[\nu^*] = p_{i,j} \odot c^{k-1}[\nu^*]$. Go to Step $(\alpha, k+1)$.

To see that the above algorithm defines a s-equivalent tight structure for each k, note that c^0 is by assumption a tight structure. Now suppose that the above algorithm generates s-equivalent tight structures $c^{k'}$ for all $k' < k$ for some fixed $k \geq 1$. Step (α, k) prescribes a continuation for any possible case. The submatching ν^*, defined in either Step (β, k) or (γ, k) is c^{k-1}-dictatorially chosen. When Step (β, k) is used Lemmas 4 and 8 guarantee the existence of a tight structure $c^k[\nu^*]$ that is s-equivalent to $c^{k-1}[\nu^*]$ and starts with a dictator. When Step (γ, k) is used, $c^k[\nu^*]$ and $c^{k-1}[\nu^*]$ both define serial dictatorships and are therefore s-equivalent. So c^k is by Lemma 10 in either case a s-equivalent tight structure.

To see that the sequence terminates with a tight structure c^K that defines a serial dictatorship, consider Step k where Step (α, k) prescribes to follow Step (β, k) and say that ν^* is chosen at Step (β, k). So ν^* is c^{k-1}-dictatorially chosen, but not c^{k-1}-dictatorial, implying that $\nu^* \not\subset \nu'$ holds for any c^{k-1}-dictatorially chosen ν' other than ν^*. Since $c^k_\nu = c^{k-1}_\nu$ holds for all c^{k-1}-relevant submatchings ν with $\nu^* \not\subset \nu$, the set of c^{k-1}-dictatorially chosen submatchings is a subset of the set of c^k-dictatorially chosen submatchings. To see that this subset relation is strict, note that $\nu^* \cup \{(i, h)\}$ with i the c^k-dictator at ν^* and h a house in $\overline{H}_{\nu^*}$, is c^k- but not c^{k-1}-dictatorially chosen. So the set of c^k-dictatorially chosen submatchings strictly increases with each Step (β, k). Since there are only finitely many submatchings, there exists a number K' such that $c^{K'}$ defines a path dependent dictatorship. The reordering prescribed by Steps (γ, k) then achieves a serial dictatorship since all re-orderings of agents as dictators go into the same direction: for i to swap with a later dictator j in some branch of a path dependent dictatorship c^k we must have $j < i$.

Example 13 To illustrate the proof of Theorem 1 let c^0 be the tight structure c defined in Example 2. There exists at least one c^0-relevant submatching that is not c^0-dictatorial: c for example prescribes for all three remaining houses at $\{(6, h_6), (5, h_5), (4, h_4)\}$ to be brokered. So Step $(\alpha, 1)$ determines that the algorithm should proceed with Step $(\beta, 1)$. To apply this step choose a minimal c^0-relevant but not c^0-dictatorial submatching ν^*, say $\nu^* := \{(6, h_6), (5, h_5), (4, h_4)\}$. Step $(\beta, 1)$ then results in a s-equivalent tight structure c^1 as the one defined in Example 12. There are strictly more c^1-dictatorially chosen submatchings than there are c^0-dictatorially chosen ones, $\{(6, h_6), (5, h_5), (4, h_4), (1, h_1)\}$, for example, is c^1- but not c^0-dictatorially chosen.

There exists at least one c^1-relevant submatching that is not c^1-dictatorial. According to c^1 agents 1 and 2 are both owners at $\{(6, h_5)\}$. So Step $(\alpha, 2)$ determines that the algorithm should proceed with Step $(\beta, 2)$. To apply this step choose a minimal c^1-relevant but not c^1-dictatorial submatching, say $\nu^* := \{(6, h_5)\}$. A tight structure c^2 is the defined by letting $c^2_\nu = c^1_\nu$ for all c^1-relevant ν with $\nu^* \not\subset \nu$ and $c^2[\nu^*]$ a tight structure that is s-equivalent to $c^1[\nu^*]$ and starts with a dictator. Since there are at least two owners at the start of $c^1[\nu^*]$ such a s-equivalent tight structure $c^2[\nu^*]$ exists by Lemma 4 and was constructed in Example 10.

For an example of a c^2- but not c^1-dictatorially chosen submatching consider $\{(6, h_5), (1, h_1)\}$. Since this submatching $\{(6, h_5), (1, h_1)\}$ is not itself c^2-dictatorial, Step $(\alpha, 3)$ directs the algorithm to Step $(\beta, 3)$ which could then, for example, convert $\{(6, h_5), (1, h_1)\}$ into a c^3-dictatorial submatching. Steps (α, k) and (β, k) are then applied until some K' where any $c^{K'}$-relevant submatchings is $c^{K'}$-dictatorial, so that $c^{K'}$ defines a path dependent dictatorship. To further transform this path dependent dictatorship into a serial dictatorship, the algorithm proceeds by the reordering of agents as dictators. This reordering stops since Step (γ, k) inverts the order to two agents i, j as dictators only if $i < j$.

12 Conclusion

The main result of the present paper shows that random serial dictatorship is identical to the symmetrization of any good matching mechanism for sets of agents and houses of any size. Special cases of this result were already established by Abdulkadiroglu and Sönmez [1], Knuth [9], Carroll [5], Pathak and Sethuraman [12], and Lee and Sethuraman [10].

Since Lee and Sethuraman [10] contains the most general results among the above list, I use Lee and Sethuraman [10] as a benchmark to put the present results into context. The present paper covers a larger class of mechanisms: Lee and Sethuraman [10] show that any hierarchical exchange mechanism defined by an inheritance tree following Papai [11] is s-equivalent to serial dictatorship. Conversely I show that this hypothesis applies to the all good mechanisms, not just to hierarchical exchange mechanisms.[13] Lee and Sethuraman [10] furthermore considers only matching problems with equally many agents and houses, whereas the only restriction I impose is that all agents prefer begin matched to homelessness.[14] In a third dimension, Lee and Sethuraman's [10] results are more general: While I am only concerned with the case of full symmetry between all agents, Lee and Sethuraman [10] also consider the case of partial symmetries. They use the notion of G-invariance to describe random matching problems in which all agents within some groups are to be treated symmetrically without the imposition of any symmetry requirements across groups.

The development of this literature suggests an even more general hypothesis: Is it true that any Pareto optimal and strategyproof mechanisms is s-equivalent to serial dictatorship? The problem here is that the present

[13] A trading and braiding mechanism is such a hierarchical exchange mechanism following Papai [11] if and only if it does not involve and brokers or braids. Papai [11] characterizes hierarchical exchange mechanisms as the set of reallocation proof good mechanisms.

[14] While Lee and Sethuraman's [10] strategy of proof easily extends to the case of more agents than houses, it is not clear how to apply it to the alternative case of more houses than agents.

approach just like the ones by Abdulkadiroglu and Sönmez [1], Knuth [9], Carroll [5], Pathak and Sethuraman [12], and Lee and Sethuraman [10] all build on trading algorithms. To extend any of these approaches to the set of all Pareto optimal and strategyproof mechanisms, we would need a trading algorithm that characterizes all such mechanisms. But the set of all Pareto optimal and strategyproof matching mechanisms has yet to be characterized. Since the symmetrization of any Pareto optimal and strategyproof mechanism is a ex post Pareto optimal and ordinally strategyproof mechanism that treats equals equally one could also try to directly show that any random matching mechanism with these three properties is random serial dictatorship. Bogomolnaia and Moulin [4] indeed do not refer to a trading algorithm to prove the uniqueness of random serial dictatorship with three agents and three objects. The problem with the extension of this approach to larger sets of agents and houses, is that set of Pareto optima can can be hard to find: Saban and Sethuraman [15] have shown that finding all house-agent matches that do not form in any Pareto optimum at some R is an NP-complete problem.

Some papers, such as Bogomolnaia and Moulin [4] have presented possible tradeoffs between Pareto optimality and strategy proofness while maintaining equal treatment of equals and non-bossiness. In this context, random serial dictatorship is typically used as the benchmark of a mechanism that is best in terms of its incentive properties (ordinally strategy proof) and worst in terms of its welfare properties (only ex post Pareto optimal). This paper strengthens the case for using random serial dictatorship as the benchmark. While initially one could have criticized the choice of a particular good mechanism as the base of the symmetrization, I have shown that this choice does not matter: the symmtrization of any good mechanism leads to random serial dictatorship.

References

[1] Abdulkadiroglu , A. and T. Sönmez: "Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems", *Econometrica*, 66, (1998), 689 - 701.

[2] Bade, S. "Pareto optimal, strategy proof and non-bossy matching mechanisms" *mimeo*, Royal Holloway, 2014.

[3] Bade, S. "Fairness and group-strategyproofness clash" *Journal of Economic Theory*, 165, (2016), 257-262.

[4] Bogomolnaia, A. and H. Moulin: "A New Solution to the Random Assignment Problem," *Journal of Economic Theory*, 100, (2001), 295-328.

[5] Carroll, G.: "A General Equivalence Theorem for Allocation of Indivisible Objects," *Journal of Mathematical Economics*, 51, (2014), 163-177.

[6] Ekici, O.: 'H'ouse Allocation with Existing Tenants: Two Equivalence Results"," *mimeo* Ozyegin University, 2017.

[7] Erdil, A.: "Strategy-proof Stochastic Assignment," *Journal of Economic Theory*, 151 (2014) 146-162.

[8] Kurino, M. and O. Kesten: "Strategy-proof Improvements upon Deferred Acceptance: A Maximal Domain for Possibility", *mimeo*, Tsukuba University, June 2017.

[9] Knuth, D.: "An exact analysis of stable allocation", *Journal of Algorithms*, 20, (1996), 431-442.

[10] Lee, T. and J. Sethuraman: "Equivalence results in the allocation of indivisible objects: A unified view", *mimeo*, Columbia University, 2011.

[11] Papai, S.: "Strategyproof Assignment by Hierarchical Exchange", *Econometrica*, 68, (2000), 1403 - 1433.

[12] Pathak, P and J. Sethuraman.: "Lotteries in student assignment: An equivalence result", *Theoretical Economics*, 6, (2011), 1-17.

[13] Pycia, M. and U. Ünver: "Incentive Compatible Allocation and Exchange of Discrete Resources," *mimeo* UCLA, February 2014.

[14] Pycia, M. and U. Ünver: "Incentive Compatible Allocation and Exchange of Discrete Resources," *Theoretical Economics*, 12 2017, 287-329.

[15] Saban, D. and J. Sethuraman: “The Complexity of Computing the Random Priority Allocation Matrix” *Mathematics of Operations Research*, 40, (2015), 1005-1014.

[16] Shapley, L. and H. Scarf: “On Cores and Indivisibility”, *Journal of Mathematical Economics*, 1, (1974), 23 - 37.