2D PROBLEMS IN GROUPS

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ABSTRACT. We investigate a conjecture about stabilisation of deficiency in finite index subgroups and relate it to the D2 Problem of C.T.C. Wall and the Relation Gap problem. We verify the pro-p version of the conjecture, as well as its higher dimensional abstract analogues.

Given a finitely presented group G, the deficiency $\delta(G)$ of G is defined as the maximum of $|X| - |R|$ over all presentations $G = \langle X \mid R \rangle$. We related deficiency of a group with 2-dimensionality in [8] and proposed the following conjecture.

2D Conjecture ([8]). *Let G be a residually finite finitely presented group such that $\delta(H) - 1 = [G : H](\delta(G) - 1)$ for every subgroup H of finite index in G. Then G has a finite 2-dimensional classifying space $K(G, 1)$.*

In this paper, we relate the above conjecture with two well-known problems in topological group theory: *Wall's D2 problem* and the *Relation Gap problem*. The main purpose of the paper is to explain the implications

affirmative D2 problem $\Rightarrow$ no relation gap $\Rightarrow$ 2D conjecture

1. Background

Let G be a finitely presented group. Set $d(G)$ to be the cardinality of a minimal generating set of G.

We denote by $b_i(G) = \dim_{\mathbb{Q}} H_i(G, \mathbb{Q})$ and note that $\delta(G) \leq b_1(G) \leq d(G)$. Starting with a presentation $\langle X|R \rangle$ for G, one obtains a Schreier presentation for H with $[G : H](|X| - 1) + 1$ generators and $[G : H]|R|$ relations showing that

$$\delta(H) - 1 \geq [G : H](\delta(G) - 1).$$

We are interested in the situation when the above inequality is in fact equality for every finite index subgroup H of G.

We next introduce the invariant $\mu_n(G)$ of Swan [13]. Let $n \in \mathbb{N}$. A partial free resolution of $\mathbb{Z}$ of length n is an exact sequence

$$\mathcal{F} : \quad (\mathbb{Z}G)^{f_n} \to (\mathbb{Z}G)^{f_{n-1}} \to \cdots \to (\mathbb{Z}G)^{f_0} \to \mathbb{Z} \to 0 \tag{1}$$

and we define $\mu_n(\mathcal{F}) = \sum_{i=0}^{n} (-1)^{n-i} f_i$.

Recall the well-known Morse inequalities.

Proposition 1. *Let $n \in \mathbb{N}$ and $\mathcal{F}$ be a partial free resolution (1) as above. Then*

$$\sum_{i=0}^{n}(-1)^{n-i}b_i(G) \leq \mu_n(\mathcal{F}).$$

R. Swan [13] defined the following invariant while studying free resolutions of modules of finite groups.

Definition 2. *Let $n \in \mathbb{N}$. The invariant $\mu_n(G)$ is defined as the minimum of $\mu_n(\mathcal{F})$ as $\mathcal{F}$ ranges over all partial free resolutions $\mathcal{F}$ of $\mathbb{Z}$.*

Given a presentation of G with e_1 generators and e_2 relations one has the partial free resolution

$$(\mathbb{Z}G)^{e_2} \xrightarrow{\partial_2} (\mathbb{Z}G)^{e_1} \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} \mathbb{Z} \to 0 \tag{2}$$

arising as the cellular chain complex of the universal cover of the presentation complex of G. By taking a presentation which realizes the deficiency of G we obtain $\mu_2(G) \leq 1 - \delta(G)$. The case $n = 2$ of the Morse inequalities applied to (2), together with $b_0(G) = 1$ gives the well-known inequality $\delta(G) \leq b_1(G) - b_2(G)$.

1.1. Groups with two dimensional classifying spaces.

The deficiency is easy to compute for groups which have finite two-dimensional classifying spaces. Examples of such groups are surface groups or more generally, torsion-free one relator groups and direct products of two free groups.

Lemma 3. *If a group G has a finite two-dimensional space $K(G, 1)$, then $\delta(G) = 1 - \chi(G)$ and consequently, $\delta(H) - 1 = [G : H](\delta(G) - 1)$ for every subgroup H of finite index in G.*

For example $\delta(F_n \times F_m) = -(n - 1)(m - 1)$ while the deficiency of a torsion-free one relator group defined on d generators is $d - 2$.

The 2D Conjecture stated in the introduction proposes that the converse of Lemma 3 holds. Note that its 1-dimensional analogue is true as shown by R. Strebel [12] (see also [1, Theorem 7] for a different perspective).

Proposition 4 ([12])**.** *Let G be a finitely generated residually finite group. Then G is a free group if and only if $d(H) - 1 = |G : H|(d(G) - 1)$ for every subgroup H of finite index in G.*

Strebel proved Proposition 4 as an answer to a question of Lubotzky and van den Dries [10], who had shown that its analogue does not hold in the class of profinite groups. At the same time Lubotzky [9, Proposition 4.2] proved that the analogue of Proposition 4 is true in the class of pro-p groups. We will return to pro-p groups in section 5 below.

We remark that the 2D conjecture is closely connected with gradients in groups and their L^2 cohomology. The following basic result characterizes groups G with two dimensional classifying spaces in terms of their L^2 Betti numbers $\beta_i(G)$.

Lemma 5 ([8]). *Let G be an infinite finitely presented group. Then $\delta(G) - 1 \leq \beta_1(G) - \beta_2(G)$ with equality if and only if G has a two dimensional classifying space.*

In particular any counterexample to the 2D conjecture must be a group G with *deficiency gradient* strictly less than $\beta_1(G) - \beta_2(G)$, see [8] for more details on this connection.

2. Wall's D2 Problem

Wall's D2 problem is a generalisation of the Eilenberg Ganea Conjecture and belongs to the class of questions that explore links between homological and geometric dimensions. A finite CW-complex X is said to be a D2 complex if it has cohomological dimension 2. The D2 Problem for a finitely presented group G asks if every finite D2 complex with fundamental group G is homotopy equivalent to a finite 2-complex. If the answer is affirmative we shall say that G has the *D2 property*. The problem was proposed by C.T.C. Wall in 1965 [14] and little is known about it except in the case when G is finite, free or abelian, see [7].

The Eilenberg-Ganea Conjecture asks if every group of cohomological dimension 2 is of geometric dimension 2. Note that a group of cohomological dimension 2 does not necessarily have a finite classifying space, as famously shown by M. Bestvina and N. Brady [2]. However, if one assumes that a group G of cohomological dimension 2 has a finite classifying space X, then X is a D2 complex. If in addition G has the D2 property, then X is homotopy equivalent to a finite 2-complex. So, G has geometric dimension two, as predicted by Eilenberg-Ganea.

3. The Relation Gap problem

Suppose that a finitely presented group G is given by the quotient F/N where F is free on the group generators X and N is normally generated in F by the relators $R \subset F$. The action of F by conjugation on N induces an action of G on the abelianisation N^{ab} of N. This makes N^{ab} into a G-module called the relation module of the presentation. Evidently, the G-module N^{ab} can be generated by $|R|$ elements and so the G-rank of N^{ab}, written $d_G(N^{ab})$, satisfies $d_G(N^{ab}) \leq d_F(N)$, where $d_F(N)$ is the minimum number of normal generators required for N.

A presentation is said to have a relation gap if $d_G(N^{ab}) \neq d_F(N)$ and the relation gap problem asks, if there exists a finitely presented group with a relation gap. As with the D2 problem, very little is known about the relation gap problem and most proposed counterexamples are not torsion-free, see [5].

We give a proof to the following.

Theorem 6. *A finitely presented group G with the D2 property does not have a relation gap for presentations realizing $\delta(G)$.*

This may be known to topological group theorists but we have not found it in the literature. There is a result of Dyer [4, Theorem 3.5] with the same statement but with the additional hypothesis $H^3(G, \mathbb{Z}G) = 0$.

We need the following.

Proposition 7 ([6] Proposition 4.3, or [3], Remark 1.3)**.** *Let G be a finitely presented group with the D2 property. Then $\mu_2(G) = 1 - \delta(G)$.*

For completeness we give a proof of Proposition 7 following [3], based on the following theorem of Wall.

Theorem 8 ([14], Theorem 4)**.** *Let X be a connected CW-complex, $G = \pi_1(X)$ and let A_* be a positive free chain complex equivalent to the cellular chain complex $C^c_*(X)$ of the universal cover of X. Let K^2 be a connected CW-complex with fundamental group G. There exists another CW complex Y and a homotopy equivalence $h : Y \to X$ such that Y is obtained from K^2 by adding 2-cells and 3-cells at the base point to obtain a D2 complex Y_0 and then further cells such that $C^c_*(Y, Y_0)$ is the part of A_* in dimension ≥ 3.*

If the symbol α_i denotes the number of i-cells or of generators in degree i then

$$\alpha_2(Y_0 - K^2) = \alpha_2(A) + \alpha_1(K) + \alpha_0(A),$$
$$\alpha_3(Y^0 - K^2) = \alpha_2(K) + \alpha_1(A) + \alpha_0(K).$$

Proof of Proposition 7. Let

$$(\mathbb{Z}G)^{f_2} \to (\mathbb{Z}G)^{f_1} \to (\mathbb{Z}G)^{f_0} \to \mathbb{Z} \to 0$$

be a partial free resolution of $\mathbb{Z}$ with $f_2 - f_1 + f_0 = \mu_2(G)$. Extend this to a free resolution A_* and let X be a CW complex which is a classifying space for G. Now A_* is homotopy equivalent to the cellular complex $C^c_*(X)$ of $\tilde{X}$ and therefore starting with any finite presentation complex K^2 for G we can apply Theorem 8 above. In particular there exists a finite 3-dimensional $D2$ complex Y_0 with $\pi_1(Y_0) = G$ and we compute

$$\chi(Y_0) = \sum_{i=0}^{3} (-1)^i \alpha_i(Y_0) = \sum_{i=0}^{2} (-1)^i \alpha_i(A_*) = \mu_2(G).$$

We are assuming that the $D2$ Problem has positive solution for G, therefore Y_0 is homotopy equivalent to a finite 2-dimensional complex L. We have $G = \pi_1(K) = \pi_1(L)$ and $\chi(L) = \chi(Y_0) = \mu_2(G)$. Hence

$$\delta(G) - 1 \geq \alpha_1(L) - \alpha_0(L) - \alpha_2(L) = -\chi(L) = -\mu_2(G).$$

Therefore $1 - \delta(G) \leq \mu_2(G)$. Since the opposite inequality $\mu_2(G) \leq 1 - \delta(G)$ always holds we have equality. □

Proof of Theorem 6. Let G be a group with the D2 property. Take a presentation $\langle X \mid R \rangle$ for G with e_1 generators and e_2 relations such that $e_1 - e_2 = \delta(G)$. We have $G \cong F/N$ where F is a free group of rank e_1

on X and N is the normal closure of the relations R. Since $e_1 - e_2$ realises the deficiency of G it follows that $e_2 = d_F(N)$. Let $M = N^{ab}$ be the relation module of this presentation. Recall the chain complex (2) above. We have $M \cong \ker \partial_1 = \mathrm{im}\partial_2$. If M has relation gap then $u := d_G(M) < e_2$ and in particular there is a surjection of $\mathbb{Z}G$ modules $f : (\mathbb{Z}G)^u \to \ker \partial_1$. Therefore we can amend the partial resolution above to

$$(\mathbb{Z}G)^u \xrightarrow{f} (\mathbb{Z}G)^{e_1} \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} \mathbb{Z} \to 0.$$

This gives $\mu_2(G) \leq 1 + u - e_2 < 1 - \delta(G)$ contradicting Proposition 7. Therefore presentations of G which realize $\delta(G)$ have no relation gap.

□

4. Relation Gap problem v.s. 2D Conjecture

Theorem 9. *If G is a counterexample to the 2D conjecture then there exists a finite index subgroup H of G such that H has a presentation with relation gap.*

Proof. Suppose that G is a finitely presented group; assume that X is a presentation 2-complex for G realising the deficiency $\delta(G)$. If X is not aspherical, then by Whitehead's Theorem, $H_2(\tilde{X}) \neq 0$. Let e_i denote the number of i-cells in X. So $\delta(G) - 1 = e_1 - e_2 - 1$. We have the exact sequence of G-modules

$$\mathcal{F}:\ 0 \longrightarrow H_2(\tilde{X}) \longrightarrow \mathbb{Z}G^{e_2} \xrightarrow{\partial_2} \mathbb{Z}G^{e_1} \xrightarrow{\partial_1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $H_2(\tilde{X}) = \ker \partial_2$. The relation module R associated to X is isomorphic to $\ker \partial_1 = \mathrm{im}\partial_2 \cong \mathbb{Z}G^{e_2}/H_2(\tilde{X})$. Take a non-zero element ρ of $H_2(\tilde{X})$. As an element of $\mathbb{Z}G^{e_2}$, ρ has a representation as a non-zero tuple $(a_1, \ldots, a_{e_2})$, where each a_i is a linear combination in $\mathbb{Z}G$ with support C_i as follows:

$$a_i = \sum_{g \in C_i} a^i_g g$$

Let $C = \cup_i C_i$; this is a finite collection of elements of G. There exists a finite index normal subgroup of G, say H such that the elements of C project to distinct cosets in G/H. The natural structure of $\mathbb{Z}G$ as a $\mathbb{Z}H$-module makes $\mathcal{F}$ into the chain complex for the action of H on $\tilde{X}$. Let E be a collection of coset representatives for H in G such that $C \subseteq E$. Consider

$$\mathbb{Z}G^{e_2} = \left(\bigoplus_{g \in E} \mathbb{Z}H.g \right)^{e_2} \cong \mathbb{Z}H^{e_2[G:H]}$$

Let d be the greatest common divisor of the integers $\{a^i_g \mid g \in C_i,\ i = 1, 2, \ldots, e_2\}$. Then $\rho = d\rho'$, where $\rho' \in \mathbb{Z}G^{e_2}$ and all its coefficients are co-prime. As ρ is an element of $\ker \partial_2$ and ∂_2 is a homomorphism of torsion-free abelian groups, we deduce that ρ' is also an element of $\ker \partial_2$. Therefore, we can assume that $d = 1$.

Consider the presentation for H arising from the action of H on $\tilde{X}$: this presentation has $(e_1 - 1)[G : H] + 1$ generators and $e_2[G : H]$ relations. The relation module R' for this presentation of H is the restriction $R \downarrow_H^G$ of the relation module R, wherein ρ represents the zero element. We have assumed that the coefficients of ρ are co-prime and so ρ is a primitive element in the abelian group $(\mathbb{Z}E)^{e_2}$ containing its support in $\mathbb{Z}G^{e_2} \cong \mathbb{Z}H^{e_2[G:H]}$. Consequently $R' \cong \mathbb{Z}H^{e_2[G:H]}/H_2(\tilde{X})$ can be generated by fewer than $e_2[G : H]$ elements as an H-module. If the above presentation of H has no relation gap then it needs strictly fewer than $e_2[G : H]$ relations and hence $\delta(H)-1 > [G : H](e_1 - e_2 - 1) = [G : H](\delta(G) - 1)$, contradiction.

Therefore if X is not aspherical some finite index subgroup of G has a relation gap. □

We note that the argument above gives the following general criterion for freeness of $\mathbb{Z}G$-modules.

Proposition 10. *Let G be a residually finite group and let M be a finitely generated $\mathbb{Z}G$-module. Assume that M is torsion free as an abelian group and let $f : (\mathbb{Z}G)^r \to M$ be a surjective homomorphism of $\mathbb{Z}G$ modules. Then f is an isomorphism if and only of $d_H(M) = r[G : H]$ for each subgroup H of finite index in G.*

In particular M is a free module if and only if $d_H(M) = [G : H]d_G(M)$ for each subgroup H of finite index in G.

Proof. If f is not injective we can find an element $\rho = (a_1, \dots, a_r) \in \ker f$ with support $C = \cup_{i=1}^r C_i$ and coefficients $a_g^i \in \mathbb{Z}$ defined by $a_i = \sum_{g \in C_i} a_g^i g$. Since M is torsion free we can assume that the greatest common divisor of all integers a_g^i is 1. There is a finite index subgroup H of G such that C projects injectively into G/H and arguing in the same way as in the proof of Theorem 9 we deduce $d_H(M) < r[G : H]$, contradiction. Therefore f is a bijection and M is a free module. □

5. The 2D conjecture for pro-p groups.

In this section G denotes a finitely presented pro-p group, where we consider presentations in the category of pro-p groups. We keep the notation $\delta(G)$ for the maximum of $|X| - |R|$ over all pro-p presentations $\langle X, R\rangle$ of G.

Below we prove the analogue of the $2D$ conjecture for G:

Theorem 11. *Let G be a finitely presented pro-p group. The following are equivalent:*

(i) $\delta(G) - 1 = [G : H](\delta(H) - 1)$ for every open subgroup H of G.

(ii) $cd_p(G) \leq 2$.

It will be interesting to find a characterizetion of the finitely presented profinite groups G for which the condition *(i)* above holds. Note that already the 1-dimensional situation for profinite groups is quite different. See [10] for examples of profinite groups which satisfy Schreier's rank-index formula for all open subgroups, but are not projective.

Proof. For pro-p groups $\delta(G) = \dim_{\mathbb{F}_p} H^1(G) - \dim_{\mathbb{F}_p} H^2(G)$ where we write $H^i(G) = H^i(G, \mathbb{F}_p)$, see [11, I.4.2 & I.4.3]. Hence, if $cd_p(G) \leq 2$ then $\delta(G) - 1 = -\chi(G)$, the pro-p Euler characteristic of G and therefore (1) holds.

Conversely, suppose that (1) holds and let $e_i = \dim_{\mathbb{F}_p} H^i(G)$ for $i = 1, 2$. We have the partial free resolution

$$\mathbb{F}_p[[G]]^{e_2} \xrightarrow{d_2} \mathbb{F}_p[[G]]^{e_1} \xrightarrow{d_1} \mathbb{F}_p[[G]] \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

arising from the presentation of G with e_1 generators and e_2 relations. We claim that $J := \ker d_2$ must be zero. Suppose not. Then we can find an open normal subgroup N of G such that the image $\bar{J}$ of J under the reduction $(\mathbb{F}_p[[G]])^{e_2} \to (\mathbb{F}_p[G/N])^{e_2}$ is non-zero.

Note that the free $\mathbb{F}_p[[G]]$ resolution above is also a partial free resolution of $\mathbb{F}_p[[N]]$ modules. We apply the functor $\mathrm{Hom}_N(-, \mathbb{F}_p)$ to the above resolution, using $\mathrm{Hom}_N(\mathbb{F}_p G, \mathbb{F}_p) \simeq (\mathbb{F}_p[G/N])^*$, where by V^* we denote the dual of the vector space V over $\mathbb{F}_p$. We obtain the chain complex

$$0 \leftarrow \bar{J}^* \xleftarrow{d_3'} (\mathbb{F}_p[G/N]^*)^{e_2} \xleftarrow{d_2'} (\mathbb{F}_p[G/N]^*)^{e_1} \xleftarrow{d_1'} \mathbb{F}_p[G/N]^* \leftarrow 0.$$

which is exact at $\bar{J}^*$ and whose homology group in degree i is $H^i(N)$ Therefore

$$\delta(N) - 1 = \sum_{i=0}^{2} (-1)^{i+1} \dim H^i(N) =$$

$$= (e_1 - e_2 - 1)[G : N] + \dim \bar{J}^* > [G : N](\delta(G) - 1),$$

since $\bar{J}^* \neq \{0\}$, a contradiction to (i). Therefore $J = \{0\}$ and $cd_p(G) \leq 2$. □

6. Higher dimensional analogues

Deficiency can be viewed as one of the partial Euler characteristics, which are defined as follows:

Let $n \geq 2$ be an integer and let G be a group of type F_n. Define $\nu_n(G)$ to be the minimum of $(-1)^n \chi(X)$ where X is a finite CW complex of dimension n such that $\pi_1(X) = G$ and $\pi_i(X) = \{0\}$ for $i = 2, 3, \ldots, n-1$ (i.e its universal cover $\tilde{X}$ is $(n-1)$-connected. Note that $\nu_2(G) = 1 - \delta(G)$ and for completeness we define $\nu_1(G) = d(G) - 1$. From the definition of ν_n and μ_n we have $\nu_n(G) \geq \mu_n(G)$ for all n. We note that Theorem 8 above implies

Proposition 12. *$\nu_n(G) = \mu_n(G)$ when $n \geq 3$.*

Here we prove the higher dimensional analogue of the 2D conjecture.

Theorem 13. *Let $n > 2$ be an integer and let G be a residually finite group of type F_n. Then G has finite classifying space of dimension n if and only if $\nu_n(H) = \nu_n(G)[G : H]$ for every subgroup H of finite index in G.*

Proof. Suppose that X is an n-dimensional $K(G,1)$ complex for G, then $\nu_n(G) \leq (-1)^n\chi(X)$ from the definition of $\nu_n(G)$. On the other hand the Morse inequalities give $\nu_n(G) \geq \sum_{i=0}^{n}(-1)^{n-i}b_i(G) = (-1)^n\chi(X)$. Therefore $\nu_n(G) = (-1)^n\chi(X)$ and in the same way $\nu_n(H) = (-1)^n\chi(X')$, where X' is the cover of X corresponding to H. Since $\chi(X') = [G : H]\chi(X)$ the equality follows.

For the other direction we could use Proposition 12. Instead we take a more elementary approach and argue directly using Proposition 10.

Suppose that $\nu_n(H) = \nu_n(G)[G : H]$ for every subgroup H of finite index in G. Let X be the n-dimensional CW complex which realises $\nu_n(G)$. Let e_i be the number of i-dimensional cells of X and let

$$F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with $F_i = (\mathbb{Z}G)^{e_i}$ be the chain complex of the universal cover $\tilde{X}$. By the Hurewicz theorem $\pi_n(X) \simeq H_n(X) = \ker\partial_n$ and thus X is aspherical if and only if ∂_n is injective.

Suppose $\ker\partial_n \neq \{0\}$ and consider $M = \ker\partial_{n-1} = \mathrm{im}\partial_n$. We apply Proposition 10 to the $\mathbb{Z}G$- homomorphism $\partial_n : F_n \to M$, where $F_n = (\mathbb{Z}G)^{e_n}$ to deduce that $u := d_H(M) < e_n[G : H]$ for some subgroup H.

Choose a set of generators $\alpha_1, \ldots, \alpha_u$ of the $\mathbb{Z}H$-module M. Let Y be the cover of X with degree $[Y : X] = [G : H]$ and $\pi_1(Y) = H$. Let $p : \tilde{X} \to Y$ be the universal covering map. Denote by Y^{n-1} and $\tilde{X}^{n-1}$ the $(n-1)$-skeleta of Y and $\tilde{X}$ respectively and observe that $\pi_{n-1}(Y^{n-1}) \simeq H_{n-1}(\tilde{X}^{n-1}) = \ker\partial_{n-1} = M$ by the Hurewicz theorem. Therefore for each $i = 1, \ldots, u$ we can find a cellular map $j_i : S^{n-1} \to \tilde{X}^{n-1}$ representing α_i. This means that $H_{n-1}(j_i)$ sends the generator of $H_{n-1}(S^{n-1})$ to the element $\alpha_i \in H_{n-1}(\tilde{X}^{n-1}) = M$.

We now attach n-dimensional cells σ_i^n to Y^{n-1} for $i = 1, \ldots u$ with boundary attaching maps

$$S^{n-1} \xrightarrow{j_i} \tilde{X}^{n-1} \xrightarrow{p} Y^{n-1}$$

and define $Z := Y^{n-1} \cup_{i=1}^{u} \sigma_i^n$. Note that since $Y^{n-1} = Z^{n-1}$ we have $\pi_i(Z) = \pi_i(Y)$ for $i = 1, \ldots, n-2$. We claim that $\pi_{n-1}(Z) = \{0\}$. It is sufficient to prove that $H_{n-1}(\tilde{Z}) = \{0\}$ for the universal cover $\tilde{Z}$ of Z. Since the $(n-1)$-skeleta of Z and X coincide, the boundary maps ∂_{n-1} on the chain complex of $\tilde{Z}$ and $\tilde{X}$ are the same and hence $\ker\partial_{n-1} = M$. On the other hand the boundary map $\partial'_n : (\mathbb{Z}H)^u \to M$ of degree n of the chain complex of $\tilde{Z}$ is surjective since by construction its image contains the generators α_i. Therefore $H_{n-1}(\tilde{Z}) = \{0\}$ and so $\tilde{Z}$ is $(n-1)$-connected as claimed.

Note that Z has $[G : H]e_i$ cells in dimension i for $i = 0, 1 \ldots, n-1$ and u cells in dimension n. Since $u < e_n[G : H]$ it follows that

$$\nu_n(H) \leq (-1)^n\chi(Z) = u + \sum_{i=0}^{n-1}(-1)^{n-i}e_i[G : H] < \nu_n(G)[G : H],$$

contradiction. Therefore $H_n(\tilde{X}) = \{0\}$ and X is a finite $K(G,1)$-complex of dimension n. □

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