Context Dependent Beliefs*

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Abstract

This paper examines a model where the set of available outcomes from which a decision maker must choose alters his perception of uncertainty. Specifically, this paper proposes a set of axioms such that each menu induces a subjective belief over an objective state space. The decision maker's preferences are dependent on the realization of the state. The resulting representation is analogous to state-dependent expected utility within each menu; the beliefs are menu dependent and the utility index is not. Under the interpretation that a menu acts as an informative signal regarding the true state, the paper examines the behavioral restrictions that coincide with different signal structures: elemental (where each element of a menu is a conditionally independent signal) and partitional (where the induced beliefs form a partition of the state space).

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1 Introduction

Both intuition and psychological evidence insist that a decision maker's (DM's) preference over alternatives is affected by the environment in which the decision is made (Kahneman and Tversky, 1984; Simonson and Tversky, 1992; Sen, 1993). While there are many external factors that potentially exert influence on the decision making process, this paper examines a model in which the set of alternatives that is *currently* available acts as a frame –a process often differentiated from general framing effects under the moniker *context dependence*. I identify the behavioral conditions for context dependent beliefs, when the DM's subjective assessment of the likelihood of events depends on the available alternatives (the menu) from which he must choose, and consider additional restrictions that correspond to particular subjective information structures.

Context dependence is often associated with notions of bounded rationality or psychological heuristics Tversky and Simonson (1993). This paper, however, interprets menu-induced framing as rational, exploring how and when such behavior exists within the subjective expected utility paradigm. If the DM believes the menu itself contains information regarding payoff relevant uncertainty, conditioning his preference on such information is a rational action. Specifically, the model assumes the payoff associated with each alternative is ex-ante uncertain. The DM's utility from consumption depends not only on the chosen outcome, but also on which *state of the world* is realized. The DM, before consumption, is uncertain about the state of the world, but holds a belief (a probability distribution) over the state space; in a given decision problem, the DM maximizes his expected utility according to his belief. When the DM interprets the current selection of alternatives as a signal about the state of the world, his preferences will change across different decision problems in response to his updated beliefs.

Before expounding the finer points of the model, it is worth considering two examples to better illustrate why menu dependent preferences are indeed necessary to explain many decision making scenarios.

Example 1.A (Luce and Raiffa's diner). *On a first date, Katya finds herself in a restaurant at which she has previously never eaten, and which offers chicken (c) or steak (s). She states her strict preference for chicken ($c \succ s$). However, upon seeing the restaurant also serves frog legs (f), she now states her strict preference for steak ($s \succ c \succ f$).*

While Katya's preference reversal in the face of a (seemingly) irrelevant alternative cannot be accommodated by the standard theory (as it violates the weak axiom of revealed preference (WARP)), it has a simple, intuitive explanation. She prefers steak when the food is well prepared, but considers chicken more resilient to the inept chef. In the typical restaurant, she believes it is unlikely the food will be well cooked, and hence, has a preference for chicken. However, in the presence of an exotic dish, she deems it is more likely the restaurant employs an expert chef and so, reverses her preference.

Example 2.A (Sen's date). *After dinner, Katya's date, Mitya, asks whether she would like to end the date and go home (h) or go next door and get a drink (d). Thinking the date a success, Katya strictly prefers getting a drink ($d \succ h$). However, before she can respond, Mitya offers a third option: the acquisition and consumption of crystal methamphetamine (m). Katya now strictly prefers going home ($h \succ d \succ m$).*

Here, again, Katya's rather intuitive behavior cannot be explained by standard theory. She understands the offer of methamphetamine as a signal regarding Mitya's character. So, while she would prefer to continue the date as long as it is likely Mitya is reputable, his proposition is sufficient to sway her beliefs away from such a outcome.

These vignettes exemplify two main components of the model. First, it is only the DM's perception of uncertainty that is changing; ex-post tastes are fixed. In other words, if the DM knew with certainty which state of the world was to be realized, he would exhibit a constant preference across menus. Second, the uncertainty is *local*. The realization regarding the quality of the food in one restaurant is not informative about the quality in a different restaurant; that a previous date was virtuous is not evidence that a future date will be.[1]

The first part of this paper axiomatizes a particular type of context-dependence which adheres to these two restrictions. As in Anscombe and Aumann (1963), I examine a DM who ranks *acts* (i.e., functions) from a state space, S, into lotteries over consumption, $\Delta(X)$.[2] Naturally, given the motivation, not all of X will always be available. The DM's entertains a family of preferences over acts, indexed by the subset of X that is currently available. Therefore, for each $A \subseteq X$, we see the decision maker's preference, $\succcurlyeq_A$, over $\{f : S \to \Delta(A)\}$. Then, a *menu-induced belief representation* (MBR) is a single utility index, $u : S \times X \to \mathbb{R}$, and a menu-indexed family of beliefs $\{\mu_A\}_{A\subseteq X} \subseteq \Delta(S)$ such that

$$U_A(f) = \mathbb{E}_{\mu_A}\left(\mathbb{E}_{f(s)}(u(s,x))\right) \tag{MBR}$$

represents $\succcurlyeq_A$, where $\mathbb{E}_\pi(\varphi)$ denotes the expectation of the random variable φ with respect to the distribution π. Fixing the menu, the DM acts as a subjective expected utility maximizer. The utility index, u, is the same across menus. This is the consequence of the main axiom, *menu consistency*. Menu consistency dictates, conditional on the realization of a particular state, the DM's preference for alternatives is fixed across menus. Therefore, the context effect is entirely characterized by the change in the DM's beliefs regarding the state space. This places clear limits on the type of context effects that can be accommodated by a MBR. Since any change in preference is the consequence of shifting beliefs, context dependence cannot reverse preference over outcomes for which the resolution of the state is payoff irrelevant (note, because the tastes are state-dependent, constant acts are not necessarily certain outcomes). The general model also imposes a *continuity condition*[3] –if two menus differ only slightly, then so do their associated beliefs.

Since this paper interperpates context effects as being entirely driven by informational concerns,[4] it is of interest to understand how the DM uses the context to update his belief. A modeler, who has access to a DM's preferences in a variety of contexts may want to understand what kind of subjective information the DM believes is encoded by each context. Identifying the connection between a context and its induced belief allows a modeler to make counterfactual arguments. For example, understanding that the DM believes frog legs are the mark of a good restaurant (rather than, say, having 3 items on the menu) allows the modeler to predict what the DM would do at a new restaurant. The second part of this paper explores how menus might correspond to the beliefs they induce. In particular, what restrictions indicate that the DM, acting as a Bayesian, holds a prior belief regarding the state space, and interprets each menu as a collection of signals

[1]Of course, one could tell a different story where there is a dynamic component by which the DM learns about the likelihood of states from experience. This is well outside of the current model.

[2]For a set Y, $\Delta(Y)$ is the set of distributions thereover.

[3]This is a vacuous assumption when X is a discrete space.

[4]In contrast to, for example, Kalai *et al.* (2002) in which the change in the rationalizing preference may be the result of changing tastes.

regarding the relative likelihood of each state? What further restrictions allow us to identify the prior and the structure of these signals?

Following the *anything goes* result of Shmaya and Yariv (2016), any MBR can be rationalized by some prior and set of signals. Without imposing any additional structure, Bayesianism imparts no falsifiable restrictions. Thus, a modeler cannot disentangle an irrational DM who chooses a belief at random in each context from a DM who acts rationally according to his prior belief and information encoded by the context. Moreover, and equally damningly, the rationalizing Bayesian model is highly non-unique. In light of this, I consider two more restrictive signal structures and their corresponding behavioral restrictions.

In the first signal structure, an *elemental signal structure*, the DM takes the elements of the menus as signals. Specifically, he assumes that in each state, s, element x is included with probability $l(x,s)$ and excluded with $1-l(x,s)$. Therefore, the collection of included elements (the menu) is the result of a series of conditionally independent random draws. If $l(x,s) > l(x,s')$ then x is more likely to be available in state s than s', so observing x will increase the relative likelihood of state s.

Example 2.B (Sen's date, revisited). *Let $S = \{r, d\}$ indicate reputable and depraved characters, respectively. Katya's has MBR preferences and the following utility index:*

$$u(r,h) = 1 \quad u(r,d) = 5 \quad u(r,m) = -10$$
$$u(d,h) = 1 \quad u(d,d) = -5 \quad u(d,m) = -10$$

She initially believes $\mu(r) = \frac{9}{10}$ and $\mu(d) = \frac{1}{10}$. She also believes that, while all dates will offer going home and getting a drink, depraved characters offer meth with probability $\frac{1}{10}$, with reputable characters with only probability $\frac{1}{100}$.

After updating upon seeing the menu $\{h,d\}$, she holds the beliefs $\mu(r) = \frac{891}{981}$ and $\mu(d) = \frac{90}{981}$; her preference is given by $U_{\{h,d\}}(d) = \frac{5(801)}{981} > 1 = U_{\{h,d\}}(h)$. After the menu $\{h,d,m\}$, she holds the beliefs $\mu(r) = \frac{9}{19}$ and $\mu(d) = \frac{10}{19}$; her preference is given by $U_{\{h,d,m\}}(h) = 1 > \frac{-5}{19} = U_{\{h,d,m\}}(d)$.

I show that this behavior is captured axiomatically by the restriction that the same element, included in two different menus, must have the same proportional effect of beliefs. Moreover, given that a DM entertains an elemental signal structure, the effect of each element on relative likelihoods can be identified uniquely. Next, I consider a *partitional signal structure* (a special case of an elemental signal structure). Here, the DM entertains a partition of the state space and each menu indicates a particular event of the partition has obtained. In other words, the DM believes each menu can only occur in a particular subset of the state space.

Example 1.B (Luce and Raiffa's diner, revisited). *Let $S = \{h, m, l\}$ indicate high and medium and low quality food, respectively. Katya's has MBR preferences and the following utility index:*

$$u(h,c) = 12 \qquad u(h,s) = 16 \qquad u(h,f) = 6$$
$$u(m,c) = 9 \qquad u(m,s) = 8 \qquad u(m,f) = 5$$
$$u(l,c) = 7 \qquad u(l,s) = 4 \qquad u(l,f) = 3$$

She initially believes each of the three types are equally likely: $\mu(h) = \mu(m) = \mu(l) = \frac{1}{3}$. She also believes only (and all) high quality restaurants offer a three dishes, while medium and low quality restaurants offer *only chicken and steak. Katya entertains the partition* $\{\{h\},\{m,l\}\}$, *where the frog legs' inclusion stipulates the first event obtains and their exclusion stipulates the second.*

So, after seeing $\{c,s\}$ *she believes food is either medium or low (with equal probability), so* $U_{\{c,s\}}(c) = 8 > 6 = U_{\{c,s\}}(s)$, *while the menu* $\{c,s,f\}$ *indicates with certainty the food is good, so* $U_{\{c,s,f\}}(s) = 16 > 12 = U_{\{c,s,f\}}(c)$.

Given the nestedness of these statistical models, Example 1B can also be understood as being rationalized by a elemental signal structure, where the $l(h,f) = 1$ and the $l(m,f) = l(l,f) = 0$. On the other hand, Example 1 cannot be rationalized with a partitional structure because in/exclusion of methamphetamine does not change the set of states Katya considers possible but does change her preferences.[5]

A different potential resolution for the context effect described in Example 1A would be the following: a steak in a restaurant that offers frog legs, s_1, is simply not the same object as a steak in a restaurant without frog legs, s_2: the DM has the completely standard and rationalizable preference $s_1 \succ c \succ s_2$. Any apparent preference reversal is the result of the modelers conflation of the two distinct alternatives s_1 and s_2. While this approach has the appeal of being trenchant and simple, it has two serious limitations. First, the consolidation of s_1 and s_2 happens for good reason; both alternatives are described by the same objective labels. So, from the modelers perspective, the only reason s_1 and s_2 should be differentiated is because of the DM's preference reversal. Of course, it could be equally likely that 'chicken' is the variable entity, rather than steak. An unrestricted expansion of the choice set can rationalize any preference, but there is no unique way of doing so.[6] Because of this, such an approach cannot provide any explanation as to the mechanism by which the DM changes his preference (for example, that frog legs are always an indication of high quality food, regardless of the menu) and has no falsifiable predictions.

Second, expanding the choice set, without any restriction on how the expansion relates to larger patterns in choice, eschews the informational channel—the DM himself only observes the label s. That is to say, while the DM *does* regard a high-quality steak as different from a low quality steak, he cannot choose between these options. Rather, he must choose s, which may turn out to be of either variety. The inclusion of frog legs does not change the final (ex-post) outcomes, but rather the probabilities they occur. This paper, by looking at acts, captures exactly this phenomenon. By understanding the change in preference to be related to the change in subjective beliefs, this paper is able to identify the type of information the DM learns from the in/exclusion of alternatives.

1.1 Organization

This paper is organized as follows. The general model is presented in Section 2, with the representation theorem for the main result contained in Section 2.3. Section 2.4 discusses the shortcomings of a variant model with state-independent utilities. Section 3 explores the additional restrictions necessary to capture particular signal structures. Finally, a survey of relevant literature is found in Section 4. Appendix A provides an example of how menu dependent beliefs could arise naturally in a strategic environment. All proofs are contained in the appendices B and C.

[5] Of course, this observation assumes we have identified the posterior distributions. Had we only had the sparse preferences outlined in Example 1, we would not have enough data to identify the signal structure.

[6] One could ask what is the minimal expansion necessary for rationalization—this is essentially the motivation behind Kalai *et al.* (2002)

2 General Model

2.1 Structure and Primitives

Let X be a separable and metrizable topological space, representing the grand set of consumption alternatives, and with typical elements x, y, z. Define $x^\star$ and $x_\star$ to be two distinguished elements of X, referred to as universal alternatives, and set $\bigstar = \{x^\star, x_\star\}$. Let $cl(X)$ denote the set of closed subsets of X; endow $cl(X)$ with the Hausdorff metric topology. Let $\mathcal{M}(X)$ denote the set of all subsets of X which contain $\bigstar$.[7] Typical elements are A, B, C. Elements of $\mathcal{M}(X)$ are called menus, with the interpretation that they represent the set of *currently available* consumption alternatives.

For any topological space Y, let $\Delta(Y)$ denote the set of all probability measures on $(Y, \mathcal{B}(Y))$, where $\mathcal{B}(Y)$ denotes the Borel σ-algebra on Y. If $\mu \in \Delta(Y)$, and $\varphi : Y \to \mathbb{R}$ is bounded and continuous, let

$$\mathbb{E}_\mu(\varphi) = \begin{cases} \int_Y \varphi(y)\ d\mu(y) & \text{whenever } Y \text{ is infinite, and,} \\ \sum_Y \varphi(y)\mu(y) & \text{whenever } Y \text{ is finite,} \end{cases}$$

denote the expectation of φ with respect to μ. Endow $\Delta(Y)$ with the weak* topology, i.e., so that the $\mathbb{E}_\mu \varphi$ is continuous in μ.

Notice, for each $A \in \mathcal{M}(X)$, A is the subset of a separable metric space, and thus, separable itself, so $\Delta(A)$ is metrizable. In the standard abuse of notation, identify $x \in X$ with the degenerate distribution on x. Typical elements of $\Delta(X)$ are denoted π, ρ, τ.

Let S denote a finite state space. Endow $\Delta(X)^S$ with the product topology. The objects of choice will be menu-induced acts: for each $A \in \mathcal{M}(X)$ define $\mathcal{F}_A = \Delta(A)^S = \{f : S \to \Delta(X) | f(s) \in \Delta(A), \forall s \in S\}$. An act is a commitment to a particular consumption conditional on the realization of the state space, and so, $\mathcal{F}_A$ corresponds to the acts available given the menu A. In other words, when the set of available alternatives is A, then the only feasible acts are those that provide only lotteries over A.

For each act, $f(s)$ is the distribution over X obtained for realization s. Again, abusing notation, identify each $\pi \in \Delta(X)$ with the degenerate act such that $\pi(s) = \pi$ for all s. For any $f, g \in \mathcal{F}_X$, and event $E \subseteq S$, let $f_{-E}g$ be the act that coincides with f everywhere except on E, where it coincides with g. Further, for some $\alpha \in (0, 1)$ let $\alpha f + (1-\alpha)g$ be the point-wise mixture of f and g (i.e., $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$ for each $s \in S$). It is immediate that if $f \in \mathcal{F}_A$ and $g \in \mathcal{F}_B$ then $f_{-E}g$ and $\alpha f + (1-\alpha)g$ both belong to $\mathcal{F}_{A\cup B}$ (in particular, note the case when $A = B$).

The primitive of the model is the family of preference relations $\{\succcurlyeq_A \subset \mathcal{F}_A \times \mathcal{F}_A\}_{A \in \mathcal{M}(X)}$. That is, the DM's preference over the acts which are available given each possible menu. With regards to notation, it is assumed whenever '$f \succcurlyeq_A g$' is written both f and g belong to $\mathcal{F}_A$. For any relation $\succcurlyeq$, let $\succ$ and $\sim$ denote the asymmetric and symmetric components, respectively.

2.2 Axioms

The goal of the most general representation is to provide the basic framework in which a DM might condition his beliefs regarding the state space on the menu at hand. That is, the DM treats the set of currently available

[7] I will interpret $\bigstar$ as a set of outside options, which explains their universal availability.

consumption alternatives as a signal regarding the likelihood of different states. Given the menu, the DM acts as a subjective expected utility maximizer, with respect to his menu-induced beliefs. Clearly, each menu dependent preference should satisfy the expected utility axioms:

[A1: Expected Utility (EU)] *For each $A \in \mathcal{M}(X)$, $\succcurlyeq_A$ satisfies the expected utility axioms, namely,*

1. ***Weak Order.*** *$\succcurlyeq_A$ is a non-trivial weak order.*
2. ***Independence.*** *For all $f, g, h \in \mathcal{F}_A$ and $\alpha \in (0,1)$, $f \succcurlyeq_A g \iff \alpha f + (1-\alpha)h \succcurlyeq_A \alpha g + (1-\alpha)h$.*
3. ***Continuity.*** *For all $f \in F_A$, the sets $\{g \in \mathcal{F}_A | g \succcurlyeq_A f\}$ and $\{g \in \mathcal{F}_A | f \succcurlyeq_A g\}$ are closed in $\mathcal{F}_A$.*

The following well known result (so well known in fact, that it is included only to fix notation for expositional purposes[8]) shows EU provides the expected utility structure for each menu dependent preference.

Proposition 2.1 (Expected Utility)**.** *$\{\succcurlyeq_A \subset \mathcal{F}_A \times \mathcal{F}_A\}_{A \in \mathcal{M}(X)}$ satisfies EU if and only if for each $A \in \mathcal{M}(A)$ there exists some continuous and bounded $w : S \times X \to \mathbb{R}$ such that*

$$U_A^{VNM}(f) = \sum_s \left(\mathbb{E}_{f(s)}(w_A(s,x)) \right),$$

represents $\succcurlyeq_A$. Moreover, if $w_A(s,x)$ and $\hat{w}_A(s,x)$ both represent $\succcurlyeq_A$, then $w_A(s,x) = a_A \hat{w}_A(s,x) + b_A(s)$ where $a_A \in \mathbb{R}_{++}$ and $b_A(s) \in \mathbb{R}$ for all $s \in S$.

Recall that $f(s)$ and $g(s)$ are given, objective probability measures. The index $w(\cdot)$ can be decomposed into tastes (the utility of consuming an object given the state) and beliefs (the subjective likelihood of each state). Indeed, choose some probability distribution $\mu \in \Delta(S)$ such that $\mu(s) > 0$ and let $u_A(s,x) = \frac{w_A(s,x)}{\mu(s)}$; it is clear that

$$\mathbb{E}_\mu \left(\mathbb{E}_{f(s)}(u_A(s,x)) \right)$$

represents $\succcurlyeq_A$. Of course, this creates the classic problem of multiple rationalizing beliefs: if we consider some other $\nu \in \Delta(S)$ such that $\nu(s) > 0$, then ν and $u'_A(s,x) = \frac{w_A(s,x)}{\nu(s)}$ also represent the same preference. We cannot identify the DM's tastes for ex-post outcomes or his beliefs; the two are jointly determined.

The motivation for expanding our data to include the family of menu-induced preferences is to understand how the menu can alter the beliefs of the DM. In light of this, it becomes obvious further structure is needed to separate the effect on the perception of uncertainty (i.e., menu induced changes in belief) from other internal changes in preference (i.e., a change in tastes).

The first novel axiom, *menu-consistency*, is the first step towards such a disentanglement, and, captures the main behavior behind menu-induced beliefs. It states that the DM's tastes for outcomes do not depend on the menu at hand. This implies, any difference in preferences across menus must be the result of a change in perception of the underlying uncertainty.

Of course, the DM only cares about the assignment to state s if he believes there is a possibility s will be realized. Therefore, menu-consistency should only hold after realizations assigned positive probability according to the DM's subjective assessment. To make such ideas precise, I first need to consider null events.

[8] Of course, for a reference with the exact set up see Grandmont (1972), Theorem 2 and its Corollary.

Definition 1. *An event, $E \subset S$, is* ***null for menu A*** *(hereafter, null-A) if for all $f, g \in \mathcal{F}_A$,*

$$f_{-E}g \sim_A f.$$

Let N_A denote the set of states that are null-A, and N denote the set of everywhere null states: $N = \bigcap_{A \in \mathcal{M}(A)} N_A$.[9]

Null events, in general, have two indistinguishable interpretations. First, that the DM is indifferent between all available options conditional on the realization of the null event, E; second, that the DM places zero probability on E occurring. However, assuming the DM's tastes are consistent across different menus (the assumption that will be formalized shortly), it is possible to differentiate these two interpretations of null events. If a state, s, is null-A, but there exists a different menu, B, for which the DM displays a strict preference over elements of A (given realization of s), it must be that s was assigned zero probability when facing A. This is because the DM cannot be indifferent to all elements of A (contingent on s) since he displays strict preference in the menu B. This is formalized by evidently-null events, first considered in Karni *et al.* (1983).

Definition 2. *An event, $E \subset S$, is* ***evidently null for menu A*** *(hereafter, e-null-A) if E is null-A and for all $s \in E$ there exists some menu B such that*

$$(f_{-s}g) \succ_B f$$

for some f, g in $\mathcal{F}_{A \cap B}$. Let E_A denote the union of all e-null-A events.[10]

With this definition in mind we can now define menu consistency.

[A2: Menu Consistency (MC)] *For all $A, B \in \mathcal{M}(X)$ and $s \in S$ with $s \notin E_A \cup E_B$, and all $f \in \mathcal{F}_A$, $g \in \mathcal{F}_B$, $h \in \mathcal{F}_{A \cap B}$, and such that $f(s) = g(s)$,*

$$f_{-s}h \succcurlyeq_A f \iff g_{-s}h \succcurlyeq_B g.$$

If $\{\succcurlyeq_A\}_{A \in \mathcal{M}(X)}$ is menu-consistent, the DM's tastes for outcomes are identical across menus. To see this, let $\pi = f(s) = g(s)$ and $\rho = h(s)$. Then MC states that the DM's preference between ρ and π, in state s, does not depend on the context in which the decision is made (i.e., does not depend on the menu from which the acts were constructed). Behaviorally, this indicates that any context effect does not alter the DM's preferences *conditional* on the realization of a particular state. In other words, if the DM knew the true state, there would be no context effect. It is this restriction that differentiates this model from a more general interpretation of context effects as psychological biases without foundation in rational behavior. The change in behavior across menus is *not* the result of a change in the state-dependent preference for outcomes (objects about which the DM is ostensibly certain) but of a change in his perception of the between-state-tradeoffs (the domain of uncertainty).

By the very nature of the problem at hand, we are losing structure in comparison to the standard model and so the axioms are weaker in comparison. As such, MC does not characterize a *new* behavior that is the

[9] Notice, the set of null-A events form a lattice with respect to set inclusion, with N_A the maximal element.

[10] Notice, the set of e-null-A events form a sub-lattice of the lattice of null-A events, with E_A the corresponding maximal element.

result of context dependent beliefs, but rather places limits on how much structure is lost. What structure is retained by MC guarantees we can find a family of representation for $\{\succcurlyeq_A\}_{A\in\mathcal{M}(X)}$ that shares a common utility index. In other words, the primitive is represented by a single utility index and a family of menu-induced beliefs. It is important to note that this does not rule out preference reversals, even over constant acts. Each menu carries with it a perception of uncertainty, and can therefore change the DM's preferences for acts. However, given menu-consistency, any preference reversal is due entirely to the change in beliefs, and not because of changes in ex-post tastes. Setting $f = g$ in the definition, consistency guarantees that the ordering between $f_{-s}\rho$ and $f_{-s}\pi$ hold regardless of the ambient menu.

Under the definition of a *frame* as (seemingly irrelevant) information which alters the DM's perception of uncertainty, then EU and MC exactly capture the behavior where the DM uses the menu as a frame. Unfortunately, from a practical vantage, this is insufficient, as the problem of non-uniqueness of beliefs persists. When tastes and beliefs cannot be separated, we cannot identify the avenue by which context effects alter the DM's choice process.

To overcome the issue of multiple rationalizing beliefs, Anscombe and Aumann (1963) restricted preferences to be state independent (i.e., in every non-null state, the ranking over distributions is the same). State dependency is a very restrictive assumption; it interprets states as abstract probabilistic events that have no meaning outside of their use as betting devices. Beyond this philosophical issue, state-dependence is a necessary requirement to capture the full gamut of context effects (this necessity is made precise in Remark 3.1). For these reasons, this model weakens state-independence to apply only over $\star$. UV plays the same roles as state independence (equivalently, monotonicity). By ensuring, over the relatively small domain $\star$, that preferences in each state coincide, beliefs can be uniquely recovered from choice data. Under the interpretation of universal elements as outside options, it is natural that the ranking of these elements does not change across different menus.

[A3: Universality (UV)] *For all $A \in \mathcal{M}(X)$ and $s \in S$ with $s \notin N_A$*

$$f_{-s}x^\star \succ_A f_{-s}x_\star.$$

for all $f \in \mathcal{F}_A$.

It is also of interest (when X is indiscrete) to understand when the context effect acts in a continuous manner.

[A4: Continuity of Context (CC)] *If $\{A_n\}_{n\in\mathbb{N}}$ converges to A in $\mathcal{M}(X)$, then for all $f, g \in \mathcal{F}_\star$, if $f \succcurlyeq_{A_n} g$ for all n then $f \succcurlyeq_A g$.*

In other words, the DM's preference over universal acts is continuous with respect to changes in the menu.[11] If CC were violated, then there would exist a menu A such that for any arbitrarily small distance, ϵ, there exists a menu, A^ϵ, that (strictly) reverses some strict preference over universal acts f and g. Thus, there would be a discrete jump in the DM's preference. Of course, CC applies only to universal acts so such a jump in preferences corresponds to discrete jump in the DM's beliefs about the likelihood of states. Hence, CC requires that the information encoded by a context changes continuously with the context.

[11] Recall, convergence, of the sequence $\{A_n\}_{n\in\mathbb{N}}$, is with respect to the Hausdorff metric on $\mathcal{M}(X)$.

2.3 Menu Induced Belief Representation

Theorem 2.2 (Menu Induced Belief Representation). *(a) $\{\succcurlyeq_A\}_{A\in\mathcal{M}(X)}$ satisfies EU, MC, UV, and CC if and only if there exists a state-dependent utility index, $u: S\times X\to\mathbb{R}$, such that $u(\cdot,x^\star)\equiv 1$ and $u(\cdot,x_\star)\equiv 0$, and such that the projections $u|_A$ are bounded and continuous for all $A\in\mathcal{M}(X)$, and a continuous function,*[12] *$\{\mu_A\in\Delta(S)\}_{A\in\mathcal{M}(X)}$, such that for all $A\in\mathcal{M}(X)$,*

$$U_A(f)=\mathbb{E}_{\mu_A}\left(\mathbb{E}_{f(s)}(u(s,x))\right) \tag{MBR}$$

represents $\succcurlyeq_A$, and $\mu_A(s)=0$ if and only if $s\in E_A\cup N$.
(b) Moreover, the family of beliefs $\{\mu_A\in\Delta(S)\}_{A\in\mathcal{M}(X)}$ is unique and the utility index, $u(\cdot)$, is unique up to null states.

Proof. In appendix C. ■

The proof is quite straightforward. First, EU provides a linear representation for each $\succcurlyeq_A$. By UV these representations can be decomposed uniquely into tastes (over A) and beliefs, where the utility index is normalized as in the statement of the theorem. Then, these utility indexes can be stitched together to provide a single u over the whole of X. Finally, MC ensures that this common index will jointly represent each $\succcurlyeq_A$ and CC that beliefs will change continuously.

Because the utility index is fixed across decision problems, the shifting of probabilities is the only avenue for preferences to change. Thus, if an act f is preferred to g on a state-by-state basis, then it is preferred to g in every menu (this is, of course, precisely the content of MC). It is through the menu dependent beliefs that this structure allows for framing effects, were by the DM's preferences change in the face of new alternatives. It follows that the types of preference reversals that are allowable is limited.

2.4 State-Independence

In light of axiom UV, it may seem parsimonious to quit worrying about the distinguished elements, $x^\star$ and $x_\star$, and require state independence outright. This can, in fact, be accomplished by strengthening MC.

[A2*: Strong Menu Consistency (SMC)] *For all $A,B\in\mathcal{M}(X)$ and $s\in S$ with $s\notin E_A$ and $s'\notin E_B$, and all $f\in\mathcal{F}_A$, $g\in\mathcal{F}_B$, $h\in\mathcal{F}_{A\cap B}$, and such that $f(s)=g(s')$,*

$$f_{-s}h\succcurlyeq_A f \iff g_{-s'}h\succcurlyeq_B g.$$

SMC states that tastes are consistent, not only across menus (if $A\neq B$) but also across states (if $s\neq s'$). As such, it implies the canonical form of state independence for each $\succcurlyeq_A$. When MC is replaced by SMC in Theorem 2.2, the resulting representation coincides except the utility index, $u: X\to\mathbb{R}$ is *state-independent*:

$$U_A^{SI}(f)=\mathbb{E}_{\mu_A}\left(\mathbb{E}_{f(s)}(u(x))\right), \tag{SI-MBR}$$

represents $\succcurlyeq_A$.[13] The existence of the family of beliefs, their uniqueness, and the uniqueness of the utility

[12] i.e., whenever A_n converges to A, μ_{A_n} converges to μ_A.
[13] Notice, UV is somewhat redundant in the presence of SMC. In fact, if we are willing to entertain a bit of notational juggling, we can forego it entirely.

index are all the same as in Theorem 2.2. While this approach is only a small deviation from the general representation, it implies that there is no uncertainty regarding the preference of constant acts. As discussed before, in order for context effects to have observable content, it must be that the underlying uncertainty is payoff relevant. Together, these facts imply that SMC prohibits the DM from changing his preference over constant acts between different menus.

Remark 2.3. *Let* $\{\succcurlyeq_A\}_{A \in \mathcal{M}(X)}$ *be represented by* (SI-MBR). *Then for all* $A, B \in \mathcal{M}(X)$, *and* $\pi, \rho \in \Delta(A \cap B)$, $\pi \succcurlyeq_A \rho \iff \pi \succcurlyeq_B \rho$.

Remark 2.3 can be seen by noting that $U_A^{SI}(\pi) = \mathbb{E}_\pi(u(x))$, which does not depend on A.

3 Bayesian Frames

The general representation, (MBR), assumes that context effects arise only though an informational channel—that the DM changes his beliefs about the relevant uncertainty after seeing the available alternatives. It does not, however, offer any insight into the connection between the menu at hand and the effect it exerts on beliefs. This section provides an exploration into the behavioral implications, and identification, of particular context effects.

It is of interest to identify the restrictions on behavior that ensure the DM is acting rationally with respect to some *information structure* that gives rise to the family of menu-induced beliefs. Consider the interpretation that the DM entertains a prior belief over the state space, $\mu \in \Delta(S)$, and observes, along with the menu, some signals, drawn from a (finite) signal space, Θ. The DM also entertains a likelihood function that specifies the likelihood of a given signal, contingent on the true state, $l : \Theta \times S \to \mathbb{R}_+$. Under this interpretation, we say the information structure (μ, l, Θ) *generates* $\{\mu_A | A \in \mathcal{M}(X)\}$, if the DM's menu-induced beliefs are the posteriors generated by observing the signals. To keep things notationally clean, through this subsection, I assume that X is finite and $N = \varnothing$. I always assume the prior, μ, has full support. These assumptions ensure the updating procedures are binding everywhere, as it alleviates the concern regarding 0 probability events.

Of course, for the posteriors to be indexed by menus there must be a connection between the signals and the menu. At the most general level, the two coincide: $\Theta = \mathcal{M}(X)$.

Definition 3. *An* ***information structure based on menus****,* $(\mu, l, \mathcal{M}(X))$*, generates* $\{\mu_A | A \in \mathcal{M}(X)\}$*, if*

$$\mu_A(s) = \frac{\mu(s) l(A, s)}{\mathbb{E}_\mu(l(A, s'))} \tag{3.1}$$

and $\sum_{\mathcal{M}(X)} l(A, s) = 1$, $\sum_s l(A, s) > 0$ *for all* $A \in \mathcal{M}(X)$ *and* $s \in S$.[14]

The requirement that the DM entertains some generating $(\mu, l, \mathcal{M}(X))$, provides no testable implications. In other words, *every* MBR can be described by some prior and likelihood function over menu realizations. The ability to choose both the signals and the prior provides enough degrees of freedom so that it is always possible to rationalize the family of menu induced beliefs.

[14] The first requirement is equivalent to $\sum_{\mathcal{M}(X)} l(A, s) > 0$, and is included in the current form only for its interpretational content. Under this normalization, we can think of $l(A, s)$ as the probability of seeing menu A in state s. The second requirement states that all menus are ex-ante possible. This ensures that 3.1 is always well defined.

Proposition 3.1. *Let $\{\succcurlyeq_A\}_{A\in\mathcal{M}(X)}$ be represented by some (MBR), with beliefs $\{\mu_A|A\in\mathcal{M}(X)\}$. Then there exists some $(\mu,l,\mathcal{M}(X))$ that generates $\{\mu_A|A\in\mathcal{M}(X)\}$ as in (3.1).*

Proof. In appendix C. ■

This result is a corollary of Lemma 1 in Shmaya and Yariv (2016). To understand the economic pertinence of the above result, consider the following two hypotheses regarding a MBR decision maker: (i) that the DM believes that likelihood of different alternatives being available depends on the underlying state of the world, and so rationally revises his beliefs after observing the available menu, and (ii) the DM associates at random a belief regarding the state space to each menu. In the first scenario, the DM could be considered a Bayesian, and there are many scenarios where identifying the DM's subjective information would be economically relevant. In the second, the DM could be considered irrational, deciding randomly which beliefs to hold. The above result indicated that a modeler who wishes to test if WARP violations present in a data set arise from hypothesis (i) or (ii) cannot do so without making more stringent assumption on the structure of the information.

Setting $\Theta=\mathcal{M}(X)$ assumes no relation between the signals associated with different menus, and it is this generality that renders behavior wholly unconstrained. However, under more specific assumptions regarding the structure of the signals, there are falsifiable restrictions on observable preference. Thus, while we can never rule out the possibility that the DM is acting in a Bayesian manner with respect to *some* signal space, we can rule out particular models of information.

Recall in example 1B, Katya believes, it is much less likely that a reputable date, rather than depraved one, offers methamphetamine, m. Moreover, she believes this is independent of whatever else was available. So, for any menu A, Katya will believe her date is more likely to be depraved when $A\cup m$ is offered rather than A alone. More generally, we could think that each element is offered with some probability that depends on the state. Under this interpretation, if the true state is s, then x will be available with probability $l(x,s)\in[0,1]$. Whenever x is more likely to be available in state s than in state s' the observation that x is available increases the relative likelihood of s compared to s'; whenever it is excluded, beliefs shift in the other direction. This corresponds to the signal structure where $\Theta=X$, and where signals are conditionally independent.

Definition 4. *An* ***information structure based on elements****, (μ,l,X), with conditionally independent signals generates $\{\mu_A|A\in\mathcal{M}(X)\}$ if*

$$\mu_A(s)=\frac{\mu(s)\prod_{x\in A}l(x,s)\prod_{y\notin A}(1-l(y,s))}{\mathbb{E}_\mu\left(\prod_{x\in A}l(x,s')\prod_{y\notin A}(1-l(y,s'))\right)},\tag{3.2}$$

and $l(x,s)\in[0,1]$ for all $(x,s)\in X\times S$, and $l(x^\star,s)=l(x_\star,s)=1$ for all $s\in S$.[15]

The fact that signals are independent, indicates that the inclusion or exclusion of a particular element carries the same informational content regardless of the composition of the menu. Of course, even though the informational value is the same, the magnitude of the effect of this information is relative to the information provided by the other elements included (or excluded) from the menu. This behavior is captured by the following axiom.

[15]The requirement regarding $x^\star$ and $x_\star$, stems from the fact that they are necessarily realized in every state, and hence, uninformative.

[A5: Independent Signals (IS)] *For all* $x \in X$*, and* $A, B \in \mathcal{M}(X)$*, such that* $x \notin A \cup B$*, and states* $s, s' \notin N_A \cup N_B$*, if for some distributions* $\pi^A, \rho^A, \in \Delta(A)$ *and* $\pi^B, \rho^B, \in \Delta(B)$*:* $(x_\star)_{-s}\pi^A \sim_A (x_\star)_{-s'}\rho^A$ *and* $(x_\star)_{-s}\pi^B \sim_B (x_\star)_{-s'}\rho^B$*, then for all* $\alpha \in (0,1)$*,*

$$(x_\star)_{-s}\pi^A \succcurlyeq_{A\cup x} (x_\star)_{-s'}(\alpha\rho^A + (1-\alpha)x_\star) \iff (x_\star)_{-s}\pi^B \succcurlyeq_{B\cup x} (x_\star)_{-s'}(\alpha\rho^B + (1-\alpha)x_\star). \tag{3.3}$$

IS states that the proportional change in belief, in response to the inclusion of an element x, is the same across all menus. Without x, obtaining π^A in state s and ρ^A in state s' (and $x_\star$ everywhere else) are equally appealing, given menu A. When x is included, the beliefs change, and therefore, so do preferences. IS states that the same proportional change in preferences must occur, regardless of the initial menu. So if the change in preferences is such that, π^A in state s is now indifferent to $\alpha\rho^A + (1-\alpha)x_\star$ in state s' (and $x_\star$ everywhere else) given A, then the same α proportional shift preserves indifference when moving from B to $B \cup x$. This behavior, along with the general representation, exactly captures the updating procedure given by (3.2).

Theorem 3.2. *Let* $\{\succcurlyeq_A\}_{A\in\mathcal{M}(X)}$ *be represented by some* (MBR)*, with beliefs* $\{\mu_A | A \in \mathcal{M}(X)\}$*, all of which have full support. Then, there exists some* (μ, l, X) *that generates* $\{\mu_A | A \in \mathcal{M}(X)\}$ *as in* (3.2) *if and only if* $\{\succcurlyeq_A\}_{A\in\mathcal{M}(X)}$ *satisfies* IS*.*

Proof. In appendix C. ■

The requirement that all beliefs have full support is tantamount to assuming $l(x,s) \in (0,1)$ for all $x \notin \star$ and $s \in S$, and ensures that (3.2) is well defined for all menus and states. To include null states in such a set up adds little intuition and greatly increases the level of attention to technical detail that needs to be paid.

In example 2B, Katya believes all high quality (and only high quality) restaurants serve frog legs, f; the inclusion of f in the menu indicates with certainty that the restaurant is high quality: the event $\{h\} \subset S$. Alternatively, she believes that medium and low quality restaurants always offer chicken and steak $\{c, s\}$. Observing a menu excluding f is indicative of the event $\{m, l\} \subset S$. She entertains the partition of S, $\{\{h\}, \{m, l\}\}$ with the composition of the menu signifying which event has obtained. This is captured by a special case of an elemental signal structure, where $l(x, \cdot)$ is constant within each event of the partition.

Definition 5. *Let* $\dot\sim$ *be an equivalence relation on* $S \times S$*. An* ***information structure based on*** $\dot\sim$ *is a information structure based on elements in which, for any* $s, s' \in S$*, (i)* $s \dot\sim s'$ *implies* $l(x,s) = l(x,s')$ *for all* $x \in X$ *and (ii)* $\neg(s \dot\sim s')$ *and* $\prod_{x\in A} l(x,s) \prod_{y\notin A}(1-l(y,s)) > 0$ *jointly imply* $\prod_{x\in A} l(x,s') \prod_{y\notin A}(1-l(y,s')) = 0$*.*

The first additional requirement dictates that within each event of the partition induced by $\dot\sim$, each menu is equally likely in each state (e.g., the menu $\{c, s\}$ is equally likely when the state is m as when it is l). The second requirement ensures each menu appears only within a single event of the partition (e.g., the menu $\{c, s\}$ is possible only in the event $\{m, l\}$ and not in the event $\{h\}$). Hence, each menu indicates a particular event of the the partition of the state space given by $\dot\sim$. If two menus, A and B are both possible in event E, then they carry exactly the same informational content (that E has obtained), and so induce the same preferences over common acts.

[A6: Partitional Signals (PS)] *For all $A \in \mathcal{M}(X)$, let $\mathcal{N}(A) = \{B \in \mathcal{M}(X) | (N_B)^c \cap (N_A)^c \neq \varnothing\}$. Then (i) for all $f, g \in \mathcal{F}_\star$, $B \in \mathcal{N}(A)$ implies,*

$$f \succcurlyeq_A g \iff f \succcurlyeq_B g.$$

and (ii) there exits an x such that $x \in B \iff B \in \mathcal{N}$ or $x \notin B \iff B \in \mathcal{N}$.

PS dictates that any two menus sharing a non-null state must induce the same preference over acts. Because the general representation fixes tastes across different menus, PS implies that if the two menus induce beliefs with a overlapping supports, those beliefs must coincide completely. It is clear that this captures the behavior generated by a partitional signal structure. Notice that PS implies IS.

Theorem 3.3. *Let $\{\succcurlyeq_A\}_{A \in \mathcal{M}(X)}$ be represented by some (MBR), with beliefs $\{\mu_A | A \in \mathcal{M}(X)\}$. Then, there exists a unique $\dot{\sim} \in S \times S$ and an information structure (μ, l, X) based on $\dot{\sim}$ that generates $\{\mu_A | A \in \mathcal{M}(X)\}$ if and only if $\{\succcurlyeq_A\}_{A \in \mathcal{M}(X)}$ satisfies PS.*

Proof. In appendix C. ■

4 Literature Review

Tversky and Kahneman (1981) developed the notion of framing –the idea that decisions are influenced by their surrounding context. Framing has a large literature, both in the theoretical, experimental, and psychological settings (Kahneman and Tversky, 1984; Rubinstein and Salant, 2008; Tversky and Shafir, 1992). A particular type of framing concerns the consideration of menu, or currently available alternatives, referred to in the literature as context dependence. In contrast to this model, context dependence is often associated with particular psychological heuristics such as a basing choice on the difference between the attributes of outcomes or reluctance to choose extreme outcomes (Simonson and Tversky, 1992).

That a menu may contain information relevant to the DM's choice over the objects it contains was first articulated by Luce and Raiffa (1957) and expounded upon by Sen (1993, 1997). Sen describes the notion of the *epistemic value* of a menu with more tact than I could hope to achieve: "What is offered for choice can give us information about the underlying situation, and can thus influence our preference over the alternatives, as we see them. For example, the chooser may learn something about the person offering the choice on the basis of what he or she is offering." It is by paraphrasing/formalizing the vignettes in Luce and Raiffa (1957) and Sen (1997) that I constructed the examples that run throughout this paper.

There are several other models that account for preference reversals and WARP violations by appealing to a DM who optimizes his choice relative to multiple (different) preferences. Kalai *et al.* (2002) consider a model in which the DM's preference ordering depends on the menu. Given a slight bit of conceptual juggling, this paper can be though of as a generalization of that model by allowing the DM to hold probabilistic beliefs about his preference. In other words, if for each menu, the DM placed probably 1 on some state, then his preferences over constant acts would be described by the Kalai *et al.* (2002) model. In Kalai *et al.* (2002), the authors seek to identify the minimal set of preferences required to rationalize data, which generically will *not* be the unique set of rationalizing preferences. The use of acts in this paper (rather than constant outcomes) is to facilitate the identification of this probabilistic belief—the representation, and specifically the set of state-dependent preferences, is unique. Moreover, the restriction that set of rationalizing preferences share a common utility index, allows for the identification of the information the DM believes is encoded in each menu (under the further restrictions outlined in Section 3).

Manzini and Mariotti (2007) also consider a model wherein the DM considers multiple preferences (rationales). There, the DM applies the same set of rationales to each choice set, in a sequential manner. WARP violations are the result of non-unique selections by earlier rationales. Interestingly, when all the rationales are binary equivalence relations (partition X into good and bad subsets) then such choice behavior is rationalizable Mandler *et al.* (2012). Apesteguia and Ballester (2010) contemplate on the complexity of identification in these models.

There have been several decision theory papers which deal with characterizing framing effects that stem from informational sources. Ahn and Ergin (2010) considers a DM whose beliefs, and hence preferences, depend on the description of the state space. There a depiction of the state space is a partition of it, and preferences are defined over all acts measurable with respect to the partition. The interpretation is that different descriptions of the state space might alert the DM of contingencies which he would otherwise be unaware. Bourgeois-Gironde and Giraud (2009) construct a model of "rational" framing in the domain of Bolker–Jeffrey decision theory. They take as motivation, and provide an axiomatic foundation for, the observation of Sher and McKenzie (2006) that (seemingly) logically equivalent statements might in fact contain different information because the choice to use one description over another might itself impart information. As such, Bourgeois-Gironde and Giraud (2009) consider a set of frames and allow two different, but logically equivalent, statements that belong to different frames to induce different beliefs of the DM.

The epistemic aspect of decision problems has been studied by Kochov (2010) in a model that shares many philosophical motivations with this one. Kochov's model defines a decision problem as a collection of menus, and imposes the canonical axioms (i.e., Dekel *et al.* (2001)) on a preference relation over each decision problem to back out a problem-specific subjective state space. The primary mechanism by which epistemic content alters the decision makers preference in Kochov's model is by changing the composition of the subjective state space (i.e., the difference in preference is mitigated through a change in tastes, rather than beliefs). The interpretation of menus revealing different unforeseen contingencies is problematic from the modelers point of view: it is impossible to observe a decision maker who is both aware and unaware of a particular contingency. This paper, on the other hand, explains the same behavior by confining the context effect to be a local one.

The appeal to a DM who holds multiple beliefs in contexts has been explored in different setting, including the elicitation of state dependent preferences, psychological states, and modeling growing awareness. Karni *et al.* (1983) propose a DM who ranks alternatives after some hypothetical event with a known probability. Like this paper, there is an imposed consistency in ex-post tastes across different decision problems. Technically similar issues arise in Karni and Vierø (2013, 2017), which model a decision maker who discovers novel information, and so expands the state space he entertains. Naturally, the authors contemplate the connection between preferences before and after the expanded state space, and like this paper and Karni *et al.* (1983), require that the DM's tastes for ex-post outcomes are fixed.[16] However, the authors further impose, in what they term "reverse bayesianism", that the proportional likelihood of states remains fixed, so that the (chronologically) first belief coincides with the posterior belief after conditioning the (chronologically) second

[16] It should be noted that the authors assume a monotonicity condition, so that preferences are assumed to be state-*in*dependent.

belief on the event that the true state is in the (chronologically) first state space. This is reminiscent of the updating procedure that underlies partitional information structures, albeit without the additional issues arising from the incorporation of novel states.[17] Also related, Karni and Safra (2016) take a somewhat converse approach. There, the decision maker has beliefs regarding his state dependent preferences, or *states of mind* which induce a preference over menus, rather than the menu inducing the belief about the state space. As such, it is the DM's beliefs regarding a subjective state space (his *state of mind*) that is invariant across decision problems.

It is also worth noting that models of endogenous reference dependence can be interpreted as context dependence. In these models the decision problem is associated with a reference level of utility by which the DM evaluates each outcome (Koszegi and Rabin, 2006; Ok *et al.*, 2015). As such, adding outcomes that will effect the reference point will thereby change the DM's preferences. These models can be thought of as a specific case of epistemic concerns; the reference point is information about some underlying state variable. A decision problem associated with reference point, r, is an indicator that the state-of-the-world is s_r.

Finally, this paper is related to the decision theoretic literature on identifying the conditions under which a decision maker is Bayesian updating with respect to subjective (and hence unseen) signals, for example, Lehrer and Teper (2015). In particular, the general model can be seen as a special case of the subjective signal structure discussed in Shmaya and Yariv (2016).

A From Equilibrium to MBR

If DM's entertain beliefs dependent on the available options, but these beliefs do not respond to the true probability of events given the menu, then the economic agents who construct menus (i.e., restaurant owners and potential dates) will be able to manipulate the DM's beliefs for personal gain. For example, low quality restaurants start to offer frog legs, to entice the DM to order a steak. In any repeated or large scale interaction, this would lead a rational DM to change his perception of the information contained in a given context, in turn leading the supply side to change its behavior, etc. The present section shows that MBR preferences can arise as the equilibrium of such a strategic exchange, where the different types of suppliers can differentiate themselves by offering different menus, and the DM correctly understands the signaling mechanism. Further, the signal structure is a partitional one. The partitional signal structure arises naturally as the consequence of signaling equilibria because it is the inclusion of particular elements acts as a signal, so different types of sellers select different compositions of goods to offer. In a partial pooling equilibrium, not all types of sellers can be distinguished, so menu chosen in equilibrium is indicative of a set of types—precisely the behavior in a partitional structure.

Consider the environment where, first, a seller constructs a menu of goods to offer the buyer at posted prices, and then, the buyer decides whether or not to buy any of the offered goods. In other words, the sellers act as *stores*, who can curate their selections. Sellers are privately endowed with a type (read: the seller's quality or ability), and this type governs both the cost of *stocking* a particular good, and also, the utility a buyer derives from its consumption. In this environment, under standard single-crossing conditions, different types of sellers might differentiate themselves in equilibrium by offering different menus of goods. In such an equilibrium, the seller's beliefs regarding the type of seller, and hence, the value of the offered

[17]Notice the resemblance between PS and MC and Karni and Vierø's 'invariant risk preferences' and 'awareness consistency' axioms, respectively.

goods, is dependent on the offered menu. Specifically, if the equilibrium is in pure strategies, this induces a MBR with partitional signal structure. In the example below, high type restaurants want to distinguish themselves as such, in order to sell steak at a high profit. In oder to do so, they offer frog legs, which are expensive for worse restaurants to carry. Because the worse types do not find it profitable to carry frog legs, in equilibrium, the diner who observes frog legs knows with certainty he is in a high quality restuarant.

Example 1.C (Luce and Raiffa's diner, one last time). *There are four types of restaurants, high (h), medium (m), and low (l) and bad (b) quality. Each can offer any selection of chicken (c), steak (s), or frog legs (f). The cost for a particular restaurant to keep an item on the menu (train the chef, provide a wine pairing, keep fresh ingredients, etc), is given by the following matrix:*

$$\begin{array}{ccc} c_h(c) = 1 & c_h(s) = 2 & c_h(f) = 3 \\ c_m(c) = 1 & c_m(s) = 2 & c_m(f) = 9 \\ c_l(c) = 1 & c_l(s) = 3 & c_l(f) = 9 \\ c_b(c) = 1 & c_b(s) = 10 & c_b(f) = 9 \end{array}$$

A patron, given that the quality of the food is known, has preferences (in dollar terms) according to

$$\begin{array}{ccc} u(h,c) = 12 & u(h,s) = 16 & u(h,f) = 6 \\ u(m,c) = 9 & u(m,s) = 8 & u(m,f) = 5 \\ u(l,c) = 7 & u(l,s) = 4 & u(l,f) = 3 \\ u(b,c) = 1 & u(b,s) = 0 & u(b,f) = 0 \end{array}$$

Each type of restaurant can select any subset of the main courses (along with posted prices) to offer potential diners. Given the observed menu and the subsequently updated beliefs, a diner will select the course that maximizes her utility (her expected utility from consumption less the posted price). All diners can take an outside option with utility 0.

Assume, initially, the diner has a uniform prior over the different types of restaurants. Then the following is a Bayes-Nash equilibrium. The high type offers $\{c, s, f\}$ (at prices \$8, \$16, and \$8, respectively), the medium and low types both offers $\{c, s\}$ (at \$8, and \$6, respectively), and the bad type offers $\{c\}$ (at price of \$1). This is a partial pooling equilibrium, the diner places probability 1 on h after seeing $\{c, s, f\}$ and chooses s, places probability $\frac{1}{2}$ on both medium and low when he sees $\{c, s\}$ and chooses c, and, places probability 1 on b after seeing $\{c\}$ and chooses c. When seeing any other menu, she places probability 1 on b, and takes the outside option.

Notice that in this example, both the utilities for outcomes and the beliefs after the observation $\{c, s, f\}$ and $\{c, s\}$ map exactly to Katya's tastes and beliefs given the same observations. As such, the behavior of buyers in such an equilibrium would correspond exactly to the MBR with the partitional information structure described in Example 1B.

B LEMMAS

Lemma 1. *If* $\{\succcurlyeq_A\}_{A\in\mathcal{M}(X)}$ *satisfies* UV*, then for all* $A \in \mathcal{M}(X)$*,* $N_A = E_A \cup N$*.*

Proof. Fix some $A \in \mathcal{M}(X)$. By definition both E_A and N are subsets of N_A, so, $E_A \cup N \subseteq N_A$. Towards the opposite inclusion, let $s \in N_A$. We will show that if $s \notin E_A$ then $s \in N$. So assume further, that $s \notin E_A$. Since s is null-A, $x^\star \sim_A x_\star$. Since s is not e-null-A, for every $B \in \mathcal{M}(X)$, $x^\star \sim_B x_\star$. By the contrapositive of UV we have $s \in N_B$. Since this holds for all B, $s \in N$. ■

Definition 6. *For a menu* $A \in \mathcal{M}(X)$*, define the **equalizer** of* A*,* $e_A : (N_A)^c \times (N_A)^c \to R_{++}$ *as*

$$e_A(s,s') \mapsto \begin{cases} \frac{1}{\alpha} \text{ such that } (x_\star)_{-s}(\alpha x^\star + (1-\alpha)x_\star) \sim_A (x_\star)_{-s'}x^\star & \text{if } (x_\star)_{-s}x^\star \succcurlyeq_A (x_\star)_{-s'}x^\star \\ \alpha \text{ such that } (x_\star)_{-s'}(\alpha x^\star + (1-\alpha)x_\star) \sim_A (x_\star)_{-s}x^\star & \text{if } (x_\star)_{-s'}x^\star \succ_A (x_\star)_{-s}x^\star \end{cases}$$

That e_A is well defined follows from the following observation.

Lemma 2. *Let* $\{\succcurlyeq_A\}_{A\in\mathcal{M}(X)}$ *be represented by some* (MBR)*, with beliefs* $\{\mu_A | A \in \mathcal{M}(X)\}$*, all of which have full support. Then, for all* $A \in \mathcal{M}(X)$*,* $e_A(s,s') = \frac{\mu_A(s)}{\mu_A(s')}$*.*

Proof. If $(x_\star)_{-s}x^\star \succcurlyeq_A (x_\star)_{-s'}x^\star$, then for some $\alpha \in (0,1)$, we have $(x_\star)_{-s}\big(\alpha x^\star + (1-\alpha)x_\star\big) \sim_A (x_\star)_{-s'}x^\star$. Using (MBR), we have that $U_A\Big((x_\star)_{-s'}x^\star\Big) = \mu_A(s')$, and $U_A\Big((x_\star)_{-s}\big(\alpha x^\star + (1-\alpha)x_\star\big)\Big) = \alpha\mu_A(s)$. Setting $e_A(s,s') = \frac{1}{\alpha}$, delivers ther result. The other case is similar. ■

Lemma 3. *Let* $\{\succcurlyeq_A\}_{A\in\mathcal{M}(X)}$ *be represented by some* (MBR) *with beliefs* $\{\mu_A | A \in \mathcal{M}(X)\}$*, all of which have full support. Then* $\{\succcurlyeq_A\}_{A\in\mathcal{M}(X)}$ *satisfies* IS *if and only if, for all* $x \in X$ *and* $A, B \in \mathcal{M}(X)$ *with* $x \notin A \cup B$*, and states* $s, s' \in S$*, we have*

$$\frac{e_A(s,s')}{e_{A\cup x}(s,s')} = \frac{e_B(s,s')}{e_{B\cup x}(s,s')}. \tag{B.1}$$

Proof. **Necessity.** Assume that (B.1) holds, with $x \in X$, $A, B \in \mathcal{M}(X)$, and $s, s' \in S$ satisfying the relevant constraints. Denote by A' and B', $A \cup x$ and $B \cup x$, respectively. Towards a contradiction, assume that there exists some $\pi^A, \rho^A, \in \Delta(A)$, $\pi^B, \rho^B, \in \Delta(A)$, and $\alpha = (0,1)$ be such that,

$$(x_\star)_{-s}\pi^A \sim_A (x_\star)_{-s'}\rho^A, \qquad \text{implying} \quad \frac{\mu_A(s)}{\mu_A(s')} = \frac{(\rho^A \cdot u)}{(\pi^A \cdot u)}, \tag{B.2}$$

$$(x_\star)_{-s}\pi^B \sim_B (x_\star)_{-s'}\rho^B, \qquad \text{implying} \quad \frac{\mu_B(s)}{\mu_B(s')} = \frac{(\rho^B \cdot u)}{(\pi^B \cdot u)}, \tag{B.3}$$

$$(x_\star)_{-s}\pi^A \succcurlyeq_{A'} (x_\star)_{-s'}(\alpha\rho^A + (1-\alpha)x_\star), \qquad \text{implying} \quad \frac{\mu_{A'}(s)}{\mu_{A'}(s')} \geqslant \alpha\frac{(\rho^A \cdot u)}{(\pi^A \cdot u)}, \tag{B.4}$$

$$(x_\star)_{-s}\pi^B \prec_{B'} (x_\star)_{-s'}(\alpha\rho^B + (1-\alpha)x_\star), \qquad \text{implying} \quad \frac{\mu_{B'}(s)}{\mu_{A'}(s')} < \alpha\frac{(\rho^B \cdot u)}{(\pi^B \cdot u)} \tag{B.5}$$

Dividing the implications of (B.2) by (B.4) and (B.3) by (B.5), and applying Lemma 2, we get a direct contradiction to (B.1).

Sufficiency. Assume IS holds. Let $x \in X$, $A, B \in \mathcal{M}(X)$, and $s, s' \in S$ satisfy the relevant constraints for IS. Let $M = \max\{\frac{\mu_A(s)}{\mu_A(s')}, \frac{\mu_B(s)}{\mu_B(s')}, 1\}$. Finally, for any $\beta \in [0, M]$, and $s \in S$, let $f^*(s,\beta) = (x_\star)_{-s}(\frac{\beta}{M}x^\star + (1-$

$\frac{\beta}{M})x_\star)$. Using (MBR), we have

$$U_A\big(f^*(s,1)\big) = U_A\big(f^*(s', \frac{\mu_A(s)}{\mu_A(s')})\big) = \frac{\mu_A(s)}{M}, \text{ and} \tag{B.6}$$

$$U_B\big(f^*(s,1)\big) = U_B\big(f^*(s', \frac{\mu_B(s)}{\mu_B(s')})\big) = \frac{\mu_B(s)}{M}. \tag{B.7}$$

Let $\alpha = \frac{\mu_A(s')\mu_{A'}(s)}{\mu_A(s)\mu_{A'}(s')}$. Case: $\alpha \leqslant 1$. Applying (MBR) again delivers,

$$U_{A'}\big(f^*(s,1)\big) = U_{A'}\big(f^*(s', \alpha\frac{\mu_A(s)}{\mu_A(s')})\big) = \frac{\mu_{A'}(s)}{M}.$$

By (B.6) and (B.7), we can apply IS, so,

$$U_{B'}\big(f^*(s,1)\big) = U_{B'}\big(f^*(s', \alpha\frac{\mu_B(s)}{\mu_B(s')})\big). \tag{B.8}$$

Expanding (B.8) according to (MBR):

$$\mu_{B'}(s) = \mu_{B'}(s')\frac{\mu_A(s')\mu_{A'}(s)}{\mu_A(s)\mu_{A'}(s')}\frac{\mu_B(s)}{\mu_B(s')},$$

which by Lemma 2, is equivalent to (B.1). In the case where $\alpha > 1$, consider $f^*(s, \frac{1}{\alpha})$ and $f^*(s', \frac{\mu_A(s)}{\mu_A(s')})$, and proceed in a similar manner. ■

C Proofs

Proof of Theorem 2.2. **Part (a), necessity.** The necessity of EU, MC, UV are obvious from the inspection of the representing functionals. CC follows from the continuity of $\mu_{(\cdot)}$. Fix some $\{A_n\}_{n\in\mathbb{N}}$ with limit point A. For each $f \in \mathcal{F}_\star$, identify $f = \big((\alpha_1 x^\star + (1-\alpha_1)x_\star), \ldots, (\alpha_{|S|}x^\star + (1-\alpha_{|S|})x_\star)\big)$ with $\hat{f} = (\alpha_{s_1}, \ldots, \alpha_{s_{|S|}})$. So let $f \succcurlyeq_{A_n} g$ for all $n \in \mathbb{N}$. Then, by the representation, we have, for every n,

$$\mathop{\mathbb{E}}_{\mu_{A_n}}(\hat{f} - \hat{g}) \geqslant 0. \tag{C.1}$$

Since $\hat{f} - \hat{g}$ is bounded and continuous, and since μ_{A_n} converges to μ_A, we have $\mathbb{E}_{\mu_A}(\hat{f} - \hat{g}) \geqslant 0$. Appealing again to the representation and the above identification delivers that $f \succcurlyeq_A g$.

Part (a), sufficiency. It is a direct application of the expected utility theorem that EU delivers for each A the existence of some continuous and bounded $w : S \times X \to \mathbb{R}$ such that

$$U_A^{VNM}(f) = \sum_s \Big(\mathop{\mathbb{E}}_{f(s)}(w_A(s,x))\Big),$$

represents $\succcurlyeq_A$. Moreover, if $w_A(s,x)$ and $\hat{w}_A(s,x)$ both represent $\succcurlyeq_A$, then $w_A(s,x) = a_A\hat{w}_A(s,x) + b_A(s)$ where $a_A \in \mathbb{R}_{++}$ and $b_A(s) \in \mathbb{R}$ for all $s \in S$.[18]

By exploiting the degrees of freedom from the scalars $b_A(s)$, we can set $w_A(s, x_\star) = 0$, for all A and all $s \in S$. The resulting functionals are unique up to linear transformations. Note, this implies that for all $s \in N_A$, $w_A(s,\cdot)$ is identically 0.

[18] For a reference using the same framework, see "NM Theorem" of Karni *et al.* (1983).

For each $A \in \mathcal{M}(X)$, let $u_A(s,x) : (N_A)^c \times A \to \mathbb{R}$ be the mapping

$$u_A : (s,x) \mapsto \frac{w_A(s,x)}{w_A(s,x^\star)},$$

and $\mu_A \in \Delta((E_A \cup N)^c)$ as the distribution defined by

$$\mu_A(s) = \frac{w_A(s,x^\star)}{\sum_s w_A(s,x^\star)}.$$

Notice, μ_A is well defined and has full support, since by the non-triviality of $\succcurlyeq_A$, $N_A \neq S$, and for each $s \in (E_A \cup N)^c$, $s \in N_A^c$ (Lemma 1), and so by UV, $w_A(s,x^\star) > w_A(s,x_\star) = 0$. Define,

$$U_A^{MD}(f) = \mathbb{E}_{\mu_A}\left(\mathbb{E}_{f(s)}(u_A(s,x))\right). \tag{C.2}$$

Following standard algebraic manipulations, we can see $\mu_A(s)u_A(s,x) = \frac{1}{\sum_s w_A(s,x^\star)} w_A(s,x)$, and therefore U_A^{MD} represents $\succcurlyeq_A$.

Let $D = \{(s,x) \in S \times X \mid \exists A \in \mathcal{M}(X), x \in A, s \notin N_A\}$. For each $(s,x) \in D$, let $A_{s,x}$ be any menu such that $x \in A_{s,x}$ and $s \notin N_{A_{s,x}}$. Define the mapping $u : D \to \mathbb{R}$ as,

$$u : (s,x) \mapsto u_{A_{s,x}}(s,x).$$

and extend u to $S \times X$, by defining $u(s,x) = 0$ for all $(x,s) \in D^c$.

We now claim, for any $A \in \mathcal{M}(X)$, $s \notin N_A$ and $x \in A$, we have $u(s,x) = u_A(s,x)$. Indeed, for every such $A, B \in \mathcal{M}(X)$ and $s \notin N_A \cup N_B$. Let $\succcurlyeq_{A|B|s} \subseteq (\Delta(A \cap B))^2$ be defined by:

$$\pi \succcurlyeq_{A|B|s} \rho \iff \mathbb{E}_\pi(u_A(s,x)) \geqslant \mathbb{E}_\rho(u_A(s,x)).$$

Since $\succcurlyeq_{A|B|s}$ is represented by a linear utility function, it satisfies EU, and so, by the expected utility theorem, $u_A(s,\cdot)$ is the unique utility index, up to affine transformations.

Fix some A and $s \notin N_A$, and $x \in A$. By (C.2), $\mathbb{E}_\pi(u_A(s,x)) \geqslant \mathbb{E}_\rho(u_A(s,x))$ holds if and only if, for all $f \in F_A$, $f_{-s}\pi \succcurlyeq_A f_{-s}\rho$. Applying MC, we immediately have $g_{-s}\pi \succcurlyeq_{A_{s,x}} g_{-s}\rho$ for any $g \in F_{A_{s,x}}$ (here we use the fact that $s \notin N_{A_{s,x}}$). From (C.2) again, $\succcurlyeq_{A|A_{s,x}|s} = \succcurlyeq_{A_{s,x}|A|s}$. So $u_A(s,\cdot)$ is an affine transformation of $u_{A_{s,x}}(s,\cdot)$. Moreover, both are twice normalized: $u_A(s,x^\star) = u_{A_{s,x}}(s,x^\star) = 1$ and $u_A(s,x_\star) = u_{A_{s,x}}(s,x_\star) = 0$. Thus, they must coincide on $A \cap A_{s,x}$. Finally, since $x \in A \cap A_{s,x}$, we have $u_A(s,x) = u_{A_{s,x}}(s,x) = u(s,x)$. Clearly, since $u_A = u|_A$ and u_A is continuous and bounded, $u|_A$ is continuous and bounded.

Because it eases exposition, we well prove that $\mu_{(\cdot)} : \mathcal{M}(X) \to \Delta(S)$ is continuous after we have shown that it is unique.

Part (b). Uniqueness results are standard. It is clear from the argument above that $u(\cdot,\cdot)$ is unique (given the normalization on $\star$), as it must represent $\succcurlyeq_{A_{s,x}|A_{s,x}|s}$. With regards to beliefs, assume to the contrary, for some $A \in \mathcal{M}(X)$, μ and ν both represent (in conjunction with u, as in (MBR)) $\succcurlyeq_A$. Then there must be some s, s', such that $\mu(s) < \nu(s)$ and $\mu(s') > \nu(s')$. Assume (*with* loss of generality, but the other case follows from the reflected argument) that $\mu(s) \leqslant \mu(s')$. Set π as the probability distribution given by,

$$\pi(x) = \begin{cases} \frac{\mu(s)}{\mu(s')} & \text{if } x = x^\star, \\ 1 - \frac{\mu(s)}{\mu(s')} & \text{if } x = x_\star, \\ 0 & \text{otherwise.} \end{cases}$$

Given that (μ, u) represents $\succcurlyeq_A$, it follows from (MBR) that $(x_\star)_{-s'}\pi \sim_A (x_\star)_{-s}x^\star$. But, since (ν, u) also represents $\succcurlyeq_A$: $(x_\star)_{-s'}\pi \prec_A (x_\star)_{-s}x^\star$, a clear contradiction.

Part (a), sufficiency continued. Let A_n converge to A but assume that μ_{A_n} does not converge to μ_A. Then there must exist some (continuous and bounded) $\hat{f} : S \to R$, such that $\lim \mathbb{E}_{\mu_{A_n}}(\hat{f}) \neq \mathbb{E}_{\mu_A}(\hat{f})$. Set $\beta = \mathbb{E}_{\mu_A}(\hat{f})$. Since $\hat{f}$ is bounded, it is without loss of generality that $f(s) \in (0,1)$, for all $s \in S$. Partition $\mathbb{N}$ into $N^+ = \{n \in \mathbb{N} | \, \mathbb{E}_{\mu_{A_n}}(\hat{f}) > \beta\}$ and N^- analogously. At least one of these sets that is infinite *and* the corresponding subsequence constructed from the entries does not converge to β. WLOG, assume it is N^+. Let n_k denote the corresponding subsequence. Since $\lim \mathbb{E}_{\mu_{A_{n_k}}}(\hat{f}) \neq \beta$ and since $\mathbb{E}_{\mu_{A_{n_k}}}(\hat{f}) > \beta$ for all k, there exists an $\epsilon > 0$ such that we can extract a further subsequence (also labeled n_k) with $\mathbb{E}_{\mu_{A_{n_k}}}(\hat{f}) > \beta + \epsilon$ for all k.

Consider the act f which assigns $\hat{f}(s)x^\star + (1-\hat{f}(s))x_\star$ to each state and the act $g = (\beta+\epsilon)x^\star + (1-(\beta+\epsilon))x_\star$ to each state (of course, we can choose an epsilon small enough to ensure $(\beta + \epsilon) < 1$. By the representation, and the above analysis, it is clear that $f \succcurlyeq_{A_{n_k}} g$ for all k, and so by CC, $f \succcurlyeq_A g$, contradicting the fact that $U_A(f) = \beta < \beta + \epsilon = U_A(g)$.

■

Proof of Proposition 3.1. This follows directly from Lemma 1 of Shmaya and Yariv (2016) which states (in the language of this paper) that given a prior μ and a set of posteriors $\{\mu_A\}_{A \in \mathcal{M}(X)}$ one can find a generating signal structure, that transforms μ into $\{\mu_A\}_{A \in \mathcal{M}(X)}$, so long as the prior beliefs lie in the relative interior of the convex hull of the set of posteriors, i.e., $\mu \in \text{ri}(\text{Conv}(\{\mu_A\}_{A \in \mathcal{M}(X)}))$. Given the additional flexibility in choosing the prior, and the fact the relative interior of a non-empty convex set is non-empty, we can always find such a μ. ■

Proof of Theorem 3.2. **Necessity.** Assume there exists some (μ, l, X), with $l(x,s) \in (0,1)$ for all $(x,s) \in X \times S$, that generates $\{\mu_A | A \in \mathcal{M}(X)\}$. For some A that does not contain x and $s, s' \in S$, we have $e_A(s,s') = \frac{\mu_A(s)}{\mu_A(s')}$, and $e_{A\cup x}(s,s') = \frac{\mu_{A\cup x}(s)}{\mu_{A\cup x}(s')}$. Using (3.2), we have

$$\mu_{A\cup x}(s) = \frac{\mu_A(s)\frac{l(x,s)}{1-l(x,s)}}{\mathbb{E}_{\mu_A}\left(\frac{l(x,s')}{1-l(x,s')}\right)}$$

for all $s \in S$. So,

$$\begin{aligned} e_{A\cup x}(s,s') &= \frac{\mu_A(s)\frac{l(x,s)}{1-l(x,s)}}{\mu_A(s')\frac{l(x,s')}{1-l(x,s')}}, \\ &= \frac{\frac{l(x,s)}{1-l(x,s)}}{\frac{l(x,s')}{1-l(x,s')}} e_A(s,s'). \end{aligned}$$

Hence the ratio of equalizers does not depend on the menu. By Lemma 3, IS holds.

Sufficiency. Assume IS holds. Let $\alpha(x,s) = \frac{\mu_{\{\star\cup x\}}(s)}{\mu_\star(s)}$, set

$$l(x,s) = \frac{\alpha(x,s)}{1+\alpha(x,s)}, \tag{C.3}$$

for all $(x,s) \in X\backslash \star \times S$ and $l(x^\star,\cdot) \equiv l(x_\star,\cdot) \equiv 1$. Let $\gamma(s) = \prod_{x \in X\backslash\star} (1 - l(x,s))$. Define $\mu \in \Delta(S)$ by,

$$\mu(s) = \frac{\frac{\mu_\star(s)}{\gamma(s)}}{\mathbb{E}_{\mu_\star}\left(\frac{1}{\gamma(s')}\right)}$$

By construction, $\mu_\star$ is generated according to (3.2).

We will now show that as defined, (μ, l, X) generates the remainder of $\{\mu_A | A \in \mathcal{M}(X)\}$. We proceed by induction on the cardinality of A.

Define $\nu_x \in \Delta(S)$

$$\nu_x(s) = \frac{\mu_\star(s)\frac{l(x,s)}{1-l(x,s)}}{\mathbb{E}_{\mu_\star}\left(\frac{l(x,s')}{1-l(x,s')}\right)}$$

Now, using the algebraic identity $\frac{\frac{\alpha}{1+\alpha}}{1-\frac{\alpha}{1+\alpha}} = \alpha$, we have $\frac{l(x,s')}{1-l(x,s')} = \alpha(x,s) = \frac{\mu_{\{\star\cup x\}}(s)}{\mu_\star(s)}$. Therefore $\nu_x(s) = \mu_{\{\star\cup x\}}(s)$. This completes the base case (for $|A| = 3$).

Now assume that (μ, l, X) generates $\{\mu_A | A \in \mathcal{M}(X), |A| \leqslant n\}$. Fix any A with n elements, and $x \notin A$. Let A' denote $A \cup x$. Set,

$$\nu_{A'} = \frac{\mu_A(s)\frac{l(x,s)}{1-l(x,s)}}{\mathbb{E}_{\mu_A}\left(\frac{l(x,s')}{1-l(x,s')}\right)}$$

Towards a contradiction, assume that $u_{A'} \neq \nu_{A'}$. Therefore, there must exist some s such that $u_{A'}(s) > \nu_{A'}(s)$, and s' such that $u_{A'}(s') < \nu_{A'}(s')$. Therefore we have:

$$\begin{aligned}\frac{e_A(s,s')}{e_{A\cup x}(s,s')} = \frac{\frac{\mu_A(s)}{\mu_A(s')}}{\frac{\mu_{A'}(s)}{\mu_{A'}(s')}} &< \frac{\frac{\mu_A(s)}{\mu_A(s')}}{\frac{\nu_{A'}(s)}{\nu_{A'}(s')}} \\ &= \frac{\frac{l(x,s')}{1-l(x,s')}}{\frac{l(x,s)}{1-l(x,s)}} = \frac{e_\star(s,s')}{e_{\star\cup x}(s,s')}\end{aligned}$$

Which, by Lemma 3 is a contradiction to IS. Therefore, the inductive step holds, and (μ, l, X) generates $\{\mu_A | A \in \mathcal{M}(X)\}$. ■

Proof of Theorem 3.3. **Necessity.** Assume there was some $\dot\sim \in S \times S$ and (μ, l, X), satisfying the relevant assumptions, that generates $\{\mu_A | A \in \mathcal{M}(X)\}$. Let $A, B \in \mathcal{M}(X)$ such that $(N_A)^c \cap (N_B)^c \neq \varnothing$. Let $s \in (N_A)^c \cap (N_B)^c$. Let $E = \{s' \in S | s' \dot\sim s\}$. For all $s' \in E$,

$$\begin{aligned}\mu_A(s') &= \frac{\mu(s')\prod_{x\in A} l(x,s')\prod_{y\notin A}(1-l(y,s'))}{\sum_{s''\in S}\mu(s'')\prod_{x\in A} l(x,s'')\prod_{y\notin A}(1-l(y,s''))} \\ &= \frac{\mu(s')\prod_{x\in A} l(x,s')\prod_{y\notin A}(1-l(y,s'))}{\sum_{s''\in E}\mu(s'')\prod_{x\in A} l(x,s'')\prod_{y\notin A}(1-l(y,s''))} \\ &= \frac{\mu(s')\prod_{x\in A} l(x,s)\prod_{y\notin A}(1-l(y,s))}{\sum_{s''\in E}\mu(s'')\prod_{x\in A} l(x,s)\prod_{y\notin A}(1-l(y,s))} \\ &= \frac{\mu(s')}{\mu(E)}.\end{aligned}$$

The first equality comes from property (ii) of partitional signal structures and the second comes from property (i). By property (ii), for any $s' \notin E$, $\mu_A(s') = 0$. Of course the exact same calculation can be made for B, so, $\mu_A = \mu_B$. Therefore the two menus induce the same preference. Further since for $s' \notin E$, $\prod_{x \in A} l(x, s') \prod_{y \notin A}(1 - l(y, s')) = 0$ it must be that for some $x \in A$, $l(x, s') = 0$ or for some $y \notin A$, $l(y, s') = 1$. Hence, PS is satisfied.

Sufficiency. Assume PS holds. By assumption $N = \varnothing$, so, for each $s \in S$, choose some menu $A(s)$ such that $s \notin N_{A(s)}$. Define $\dot{\sim}$ over $S \times S$, by $s \dot{\sim} s'$ if there exists some s'' such that $s, s' \notin N_{A(s'')}$. It is obvious that $\dot{\sim}$ is a equivalence relation. The choice of $A(s)$ was irrelevant, since by PS, if $s \in N_B$, then the projections of $\succcurlyeq_A$ and $\succcurlyeq_B$ onto $\mathcal{F}_\star$ must coincide and so $\mu_A = \mu_B$, implying that $N_{A(s)} = N_B$. Let K denote the number of partitions in $S/\dot{\sim}$.

Define $\mu(s) = \frac{\mu_{A(s)(s)}}{K}$. Recall, $\mathcal{N}(A) = \{B \in \mathcal{M}(X) | N_B \cap N_A \neq \varnothing\}$. For each s, let $x(s)$ denote the dignified element (which exists by PS) for $\mathcal{N}(A(s))$. For each $(x(s), s)$ let $l(x(s), s) = 1$ if $x(s) \in A(s)$ and $l(x(s), s) = 0$ if $x(s) \notin A$. Define the rest of $l(x, s) = \frac{1}{2}$. It is straightforward to see that $\mu_A(s) = \mu(s | \{s \in S | s \notin A(s)\})$. ■

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