Exactly solvable many-particle quantum graphs

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Declaration of authorship

I, George William Garforth, hereby declare that this thesis and the work presented in it is entirely my own. Where I have consulted the work of others, this is always clearly stated.

Signed:

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Date:

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Abstract

In this thesis, we construct exactly solvable many-particle quantum graphs in order to calculate and analyse their spectra. We begin by constructing two-particle quantum graphs with two-particle interactions, establishing appropriate boundary conditions via suitable self-adjoint realisations of the two-particle Laplacian. For certain non-local particle interactions, we show that explicit Laplace eigenfunctions can be constructed using the Bethe ansatz. Imposing appropriate boundary conditions on these eigenfunctions, we arrive at exact expressions for the spectra of two-particle quantum graphs given by solutions to a pair of secular equations. Performing numerical eigenvalue searches, we compare the spectral statistics of certain examples to well known results in random matrix theory, analysing the chaotic properties of their classical counterparts. We finish by generalising the approach to n particles, arriving at exact expressions for the spectra of n-particle quantum graphs given by solutions to a set of n secular equations.

Contents

Chapter 1

Introduction

In this thesis we investigate the properties of many-particle quantum graphs with particular focus on the acquisition and analysis of their spectra. A quantum graph is a collection of vertices and edges of finite or infinite length equipped with a differential operator. The first theoretical model of a quantum graph was devised by Pauling [Pau36]. His motivation was to study the dynamics of free electrons in hydrocarbons by modelling carbon molecules as vertices and carbon-carbon bonds as edges. This idea was later adopted by Ruedenberg and Scherr [RS53] who used quantum graphs to describe free electrons donated by covalent bonds confined to entire quasi-one-dimensional molecules. Since then there have been multiple applications of quantum graphs in a variety of fields including quantum waveguides [FJK87], quantum chaos [KS97], quantum computation [Lov10, KS00] and mesoscopic systems [TM05]. For a review of quantum graphs, see [EKK+08, BK13a].

Typically, quantum graphs are constructed by establishing boundary conditions characterised by self-adjoint realisations of the one-body Laplacian [KS99, Kuc04]. The corresponding Laplace eigenvalues then play a major role in their study. An important aspect of quantum graphs is that they may serve as models for quantum systems with corresponding complex classical dynamics. Kottos and Smilansky [KS97] demonstrated that eigenvalue correlations in quantum graphs can be described with random matrix models, therefore providing an example for the celebrated Bohigas-Giannoni-Schmit conjecture [BGS84] which is a central topic in quantum chaos.

While the majority of quantum graphs literature is focussed on one-particle models, there have been a number of studies of many-particle quantum graphs. The first of these, by Melnikov and Pavlov [MP95], investigated the dynamics of two interacting particles on a connected graph with three infinite edges. Under certain restrictions of the system, they were able to find self-adjoint realisations of the two-body Laplacian corresponding to particle-particle and particle-vertex interactions. The resulting two-body wave function allowed the calculation of the conductivity of the system. More recently, Bolte and Kerner constructed two-particle quantum graphs, initially with interactions localised at the vertices [BK13b], and later with singular contact interactions [BK13c]. Boundary conditions via suitable self-adjoint realisations of the two-particle Laplacian were ascertained using quadratic forms. These results were then used to study Bose-Einstein condensation [BK14].

To some extent, the success of one-particle quantum graph models relies upon the fact that their spectra are determined by a secular equation [KS97], that is, Laplace eigenvalues are given by zeros of a finite-dimensional determinant. This leads to very efficient methods of calculating eigenvalues, and also allows one to prove exact trace formulae for spectral densities [Rot83, KS97, BE09]. The acquisition of the spectra in this way is possible since, locally, the classical configuration space of a one-particle graph is one-dimensional. For the quantum model this means that every eigenfunction must be a linear combination of left- and right-moving, one-dimensional plane waves. Many-particle quantum graphs have higher dimensional classical configuration spaces, in general prohibiting a finite-dimensional secular equation that determines the eigenvalues. This obstacle can be overcome under specific circumstances when symmetries lead to an exactly solvable model.

The first model of an exactly solvable many-body quantum system confined to a single dimension was developed by Lieb and Liniger [LL63]. They determined the exact spectra of a repulsively δ-interacting Bose gas on a circle, a result which was later generalised to distinguishable particles by Yang [Yan67], and extended to systems confined to an interval by Gaudin [Gau71]. Each of these results were formalised by the use of the Bethe ansatz, a sum of two-particle plane waves over possible particle configurations. Implicit in the use of the Bethe ansatz is the requirement for certain symmetries brought about by the interactions in the model. The consequence of increasing the complexity to systems of particles on general graphs is that these symmetries are destroyed. By imposing certain non-local particle interactions, however, Caudrelier and Crampé [CC07] showed that, for systems of particles on two-edge star graphs, compatibility with the Bethe ansatz is recovered. They were then able to calculate the exact spectra of these systems. Extending this method to general many-particle quantum graphs is the main aim of this thesis. The main results established in this thesis regarding two-particle quantum graphs are summarised in [BG16].

One-particle quantum graphs

A combinatorial, oriented graph $\Gamma(\mathcal{V},\mathcal{I},\mathcal{E},f)$ is a set of vertices $\mathcal{V}=\{v_1,\dots,v_{|\mathcal{V}|}\}$, connected by a set of internal edges $\mathcal{I}=\{i_1,\dots,i_{|\mathcal{I}|}\}$ and external edges $\mathcal{E}=\{e_1,\dots,e_{|\mathcal{E}|}\}$. The map f assigns to each external edge e_j a single vertex $f(e_j)=v_\eta$, and to each internal edge i_j an ordered pair of vertices $f(i_j)=(v_\gamma,v_\lambda)$, where $v_\gamma =: f_0(i_j)$ and $v_\lambda =: f_l(i_j)$ are initial and terminal vertices respectively. A pair of edges will be called *distant* if they have no common vertex and *neighbouring* if they have at least one common vertex. The set of distant and neighbouring edge couples will be denoted $\mathcal{D}$ and $\mathcal{N}$, respectively. The degree d_η of a vertex $v_\eta\in\mathcal{V}$ is the number of edges connected to it. The combinatorial graph is turned into a metric graph by assigning a finite interval $[0,l_j]$ to each internal edge $i_j\in\mathcal{I}$ in such a way that $f_0(i_j)$ is identified with $x=0$ and $f_l(i_j)$ with $x=l_j$. To each external edge $e_j\in\mathcal{E}$, a half-line $[0,\infty)$ is assigned such that $f(e_j)$ is identified with $x=0$. A metric graph is called compact if there are no external edges, $\mathcal{E}=\varnothing$. We proceed by restricting our attention to compact metric graphs only revisiting the notion of external edges when necessary.

The relevant one-particle Hilbert space

$$\mathscr{H}_1=\bigoplus_{j=1}^{|\mathcal{I}|}L^2(0,l_j) \tag{1.0.1}$$

on a compact quantum graph Γ is the direct sum of constituent Hilbert spaces on each edge. Thus vectors

$$\Psi=(\psi_j)_{j=1}^{|\mathcal{I}|} \tag{1.0.2}$$

in $\mathscr{H}_1$ are lists of square-integrable functions $\psi_j : (0, l_j) \to \mathbb{C}$. Throughout this thesis a quantum graph will always be a metric graph with an associated Laplacian. In the one-particle setting, such a Laplacian $-\Delta_1$ acts according to

$$-\Delta_1 \Psi = (-\psi_j''(x))_{j=1}^{|\mathcal{I}|}, \tag{1.0.3}$$

where dashes denote ordinary, possibly weak, derivatives. Thus vectors Ψ will be defined in an appropriate Sobolev space

$$H^2(\Gamma) = \bigoplus_{j=1}^{|\mathcal{I}|} H^2(0, l_j) \subset \mathscr{H}_1. \tag{1.0.4}$$

One-particle observables on Γ are self-adjoint operators on $\mathscr{H}_1$. We thus look for self-adjoint realisations of $-\Delta_1$. These will be given as conditions on boundary vectors

$$\Psi_{bv} = \begin{pmatrix} (\psi_j(0))_{j=1}^{|\mathcal{I}|} \\ (\psi_j(l_j))_{j=1}^{|\mathcal{I}|} \end{pmatrix} \text{ and } \Psi_{bv}' \begin{pmatrix} (\psi_j'(0))_{j=1}^{|\mathcal{I}|} \\ (-\psi_j'(l_j))_{j=1}^{|\mathcal{I}|} \end{pmatrix} \tag{1.0.5}$$

where we denote by $\psi_j(p)$, the limit $\lim_{x\to p} \psi_j(x)$. We adopt this notation throughout the thesis. Kostrykin and Schrader [KS99] showed that $-\Delta_1$ is self-adjoint on a domain $D(A, B) \subset H^2(\Gamma)$ such that functions $\Psi \in D(A, B)$ fulfil the boundary conditions

$$A\Psi_{bv} + B\Psi_{bv}' = 0, \tag{1.0.6}$$

where the $2|\mathcal{I}| \times 4|\mathcal{I}|$ matrix (A, B) has maximal rank equal to $2|\mathcal{I}|$ and $AB^\dagger = BA^\dagger$. Matrices A and B are often interpreted as encoding external potentials localised at the vertices [KS99].

The spectra can then be acquired by finding solutions to the eigenvalue equation

$$-\Delta_1 \Psi = E\Psi. \tag{1.0.7}$$

For non-zero eigenvalues $E = k^2 \in \mathbb{R}$, constituent wave functions ψ_j of corresponding eigenfunctions Ψ are necessarily superpositions of oppositely directed plane waves

$$\psi_j(x) = \alpha_j e^{ikx} + \beta_j e^{-ikx} \tag{1.0.8}$$

on each edge i_j. Imposing on $\psi_j(x)$, the boundary conditions (1.0.6), one can show that Laplace eigenvalues are given by $E = k^2$ where values k are the solutions of the secular equation

$$\det\left[\mathbb{I}_{2|\mathcal{I}|} - S_v(k)T(k,\boldsymbol{l})\right] = 0. \tag{1.0.9}$$

Here the scattering matrix

$$S_v(k) = -(A + ikB)^{-1}(A - ikB) \tag{1.0.10}$$

contains information about particle interaction at the vertices while the metric information is encoded in

$$T(k,\boldsymbol{l}) = \begin{pmatrix} 0 & e^{ik\boldsymbol{l}} \\ e^{ik\boldsymbol{l}} & 0 \end{pmatrix}. \tag{1.0.11}$$

The blocks in (1.0.11) are the diagonal matrices $e^{ik\boldsymbol{l}} = \mathrm{diag}(e^{ikl_j})_{j=1}^{|\mathcal{I}|}$.

As well as leading to efficient ways to calculate Laplace eigenvalues, the secular equation (1.0.9) allows one to prove exact trace formulae for spectral densities [Rot83, KS97, BE09]. Thus, one can express the eigenvalue counting function

$$N(E) = \#\{n; E_n \leq E\} \tag{1.0.12}$$

in terms of classical periodic orbits of a general graph.

Spectral statistics

The spectra of quantum graphs can be used to analyse their corresponding classical dynamics. A particularly useful statistical measure is the nearest neighbour level spacings distribution

$$\int_a^b p(s)ds = \lim_{N\to\infty} \frac{1}{N} \#\{n \leq N;\ a \leq \epsilon_{n+1} - \epsilon_n \leq b\} \tag{1.0.13}$$

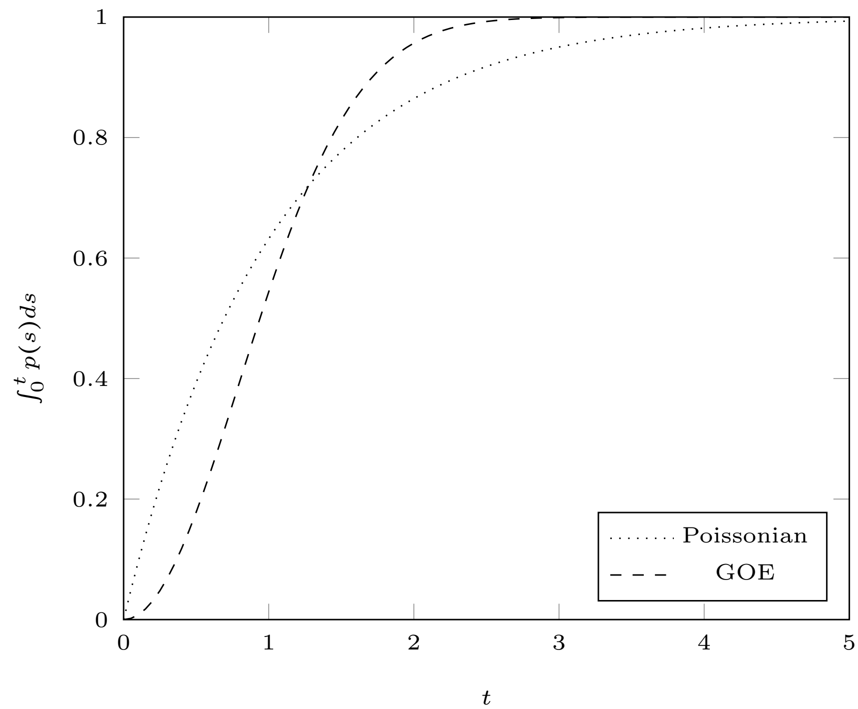

Figure 1.1: Characteristic nearest neighbour energy level distributions.

of the unfolded versions, $\epsilon_1 < \epsilon_2 < \epsilon_3 < \dots$, of the energy eigenvalues; that is the energies are rescaled such that the average spacing is equal to unity. Generic quantum systems with integrable classical limits are conjectured to have spectra with Poissonian statistics [BT77]

$$p(s) = e^{-s}, \tag{1.0.14}$$

while generic chaotic classical systems have quantum counterparts with correlations described by random matrix models. Indeed, the Bohigas-Giannoni-Schmit conjecture [BGS84] states that for such systems, with integer spin and time-reversal symmetry, Gaussian orthogonal ensemble (GOE) statistics apply. In this case, eigenvalues are known to exhibit *level repulsion*, with the level spacings distribution approximated by

$$p(s) = \frac{\pi}{2} s\, \mathrm{e}^{-\frac{\pi}{4}s^2} \tag{1.0.15}$$

(see [Haa91]). The characteristic Poissonian and GOE shapes are shown in Figure 1.1.

Two-particle quantum graphs

We have seen previously that one-particle quantum graphs can be characterised through self-adjoint realisations of $-\Delta_1$. A natural question is to ask how this approach can be extended to systems of two (or more) particles on a graph. Two-particle systems on a compact metric graph Γ are associated with the two-particle Hilbert space

$$\mathscr{H}_2 = \bigoplus_{j=1}^{|\mathcal{I}|} L^2(0,l_j) \otimes \bigoplus_{j=1}^{|\mathcal{I}|} L^2(0,l_j) \tag{1.0.16}$$

given by the tensor product of constituent one-particle Hilbert spaces. Thus vectors

$$\Psi = (\psi_{mn}(x_1,x_2))_{m,n=1}^{|\mathcal{I}|} \tag{1.0.17}$$

in $\mathscr{H}_2$ are lists of two-particle functions $\psi_{mn} : (0,l_m) \times (0,l_n) \to \mathbb{C}$. A two-particle quantum graph is then associated with the two-particle Laplacian

$$-\Delta_2\Psi = \left(-\frac{\partial^2\psi_{mn}}{\partial x_1^2} - \frac{\partial^2\psi_{mn}}{\partial x_2^2}\right)_{m,n=1}^{|\mathcal{I}|}. \tag{1.0.18}$$

The vector Ψ is again defined in an appropriate Sobolev space $H^2(D_\Gamma) \subset \mathscr{H}_2$, where D_Γ is the configuration space for two particles on Γ.

In the non-interacting case, appropriate self-adjoint realisations of $-\Delta_2$ are trivial extensions of the one-particle case. In this way, defining the two-particle boundary vectors

$$\Psi_{\mathrm{bv}}^{(v)}(y) = \begin{pmatrix} (\psi_{mn}(0,l_n y))_{m,n=1}^{|\mathcal{I}|} \\ (\psi_{mn}(l_m,l_n y))_{m,n=1}^{|\mathcal{I}|} \\ (\psi_{mn}(l_m y,0))_{n,m=1}^{|\mathcal{I}|} \\ (\psi_{mn}(l_m y,l_n))_{n,m=1}^{|\mathcal{I}|} \end{pmatrix} \text{ and } \Psi_{\mathrm{bv}}^{(v)\prime}(y) = \begin{pmatrix} (\psi_{mn,1}(0,l_n y))_{m,n=1}^{|\mathcal{I}|} \\ (\psi_{mn,1}(l_m,l_n y))_{m,n=1}^{|\mathcal{I}|} \\ (\psi_{mn,2}(l_m y,0))_{n,m=1}^{|\mathcal{I}|} \\ (\psi_{mn,2}(l_m y,l_n))_{n,m=1}^{|\mathcal{I}|} \end{pmatrix}, \tag{1.0.19}$$

with $y \in [0,1]$, where functions $\psi_{mn,1}$ and $\psi_{mn,2}$ are inward derivatives normal to the lines $x_1 = 0$ and $x_2 = 0$ respectively, $-\Delta_2$ is self-adjoint on the domain $D_2(A,B) \subset H^2(D_\Gamma)$ such that functions $\Psi \in D_2(A,B)$ fulfil the boundary conditions

$$(\mathbb{I}_2 \otimes A \otimes \mathbb{I}_{|\mathcal{I}|})\Psi_{bv}^{(v)}(y) + (\mathbb{I}_2 \otimes B \otimes \mathbb{I}_{|\mathcal{I}|})\Psi_{\mathrm{bv}}^{(v)\prime}(y) = 0, \tag{1.0.20}$$

More interesting models include singular contact interactions between particles. Such interactions take place along diagonals $x_1 = x_2$ of squares

$$(0,l_m) \times (0,l_m) \tag{1.0.21}$$

and thus additional boundary vectors are defined according to

$$\begin{aligned} \Psi_{\mathrm{bv}}^{(p)}(y) &= \begin{pmatrix} (\psi_{mm}(l_m y^+, l_m y))_{m=1}^{|\mathcal{I}|} \\ (\psi_{mm}(l_m y^-, l_m y))_{m=1}^{|\mathcal{I}|} \end{pmatrix} \\ \text{and } \Psi_{\mathrm{bv}}^{(p)\prime}(y) &= \begin{pmatrix} (\psi_{mm,d}(l_m y^+, l_m y))_{m=1}^{|\mathcal{I}|} \\ (\psi_{mm,d}(l_m y^-, l_m y))_{m=1}^{|\mathcal{I}|} \end{pmatrix}, \end{aligned} \tag{1.0.22}$$

where $\psi_{mm,d}$ are derivatives normal to the lines $x_1 = x_2$. Here and throughout this thesis we adopt the notation

$$y^\pm = \lim_{\epsilon \to 0^+} y \pm \epsilon. \tag{1.0.23}$$

Bolte and Kerner [BK13c] showed that for two-particle quantum graphs with singular contact interactions, $-\Delta_2$ is self-adjoint on the domain of sufficiently regular functions which, in addition to being subject to vertex conditions (1.0.20), also obey the boundary conditions

$$P_p(y)\Psi_{\mathrm{bv}}^{(p)}(y) = 0 \text{ and } Q_p(y)\Psi_{\mathrm{bv}}^{(p)\prime}(y) + L_p(y)Q_p(y)\Psi_{\mathrm{bv}}^{(p)}(y) = 0, \tag{1.0.24}$$

where bounded and measurable maps

$$P_p, L_p : (0,1) \to M(\mathbb{I}_{2|\mathcal{I}|}, \mathbb{C}) \tag{1.0.25}$$

are required to fulfil the conditions that

1. $P_p(y) = \mathbb{I} - Q_p(y)$ is an orthogonal projection;

2. $L_p(y)$ is a self-adjoint endomorphism on $\ker P_p(y)$.

In this way P_p and L_p prescribe the nature of the particle interactions. For such self-adjoint realisations, they proved that the counting function (1.0.12) obeys the Weyl law

$$N(E) \sim \frac{\mathcal{L}^2 E}{4\pi},\ E \to \infty \tag{1.0.26}$$

for distinguishable particles with an additional factor of one half

$$N_b(E) \sim \frac{\mathcal{L}^2 E}{8\pi},\ E \to \infty \tag{1.0.27}$$

for the bosonic case, where $\mathcal{L} = \sum_{j=1}^{|\mathcal{I}|} l_j$ is the total length of the graph.

Exactly solvable many-body systems

Ideally, we would like to be able to determine the exact spectra of many-particle quantum graphs using an analogous approach to that in the one-particle setting. It turns out that for general many-particle quantum graphs this is not possible. However, there are a number of specific examples whereby the complexity is sufficiently reduced, for which exact solutions are possible. One such example is a system of n δ-interacting bosons on a circle [LL63]. The method is centred around finding solutions to the n-particle eigenvalue equation

$$-\Delta_n \psi = E\psi \tag{1.0.28}$$

by the construction of explicit eigenfunctions

$$\psi = \psi(x_1, \dots, x_n) \tag{1.0.29}$$

of the n-particle Laplacian

$$-\Delta_n = -\sum_{j=1}^{n} \frac{\partial^2}{\partial {x_j}^2}. \tag{1.0.30}$$

The Bethe ansatz method in this context is the assumption that eigenfunctions ψ take the form

$$\psi = \sum_{Q \in S_n} \mathcal{A}^Q e^{i(k_{Q1}x_1 + \cdots + k_{Qn}x_n)} \tag{1.0.31}$$

with amplitudes $\mathcal{A}^Q$ and where elements Q of the symmetric group S_n act on the set $\{1, \ldots, n\}$.

Gaudin [Gau71] extended this approach to reflecting boundaries, that is n-particle systems confined to a box. The crucial difference in this model is the appearance of negative momenta due to reflection at the boundaries. To account for this, the appropriate Bethe ansatz becomes

$$\psi(x_1, \ldots, x_n) = \sum_{P \in \mathscr{W}_n} \mathcal{A}^P e^{i(k_{P1}x_1 + \cdots + k_{Pn}x_n)}, \tag{1.0.32}$$

where elements P of the Weyl group $\mathscr{W}_n$ act on the set $\{\pm 1, \ldots, \pm n\}$.

In each case, the eigenvalue equation (1.0.28) is satisfied with Laplace eigenvalues

$$E = k_1^2 + \cdots + k_n^2, \tag{1.0.33}$$

where values $\{k_1, \ldots, k_n\}$ are simultaneous solutions to a set of n secular equations determined by applying appropriate boundary conditions on the Bethe ansatz.

In the case of singular contact interactions, general many-particle quantum graphs are not compatible with this approach. Caudrelier and Crampé [CC07] however, noticed that the Bethe ansatz approach can be used for two-edge star graphs where certain symmetries are insured by a particular choice of non-local particle interactions. The main aim of this thesis is to extend their approach to general quantum graphs, establishing appropriate boundary conditions in the context of self-adjoint realisations of $-\Delta_n$.

In Chapter 2 we formalise the construction of general one-particle quantum graphs, establishing appropriate boundary conditions by self-adjoint realisations of the one-particle Laplacian $-\Delta_1$. The boundary conditions lead to a quantisation condition from which the spectra can be calculated. We then analyse the spectral statistics of a specific example commenting on its corresponding classical dynamics. In Chapter 3 we construct a number of exactly solvable two-particle models, calculating their spectra using the Bethe ansatz and analysing their spectral statistics. In Chapter 4 we review the construction of general two-particle quantum graphs with singular contact interactions following the method in [BK13c]. We then use this method to establish certain non-local interactions in the context of self-adjoint realisations of the two-particle Laplacian which permit exact solutions via the Bethe ansatz method. We proceed by determining a quantisation condition which yields the spectra of such graphs. We then analyse the spectral statistics of a number of examples commenting on the nature of the classical dynamics. In Chapter 5 we extend the procedure to general n-particle quantum graphs. Finally, in Chapter 6, we draw conclusions and outline possible directions for future study.

Chapter 2

One-particle quantum graphs

In this chapter, we review one-particle quantum graphs and their spectral properties. We begin by establishing boundary conditions which characterise self-adjoint realisations of the one-particle Laplacian. Then, specifying the form of the eigenfunctions of the Laplacian and imposing boundary conditions, we arrive at a quantisation condition from which one-particle quantum graph spectra can be deduced. We then discuss some properties of the spectra, focussing on the example of the quantum tetrahedron and commenting on its corresponding classical dynamics.

2.1 Preliminaries

Before we proceed, it is useful to introduce some definitions and conventions in the context of Hilbert spaces and symmetric operators. The reader is assumed to have some familiarity with the basic properties of Hilbert spaces. Material is taken from [Gri85, RS72, RS75].

In the following we denote by $\mathscr{H}$, a Hilbert space with inner product

$$\begin{aligned} \mathscr{H} \times \mathscr{H} &\mapsto \mathbb{C} \\ (\phi, \psi) &\mapsto \langle \phi | \psi \rangle, \end{aligned} \tag{2.1.1}$$

where we adopt the convention that the bra-ket notation $\langle \phi | \psi \rangle$ implies a conjugation in the first argument ϕ. In this thesis we will usually be concerned with the particular Hilbert space called the Lebesgue space.

Definition 2.1.1. The Lebesgue space $L^2(\Omega)$ is the set of complex functions ϕ with domain Ω that are square-integrable in the Lebesgue sense

$$\int_\Omega |\phi(x)|^2 dx < \infty \tag{2.1.2}$$

and with inner product defined as

$$\langle\phi|\psi\rangle = \int_\Omega \overline{\phi(x)}\psi(x)dx. \tag{2.1.3}$$

Moreover, due to the action of the Laplacian, functions defined on graphs will need to possess weak second derivatives. Thus they will be defined in an appropriate Sobolev space $H^2(\Omega)$ which is dense in $L^2(\Omega)$.

Definition 2.1.2. Let $m \in \mathbb{N}_0$. The Sobolev space $H^m(\Omega)$ consists of all functions $\psi \in L^2(\Omega)$ such that all weak derivatives up to order m are in $L^2(\Omega)$.

Typically, quantum graphs are constructed by establishing boundary conditions characterised by self-adjoint extensions of the Laplacian. Such operators are constructed by first defining a corresponding symmetric operator.

Definition 2.1.3. An operator A with dense domain $D(A) \subset \mathscr{H}$ is symmetric if for all $\phi, \psi \in D(A)$

$$\langle\phi|A\psi\rangle = \langle A\phi|\psi\rangle. \tag{2.1.4}$$

The adjoint operator A^* has domain $D(A^*)$ defined as the set of $\phi \in \mathscr{H}$ for which there exists some $\chi \in \mathscr{H}$ such that

$$\langle\phi|A\psi\rangle = \langle\chi|\psi\rangle \tag{2.1.5}$$

for all $\psi \in D(A)$. A symmetric operator A is called self-adjoint if the domains of A and A^* are equal; $D(A) = D(A^*)$.

Consider a symmetric operator A such that $D(A) \subset \mathscr{H}$. The extension B of A is an extension of $D(A)$ such that $D(A) \subset D(B)$ and

$$B\phi = A\phi \text{ for all } \phi \in D(A). \tag{2.1.6}$$

Since the domain of a symmetric operator is contained in the domain of its adjoint we have that

$$D(A) \subset D(B) \subset D(B^*) \subset D(A^*). \tag{2.1.7}$$

When self-adjoint extensions are possible there exist domains $D(H)$ which contain $D(A)$ and are subsets of $D(A^*)$ for which $D(H) = D(H^*)$. An operator H with such a domain is self-adjoint.

Another method for finding self-adjoint operators involves constructing suitable quadratic forms.

Definition 2.1.4. A sesquilinear form q with domain $D(q) \subset \mathscr{H}$ is a map

$$q : D(q) \times D(q) \to \mathbb{C} \tag{2.1.8}$$

with $D(q)$ a dense linear subspace of $\mathscr{H}$ which is conjugate linear in the first argument and linear in the second. We call q symmetric if

$$q(\phi, \psi) = \overline{q(\psi, \phi)}, \tag{2.1.9}$$

for all $\phi, \psi \in D(q)$. The corresponding quadratic form $q(\phi, \phi)$ is called semi-bounded if there exists some $\mu \geq 0$ for which

$$q(\phi, \phi) \geq -\mu ||\phi||^2_{\mathscr{H}}, \tag{2.1.10}$$

for all $\phi \in D(q)$, and closed if $D(q)$ is complete with respect to the norm

$$||\cdot||^2_q = q(\cdot) + (\lambda + 1)||\cdot||^2_{\mathscr{H}}. \tag{2.1.11}$$

Self-adjoint operators can then be identified using the following theorem from [Kat66].

Theorem 2.1.5. Every symmetric sesquilinear form $(q, D(q))$, with a corresponding quadratic form which is closed and semibounded, corresponds to a unique, semibounded and self-adjoint operator $(H, D(H))$ with $D(H) \subset D(q)$ such that

$$q(\phi, \psi) = \langle \phi | H\psi \rangle \tag{2.1.12}$$

for every $\phi \in D(q)$ and $\psi \in D(H)$.

2.2 Self-adjoint extension

Let us consider the compact metric graph $\Gamma(\mathcal{V}, \mathcal{I}, f)$. The appropriate Hilbert space

$$\mathscr{H}_1 = \bigoplus_{j=1}^{|\mathcal{I}|} L^2(0, l_j) \tag{2.2.1}$$

is the direct sum of constituent Hilbert spaces on each edge. Vectors

$$\Psi = (\psi_j)_{j=1}^{|\mathcal{I}|} \tag{2.2.2}$$

in $\mathscr{H}_1$ are lists of square-integrable functions $\psi_j : (0, l_j) \to \mathbb{C}$. A quantum graph is a metric graph Γ with an associated Laplacian $-\Delta_1$ which acts according to

$$-\Delta_1 \Psi = (-\psi_j''(x))_{j=1}^{|\mathcal{I}|}, \tag{2.2.3}$$

where dashes denote ordinary, possibly weak, derivatives. We wish to consider the eigenvalue equation

$$-\Delta_1 \Psi = E\Psi \tag{2.2.4}$$

alongside conditions which characterise interactions at the vertices.

One-particle observables on Γ are self-adjoint operators on $\mathscr{H}_1$. We thus look for self-adjoint realisations of $-\Delta_1$ with domains characterised by appropriate boundary conditions. We follow the method by Kostrykin and Schrader in [KS06b] noting that their formalism includes graphs with external edges. Restricting our attention to compact graphs will be useful when extending our approach two-particle systems.

Let us define the Sobolev space $H^2(\Gamma)$ as the set of $\Psi \in \mathscr{H}_1$ such that

$$\psi_j \in H^2(0, l_j) \tag{2.2.5}$$

for all $j \in \{1, \ldots, |\mathcal{I}|\}$, and let $\Omega : H^2(\Gamma) \times H^2(\Gamma) \to \mathbb{C}$ be the sesquilinear form

$$\begin{aligned}\Omega(\Phi, \Psi) =& \langle -\Delta_1 \Phi|\Psi\rangle - \langle \Phi| - \Delta_1 \Psi\rangle \\ =& \sum_{j=1}^{|\mathcal{I}|} \left(\overline{\phi}_j(l_j)\psi_j'(l_j) - \overline{\phi}_j(0)\psi_j'(0) - \overline{\phi}_j'(l_j)\psi_j(l_j) + \overline{\phi}_j'(0)\psi_j(0) \right).\end{aligned} \tag{2.2.6}$$

The bottom line of (2.2.6) is calculated by partial integration. Let us then define the subspace $H_0^2(\Gamma)$ as the set of $\Psi \in H^2(\Gamma)$ such that

$$\psi_j(0) = \psi_j(l_j) = \psi_j'(0) = \psi_j'(l_j) = 0 \tag{2.2.7}$$

for all $j \in \{1, \ldots, |\mathcal{I}|\}$, and let $-\Delta_1^0$ denote the Laplacian $-\Delta_1$ restricted to the domain $H_0^2(\Gamma)$.

Lemma 2.2.1. The operator $-\Delta_1^0$ is symmetric but not self-adjoint.

Proof. Symmetry is easily seen by noticing that $\Omega(\Phi, \Psi)$ vanishes for all $\Phi, \Psi \in H_0^2(\Gamma)$. The domain $D(-\Delta_1^{0*})$ of the adjoint operator $-\Delta_1^{0*}$ is the set of functions $\Phi \in \mathscr{H}_1$ which satisfy the condition

$$\langle \Phi| - \Delta_1 \Psi\rangle = \langle -\Delta_1^* \Phi|\Psi\rangle \tag{2.2.8}$$

for all $\Psi \in H_0^2(\Gamma)$. Since the action of $-\Delta_1$ is the same as its adjoint, this is the condition that $\Omega(\Phi, \Psi)$ vanishes which is clearly satisfied for all $\Phi \in H^2(\Gamma)$. We then have that $D(-\Delta_1^{0*}) = H^2(\Gamma)$ and thus that $H_0^2(\Gamma) \subset D(-\Delta_1^{0*})$. □

The aim is to find self-adjoint extensions of $-\Delta_1^0$. Since symmetric extensions of $-\Delta_1^0$ are contained in their adjoints, we look for maximal subspaces of $H^2(\Gamma)$ for which $-\Delta_1$ is symmetric, that is $\Omega(\Phi, \Psi) = 0$.

Let boundary vectors $\Psi_{bv}, \Psi'_{bv} \in \mathbb{C}^{2|\mathcal{I}|}$ be defined

$$\Psi_{\rm bv} = \begin{pmatrix} (\psi_j(0))_{j=1}^{|\mathcal{I}|} \\ (\psi_j(l_j))_{j=1}^{|\mathcal{I}|} \end{pmatrix} \text{ and } \Psi'_{\rm bv} = \begin{pmatrix} (\psi'_j(0))_{j=1}^{|\mathcal{I}|} \\ (-\psi'_j(l_j))_{j=1}^{|\mathcal{I}|} \end{pmatrix}. \tag{2.2.9}$$

Then, defining the vector

$$\underline{\Psi} = \begin{pmatrix} \Psi_{\rm bv} \\ \Psi'_{\rm bv} \end{pmatrix} \in \mathbb{C}^{4|\mathcal{I}|} \tag{2.2.10}$$

along with the symplectic matrix

$$J = \begin{pmatrix} 0 & -\mathbb{I}_{2|\mathcal{I}|} \\ \mathbb{I}_{2|\mathcal{I}|} & 0 \end{pmatrix}, \tag{2.2.11}$$

the form $\Omega(\Phi, \Psi)$ can be rewritten as the skew-Hermitian form

$$w(\underline{\Phi}, \underline{\Psi}) = \langle \underline{\Phi} | J \underline{\Psi} \rangle. \tag{2.2.12}$$

To find maximal subspaces of $H^2(\Gamma)$ for which $\Omega(\Phi, \Psi) = 0$, it is sufficient to find maximal subspaces in $\mathbb{C}^{4|\mathcal{I}|}$ for which $w(\underline{\Phi}, \underline{\Psi}) = 0$. By characterising the space $\mathcal{M} = \mathcal{M}(A, B)$ with $2|\mathcal{I}| \times 2|\mathcal{I}|$ matrices A and B, Kostrykin and Schrader [KS06b] proved a generalisation of the following theorem which we present for compact graphs only.

Theorem 2.2.2. The Laplacian $-\Delta_1$ is self-adjoint on the set of all $\Psi \in H^2(\Gamma)$ which satisfy the boundary condition

$$A\Psi_{bv} + B\Psi'_{bv} = 0, \tag{2.2.13}$$

with $2|\mathcal{I}| \times 2|\mathcal{I}|$ matrices A, B subject to

1. $\text{rank}(A, B) = 2|\mathcal{I}|$;

2. $AB^\dagger = BA^\dagger$.

It is convenient at this point to discuss another method for finding self-adjoint realisations of the Laplacian which is centred around Theorem 2.1.5. The idea is to associate with the problem, a quadratic form, show that it is closed, symmetric and semibounded and then extract the corresponding self-adjoint Laplacian. The following theorem, based on this approach, is from [Kuc04].

Theorem 2.2.3. Consider the maps P, L acting on the space $\mathbb{C}^{2|\mathcal{I}|}$ of boundary vectors, where P is an orthogonal projection and L is a self-adjoint endomorphism on $\ker(P)$. Moreover, set $Q = \mathbb{I}_{2|\mathcal{I}|} - P$. The Laplacian $-\Delta_1$ acting on a quantum graph is self-adjoint under the restriction $\Psi \in H^2(\Gamma)$ such that

$$P\Psi_{\mathrm{bv}} = 0 \text{ and } Q\Psi'_{\mathrm{bv}} + LQ\Psi_{\mathrm{bv}} = 0. \tag{2.2.14}$$

It is possible to show equivalence with the A, B parameterisation in Theorem 2.2.2 by letting P be an orthogonal projection onto $\ker(B)$ and

$$L = B^{-1}_{(\ker B)^{\perp}} AQ. \tag{2.2.15}$$

The restriction of B to $\ker(B)^{\perp}$ before taking its inverse is necessary since B in (2.2.13) need not be invertible. Moreover, Fulling, Kuchment and Wilson [FKW07] showed that there exists some invertible C such that

$$A' = CA = P + L \text{ and } B' = CB = Q \tag{2.2.16}$$

which implies

$$L = A'B'^{\dagger}. \tag{2.2.17}$$

At this point we distinguish between two important classes of boundary conditions. We note from (2.2.14) that for cases where $L = 0$, and thus for $AB^{\dagger} = 0$, boundary values of functions Ψ_{bv} and their derivatives Ψ'_{bv} do not mix. We call such boundary conditions non-Robin. Otherwise boundary conditions are called Robin.

2.3 Spectra of quantum graphs

In this section we calculate the spectra of one-particle quantum graphs by considering the eigenvalue equation

$$-\Delta_1 \Psi = E\Psi \tag{2.3.1}$$

alongside boundary conditions prescribed by Theorem 2.2.2. The starting point is the observation that the components $\psi_j(x)$ of eigenfunctions $\Psi \in \mathscr{H}_1$ with non-zero Laplace eigenvalues $E = k^2 \in \mathbb{R}$ are necessarily of the form

$$\psi_j(x) = \alpha_j e^{ikx} + \beta_j e^{-ikx}. \tag{2.3.2}$$

Each wave function ψ_j is a superposition of oppositely directed plane waves propagating along edge i_j with α_j and β_j their respective complex amplitudes. Defining vectors

$$\boldsymbol{\alpha} = (\alpha_j)_{j=1}^{|\mathcal{I}|} \text{ and } \boldsymbol{\beta} = (\beta_j)_{j=1}^{|\mathcal{I}|} \tag{2.3.3}$$

and imposing boundary conditions (2.2.13), we have that

$$(AX(k) + ikBY(k)) \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = 0, \tag{2.3.4}$$

where

$$X(k, \boldsymbol{l}) = \begin{pmatrix} \mathbb{I}_{|\mathcal{I}|} & \mathbb{I}_{|\mathcal{I}|} \\ e^{ik\boldsymbol{l}} & e^{-ik\boldsymbol{l}} \end{pmatrix} \text{ and } Y(k, \boldsymbol{l}) = \begin{pmatrix} \mathbb{I}_{|\mathcal{I}|} & -\mathbb{I}_{|\mathcal{I}|} \\ -e^{ik\boldsymbol{l}} & e^{-ik\boldsymbol{l}} \end{pmatrix}. \tag{2.3.5}$$

The blocks in (2.3.5) are the diagonal matrices

$$e^{ik\boldsymbol{l}} = \operatorname{diag}(e^{ikl_j})_{j=1}^{|\mathcal{I}|}. \tag{2.3.6}$$

By making the definitions

$$T(k, \boldsymbol{l}) = \begin{pmatrix} 0 & e^{ik\boldsymbol{l}} \\ e^{ik\boldsymbol{l}} & 0 \end{pmatrix} \tag{2.3.7}$$

and

$$S_v(k) = -(A + ikB)^{-1}(A - ikB), \tag{2.3.8}$$

and using the result in [KS99] that for $k \neq 0$, the matrices $A \pm ikB$ are invertible, we arrive at the following theorem from [KS06b].

Theorem 2.3.1. The non-zero eigenvalues of a self-adjoint Laplacian $-\Delta_1$ defined on Γ and specified through A, B are the values $E = k^2$ with multiplicity m, where $k \neq 0$ are solutions to the secular equation

$$\det\left[\mathbb{I}_{2|\mathcal{I}|} - S_v(k)T(k, \boldsymbol{l})\right] = 0 \tag{2.3.9}$$

with multiplicity m.

Proof. Condition (2.3.4) can be written

$$\det\left[AX(k, \boldsymbol{l}) + ikBY(k, \boldsymbol{l})\right] = 0 \tag{2.3.10}$$

which implies

$$\det\left[(A + ikB)\frac{X(k, \boldsymbol{l}) + Y(k, \boldsymbol{l})}{2} + (A - ikB)\frac{X(k, \boldsymbol{l}) - Y(k, \boldsymbol{l})}{2}\right] = 0. \tag{2.3.11}$$

Then, using the invertibility of $A + ikB$, and multiplying on the left by

$$\det\left[A + ikB\right]^{-1} \tag{2.3.12}$$

and on the right by

$$\det\left[\frac{X(k, \boldsymbol{l}) + Y(k, \boldsymbol{l})}{2}\right]^{-1}, \tag{2.3.13}$$

we have that

$$\det\left[\mathbb{I} + (A + ikB)^{-1}(A - ikB)T(k, \boldsymbol{l})\right] = 0, \tag{2.3.14}$$

where we have used the fact that

$$T(k,\boldsymbol{l}) = \left(\frac{X(k,\boldsymbol{l}) - Y(k,\boldsymbol{l})}{2}\right)\left(\frac{X(k,\boldsymbol{l}) + Y(k,\boldsymbol{l})}{2}\right)^{-1}. \tag{2.3.15}$$

The definition of $S_v(k)$ completes the proof. □

Clearly the matrices $S_v(k)$ have, in general, some non-trivial dependency on k. It was shown in [KPS07] that such matrices are k-independent if and only if A and B prescribe non-Robin boundary conditions. Since in [KS06b] it is shown that the number of negative Laplace eigenvalues is bounded by the number of eigenvalues of $AB^\dagger$, we have that non-Robin boundary conditions, for which $AB^\dagger = 0$ (see Section 2.2), imply no negative Laplace eigenvalues.

2.4 Scattering matrices

We have seen that the spectrum of a compact quantum graph is given by the secular equation (2.3.9) which is a function of matrices $T(k,\boldsymbol{l})$ and $S_v(k)$. The former clearly contains the metric information. The latter contains information about the interactions at the vertices prescribed by A and B. In what follows we restrict our attention to *local* boundary conditions where boundary values of functions at different vertices are not related. The significance of this is that we can consider scattering at each vertex independently. We formalise this interpretation by dissecting the compact graph into a collection of star graphs with finitely many, external edges.

Definition 2.4.1. Consider a compact graph $\Gamma(\mathcal{V},\mathcal{I},f)$. Let the map g associate to each internal edge i_j an ordered pair of external edges $g(i_j) := (e_j, e_{j+|\mathcal{I}|})$. Here $e_j =: g_0(i_j)$ and $e_{j+|\mathcal{I}|} =: g_l(i_j)$ are external edges associated with initial and terminal vertices of i_j respectively so that $f(e_j) = f_0(i_j)$ and $f(e_{j+|\mathcal{I}|}) = f_l(i_j)$. The star representation of the compact graph Γ is the collection $\Gamma^{(s)}(\mathcal{V},\mathcal{E},f)$ of star graphs $\Gamma_\eta(v_\eta,\mathcal{E}_\eta,f)$ where $\mathcal{E}_\eta$ is the set of edges e_j such that $f(e_j) = v_\eta$. Clearly we have that $2|\mathcal{I}| = |\mathcal{E}|$. The star graphs are turned into metric graphs by assigning half-lines $[0,\infty)$ to its edges.

Consider the star representation $\Gamma^{(s)}$ of a compact graph Γ. The Hilbert space associated with $\Gamma^{(s)}$ is

$$\mathscr{H}_1^{(s)} = \bigoplus_{j=1}^{|\mathcal{E}|} L^2(0,\infty). \tag{2.4.1}$$

Vectors

$$\Psi = (\psi_j^{(s)})_{j=1}^{|\mathcal{E}|} \tag{2.4.2}$$

in $\mathscr{H}_1^{(s)}$ are lists of square-integrable functions $\psi_j^{(s)} : (0,\infty) \to \mathbb{C}$. Boundary vectors are then defined

$$\Psi_{\mathrm{bv}}^{(s)} = (\psi_j^{(s)}(0))_{j=1}^{|\mathcal{E}|} \text{ and } \Psi_{\mathrm{bv}}^{(s)\prime} = (\psi_j^{(s)\prime}(0))_{j=1}^{|\mathcal{E}|}, \tag{2.4.3}$$

so that analogues of boundary conditions (2.2.13) are given by

$$A\Psi_{\mathrm{bv}}^{(s)} + B\Psi_{\mathrm{bv}}^{(s)\prime} = 0. \tag{2.4.4}$$

Let $\mathbb{P}$ be an $|\mathcal{E}|$-dimensional permutation matrix which reorders vectors Ψ according to

$$\mathbb{P}\Psi = (\Psi_\eta)_{\eta=1}^{|\mathcal{V}|}, \tag{2.4.5}$$

where each Ψ_η lists functions $\psi_j^{(s)}$ with $f(e_j) = v_\eta$. Local boundary conditions then imply the decomposition

$$A = \mathbb{P}^{-1}\left(\bigoplus_{v_\eta \in \mathcal{V}} A_\eta\right)\mathbb{P} \text{ and } B = \mathbb{P}^{-1}\left(\bigoplus_{v_\eta \in \mathcal{V}} B_\eta\right)\mathbb{P}. \tag{2.4.6}$$

With reference to Figure 2.1, we describe the propagation of particles through a graph by considering their asymptotics on infinite stars Γ_η associated with vertices $v_\eta \in \mathcal{V}$. Let us consider a particle incoming along an edge $e_i \in \mathcal{E}_\eta$ with plane wave e^{-ikx}. The scattering matrix $S_v^{(\eta)}(k)$ then defines the amplitudes of plane waves outgoing on edges $e_j \in \mathcal{E}_\eta$ according to

$$\psi_j^{(s)}(x) = \begin{cases} S_{ji}^{(\eta)}(k)e^{ikx} & \text{if } j \neq i; \\ S_{ii}^{(\eta)}(k)e^{ikx} + e^{-ikx} & \text{if } j = i. \end{cases} \tag{2.4.7}$$

By application of local boundary conditions (2.4.4), one can then read off the scattering matrix

$$S_v^{(\eta)}(k) = -(A_\eta + ikB_\eta)^{-1}(A_\eta - ikB_\eta). \tag{2.4.8}$$

The total scattering matrix

$$S_v(k) = \mathbb{P}^{-1}\left(\bigoplus_{\eta=1}^{|\mathcal{V}|} S_v^{(\eta)}(k)\right)\mathbb{P} \tag{2.4.9}$$

can can be reconstructed from sub-graphs Γ_η by considering the scattering process vertex by vertex and applying the relationship (2.4.6).

Let us retrieve the secular equation (2.3.9) by reconstructing the original compact graph from its star representation (see [KN05, KS97] for related methods). Firstly, choosing the form (2.3.2) on each external edge e_j, defining vectors

$$\boldsymbol{\alpha} = (\alpha_j)_{j=1}^{|\mathcal{E}|} \text{ and } \boldsymbol{\beta} = (\beta_j)_{j=1}^{|\mathcal{E}|}, \tag{2.4.10}$$

and imposing boundary conditions (2.4.4), we arrive at the scattering relation

$$\boldsymbol{\alpha} = S_v(k)\boldsymbol{\beta}. \tag{2.4.11}$$

Now let us consider the functions $\psi_j^{(s)}$ and $\psi_{j+|\mathcal{I}|}^{(s)}$ related to the external edges e_j and $e_{j+|\mathcal{I}|}$ respectively. Joining up the external edges to form the single internal edge i_j of length l_j (see Figure 2.2) is imposing the condition

$$\psi_j^{(s)}(x) = \psi_{j+|\mathcal{I}|}^{(s)}(l_j - x) \tag{2.4.12}$$

which yields the relation

$$\boldsymbol{\beta} = T(k, \boldsymbol{l})\boldsymbol{\alpha}. \tag{2.4.13}$$

Applying (2.4.11) and (2.4.13) successively we recover the secular equation (2.3.9) as required.

It is convenient here to define some examples of boundary conditions which will

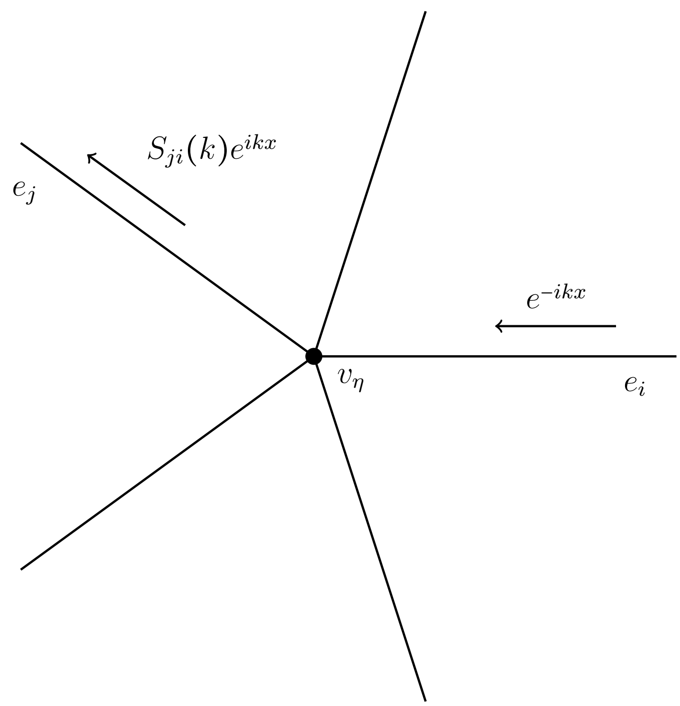

Figure 2.1: Scattering of an incoming plane wave along e_i to e_j with probability S_{ji} on a star graph Γ_η associated with vertex $v_\eta \in \mathcal{V}$.

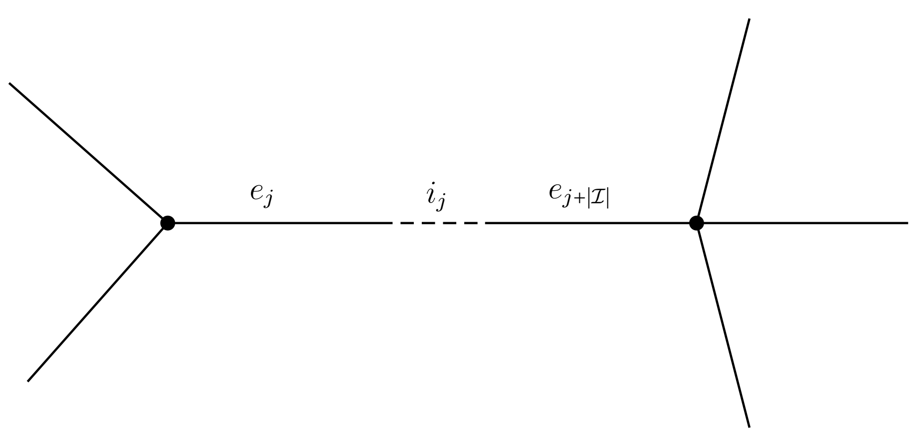

Figure 2.2: Joining two external edges e_j and $e_{j+|\mathcal{I}|}$ to reconstruct internal edge i_j.

be useful in the remainder of the thesis. Consider the vertex $v_\sigma \in \mathcal{V}$ on a graph Γ. In the star representation $\Gamma^{(s)}$, δ-type interactions at v_σ are prescribed by the conditions

$$\psi_i^{(s)}(0) = \psi_j^{(s)}(0) \tag{2.4.14}$$

for all $e_i, e_j \in \mathcal{E}_\sigma$ and

$$\sum_{e_j \in \mathcal{E}_\sigma} \psi_j^{(s)\prime}(0) = \eta \psi_i^{(s)}(0) \tag{2.4.15}$$

where $\eta \in \mathbb{R}$ parameterises the strength of interaction. These are recovered from the boundary conditions (2.4.4) by setting

$$A_\sigma = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ -\eta & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \text{ and } B_\sigma = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}. \tag{2.4.16}$$

We will call the special case for which $\eta = 0$, standard boundary conditions. We remark here that in the literature, such conditions are often referred to as Kirchhoff or Neumann conditions.

We reiterate here that boundary conditions prescribed by Theorem 2.2.2 (or equivalently Theorem 2.2.3) define self-adjoint realisations of the Laplacian. However, physicists are often satisfied with imposing the weaker restriction that each scattering matrix $S_v^{(\sigma)}(k)$, associated with a vertex v_σ, need only be unitary, in order to ensure probability conservation [GS06]. In general, such choices do not provide self-adjoint realisations of the Laplacian. As shown in [Car99] however, this problem is overcome the unitary scattering matrices are chosen to be k-independent. One such scattering matrix, which will be used later in the thesis, is the Discrete Fourier Transform (DFT) scattering matrix $S_v^{(\sigma,DFT)}$. Such a matrix has elements

$$(S_v^{(\sigma,DFT)})_{\gamma\lambda} = \frac{1}{\sqrt{d_\sigma}} e^{2\pi i \frac{n(\gamma)n(\lambda)}{d_\sigma}} \tag{2.4.17}$$

with $n(\cdot)$ a bijection of the d_σ neighbouring vertices of v_σ onto the numbers $\{0,\dots,d_\sigma-1\}$.

2.5 Periodic orbits and the trace formula

In this section we establish the connection between the spectra of quantum graphs and the dynamics of their classical counterparts by means of a trace formula. The description of the classical dynamics appears as sums over periodic orbits.

Definition 2.5.1. An orbit of possibly infinite size m on a compact graph Γ is a sequence of vertices $(v_{\eta_1},\dots,v_{\eta_m})$ such that the vertices in each subsequence $(v_{\eta_i},v_{\eta_{i+1}})$ are connected by some edge $i_j\in\mathcal{I}$. An orbit is periodic with period n if $v_{\eta_i}=v_{\eta_{i+n}}$ for all $i\in\{1,\dots,m-n\}$. A periodic orbit which cannot be written as a repetition of a smaller periodic orbit is called primitive.

The trace formula in the context of quantum graphs is concerned with expressing the count

$$N(E)=\#\{n;\ E_n\le E\} \tag{2.5.1}$$

of non-negative Laplace eigenvalues in terms of periodic orbits on the graph. The first such trace formula was deduced by Roth [Rot83] for standard boundary conditions and later generalised to non-Robin conditions in [KPS07]. More recently, by applying the argument principle to the secular equation (2.3.9), Bolte and Endres [BE09] calculated the trace formula for quantum graphs with general boundary conditions. It is sufficient in the context of this thesis to present the case for non-Robin boundary conditions given by

$$\sum_{n=0}^{\infty} g_n h(k_n) = \mathcal{L}\hat{h}(0)+(g_0-\frac{1}{2}N)h(0)+\sum_{p\in\mathcal{P}}\left((A_p+\bar{A}_p)\hat{h}(l_p)\right). \tag{2.5.2}$$

This is an analytical representation of the counting function (2.5.1) given in terms of the multiplicities g_n of eigenvalues E_n, the total length $\mathcal{L}=\sum_j l_j$ of the graph, the order N of the solution $k=0$ of (2.3.9) and a suitable test function $h:\mathbb{C}\to\mathbb{C}$ with Fourier transform defined

$$\hat{h}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} h(k)e^{ikx}dk. \tag{2.5.3}$$

Furthermore, amplitudes A_p associated with periodic orbits $p \in \mathcal{P}$ with length l_p are calculated by multiplying the local scattering matrices at each vertex along p.

To finish this section we remark that the multiplicity g_0 of the Laplace eigenvalue $E = 0$ is, in general, not equal to the order N of the value $k = 0$ as a solution to (2.3.9). For the purposes of this thesis it is sufficient to give the result in [FKW07] that, in the case of non-Robin boundary conditions where $S_v(k)$ is independent of k, the multiplicity of the eigenvalue zero is given by

$$g_0 - \frac{1}{2}N = \frac{1}{4}\lim_{k\to 0} \operatorname{tr} S_v(k). \tag{2.5.4}$$

A generalisation of this result to include Robin conditions can be found in [BES15].

2.6 The tetrahedron

In this final section we calculate and analyse the spectrum of a quantum tetrahedron with standard boundary conditions. A tetrahedron is a graph Γ with $|\mathcal{V}| = 4$ vertices and $|\mathcal{I}| = 6$ edges where each vertex $v_\eta \in \mathcal{V}$ is connected to each of the others $v_{\gamma\neq\eta} \in \mathcal{V}$ by a single edge $i_j \in \mathcal{I}$. Figure 2.3 depicts a tetrahedron on which the edges and vertices are labelled. In order to associate an explicit permutation matrix $\mathbb{P}$ (see (2.4.5)) with the tetrahedron, we specify the orientations of the edges i_j connecting a pair of vertices $\{v_\gamma, v_\lambda\}$ according to $f_0(i_j) = v_{\min(\gamma,\lambda)}$ and $f_l(i_j) = v_{\max(\gamma,\lambda)}$. Local standard boundary conditions are given by

$$A = \mathbb{P}^{-1}(\mathbb{I}_4 \otimes A_\eta)\mathbb{P} \text{ and } B = \mathbb{P}^{-1}(\mathbb{I}_4 \otimes B_\eta)\mathbb{P}, \tag{2.6.1}$$

where for each vertex v_η we have

$$A_\eta = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B_\eta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2.6.2}$$

These are clearly non-Robin conditions and thus the resultant scattering matrix

$$S_v(k) = \mathbb{P}^{-1}\left(\mathbb{I}_4 \otimes S_v^{(\eta)}(k)\right)\mathbb{P}, \tag{2.6.3}$$

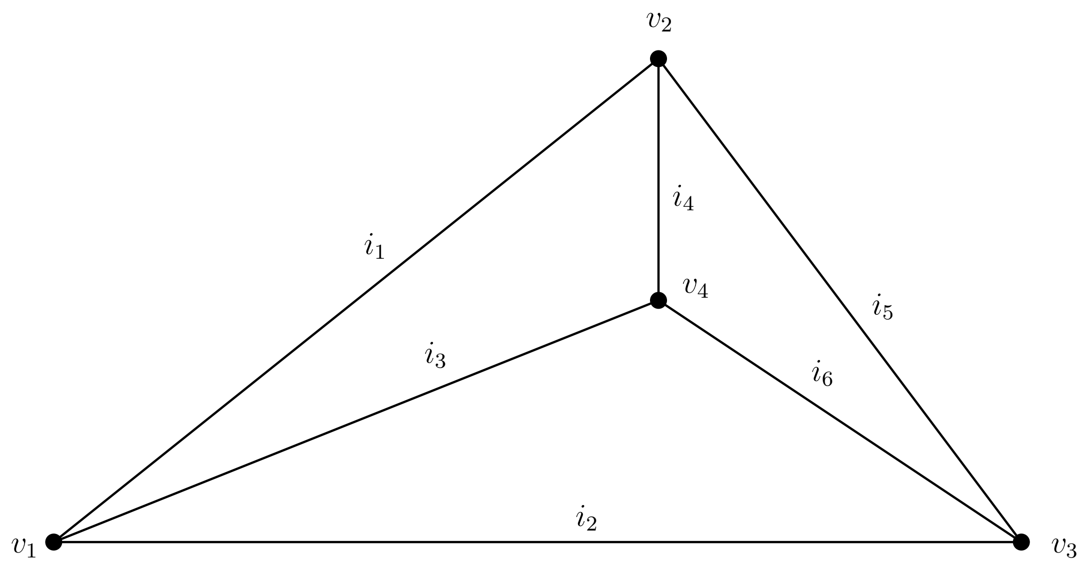

Figure 2.3: Tetrahedron with lengths and vertices arbitrarily specified.

with

$$S_v^{(\eta)}(k) = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}, \tag{2.6.4}$$

is k-independent. Choosing rationally independent lengths

$$l_1 = 1,\ l_2 = \sqrt{2},\ l_3 = \sqrt{3},\ l_4 = \sqrt{5},\ l_5 = \sqrt{7},\ l_6 = \sqrt{11}, \tag{2.6.5}$$

we ensure non-degenerate solutions $k \geq 0$ of the secular equation (2.3.9). As stated in Theorem 2.3.1 the Laplace eigenvalues $E = k^2$ exactly correspond to these solutions. The one exception is the solution $k = 0$ which has order $N = 4$ equal to the multiplicity of the eigenvalue one of

$$S_v(k) \begin{pmatrix} 0 & \mathbb{I}_6 \\ \mathbb{I}_6 & 0 \end{pmatrix}. \tag{2.6.6}$$

The multiplicity $g_0 = 1$ of the Laplace eigenvalue $E = 0$ is then calculated using (2.5.4).

Using the identity

$$\int \delta(x-a)f(x)dx = f(a) \tag{2.6.7}$$

as well as (2.5.3), the trace formula (2.5.2) can be written

$$\begin{aligned} &\int_{-\infty}^{\infty} h(k)\sum_{n=0}^{\infty} g_n\delta(k-k_n)dk \\ &= \int_{-\infty}^{\infty} h(k)\left(\frac{\mathcal{L}}{2\pi} + (g_0 - \frac{1}{2}N)\delta(k) + \sum_{p\in\mathcal{P}} \frac{A_p + \bar{A}_p}{2\pi} e^{ikl_p}\right) dk. \end{aligned} \tag{2.6.8}$$

Since the spectral density is defined

$$d(k) = \sum_{n=0}^{\infty} g_n\delta(k-k_n), \tag{2.6.9}$$

we have that

$$d(k) = \frac{\mathcal{L}}{2\pi} + (g_0 - \frac{1}{2}N)\delta(k) + \sum_{p\in\mathcal{P}} \frac{A_p + \bar{A}_p}{2\pi} e^{ikl_p}. \tag{2.6.10}$$

The counting function $N(E)$ for all eigenvalues up to E is then calculated by integrating the spectral density $d(k)$ from $-\sqrt{E}$ to $\sqrt{E}$. We then have that

$$\begin{aligned} N(E) &= \int_{-\sqrt{E}}^{\sqrt{E}} \left(\frac{\mathcal{L}}{2\pi} + (g_0 - \frac{1}{2}N)\delta(k) + \sum_{p\in\mathcal{P}} \frac{A_p + \bar{A}_p}{2\pi} e^{ikl_p}\right) dk \\ &= \underbrace{\frac{\mathcal{L}\sqrt{E}}{\pi} + g_0 - \frac{1}{2}N}_{N_s(E)} + \underbrace{\sum_{p\in\mathcal{P}} \frac{A_p + \bar{A}_p}{2\pi i l_p}\left(e^{i\sqrt{E}l_p} - e^{-i\sqrt{E}l_p}\right)}_{N_o(E)}. \end{aligned} \tag{2.6.11}$$

The first three terms comprise the smooth part $N_s(E)$ of the counting function. The final term is the oscillatory part $N_o(E)$. Until now we have not been explicit about the exact nature of amplitudes A_p. Kottos and Smilansky [KS97] showed that the oscillatory part can be written

$$N_o(E) = \frac{1}{\pi}\,\mathrm{Im}\sum_{n=1}^{\infty} \mathrm{tr}\left(S_v(k)T(k,\boldsymbol{l})\right)^n. \tag{2.6.12}$$

Figure 2.4 shows the counting function $N(E)$ plotted from numerically deduced

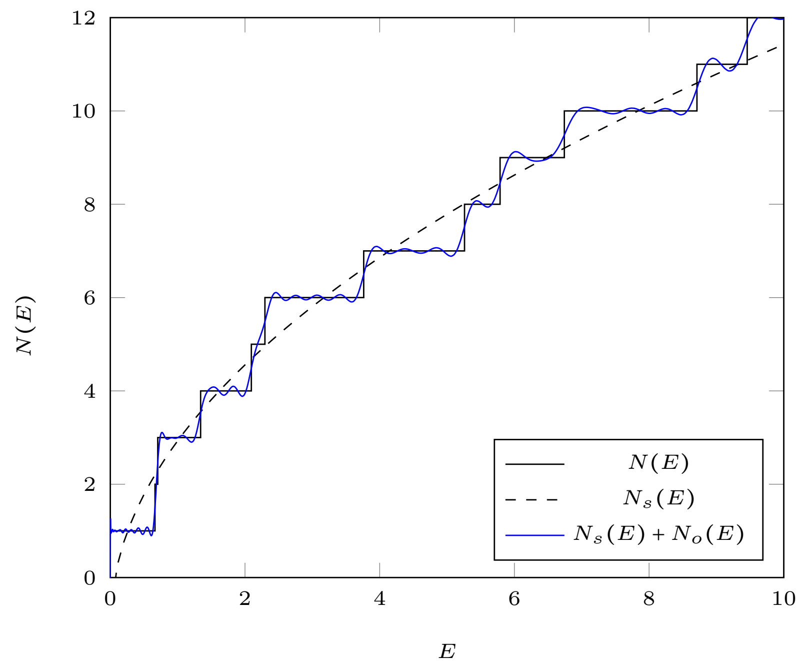

Figure 2.4: Eigenvalue count for a tetrahedron with local standard boundary conditions. $N_0(E)$ is calculated using (2.6.12) summing the first 20 terms.

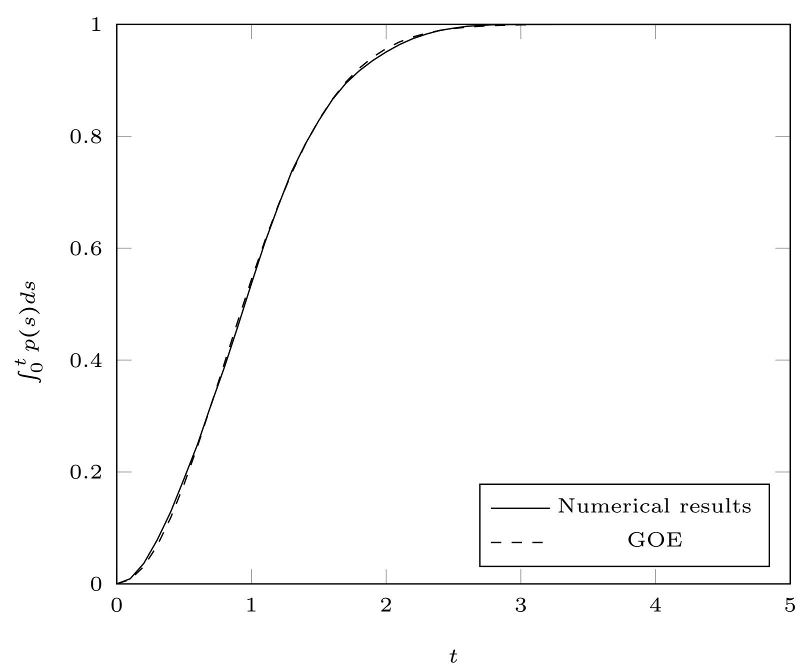

Figure 2.5: Nearest neighbour energy level distribution for quantum tetrahedron. Lowest $100,000$ energies.

eigenvalues E_n along with the corresponding analytical expression calculated from the trace formula (2.6.11). Clearly the numerical data agrees with the trend predicted by $N_s(E)$. The analytical expression converges to the numerical data as more terms in N_o are summed.

A particularly useful statistical measure is the nearest neighbour level spacings distribution

$$\int_a^b p(s)ds = \lim_{N\to\infty} \frac{1}{N}\#\{n \le N;\ a \le \epsilon_{n+1} - \epsilon_n \le b\} \tag{2.6.13}$$

of the unfolded versions, $\epsilon_1 < \epsilon_2 < \epsilon_3 < \dots$, of the energy eigenvalues, that is the energies are rescaled such that the average spacing is equal to unity. Chaotic classical systems have quantum counterparts with correlations described by random matrix models. For such systems, with integer spin and time-reversal symmetry, Gaussian orthogonal ensemble (GOE) statistics are conjectured to apply [BGS84], where the level spacings distribution can be approximated by

$$p(s) = \frac{\pi}{2} s\, \mathrm{e}^{-\frac{\pi}{4}s^2} \tag{2.6.14}$$

(see [Haa91]). Figure 2.5 shows the nearest neighbour distribution for the quantum tetrahedron with standard boundary conditions. We see that the nearest neighbour data closely follows GOE statistics. Such spectral correlations characterise quantum systems which are chaotic in the classical limit.

Chapter 3

Exactly solvable two-particle systems

The complexity of most macroscopic phenomena means that their exact treatment is impossible. In these cases meaningful, yet approximate, theoretical results can be established by means of simulation, perturbation theory or a reduction of the problem to a simpler exactly solvable model. There are however, a number of macroscopic systems which can be solved exactly by considering individual particle mechanics. In the context of this thesis we use the term exact solvability to describe systems for which exact expressions for energy can be obtained. Since our aim is to study many-particle quantum graphs, we are most interested in such systems confined to a single spatial dimension.

We begin by introducing a number of approaches devised to identify and solve n-particle systems confined to a single dimension. In 1931, Bethe [Bet31] identified the eigenfunctions and calculated the spectra of the Heisenberg-Ising anisotropic chain; a linear chain of spin-$\frac{1}{2}$ particles interacting with their nearest neighbours. At the centre of this approach is the construction and employment of what is now known as the Bethe ansatz; a superposition of possible many-particle plane wave states. Adapting this method to find solutions of the n-particle Schrödinger equation

$$\left(-\Delta_n + 2\alpha \sum_{i\neq j} \delta(x_i - x_j)\right)\psi = E\psi, \tag{3.0.1}$$

Lieb and Liniger [LL63] determined the exact spectra of a repulsively δ-interacting Bose gas on a circle, a result which was later generalised to distinguishable particles by Yang [Yan67]. Gaudin [Gau71] later employed the Bethe ansatz to describe similar systems confined to an interval. Central to the validity of the Bethe ansatz method is the existence of certain symmetries associated with the system in question. It turns out that this requirement is that interactions can be characterised in terms of Weyl groups of root systems. These symmetries become particularly intuitive when considering McGuire's optical wave analogy [McG64]. He reformulated the problem of n δ-interacting particles confined to a single dimension in terms of the propagation of a single optical ray in an n-dimensional domain subject to interaction with $\frac{1}{2}n(n-1)$ transmitting and reflecting $(n-1)$-dimensional plates.

Of course, it would be useful to extend the Bethe ansatz approach from interacting systems on an interval to general quantum graphs. Unfortunately though, the consequence of increasing complexity to systems of δ-interacting particles on a non-trivial quantum graph with more than a single edge, is that compatibility with the Bethe ansatz method is destroyed. By imposing certain non-local interactions, however, Caudrelier and Crampé [CC07] showed that, for systems of particles on two-edge star graphs, compatibility with the Bethe ansatz is recovered.

In this chapter we begin by discussing the theory surrounding the identification and solutions of systems of particles on a circle and in a box restricting the mathematical presentation to two-particle models. We refer to McGuire's optical interpretation when justifying an appropriate Bethe ansatz. We then discuss the related model in [CC07] commenting on its potential extension to systems of particles defined on general graphs. We finish by numerically deducing the spectra of certain examples and commenting on their statistics.

The reader should keep in mind that restricting the formalism to only two particles in this chapter allows us to establish a framework for identifying exactly solvable two-particle quantum graphs in Chapter 4. The full n-particle analyses will be presented in Chapter 5. For readability, we also overlook any formal discussion of self-adjoint realisations of the Laplacian, preferring to refer to literature when stating appropriate boundary conditions. We return to a rigorous formalism in the following chapter.

3.1 Preliminaries

Before we proceed, it is useful to define and explain the groups S_2 and $\mathscr{W}_2$ which we will use to characterise the symmetries of exactly solvable two-particle systems. Material is taken from [Hum72, AMP81, Ros09].

Systems of two δ-interacting bosons on a circle [LL63] exhibit symmetries which can be described by the symmetric group S_2. It is convenient to define elements $Q \in S_2$ as acting on the set $\{1, 2\}$.

Definition 3.1.1. Elements Q in the symmetric group $S_2 = \{I, T\}$ act on the set $\{1, 2\}$ according to

1. I(1,2)=(1,2);
2. T(1,2)=(2,1).

Clearly we have that $TT = I$.

Definition 3.1.2. Let V be a two-dimensional Euclidean space with Euclidean inner product $\langle\cdot|\cdot\rangle$. A root system Ω in V is a finite set of non-zero vectors (or roots) which satisfy the conditions

1. Given a root $z \in \Omega$, the scalar multiple λz is also in Ω if and only if $\lambda = \pm 1$;
2. Ω is closed under reflection in the line perpendicular to any $z \in \Omega$;
3. The roots of Ω span V;
4. The inner product $\langle x|y\rangle$ for any two roots $x, y \in \Omega$ is an integer.

Of particular importance to us will be the root system C_2 in $\mathbb{R}^2$ given by the set of eight vectors

$$\left\{ \begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 2 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix} \right\} \tag{3.1.1}$$

as depicted in Figure 3.1.

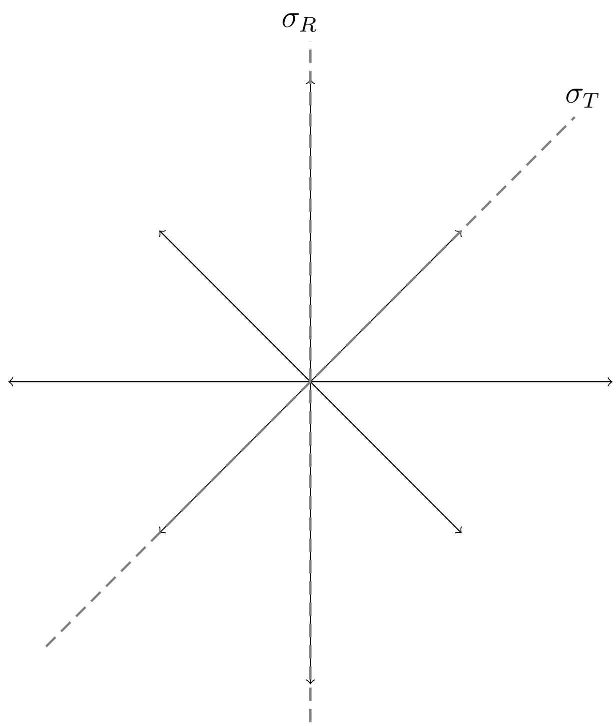

Figure 3.1: The root system C_2 and associated Weyl group $\mathscr{W}_2$ generated by reflections σ_R and σ_T.

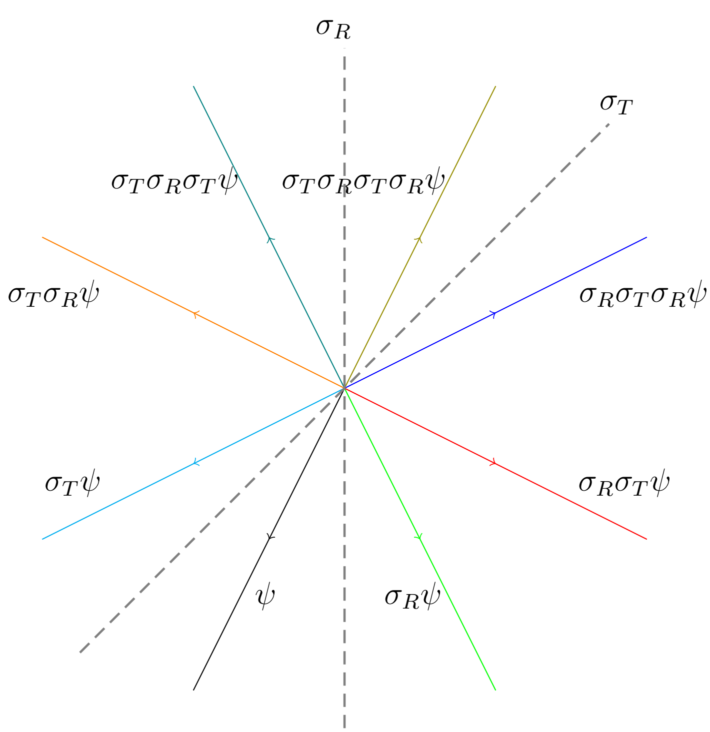

Figure 3.2: Generators σ_R and σ_T of the Weyl group $\mathscr{W}_2$ acting on a vector ψ.

Definition 3.1.3. The Weyl group of a root system Ω is the group of isometries generated by the reflections through hyperplanes perpendicular to the roots of Ω.

Systems of two δ-interacting particles in a box [Gau71, Yan67], and indeed two-particle systems with certain non-local interactions [CC07], exhibit symmetries which can be described by the Weyl group $\mathscr{W}_2$ of the root system C_2. In the context of this thesis, the generators of $\mathscr{W}_2$, which are the reflections σ_R and σ_T depicted in Figure 3.1, will act on vectors ψ defined in $\mathbb{R}_2$ as depicted in Figure 3.2. It is convenient to define elements P of $\mathscr{W}_2$ as acting on the set $\{\pm 1, \pm 2\}$ accordingly.

Definition 3.1.4. Consider a group G with identity element I and subgroups N and H which satisfy the conditions

1. N is a normal subgroup in G;
2. The intersection $N \cap H$ is the identity I;
3. G is the product of the subgroups NH.

Then $G = N \rtimes H$ is the semidirect product of N and H.

Definition 3.1.5. Elements P in the Weyl group

$$\mathscr{W}_2 := (\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2, \tag{3.1.2}$$

of order 8, acting on the set $\{\pm 1, \pm 2\}$ will be written in terms of generators T and R which act according to

1. $T(1,2) = (2,1)$;
2. $R(1,2) = (-1,2)$,

and satisfy the conditions

1. $TT = I$;
2. $RR = I$;
3. $TRTR = RTRT$.

We note that, with S_2 and $\mathscr{W}_2$ defined as above, the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^2$ in $\mathscr{W}_2$ can be written in terms of generators R and TRT. With this in mind, the conditions in Definition 3.1.4, which validate the semidirect product are easily verified.

3.2 Simple exactly solvable two-particle systems

We begin by presenting three well known examples of exactly solvable two-particle systems. Each example can be described as a two-particle quantum graph consisting of a single internal edge with certain boundary conditions applied at the end points. This simplified presentation will provide a base from which we can generalise the discussion to graphs with more than a single edge.

3.2.1 Bosons on a circle

We first consider a system of two δ-interacting bosons confined to the perimeter of a circle length l. Lieb and Liniger [LL63] formulated this problem as a search for solutions

$$\psi = \psi(x_1, x_2) \tag{3.2.1}$$

of the two-particle Schrödinger equation

$$\left(-\Delta_2 + 2\alpha\delta(x_1 - x_2)\right)\psi = E\psi, \tag{3.2.2}$$

with particle positions x_1, x_2 defined on the real line $\mathbb{R} = (-\infty, \infty)$. Here $\alpha > 0$ parameterises the repulsive interaction strength. The two-particle Laplacian acts according to

$$-\Delta_2\psi = -\frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2}. \tag{3.2.3}$$

We note that the problem of impenetrable bosons [Gir60] can be recovered by letting $\alpha \to \infty$. By requiring that $-\Delta_2$ is self-adjoint and imposing bosonic symmetry

$$\psi(x_1, x_2) = \psi(x_2, x_1), \tag{3.2.4}$$

equation (3.2.2) decomposes into the eigenvalue equation

$$-\Delta_2 \psi = E\psi \tag{3.2.5}$$

alongside the jump condition in the derivatives

$$\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right)\psi(x_1, x_2)|_{x_1 = x_2^-} = \alpha\psi(x_1, x_2)|_{x_1 = x_2^-}, \tag{3.2.6}$$

with ψ restricted to the subspace

$$D^- = \{(x_1, x_2) \in \mathbb{R}^2;\ x_1 < x_2\}. \tag{3.2.7}$$

Together with the imposition of bosonic symmetry, the problem is then also defined in

$$D^+ = \{(x_1, x_2) \in \mathbb{R}^2;\ x_1 > x_2\} \tag{3.2.8}$$

and thus all of $\mathbb{R}^2$.

The task is then to construct explicit Laplace eigenfunctions ψ in D^- which satisfy (3.2.6). To this end let us consider the two-particle plane wave state

$$\psi_I = e^{i(k_1 x_1 + k_2 x_2)} \tag{3.2.9}$$

defined with momenta $k_1, k_2 \in \mathbb{R}$. In order to justify an appropriate form of ψ let us assume $k_2 < k_1$ so that the system is approaching a point of particle interaction $x_1 = x_2$. Central to the Bethe ansatz method is the assumption that no new momenta are generated [McG64] by such interactions. In this context then, interactions between particles result either in the momenta of each particle being swapped

$$(k_1, k_2) \to (k_2, k_1), \tag{3.2.10}$$

or else remaining as they were

$$(k_1, k_2) \to (k_1, k_2). \tag{3.2.11}$$

We thus expect that any resulting two-particle state must be one of two two-particle plane waves

$$\psi_Q = e^{i(k_{Q1}x_1 + k_{Q2}x_2)}, \tag{3.2.12}$$

with elements $Q \in S_2$ as prescribed in Definition 3.1.1. We can think of each $Q \in S_2$ as corresponding to some configuration of particle momenta

$$\boldsymbol{k}_Q = (k_{Q1}, k_{Q2}). \tag{3.2.13}$$

The Bethe ansatz in this context is the sum of possible two-particle plane wave states

$$\psi(x_1, x_2) = \sum_{Q \in S_2} \mathcal{A}^Q e^{i(k_{Q1}x_1 + k_{Q2}x_2)}, \tag{3.2.14}$$

with $\mathcal{A}^Q$ the amplitudes of constituent states ψ_Q.

Using the form (3.2.14), equation (3.2.5) is satisfied with Laplace eigenvalues

$$E = k_1^2 + k_2^2. \tag{3.2.15}$$

The boundary condition (3.2.6) implies the relation

$$\mathcal{A}^{QT} = s_p(k_{Q1} - k_{Q2})\mathcal{A}^Q \tag{3.2.16}$$

for all $Q \in S_2$ with

$$s_p(k) = \frac{k - i\alpha}{k + i\alpha}. \tag{3.2.17}$$

We note here the restriction that momenta k_1 and k_2 must be distinct since for identical momenta $k_1 = k_2$, the ansatz (3.2.14) vanishes. To prove exact solvability we need only show that the relation (3.2.16) is consistent with the properties of S_2, namely that $TT = I$. This amounts to the requirement $s_p(u)s_p(-u) = 1$ which is easily verified.

Until this point we have said nothing about the geometry of the one-dimensional problem in question. Putting the particles on the perimeter of a circle length l is applying periodic boundary conditions

$$\psi(0,x) = \psi(x,l); \tag{3.2.18}$$

$$\frac{\partial}{\partial x_1}\psi(x_1,x)|_{x_1=0} = \frac{\partial}{\partial x_2}\psi(x,x_2)|_{x_2=l} \tag{3.2.19}$$

for all $x \in (0,l)$, which, by again using the form (3.2.14), imply the relations

$$\mathcal{A}^Q = \mathcal{A}^{QT} e^{ik_{Q1}l} \tag{3.2.20}$$

for all $Q \in S_2$. Finally, applying (3.2.16) and (3.2.20) successively, we arrive at the pair of quantisation conditions

$$e^{-ik_j l} = s_p(k_j - k_i), \tag{3.2.21}$$

with $j, i \neq j \in \{1,2\}$. Solutions (k_1,k_2) then constitute energies (3.2.15). It is important to note here that since, for any solution (k_1,k_2) of the quantisation conditions (3.2.21), one also has the solution (k_2,k_1), it is sufficient to search for solutions in the region $k_1 < k_2$.

3.2.2 Bosons in a box

We have seen how to construct systems of two particles on a circle by first defining particle position x_1, x_2 on the real line $\mathbb{R}$. We would now like to adapt this approach to describe particles confined to a box. Gaudin [Gau71] formulated this problem as a search for solutions ψ of the two-particle Schrödinger equation (3.2.2) with x_1, x_2 defined on the half-line $\mathbb{R}_+ = (0,\infty)$. As we will see, framing the problem in this way has a profound affect on the appropriate Bethe ansatz. Specifically, the possible configurations of particle momenta will no longer correspond to elements of the symmetric group S_2, but rather the Weyl group $\mathscr{W}_2$. We begin by again restricting our attention to systems of bosons before generalising our approach to distinguishable particles. As previously, (3.2.2) decomposes into (3.2.5) alongside the boundary condition (3.2.6) but with ψ now restricted to the subspace

$$d^- = \{(x_1,x_2) \in \mathbb{R}_+^2;\ x_1 < x_2\}. \tag{3.2.22}$$

Together with the imposition of bosonic symmetry, the problem is then also defined on

$$d^{+} = \{(x_1, x_2) \in \mathbb{R}_+^2;\ x_1 > x_2\} \tag{3.2.23}$$

and thus all of $\mathbb{R}_+^2$.

The task now to construct explicit Laplace eigenfunctions ψ in d^- which satisfy conditions (3.2.6) as well as the Dirichlet condition

$$\psi(0, x) = 0 \tag{3.2.24}$$

associated with the finite endpoint of $\mathbb{R}_+$. Let us again consider the two-particle plane wave state ψ_I, as in (3.2.9), but now defined on d^-. Let us also make the additional assumption

$$k_2 \le k_1 \le 0 \text{ and } (k_1, k_2) \neq (0, 0), \tag{3.2.25}$$

so that the system is approaching one of the two boundaries, $x_1 = x_2$ and $x_1 = 0$, of d^-. As previously, the two possible consequences of δ-type interactions at the former boundary are the momenta being swapped (3.2.10), or else remaining as they were (3.2.11). Dirichlet conditions at the latter boundary result in momentum reversal

$$(k_1, k_2) \to (-k_1, k_2). \tag{3.2.26}$$

Taking into account all possible particle interactions, we expect that any resulting two-particle state must be one of eight two-particle plane waves

$$\psi_P = e^{i(k_{P1}x_1 + k_{P2}x_2)}, \tag{3.2.27}$$

with elements $P \in \mathscr{W}_2$ as prescribed in Definition 3.1.5. We can think of each $P \in \mathscr{W}_2$ as corresponding to some configuration of particle momenta

$$\boldsymbol{k}_P = (k_{P1}, k_{P2}). \tag{3.2.28}$$

The Bethe ansatz in this context is the sum of possible plane wave states

$$\psi(x_1, x_2) = \sum_{P \in \mathscr{W}_2} \mathcal{A}^P e^{i(k_{P1}x_1 + k_{P2}x_2)}, \tag{3.2.29}$$

with $\mathcal{A}^P$ the amplitudes of constituent states ψ_P.

To gain some intuition here we refer to an interpretation of many-particle dynamics introduced by McGuire [McG64]. He reinterpreted the n-particle Schrödinger equation (3.0.1) as describing optical waves propagating in n-dimensional Euclidean space subject to reflection or transmission at the $\frac{1}{2}n(n-1)$ hyperplanes $x_i = x_j$. The superposition of the waves which result from possible transformations at these hyperplanes is the Bethe ansatz characterised by the Weyl group $\mathscr{W}_n$. To illustrate this point let us consider this optical analogy in the context of a pair of δ-interacting bosons in a box. Let us depict the state ψ_I as an optical wave propagating in the two-dimensional space d^- and track the transformations on $\boldsymbol{k}_I$ as the ray interacts with the boundaries of d^- (see Figure 3.5). Figure 3.4 depicts the collection of possible resulting states ψ_P, with $P \in \mathscr{W}_2$. Given the state ψ_I, each ψ_P can be accessed by a combination of reflections σ_T and σ_R. The set of the eight transformations generated by σ_T and σ_R is the Weyl group $\mathscr{W}_2$ of the root system C_2.

We proceed by noting that, using the form (3.2.29), equation (3.2.5) is satisfied with Laplace eigenvalues (3.2.15). Boundary conditions (3.2.6) and (3.2.24) then imply

$$\mathcal{A}^{PT} = s_p(k_{P1} - k_{P2})\mathcal{A}^P \tag{3.2.30}$$

and

$$\mathcal{A}^{PR} = -\mathcal{A}^P \tag{3.2.31}$$

for all $P \in \mathscr{W}_2$. To prove exact solvability we need only show that relations (3.2.30) and (3.2.31) are consistent with the properties of $\mathscr{W}_2$. This again amounts only to the requirement $s_p(u)s_p(-u) = 1$.

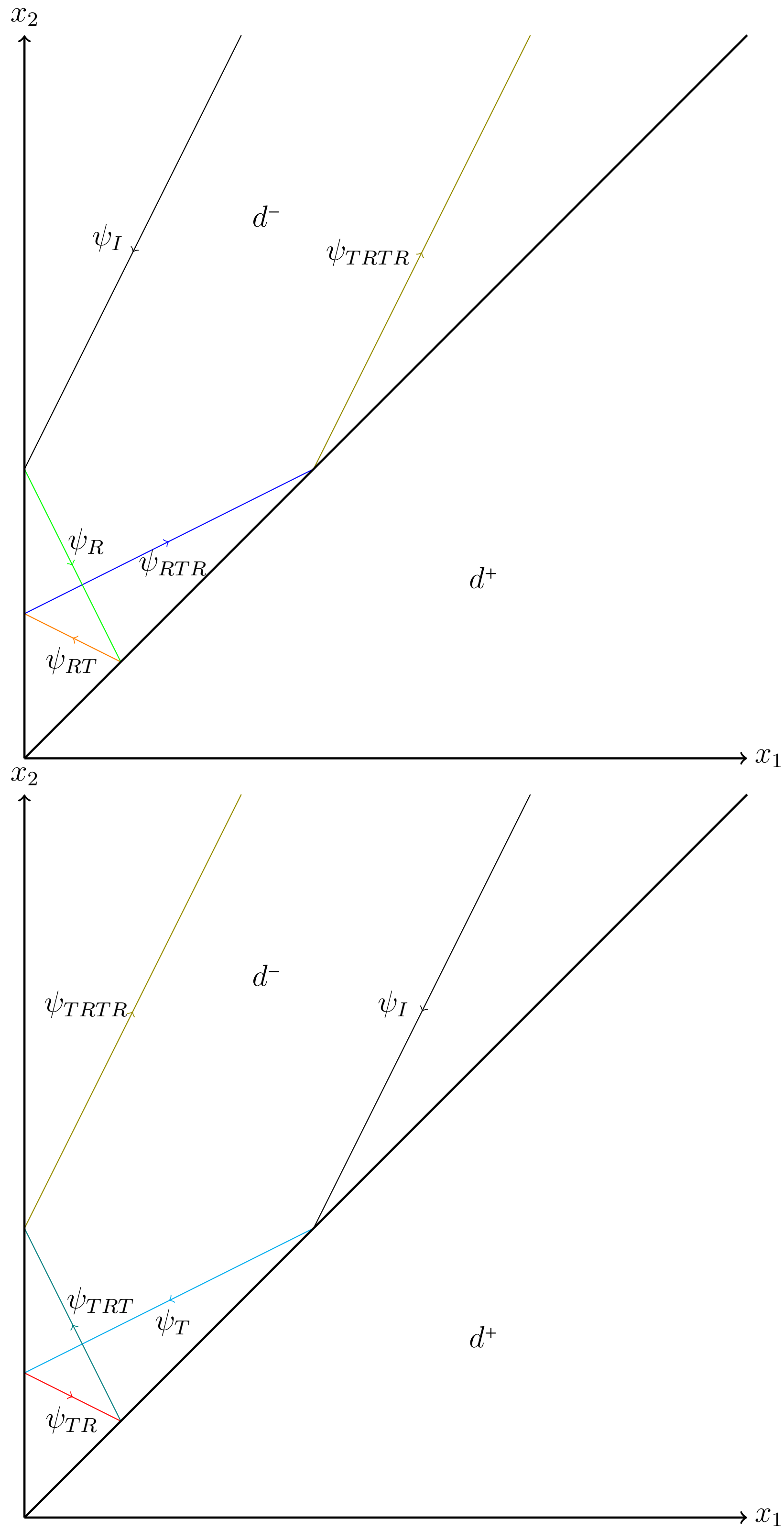

Figure 3.3: Ray tracing diagram showing scattering of two distinguishable δ-interacting particles on $\mathbb{R}_+$.

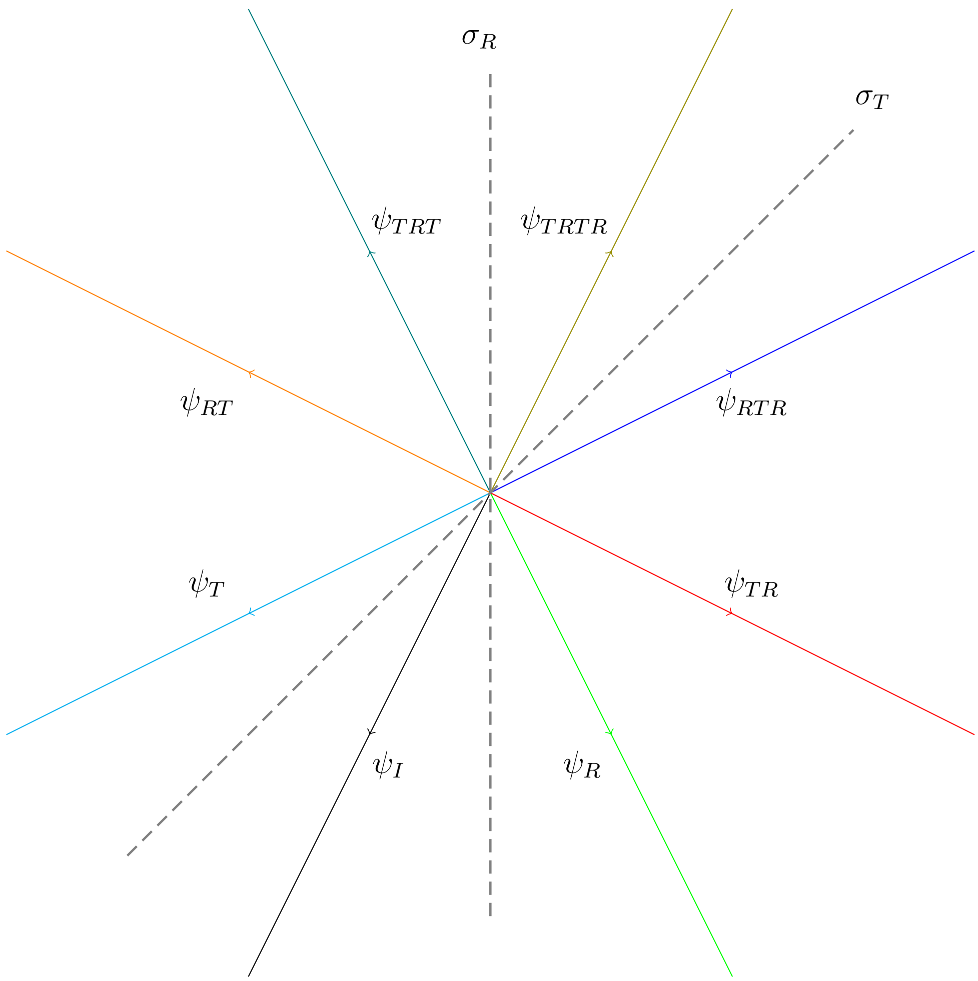

Figure 3.4: Weyl group $\mathscr{W}_2$ generated by reflections σ_R and σ_T acting on a vector ψ_I.

Bringing our attention back to the geometry of the problem at hand, we enclose the particles in a box of length l by enforcing the Dirichlet condition

$$\psi(x,l) = 0, \tag{3.2.32}$$

which implies the relation

$$\mathcal{A}^P = -e^{-2ik_{P2}l}\mathcal{A}^{PTRT}. \tag{3.2.33}$$

Finally, applying (3.2.30), (3.2.31) and (3.2.33) successively, we arrive at the condition

$$e^{-2ik_{P2}l} = s_p(k_{P1}+k_{P2})s_p(k_{P2}-k_{P1}) \tag{3.2.34}$$

for all $P \in \mathscr{W}_2$. It is clear that the form of $s_p(k)$ is such that, if (3.2.34) is satisfied for some $P \in \mathscr{W}_2$, then it is necessarily satisfied for elements PR and $PTRT$. We thus have the pair of quantisation conditions

$$e^{-2ik_j l} = s_p(k_j+k_i)s_p(k_j-k_i), \tag{3.2.35}$$

with $j, i \neq j \in \{1,2\}$. Solutions $(k_1,k_2) \neq (0,0)$, such that $0 \leq k_1 \leq k_2$, then constitute energies (3.2.15).

3.2.3 Distinguishable particles in a box

Until this point we have considered only systems of bosons. Here we present a system of two distinguishable particles confined to a box. The method closely resembles that in Section 3.2.2 with the crucial difference being that we no longer have equivalence between the subspaces d^- and d^+ of $\mathbb{R}^2_+$. To this end, we denote functions $\psi^\pm$ as the restrictions of ψ to $d^\pm$. In this setting, (3.2.2) decomposes into (3.2.5) alongside the condition of continuity

$$\psi^+(x_1,x_2)|_{x_1=x_2^+} = \psi^-(x_1,x_2)|_{x_1=x_2^-} \tag{3.2.36}$$

and the jump condition in the derivatives

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2\alpha\right)\psi^+(x_1,x_2)|_{x_1=x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi^-(x_1,x_2)|_{x_1=x_2^-} \qquad (3.2.37)$$

across the line $x_1 = x_2$, where $\alpha \in \mathbb{R}$ parameterises the strength of interaction. Again we wish to employ Dirichlet conditions at the endpoint of $\mathbb{R}_+$. These now appear as the conditions

$$\psi^-(0,x) = \psi^+(x,0) = 0. \qquad (3.2.38)$$

The appropriate Laplace eigenfunction is then required to satisfy boundary conditions (3.2.36)–(3.2.38). The choice of Bethe ansatz can be justified in the same way as in the bosonic case taking into account that we must distinguish between subdomains of $\mathbb{R}_+^2$. We thus have the ansatz

$$\psi^\pm(x_1,x_2) = \sum_{P\in\mathscr{W}_2} \mathcal{A}^{(P,\pm)} e^{i(k_{P1}x_1 + k_{P2}x_2)}, \qquad (3.2.39)$$

with $\mathcal{A}^{(P,\pm)}$ defined as the amplitudes of the restrictions $\psi_P^\pm$ of constituent plane waves ψ_P to the domains $d^\pm$. Figure 3.5 illustrates this justification in the spirit of McGuire's optical analogy. Depicting the state ψ_I^- as an optical wave propagating in d^-, and drawing a ray tracing diagram, we arrive at a collection of possible resulting plane wave states $\psi_P^\pm$, with $P \in \mathscr{W}_2$, associated with each subdomain $d^\pm$.

Using the form (3.2.39), equation (3.2.5) is again satisfied with Laplace eigenvalues (3.2.15). Let us define the vector of amplitudes

$$\mathcal{A}^P = \begin{pmatrix} \mathcal{A}^{(P,-)} \\ \mathcal{A}^{(PT,+)} \end{pmatrix}. \qquad (3.2.40)$$

The δ-type conditions (3.2.36)–(3.2.37) then imply

$$\mathcal{A}^{PT} = S_p(k_{P1} - k_{P2})\mathcal{A}^P, \qquad (3.2.41)$$

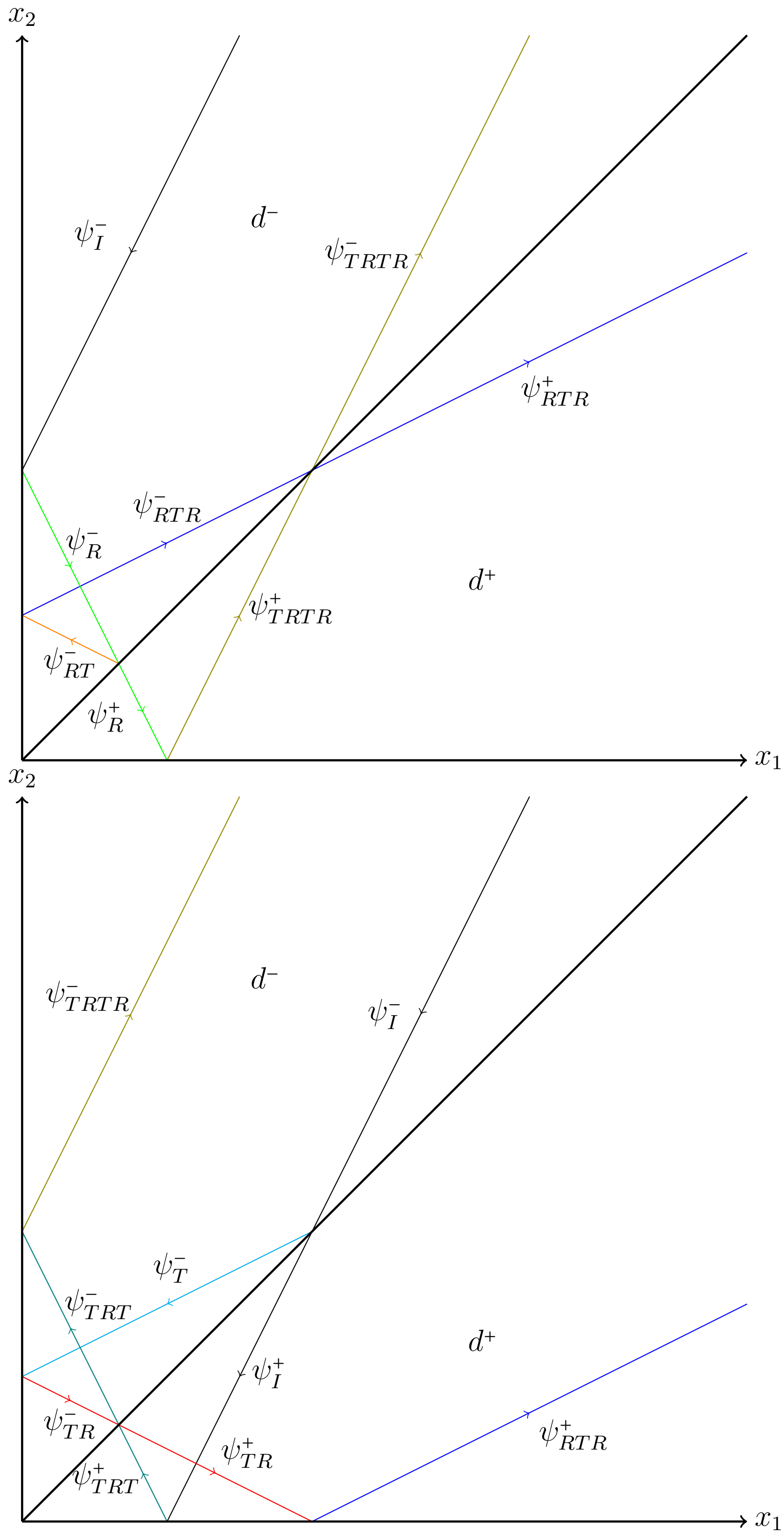

Figure 3.5: Ray tracing diagram showing scattering of two distinguishable δ-interacting particles on $\mathbb{R}_+$.

with

$$S_p(k) = \frac{1}{k + i\alpha} \begin{pmatrix} -i\alpha & k \\ k & -i\alpha \end{pmatrix}, \tag{3.2.42}$$

for all $P \in \mathscr{W}_2$. The Dirichlet condition (3.2.38) implies

$$\mathcal{A}^{PR} = -\mathcal{A}^{P} \tag{3.2.43}$$

for all $P \in \mathscr{W}_2$. To prove exact solvability we need only show that relations (3.2.41) and (3.2.43) are consistent with the properties of $\mathscr{W}_2$. This amounts to the requirements

1. $S_p(u)S_p(-u) = \mathbb{I}_2$;
2. $S_p(u)S_p(v) = S_p(v)S_p(u)$,

which are easily verified by the explicit form of $S_p(k)$.

Enclosing the particles in a box of length l is enforcing the Dirichlet conditions

$$\psi^{-}(x, l) = \psi^{+}(l, x) = 0 \tag{3.2.44}$$

which imply the relations

$$\mathcal{A}^{P} = -e^{-2ik_{P2}l}\mathcal{A}^{PTRT} \tag{3.2.45}$$

for all $P \in \mathscr{W}_2$. Finally, applying (3.2.41), (3.2.43) and (3.2.45) successively, we arrive at the condition that

$$z(k_{P1}, k_{P2}) = 0, \tag{3.2.46}$$

with

$$z(k_1, k_2) = \det\left[\mathbb{I}_2 - e^{2ik_1 l} S_p(k_1 - k_2) S_p(k_1 + k_2)\right], \tag{3.2.47}$$

is satisfied for all $P \in \mathscr{W}_2$. We note here that the form of $S_p(k)$ is such that, if (3.2.46) is satisfied for some $P \in \mathscr{W}_2$, then it is necessarily satisfied for elements

PR and $PTRT$. We thus have the pair of quantisation conditions

$$z(k_i, k_j) = 0, \tag{3.2.48}$$

with $j, i \neq j \in \{1, 2\}$. Solutions $(k_1, k_2) \neq (0, 0)$, such that $0 \leq k_1 \leq k_2$, then constitute energies (3.2.15).

We finish this section by showing how to recover the quantisation condition in Section 3.2.2 where particles are assumed to be bosons. Requiring bosonic symmetry is imposing the condition

$$\psi^-(x_1, x_2) = \psi^+(x_2, x_1) \tag{3.2.49}$$

which implies the relation

$$\mathcal{A}^{(P,-)} = \mathcal{A}^{(PT,+)} \tag{3.2.50}$$

for all $P \in \mathscr{W}_2$. As a consequence, the matrix $S_p(k)$ in (3.2.41), is replaced with the scalar form $s_p(k)\mathbb{I}_2$. Clearly then, the condition (3.2.35) is recovered from (3.2.48) as required.

3.3 Extension to general graphs

In Sections 3.2.2 and 3.2.3, models were constructed by placing a pair of particles on the half-line $\mathbb{R}_+$ and establishing an appropriate Bethe ansatz on which appropriate boundary conditions were imposed. In the language of quantum graphs, this procedure amounts to considering two particles on a single external edge. We would like to investigate how this approach can be generalised to graphs with more than a single edge. It is natural then, to begin this investigation by considering systems of two particles on a pair of external edges $\{e_1, e_2\}$ with a common vertex $v = f(e_1) = f(e_2)$, that is, a two-edge infinite star graph.

Let us begin by introducing the vector

$$\Psi = (\psi_{mn})_{m,n=1}^{2}, \tag{3.3.1}$$

with each

$$\psi_{mn} = \psi_{mn}(x_1, x_2) \tag{3.3.2}$$

defined on the domain

$$d_{mn} = (0, \infty) \times (0, \infty) \tag{3.3.3}$$

so that x_1 and x_2 correspond to the positions of the particles along edges e_m and e_n respectively.

Carrying over the procedure in previous sections, we look for solutions Ψ of the eigenvalue equation

$$-\Delta_2 \Psi = E\Psi \tag{3.3.4}$$

where the two particle Laplacian acts according to

$$-\Delta_2 \Psi = \left(-\frac{\partial^2 \psi_{mn}}{\partial x_1^2} - \frac{\partial^2 \psi_{mn}}{\partial x_2^2} \right)_{m,n=1}^{2} . \tag{3.3.5}$$

Let us also define the subspaces

$$d_{mn}^{+} = \{(x_1, x_2) \in d_{mn};\ x_2 < x_1\} \tag{3.3.6}$$

and

$$d_{mn}^{-} = \{(x_1, x_2) \in d_{mn};\ x_1 < x_2\} \tag{3.3.7}$$

with functions $\psi_{mn}^{\pm}$, the restrictions of ψ_{mn} to $d_{mn}^{\pm}$.

3.3.1 Systems of δ-interacting particles

Ideally, we would like to consider systems of δ-interacting particles as in the previous sections. In this setting, such interactions occur when particles are located at the same position on the same edge, and will be implemented according to the conditions

$$\psi^+_{mm}(x_1,x_2)|_{x_1=x_2^+} = \psi^-_{mm}(x_1,x_2)|_{x_1=x_2^-}; \tag{3.3.8}$$

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2\alpha\right)\psi^+_{mm}(x_1,x_2)|_{x_1=x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi^-_{mm}(x_1,x_2)|_{x_1=x_2^-}. \tag{3.3.9}$$

The independence of particles located on different edges is established by imposing the conditions

$$\psi^+_{mn}(x_1,x_2)|_{x_1=x_2^+} = \psi^-_{mn}(x_1,x_2)|_{x_1=x_2^-}; \tag{3.3.10}$$

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi^+_{mn}(x_1,x_2)|_{x_1=x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi^-_{mn}(x_1,x_2)|_{x_1=x_2^-}, \tag{3.3.11}$$

with $m \neq n$.

We would also like to characterise single-particle interactions at the vertices. Such interactions are governed by the single-particle boundary conditions (2.4.4) which, in the two-particle setting, can be written

$$A\begin{pmatrix}\psi^-_{1j}(0,x)\\ \psi^-_{2j}(0,x)\end{pmatrix} + B\frac{\partial}{\partial x_1}\begin{pmatrix}\psi^-_{1j}(0,x)\\ \psi^-_{2j}(0,x)\end{pmatrix} = 0; \tag{3.3.12}$$

$$A\begin{pmatrix}\psi^+_{j1}(x,0)\\ \psi^+_{j2}(x,0)\end{pmatrix} + B\frac{\partial}{\partial x_2}\begin{pmatrix}\psi^+_{j1}(x,0)\\ \psi^+_{j2}(x,0)\end{pmatrix} = 0. \tag{3.3.13}$$

The task is to construct explicit Laplace eigenfunctions Ψ which satisfy conditions (3.3.8)–(3.3.13). Naively extending the approach in the previous sections, let us assume compatibility with the Bethe ansatz

$$\psi^\pm_{mn} = \sum_{P\in\mathscr{W}_2} \mathcal{A}^{(P,\pm)}_{mn} e^{i(k_{P1}x_1 + k_{P2}x_2)}. \tag{3.3.14}$$

Defining the eight-dimensional vector of amplitudes

$$\mathcal{A}^P = \begin{pmatrix} \left(\mathcal{A}^{(P,-)}_{mn}\right)^2_{m,n=1} \\ \mathbb{T}_4\left(\mathcal{A}^{(PT,+)}_{mn}\right)^2_{m,n=1}\end{pmatrix}, \tag{3.3.15}$$

with

$$\mathbb{T}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.3.16}$$

and imposing on the form (3.3.14), the boundary conditions (3.3.8)–(3.3.11), we arrive at the relations

$$\mathcal{A}^{PT} = Y(k_{P1} - k_{P2})\mathcal{A}^{P}, \tag{3.3.17}$$

with

$$Y(k) = \begin{pmatrix} \frac{-i\alpha}{k+i\alpha} & 0 & 0 & 0 & \frac{k}{k+i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-i\alpha}{k+i\alpha} & 0 & 0 & 0 & \frac{k}{k+i\alpha} \\ \frac{k}{k+i\alpha} & 0 & 0 & 0 & \frac{-i\alpha}{k+i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{k+i\alpha} & 0 & 0 & 0 & \frac{-i\alpha}{k+i\alpha} \end{pmatrix}. \tag{3.3.18}$$

Vertex conditions (3.3.12)–(3.3.13) imply

$$\mathcal{A}^{PR} = \left(\mathbb{I}_2 \otimes S_v(-k_{P1}) \otimes \mathbb{I}_2\right) \mathcal{A}^{P}, \tag{3.3.19}$$

where $S_v(k)$ is the one-particle scattering matrix as defined in (2.3.8). To prove exact solvability we need to show that relations (3.3.17) and (3.3.19) are consistent with the properties of $\mathscr{W}_2$. This amounts to the requirements

1. $Y(u)Y(-u) = \mathbb{I}_8$;
2. $S_v(u)S_v(-u) = \mathbb{I}_2$;
3. $S_v(u)Y(u+v)S_v(v)Y(v-u) = Y(v-u)S_v(v)Y(u+v)S_v(u)$.

While the first two cases are satisfied, the third is not. We must then conclude that the two-particle star graph is not exactly solvable for general vertex conditions A, B and δ-type interactions.

We can of course choose certain vertex conditions A, B in order to describe systems which we know, from the previous sections, to be exactly solvable. For example, choosing standard boundary conditions

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \tag{3.3.20}$$

yields the scattering matrix

$$S_v(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.3.21}$$

Alternatively, choosing Dirichlet conditions

$$A = \mathbb{I}_2 \text{ and } B = 0, \tag{3.3.22}$$

yields the scattering matrix

$$S_v(k) = -\mathbb{I}_2. \tag{3.3.23}$$

In each case, the third condition above is easily seen to hold. We might also consider turning off the δ-interactions altogether, that is, setting $\alpha = 0$. In this case we have

$$Y(k)|_{\alpha=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{T}_4 \tag{3.3.24}$$

and again the third condition above is easily seen to hold. Indeed, in this setting, the model collapses to two separable one-particle systems.

3.3.2 Systems of $\tilde{\delta}$-interacting particles

The purpose of this thesis is to extend the Bethe ansatz approach to general two-particle quantum graphs. As we have seen, though, systems of δ-interacting particles on a two-edge infinite star graph are, in general, not compatible with this approach. We wish to characterise a system which is exactly solvable and thus scalable to general quantum graphs. Such a system was identified in [CC07]. The central notion therein was to impose certain non-local δ-type interactions characterised by the conditions

$$\psi^{+}_{mn}(x_1,x_2)|_{x_1=x_2^+} = \psi^{-}_{nm}(x_1,x_2)|_{x_1=x_2^-}; \tag{3.3.25}$$

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2\alpha\right)\psi^{+}_{mn}(x_1,x_2)|_{x_1=x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi^{-}_{nm}(x_1,x_2)|_{x_1=x_2^-} \tag{3.3.26}$$

for all edge couples (e_m, e_n). In the remainder of this thesis we refer to these interactions as $\tilde{\delta}$-type. We stress here that such interactions can take place when particles are located on different edges and are therefore rather less physical than the δ-type contact interactions imposed up until now. We choose $\tilde{\delta}$-type interactions since, as we will show, they permit exact solutions via the Bethe ansatz method. As we have discussed, in general, quantum graphs with δ-type interactions do not permit exact solutions. In this way there is a clear trade-off between physicality and exact solvability in the types of particle interaction we wish to consider.

Let us proceed as before by applying appropriate boundary conditions to vectors Ψ. Using the form (3.3.14), the $\tilde{\delta}$-type conditions (3.3.25)–(3.3.26) imply

$$\mathcal{A}^{PT} = \left(S_p(k_{P1} - k_{P2}) \otimes \mathbb{I}_4\right)\mathcal{A}^{P}. \tag{3.3.27}$$

Boundary conditions at the vertices are again given by (3.3.12) and (3.3.13) and imply the relation (3.3.19). To prove exact solvability we need only show that relations (3.3.19) and (3.3.27) are consistent with the properties of $\mathscr{W}_2$. This amounts to the requirements

1. $S_p(u)S_p(-u) = \mathbb{I}_2$;

2. $S_v(u)S_v(-u) = \mathbb{I}_2$;

3. $\left(\mathbb{I}_2 \otimes S_v(u) \otimes \mathbb{I}_2\right)\left(S_p(u+v) \otimes \mathbb{I}_4\right)\left(\mathbb{I}_2 \otimes S_v(v) \otimes \mathbb{I}_2\right)\left(S_p(v-u) \otimes \mathbb{I}_4\right)$
   $= \left(S_p(v-u) \otimes \mathbb{I}_4\right)\left(\mathbb{I}_2 \otimes S_v(u) \otimes \mathbb{I}_2\right)\left(S_p(u+v) \otimes \mathbb{I}_4\right)\left(\mathbb{I}_2 \otimes S_v(v) \otimes \mathbb{I}_2\right)$,

which are easily verified by using the explicit forms of $S_p(k)$ and $S_v(k)$ as well as the result in [KS06a] that, for any A, B and u, v, we have the commutation relation

$$[S_v(u), S_v(v)] = 0. \tag{3.3.28}$$

3.3.3 Calculating spectra

Now we have established exactly solvable systems of two $\tilde{\delta}$-interacting particles on two-edge infinite star graphs, we would like to calculate the spectra of certain compact analogues. As previously, this involves imposing certain conditions which restrict the particles to a finite domain. Each of the examples we present here are the two-particle versions of those given in [CC07] (see Proposition 3.1 therein).

Particles in a circle with an impurity

Let us first consider the problem of two $\tilde{\delta}$-interacting particles on a circle of length $2l$ with an impurity characterised by matrices A, B. Confining the particles to this structure is enforcing periodic conditions on each particle according to

$$\psi^-_{j1}(x, l) = \psi^-_{j2}(x, l); \tag{3.3.29}$$

$$\frac{\partial}{\partial x_2}\psi^-_{j1}(x, x_2)|_{x_2=l} = -\frac{\partial}{\partial x_2}\psi^-_{j2}(x, x_2)|_{x_2=l}; \tag{3.3.30}$$

$$\psi^+_{1j}(l, x) = \psi^+_{2j}(l, x); \tag{3.3.31}$$

$$\frac{\partial}{\partial x_1}\psi^+_{1j}(x_1, x)|_{x_1=l} = -\frac{\partial}{\partial x_1}\psi^+_{2j}(x_1, x)|_{x_1=l}, \tag{3.3.32}$$

which imply the relation

$$\mathcal{A}^P = e^{-2ik_{P2}l}\left(\mathbb{I}_4 \otimes \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)\right)\mathcal{A}^{PTRT} \tag{3.3.33}$$

for all $P \in \mathscr{W}_2$. Applying (3.3.19), (3.3.27) and (3.3.33) successively, we arrive at the condition that

$$Z_{\text{circle}}(k_{P1}, k_{P2}) = 0, \tag{3.3.34}$$

with

$$Z_{\text{circle}}(k_1, k_2) = \det\left[\mathbb{I}_8 - e^{2ik_2l}S_p(k_1 + k_2)S_p(k_2 - k_1) \otimes S_v(k_2) \otimes \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)\right], \tag{3.3.35}$$

is satisfied for all $P \in \mathscr{W}_2$. We note here that the forms of $S_p(k)$ and $S_v(k)$ are such that, if (3.3.34) is satisfied for some $P \in \mathscr{W}_2$, then it is necessarily satisfied for elements PR and $PTRT$. We thus have the pair of quantisation conditions

$$Z_{\text{circle}}(k_i, k_j) = 0, \tag{3.3.36}$$

with $j, i \neq j \in \{1, 2\}$. Solutions $(k_1, k_2) \neq (0, 0)$, such that $0 \leq k_1 \leq k_2$, then constitute energies (3.2.15).

Let us also calculate the appropriate quantisation condition where particles are assumed to be bosons. Requiring bosonic symmetry is imposing the condition

$$\psi^-_{mn}(x_1, x_2) = \psi^+_{nm}(x_2, x_1) \tag{3.3.37}$$

which implies the relation

$$\mathcal{A}^P = \begin{pmatrix} 0 & \mathbb{I}_4 \\ \mathbb{I}_4 & 0 \end{pmatrix} \mathcal{A}^P \tag{3.3.38}$$

for all $P \in \mathscr{W}_2$. As a consequence, matrices $S_p(k)$ are replaced with the scalar forms $s_p(k)\mathbb{I}_2$ and (3.3.36) reduces to the quantisation condition

$$e^{2ik_j l} s_p(k_j + k_i) s_p(k_j - k_i) = \pm \operatorname{eig}\left(S_v(-k_j)\right). \tag{3.3.39}$$

Particles in a box with a central impurity

By instead choosing Dirichlet boundary conditions

$$\psi^-_{mn}(x, l) = \psi^+_{mn}(l, x) = 0 \tag{3.3.40}$$

we confine particles to a box of length l with a central impurity. Conditions (3.3.40) imply the relations

$$\mathcal{A}^P = -e^{-2ik_{P2}l} \mathcal{A}^{PTRT} \tag{3.3.41}$$

which, in combination with (3.3.19) and (3.3.27) yields the condition that

$$Z_{\text{box}}(k_{P1}, k_{P2}) = 0, \tag{3.3.42}$$

with

$$Z_{\text{box}}(k_1, k_2) = \det\left[\mathbb{I}_4 + e^{2ik_2 l} S_p(k_1 + k_2) S_p(k_2 - k_1) \otimes S_v(k_2)\right], \tag{3.3.43}$$

is satisfied for all $P \in \mathscr{W}_2$. By following the arguments in the previous example, we arrive at the pair of quantisation conditions

$$Z_{\text{box}}(k_i, k_j) = 0, \tag{3.3.44}$$

with $j, i \neq j \in \{1, 2\}$. Again, solutions $(k_1, k_2) \neq (0, 0)$, such that $0 \leq k_1 \leq k_2$, constitute energies (3.2.15). Imposing bosonic symmetry according to (3.3.37), this condition reduces to

$$e^{2ik_j l} s_p(k_j + k_i) s_p(k_j - k_i) = -\operatorname{eig}\left(S_v(-k_j)\right). \tag{3.3.45}$$

3.4 Spectral statistics

In this chapter we have discussed a number of two-particle systems which, by application of the Bethe ansatz method, we have shown to be exactly solvable. Furthermore, we have deduced quantisation conditions which provide the spectra of the systems. In this final section, we investigate the properties of these spectra. For compactness we restrict our attention to two examples of bosonic systems. In each case, we perform numerical eigenvalue searches to obtain the smallest 10,000 energy levels, choosing the length scale $l = 1$. In particular, we pay attention to the nearest neighbour energy level distribution (2.6.13) as well as the appropriate Weyl law

$$N(E) \sim \frac{\mathcal{L}^2}{8\pi} E, \;\; E \to \infty \tag{3.4.1}$$

proved in [BK13c] for two-boson quantum graphs with singular contact interactions. To this end, we will assign a line of best fit

$$\overline{N}(E) = aE + b\sqrt{E} + c \tag{3.4.2}$$

to the counting function and calculate

$$\frac{\mathcal{L}^2}{8\pi a}, \tag{3.4.3}$$

with values close to unity signifying agreement with (3.4.1).

3.4.1 Bosons in a box

Let us take, as a first example, a system of two δ-interacting bosons in a box. The appropriate spectra are the Laplace eigenvalues (3.2.15) calculated according to the quantisation conditions (3.2.35).

Figure 3.6 plots the α-dependancy of the lowest energy levels of the system for repulsive interactions $\alpha > 0$. We observe that, in the case $\alpha \to \infty$, each eigenvalue exactly corresponds to an eigenvalue for $\alpha \to 0$. This follows from the fact that

$$s_p(k_j + k_i)s_p(k_j - k_i)|_{\alpha=0} = \lim_{\alpha\to\infty} s_p(k_j + k_i)s_p(k_j - k_i) = 1, \tag{3.4.4}$$

for $k_1 \neq k_2$, so that in each case, the quantisation conditions (3.2.35) reduce to the familiar independent conditions

$$e^{2ik_jl} = 1, \tag{3.4.5}$$

with $i, j \neq i \in \{1, 2\}$. Additional eigenvalues for $\alpha \to 0$ in Figure 3.6, appear as a consequence of the case $k_1 = k_2$ where the limit

$$\lim_{\alpha\to 0} s_p(k_j + k_i)s_p(k_j - k_i) = -1. \tag{3.4.6}$$

As illustrated by the example in Figure 3.7, at higher energies, we observe increasingly degenerate eigenvalues for $\alpha \to 0$ and $\alpha \to \infty$. To understand how these degeneracies arise let us consider the Laplace eigenvalues prescribed by (3.4.5)

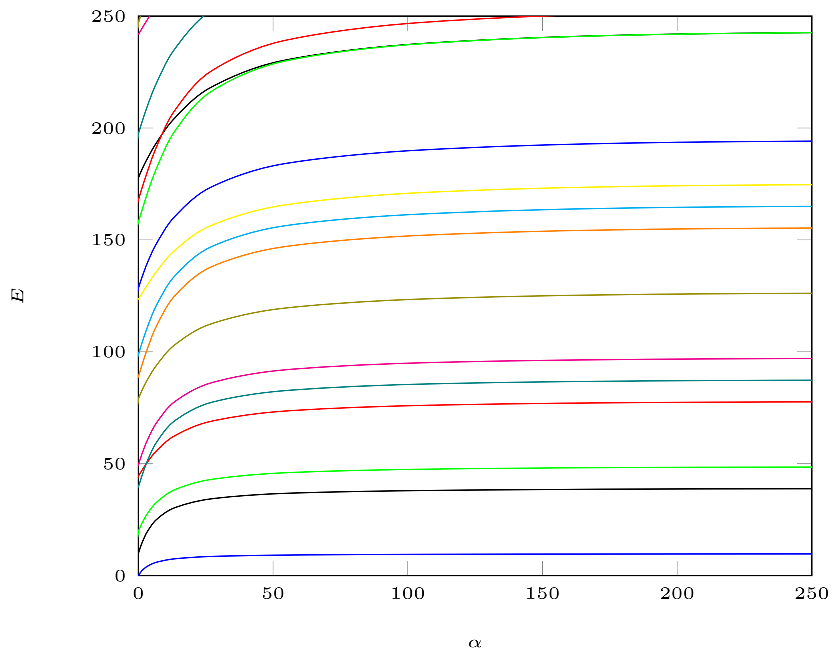

Figure 3.6: Dependency on interaction strength α of small eigenvalues of a system of two bosons in a box.

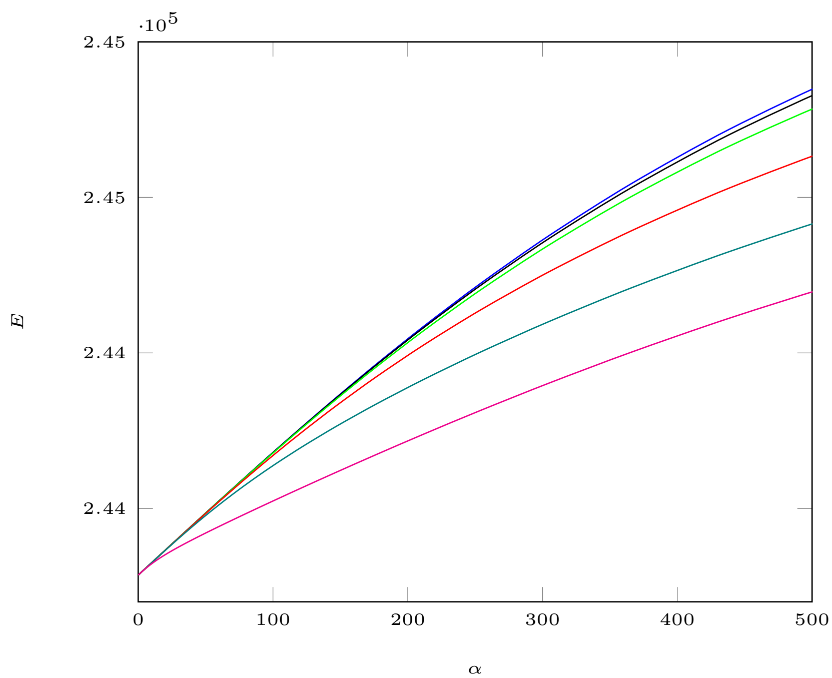

Figure 3.7: Dependency on interaction strength α of six large eigenvalues which are degenerate at $\alpha = 0$ for a system of two bosons in a box.

which can be written

$$E_{mn} = \frac{\pi^2}{l^2}(n^2 + m^2), \tag{3.4.7}$$

with $m, n \in \mathbb{N}_0$. The multiplicities of energies E_{mn} clearly correspond to the measure of degeneracy

$$r_2(d) = \#\{(m,n) \in \mathbb{N}_0^2;\ m^2 + n^2 = d\}. \tag{3.4.8}$$

The average value

$$\overline{r_2(d)} = \frac{\pi}{4} \tag{3.4.9}$$

can be calculated by counting the lattice points on a quarter of a circle [CK97]. Let us also define the related measure

$$B_2(d) = \begin{cases} 1 & \text{if } r_2(d) \geq 1; \\ 0 & \text{if } r_2(d) = 0. \end{cases} \tag{3.4.10}$$

Landau [Lan08] showed that the local average value $\overline{B_2(d)}$ is given by

$$\lim_{d\to\infty} \overline{B_2(d)} \simeq \frac{0.764}{\sqrt{\ln d}}. \tag{3.4.11}$$

Simplistically, so that the result (3.4.9) is consistent with this logarithmically increasing separation between eigenvalues, there must be a corresponding logarithmic increase of the average degeneracy of the eigenvalues [CK97].

Generic quantum systems which are integrable in the classical limit are conjectured to have spectra with Poissonian statistics [BT77]

$$p(s) = e^{-s}. \tag{3.4.12}$$

Although the system in question is integrable, degenerate eigenvalues (3.4.7) make it non-generic. In this setting we expect spectral statistics to fluctuate about the generic Poissonian background [CK99]. Figure 3.8 depicts the nearest neighbour energy level distribution. Firstly for $\alpha = 0$ and $\alpha \to \infty$ one notices a large pro-

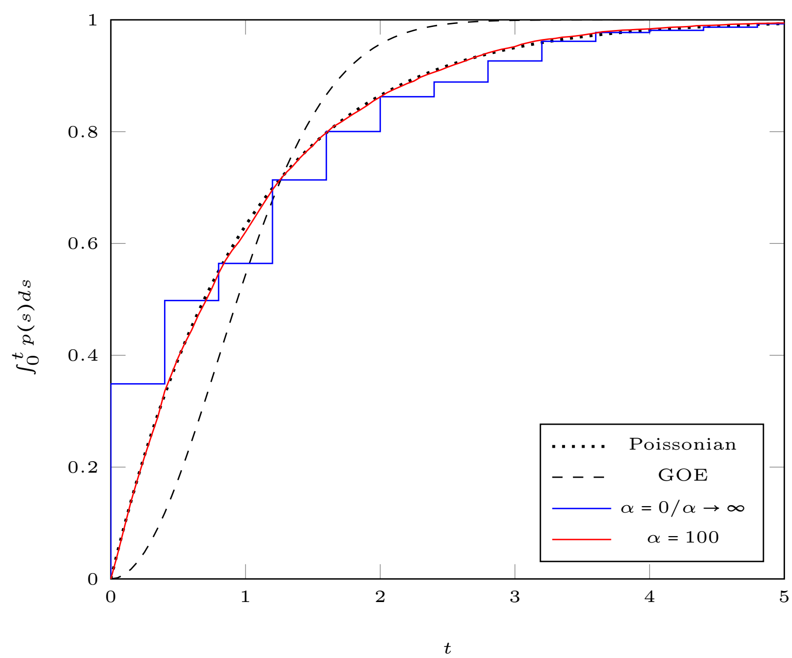

Figure 3.8: Integrated level spacings distributions for systems of two bosons in a box.

portion of spacings equal to zero corresponding to degenerate values. The step function form results from the discrete nature of possible energy level separations. As α moves away from these extremes, the system becomes more generic and thus approaches Poissonian spectral statistics.

Let us compare the eigenvalue counting function $N(E)$ as defined in (2.5.1) to the appropriate Weyl law (3.4.1). Figure 3.9 plots the counting function with $\alpha = 100$. The leading term in $\bar{N}(E)$ is consistent with the Weyl law for all interaction strengths α; in this case we have the value

$$\frac{\mathcal{L}^2}{8\pi a} = 0.995. \tag{3.4.13}$$

3.4.2 Bosons on a circle with an impurity

As a second example, let us take a system of two $\tilde{\delta}$-interacting bosons on a circle with an impurity parameterised by δ-type vertex conditions

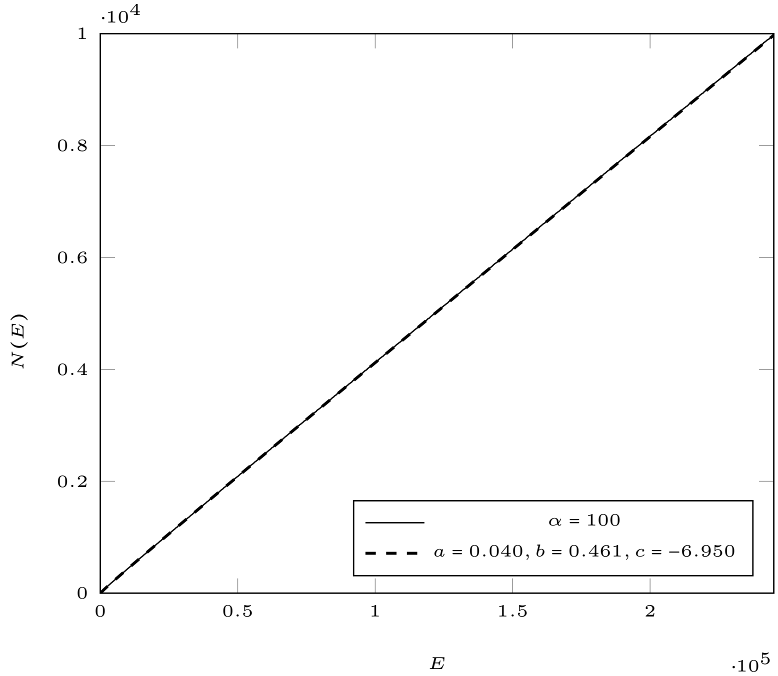

Figure 3.9: Counting functions $N(E)$ (solid line) with line of best fit $\overline{N}(E)$ (dashed line) for systems of two bosons in a box.

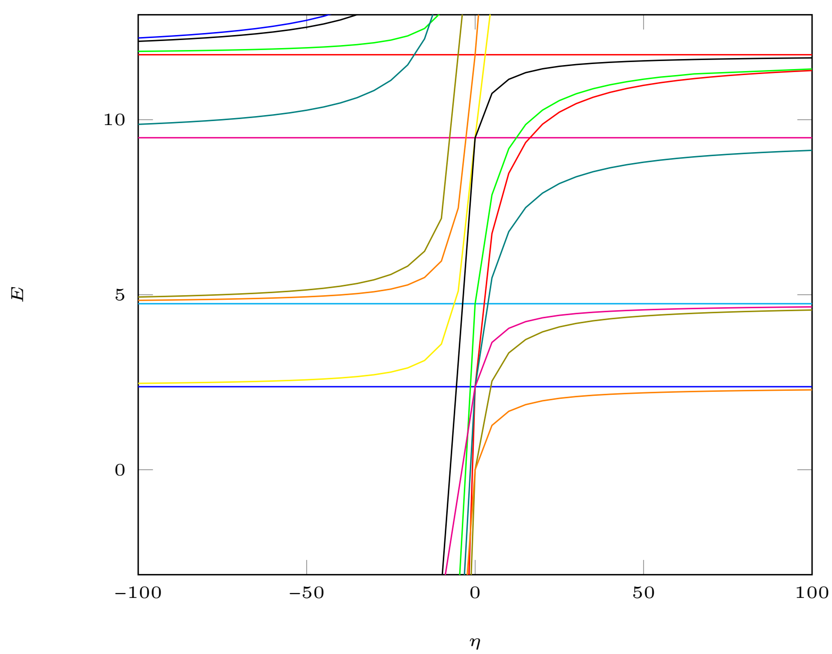

Figure 3.10: Dependency on impurity interaction strength η of small eigenvalues of a system of two bosons in a box with $\alpha = 100$.

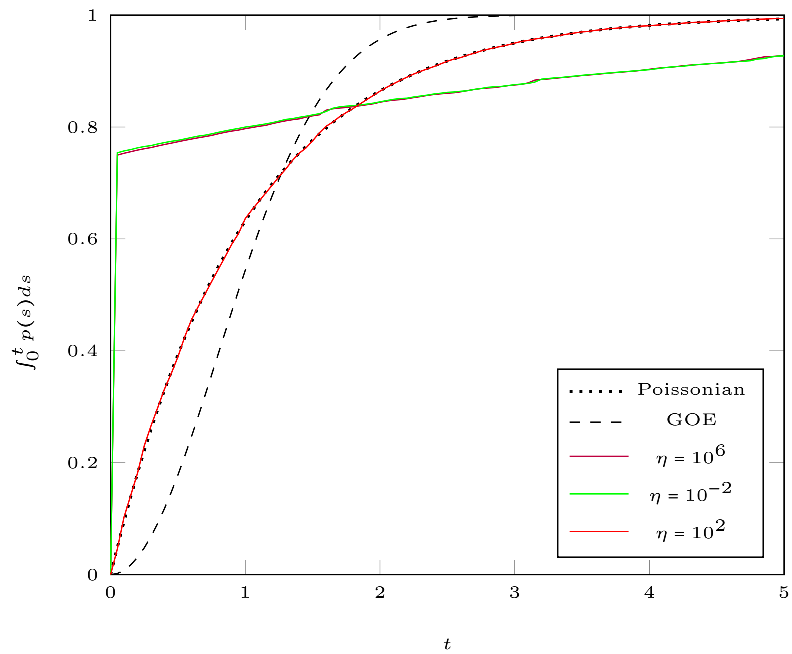

Figure 3.11: Integrated level spacings distributions for systems of two bosons on a circle with an impurity.

$$A = \begin{pmatrix} 1 & -1 \\ -\eta & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}. \tag{3.4.14}$$

The appropriate spectra are the Laplace eigenvalues (3.2.15) calculated according to the quantisation condition (3.3.39) with

$$\text{eig}\left(S_v(-k)\right) = \left\{-1, \frac{2k + i\eta}{2k - i\eta}\right\}. \tag{3.4.15}$$

We note here that the equivalent condition in [CC07] is recovered simply by reparameterising the impurity according to

$$\eta = \frac{2}{\tan(\zeta)}. \tag{3.4.16}$$

We have discussed, in the previous example, the effect of degenerate eigenvalues on the deviation of the spectra from Poissonian statistics. To progress, let us choose an interaction strength ($\alpha = 100$) which minimises this effect. Figure 3.10 shows

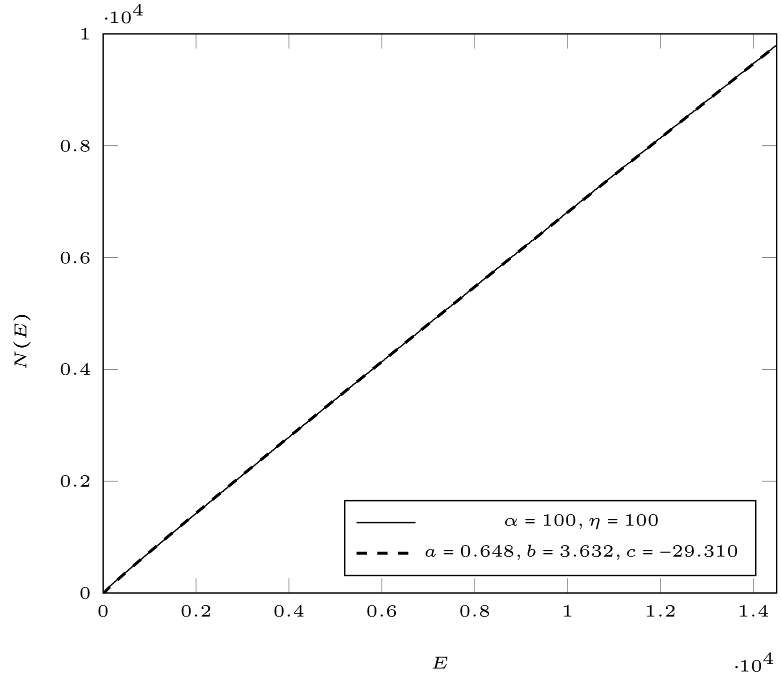

Figure 3.12: Counting functions $N(E)$ (solid line) with line of best fit $\overline{N}(E)$ (dashed line) for systems of two bosons on a circle with an impurity.

the η-dependency of a collection of low-energy levels. We observe here that the spectra for $\eta \to 0$ and for $\eta \to \pm\infty$ are identical. This follows from the fact that

$$\pm \lim_{\eta \to 0} \mathrm{eig}\,(S_v(-k)) = \pm \lim_{\eta \to \pm\infty} \mathrm{eig}\, S_v(-k) = \{\pm 1, \pm 1\}. \tag{3.4.17}$$

We also observe that negative coupling strengths $\eta < 0$ give rise to bound states with the impurity.

Again, degenerate eigenvalues play a role in the spectral statistics we expect. As can be seen in Figure 3.10, degenerate eigenvalues appear as $\eta \to \pm\infty$ and $\eta \to 0$. At these limits, we thus expect a large number of energy level separations which approach zero. Away from these extremes we expect a transition to a generic system and thus Poissonian statistics. Figure 3.11 depicts these statistics.

Finally, we would again like to compare the eigenvalue counting function $N(E)$ to the Weyl law (3.4.1) which we know to be true for contact interactions. Figure 3.12 plots the counting function with $\alpha = 100, \eta = 100$. It turns out that the leading term in $\bar{N}(E)$ does not agree with the Weyl prediction; in this case we have the value

$$\frac{\mathcal{L}^2}{8\pi a} = 0.246. \tag{3.4.18}$$

Disagreement with the Weyl prediction here is due to the character of the $\tilde{\delta}$-type interactions which invoke coupling between particles on different edges. The Weyl law (3.4.1) refers to contact interactions which occur when particles are located at the same position. Providing a correct Weyl law for $\tilde{\delta}$-interactions is beyond the scope of this thesis but serves as an interesting area for further study.

Chapter 4

Two-particle quantum graphs

In this chapter we extend the Bethe ansatz approach formalised in the previous chapter to general quantum graphs. Following [BK13c], we begin by constructing general two-particle quantum graphs with singular contact interactions, establishing appropriate boundary conditions which characterise self-adjoint two-particle Laplacians. Such graphs are, in general, not exactly solvable. With this in mind, the remainder of the chapter will focus on constructing two-particle graphs with non-local $\tilde{\delta}$-interactions introduced in Section 3.3.2, showing that corresponding boundary conditions provide self-adjoint realisations of the two-particle Laplacian. Using an appropriate Bethe ansatz we then prove that such systems are exactly solvable and calculate their spectra. We finish by discussing the spectral statistics of some examples.

4.1 General graphs with contact interactions

Consider the compact graph $\Gamma(\mathcal{V},\mathcal{I},f)$. In Section 2.2 we introduced the Hilbert space $\mathscr{H}_1$ for a single particle on Γ. The Hilbert space of a many-particle quantum system is given by the tensor product of one-particle Hilbert spaces [Bon15]. The appropriate two-particle Hilbert space for a compact two-particle quantum graph is then

$$\mathscr{H}_2 = \left(\bigoplus_{j=1}^{|\mathcal{I}|} L^2(0,l_j)\right) \otimes \left(\bigoplus_{j=1}^{|\mathcal{I}|} L^2(0,l_j)\right). \tag{4.1.1}$$

Vectors

$$\Psi = (\psi_{mn})_{m,n=1}^{|\mathcal{I}|} \tag{4.1.2}$$

in $\mathscr{H}_2$ are lists of two-particle functions $\psi_{mn} : D_{mn} \to \mathbb{C}$ in $L^2(D_{mn})$ with rectangular subdomains defined as

$$D_{mn} = (0, l_m) \times (0, l_n). \tag{4.1.3}$$

The total configuration space for two particles on Γ is the disjoint union

$$D_\Gamma = \dot{\bigcup_{m,n=1}^{|\mathcal{I}|}} D_{mn} \tag{4.1.4}$$

of these rectangles. The two-particle Hilbert space can then be written

$$\mathscr{H}_2 = L^2(D_\Gamma) = \bigoplus_{m,n=1}^{|\mathcal{I}|} L^2(D_{mn}). \tag{4.1.5}$$

At this point let us introduce the two-particle Laplacian $-\Delta_2$ which acts according to

$$-\Delta_2 \Psi = \left(-\frac{\partial^2 \psi_{mn}}{\partial x_1^2} - \frac{\partial^2 \psi_{mn}}{\partial x_2^2} \right)_{m,n=1}^{|\mathcal{I}|}. \tag{4.1.6}$$

We wish to consider the two-particle eigenvalue equation

$$-\Delta_2 \Psi = E \Psi \tag{4.1.7}$$

alongside boundary conditions which prescribe interactions at the vertices as well as singular contact interactions between particles. The latter take place along the diagonals $x_1 = x_2$ of squares D_{mm} and naturally define the *dissected* configuration space

$$D_\Gamma^* = \left(\dot{\bigcup_{m,n=1 | m \neq n}^{|\mathcal{I}|}} D_{mn} \right) \dot{\bigcup} \left(\dot{\bigcup_{m=1}^{|\mathcal{I}|}} (D_{mm}^+ \cup D_{mm}^-) \right), \tag{4.1.8}$$

with subdomains of dissected squares D^*_{mm} defined as

$$D^+_{mm} = \{(x_1, x_2) \in D_{mm};\ x_1 > x_2\} \tag{4.1.9}$$

and

$$D^-_{mm} = \{(x_1, x_2) \in D_{mm};\ x_1 < x_2\}. \tag{4.1.10}$$

The total dissected two-particle Hilbert space is then $\mathscr{H}_2^* = L^2(D^*_\Gamma)$. Thus vectors $\Psi \in \mathscr{H}_2^*$ are lists

$$\Psi = \begin{pmatrix} (\psi_{mn})_{m,n=1|m\neq n}^{|\mathcal{I}|} \\ (\psi^+_{mm})_{m=1}^{|\mathcal{I}|} \\ (\psi^-_{mm})_{m=1}^{|\mathcal{I}|} \end{pmatrix} \tag{4.1.11}$$

of square-integrable functions $\psi_{mn} : D_{mn} \to \mathbb{C}$, for $m \neq n$, and $\psi^\pm_{mm} : D^\pm_{mm} \to \mathbb{C}$. We remark here that $\mathscr{H}_2$ and $\mathscr{H}_2^*$ are in fact equivalent Hilbert spaces. We distinguish between the two in order to make it apparent when functions are defined on the dissected configuration space D^*_Γ. As in the one-particle setting, boundary conditions will be imposed on functions in an appropriate Sobolev space. To this end we define $H^2(D^*_\Gamma)$ as the set of $\Psi \in \mathscr{H}_2^*$ consisting of functions $\psi_{mn} \in H^2(D_{mn})$, for $m \neq n$, and $\psi^\pm_{mm} \in H^2(D^\pm_{mm})$.

Before we continue, it is convenient to single out certain interactions which will be of particular importance to us. In the remainder of this thesis we will always impose boundary conditions which prescribe single-particle interactions with vertices. These will be described by simple two-particle lifts of those (2.2.13) imposed in the one-particle setting. The values of $\Psi \in H^2(D^*_\Gamma)$ at the vertices are given by boundary vectors

$$\Psi^{(v)}_{\mathrm{bv}}(y) = \begin{pmatrix} (\psi_{mn}(0, l_n y))_{m,n=1}^{|\mathcal{I}|} \\ (\psi_{mn}(l_m, l_n y))_{m,n=1}^{|\mathcal{I}|} \\ (\psi_{mn}(l_m y, 0))_{n,m=1}^{|\mathcal{I}|} \\ (\psi_{mn}(l_m y, l_n))_{n,m=1}^{|\mathcal{I}|} \end{pmatrix} \text{ and } \Psi^{(v)\prime}_{\mathrm{bv}}(y) = \begin{pmatrix} (\psi_{mn,1}(0, l_n y))_{m,n=1}^{|\mathcal{I}|} \\ (\psi_{mn,1}(l_m, l_n y))_{m,n=1}^{|\mathcal{I}|} \\ (\psi_{mn,2}(l_m y, 0))_{n,m=1}^{|\mathcal{I}|} \\ (\psi_{mn,2}(l_m y, l_n))_{n,m=1}^{|\mathcal{I}|} \end{pmatrix} \tag{4.1.12}$$

for all $y \in (0,1)$, where for compactness, the labels $\pm$ on functions $\psi^\pm_{mm} \in D^\pm_{mm}$ are dropped. Here, functions $\psi_{mn,1}$ and $\psi_{mn,2}$ denote inward derivatives normal to the lines $x_1 = 0$ and $x_2 = 0$ respectively. Boundary conditions at the vertices are then

$$\left(\mathbb{I}_2 \otimes A \otimes \mathbb{I}_{|\mathcal{I}|}\right) \Psi_{\mathrm{bv}}^{(v)} + \left(\mathbb{I}_2 \otimes B \otimes \mathbb{I}_{|\mathcal{I}|}\right) \Psi_{\mathrm{bv}}^{(v)\prime} = 0, \tag{4.1.13}$$

with A, B defined as in Theorem 2.2.2. Let us also define boundary conditions which prescribe δ-interactions. These are exact analogues of those imposed in Section 3.2, where particles are confined to an interval. On a general graph, such interactions are characterised by the conditions

$$\psi^{+}_{mm}(x_1, x_2)|_{x_1 = x_2^+} = \psi^{-}_{mm}(x_1, x_2)|_{x_1 = x_2^-}; \tag{4.1.14}$$

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2\alpha\right) \psi^{+}_{mm}(x_1, x_2)|_{x_1 = x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) \psi^{-}_{mm}(x_1, x_2)|_{x_1 = x_2^-} \tag{4.1.15}$$

for all $x_1, x_2 \in (0, l_m)$.

4.1.1 Self-adjoint realisation

Two-particle observables are self-adjoint operators on the Hilbert space $\mathscr{H}_2^*$. We thus look for self-adjoint realisations of $-\Delta_2$ with domains characterised by boundary conditions which prescribe interactions at the vertices as well as singular contact interactions between particles.

Let $H_0^2(D_\Gamma^*) \subset H^2(D_\Gamma^*)$ carry the extra condition that functions ψ_{mn} and $\psi^{\pm}_{mm}$ vanish at the boundaries of their respective subdomains along with their inward derivatives. This is the requirement that, for all $y \in [0, 1]$,

$$\begin{aligned} &\psi_{mn}(0, l_n y) = \psi_{mn}(l_m, l_n y) = \psi_{mn}(l_m y, 0) = \psi_{mn}(l_m y, l_n) \\ =&\psi_{mn,1}(0, l_n y) = \psi_{mn,1}(l_m, l_n y) = \psi_{mn,2}(l_m y, 0) = \psi_{mn,2}(l_m y, l_n) = 0 \end{aligned} \tag{4.1.16}$$

in rectangles D_{mn}, with $m \neq n$, and

$$\begin{aligned} &\psi^{-}_{mm}(0, l_m y) = \psi^{+}_{mm}(l_m, l_m y) = \psi^{+}_{mm}(l_m y, 0) = \psi^{-}_{mm}(l_m y, l_m) \\ =&\psi^{-}_{mm,1}(0, l_m y) = \psi^{+}_{mm,1}(l_m, l_m y) = \psi^{+}_{mm,2}(l_m y, 0) = \psi^{-}_{mm,2}(l_m y, l_m) \\ =&\psi^{-}_{mm}(l_m y, l_m y) = \psi^{+}_{mm}(l_m y, l_m y) \\ =&\psi^{-}_{mm,d}(l_m y, l_m y) = \psi^{+}_{mm,d}(l_m y, l_m y) = 0 \end{aligned} \tag{4.1.17}$$

in dissected squares D_{mm}^*. Let $-\Delta_2^0$ denote the Laplacian $-\Delta_2$ with domain $H_0^2(D_\Gamma^*)$. Integrating by parts, and using the boundary conditions (4.1.16)–(4.1.17), it is easy to show that the expression

$$\langle \Phi | -\Delta_2 \Psi \rangle - \langle -\Delta_2 \Phi | \Psi \rangle \tag{4.1.18}$$

vanishes for all $\Phi, \Psi \in H_0^2(D_\Gamma^*)$ and therefore that the operator $-\Delta_2^0$ is symmetric. As shown in [BK13c], it is not however, self-adjoint. Furthermore, in contrast to the one-particle setting, the domain $D(-\Delta_2^{0*})$ of the adjoint operator $-\Delta_2^{0*}$ is not generally known to be contained within $H^2(D_\Gamma^*)$. In the one-particle setting (see Section 2.2), self-adjoint realisations of $-\Delta_1$ can be found by searching for maximal symmetric extensions of $-\Delta_1^0$ with domain $D(-\Delta_1^0) \subset H^2(\Gamma)$. Crucially, the domain of the adjoint operator $-\Delta_1^{0*}$ is also a subset of $H^2(\Gamma)$. We can therefore be assured that the domain of any self-adjoint realisation of $-\Delta_1$ is itself a subset of $H^2(\Gamma)$ and thus consists of functions with valid second derivatives. In the two-particle case, since it could be the case that $H^2(D_\Gamma^*) \subset D(-\Delta_2^{0*})$, we cannot be sure that domains of self-adjoint realisations of $-\Delta_2$ are themselves contained in $H^2(D_\Gamma^*)$ and thus consist of functions with valid second order partial derivatives. For this reason, a straightforward generalisation of the one-particle method of finding maximal symmetric extensions cannot be made. In particular we have the additional requirement that self-adjoint extensions have domains which are subsets of $H^2(D_\Gamma^*)$. We will call this property H^2-regularity.

Self-adjoint realisations of $-\Delta_2$ will be extensions of $-\Delta_2^0$ with domains characterised by conditions on boundary values of functions $\Psi \in H^2(D_\Gamma^*)$ and their derivatives. To this end, we define the boundary vectors

$$\Psi_{\mathrm{bv}}(y) = (\psi_{mn,\mathrm{bv}}(y))_{m,n=1}^{|\mathcal{I}|} \text{ and } \Psi'_{\mathrm{bv}}(y) = \left(\psi'_{mn,\mathrm{bv}}(y)\right)_{m,n=1}^{|\mathcal{I}|} \tag{4.1.19}$$

for all $y \in (0,1)$, where $\psi_{mn,\mathrm{bv}}, \psi'_{mn,\mathrm{bv}}$, with $m \neq n$, and $\psi_{mm,\mathrm{bv}}, \psi'_{mm,\mathrm{bv}}$ list values at the boundaries of D_{mn} and D_{mm}^* respectively. Specifically, for $m \neq n$, there are no interactions between particles and we set

$$\psi_{mn,\mathrm{bv}}(y) = \begin{pmatrix} \psi_{mn}(0,l_n y) \\ \psi_{mn}(l_m,l_n y) \\ \psi_{mn}(l_m y,0) \\ \psi_{mn}(l_m y,l_n) \end{pmatrix} \text{ and } \psi'_{mn,\mathrm{bv}}(y) = \begin{pmatrix} \psi_{mn,1}(0,l_n y) \\ \psi_{mn,1}(l_m,l_n y) \\ \psi_{mn,2}(l_m y,0) \\ \psi_{mn,2}(l_m y,l_n) \end{pmatrix}. \tag{4.1.20}$$

For $m = n$, we must include boundary vectors along the diagonals $x_1 = x_2$ to accommodate singular contact interactions. We thus set

$$\psi_{mm,\mathrm{bv}}(y) = \begin{pmatrix} \psi^-_{mm}(0,l_m y) \\ \psi^+_{mm}(l_m,l_m y) \\ \psi^+_{mm}(l_m y,0) \\ \psi^-_{mm}(l_m y,l_m) \\ \psi^+_{mm}(l_m y,l_m y) \\ \psi^-_{mm}(l_m y,l_m y) \end{pmatrix} \text{ and } \psi'_{mm,\mathrm{bv}}(y) = \begin{pmatrix} \psi^-_{mm,1}(0,l_m y) \\ \psi^+_{mm,1}(l_m,l_m y) \\ \psi^+_{mm,2}(l_m y,0) \\ \psi^-_{mm,2}(l_m y,l_m) \\ \psi^+_{mm,d}(l_m y,l_m y) \\ \psi^-_{mm,d}(l_m y,l_m y) \end{pmatrix}. \tag{4.1.21}$$

Here functions $\psi_{mm,d}$ are inward derivatives normal to the lines $x_1 = x_2$. Clearly we have that $\Psi_{\mathrm{bv}}(y), \Psi'_{\mathrm{bv}}(y) \in \mathbb{C}^{n(\mathcal{I})}$ with $n(\mathcal{I}) = 4|\mathcal{I}|^2 + 2|\mathcal{I}|$. It is easy to see that, with these definitions, the domain $H^2_0(D^*_\Gamma)$ of the symmetric Laplacian $-\Delta^0_2$ can be characterised by the condition $\Psi_{\mathrm{bv}}(y) = \Psi'_{\mathrm{bv}}(y) = 0$.

After having defined boundary vectors $\Psi_{\mathrm{bv}}(y)$ and $\Psi'_{\mathrm{bv}}(y)$, one can characterise self-adjoint realisations of $-\Delta_2$ in an analogous way to the approach in [Kuc04] for one-particle quantum graphs (see Theorem 2.2.3), that is by defining a symmetric, semibounded and closed form q with domain $D(q)$ and then, using Theorem 2.1.5, extracting the associated self-adjoint operator H with domain $D(H) \in D(q)$. In this way, by first assuming that domains $D(H)$ possess H^2-regularity, Bolte and Kerner [BK13c] established self-adjoint realisations of $-\Delta_2$ which we present in the following theorem.

Theorem 4.1.1. Let bounded and measurable maps $P, L : [0,1] \to M(n(\mathcal{I}), \mathbb{C})$ be such that

1. $P(y) = \mathbb{I}_{n(\mathcal{I})} - Q(y)$ is an orthogonal projector of class C^1;

2. $L(y)$ a self-adjoint endomorphism on $\ker P(y)$,

for almost every $y \in [0,1]$. Additionally let bounded and self-adjoint operators Π and Λ on $L^2(0,1) \otimes \mathbb{C}^{n(\mathcal{I})}$ act according to $\Pi\chi(y) := P(y)\chi(y)$ and $\Lambda\chi(y) := L(y)\chi(y)$ on $\chi \in L^2(0,1) \otimes \mathbb{C}^{n(\mathcal{I})}$. Finally let us define the domain $D_2(P,L)$ as the set of $\Psi \in H^2(D_\Gamma^*)$ such that

$$P(y)\Psi_{\mathrm{bv}}(y) = 0 \text{ and } Q(y)\Psi'_{\mathrm{bv}}(y) + L(y)Q(y)\Psi_{\mathrm{bv}}(y) = 0. \tag{4.1.22}$$

The two-particle Laplacian $-\Delta_2$ with domain $D_2(P,L)$ is self-adjoint.

For completeness, let us present an outline of the proof of Theorem 4.1.1, which we reiterate is found in [BK13c]. The starting point is the definition of the symmetric sesquilinear form

$$Q_{PL}^{(2)}[\Phi,\Psi] = \langle \nabla\Phi | \nabla\Psi \rangle_{\mathscr{H}_2^*} - \int_0^1 \langle \Phi_{\mathrm{bv}}(y) | L(y)\Psi_{\mathrm{bv}}(y) \rangle_{\mathbb{C}^{n(\mathcal{I})}} dy \tag{4.1.23}$$

defined on the domain

$$D_Q^{(2)} = \{\Psi \in H^1(D_\Gamma^*);\ P(y)\Psi_{\mathrm{bv}}(y) = 0\} \tag{4.1.24}$$

for almost every $y \in [0,1]$. The fact that the corresponding quadratic form $Q_{PL}^{(2)}[\Psi,\Psi]$ is semibounded and closed (see Definition 2.1.4) is established in [BK13b] for the un-dissected configuration space D_Γ and is easily adapted to account for triangular subspaces $D_{mm}^{\pm}$ in the dissected space D_Γ^*. Using Theorem 2.1.5, the form $Q_{PL}^{(2)}$, defined on the domain $D_Q^{(2)}$, corresponds to a self-adjoint operator H with domain $D(H) \subset D_Q^{(2)}$ such that

$$Q_{PL}^{(2)}[\Phi,\Psi] = \langle \Phi | H\Psi \rangle \tag{4.1.25}$$

for every $\Phi \in D_Q^{(2)}$ and $\Psi \in D(H)$. At this point, we make the additional assumption that the quadratic form $Q_{PL}^{(2)}$ leads to operators H with domains $D(H)$ which possess H^2-regularity, that is $D(H) \subset H^2(D_\Gamma^*)$. In order to determine the action of H, let us restrict our attention to the set of functions $\Phi \in D_Q^{(2)}$ for which $\Phi_{\mathrm{bv}}(y) = 0$. Then, by partial integration, we have that

$$Q_{PL}^{(2)}[\Phi,\Psi] = \langle \Phi | -\Delta_2\Psi \rangle \tag{4.1.26}$$

so that, by comparison to (4.1.25), the operator H acts as the two-particle Laplacian $-\Delta_2$. In order to establish the domain $D(H)$, we now extend our consideration to all functions $\Phi \in D_Q^{(2)}$. Again, by partial integration, one has that

$$-\int_0^1 \langle \Phi'_{\mathrm{bv}}(y) + L(y)\Phi_{\mathrm{bv}}(y) | \Psi_{\mathrm{bv}}(y) \rangle_{\mathbb{C}^{n(\mathcal{I})}} dy = 0. \tag{4.1.27}$$

Finally, recalling that, since $\Psi \in D_Q^{(2)}$, Ψ_{bv} is an arbitrary vector in $\ker \Pi$, we must have that

$$\Phi'_{\mathrm{bv}}(y) + L(y)\Phi_{\mathrm{bv}}(y) \in \ker \Pi^{\perp}, \tag{4.1.28}$$

which implies

$$Q(y)(\Psi'_{\mathrm{bv}}(y) + L(y)\Psi_{\mathrm{bv}}(y)) = 0. \tag{4.1.29}$$

Using the properties of P and L we arrive at the second condition in (4.1.22) which completes the proof.

Before we address the issue of H^2-regularity, let us show how to recover boundary conditions (4.1.13)–(4.1.15) by choosing P and L appropriately. Firstly, to distinguish boundary values relating to vertex interactions from those relating to particle interactions we assume the decomposition

$$\mathbb{C}^{n(\mathcal{I})} = W_v \oplus W_p \tag{4.1.30}$$

where W_v and W_p have dimension $4|\mathcal{I}|^2$ and $2|\mathcal{I}|$ respectively. The subspace W_v is then composed of all components in vectors (4.1.20) as well as the top four components in vectors (4.1.21). The subspace W_p is composed of the bottom two components in vectors (4.1.21). Choosing block diagonal forms

$$P = \begin{pmatrix} P_v & 0 \\ 0 & P_p \end{pmatrix} \text{ and } L = \begin{pmatrix} L_v & 0 \\ 0 & L_p \end{pmatrix} \tag{4.1.31}$$

with respect to this decomposition, we impose that vertex and particle interactions are independent of each other.

For vertex interactions we first note that the restrictions of Ψ_{bv} and Ψ'_{bv} to W_v

are the boundary vectors (4.1.12). In order to recover the boundary conditions (4.1.13) which prescribe single-particle interactions with the vertices, we must establish correspondence between maps P_v, L_v and matrices A, B. To this end, in exact analogy to the approach in Section 2.2, let P_v be an orthogonal projection onto $\ker(\mathbb{I}_2 \otimes B \otimes \mathbb{I}_{|\mathcal{I}|})$ and

$$L_v = (\mathbb{I}_2 \otimes B^{-1}_{(\ker B)^\perp} A \otimes \mathbb{I}_{|\mathcal{I}|}) Q_v. \tag{4.1.32}$$

Substituting into (4.1.22), we recover boundary conditions (4.1.13) as required.

In W_p we would like to distinguish clearly between contact interactions on different edges. Therefore we define the decomposition

$$W_p = \bigoplus_{m=1}^{|\mathcal{I}|} W_{p,m}, \tag{4.1.33}$$

where each $W_{p,m}$ is composed of the bottom two components of $\psi_{mm,\mathrm{bv}}$. Then, by fixing the block diagonal forms

$$P_p = \bigoplus_{m=1}^{|\mathcal{I}|} P_{p,m} \text{ and } L_p = \bigoplus_{m=1}^{|\mathcal{I}|} L_{p,m} \tag{4.1.34}$$

with respect to this decomposition, we impose that there are no interactions between particles on different edges. Of particular interest to us will be δ-type contact interactions prescribed by conditions (4.1.14)–(4.1.15). It is easy to see that these conditions are retrieved by setting

$$P_{p,m}(y) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } L_{p,m}(y) = -\alpha \mathbb{I}_2. \tag{4.1.35}$$

Finally, let us bring our attention back to the problem of H^2-regularity. For certain additional conditions on the maps P and L, Bolte and Kerner [BK13c] proved that the quadratic form $Q^{(2)}_{PL}$ does indeed lead to domains $D_2(P,L)$ which are contained in $H^2(D^*_\Gamma)$. We choose not to present the general theorem here, but remark that the maps include those prescribed above for δ-interactions. It will be useful later in this chapter to mention that, in order to show $D_2(P,L) \subset H^2(D^*_\Gamma)$, it is enough to show H^2-regularity on each subspace D_{mn} and $D^\pm_{mm}$. The former is established in [BK13b], the latter in [BK13c]. In subsequent sections we will define variants of δ-type interactions which will appear as boundary conditions along new dissections of D_Γ. We argue here that H^2-regularity therein can always be assured by imposing further dissections of D_Γ into constituent subspaces which are equivalent to D_{mn} and $D^{\pm}_{mm}$.

4.1.2 Spectra

In Section 2.3, we calculated the spectra of one particle quantum graphs by specifying the form (2.3.2) of eigenfunctions of $-\Delta_1$ and applying boundary conditions (2.2.13). We would like to extend this approach to the two-particle quantum graph setting. As we are dealing with a two dimensional configuration space, the difficulty is that, in general, there does not exist a suitable analogue of the general form of an eigenfunction (2.3.2). As shown in Chapter 3, however, in particular cases, a Bethe ansatz can be used in this way. We present two such examples next.

The task will be to specify eigenvectors $\Psi \in H^2(D^*_\Gamma)$ of $-\Delta_2$ which satisfy vertex conditions (4.1.13) as well as the δ-type interaction conditions (4.1.14)–(4.1.15). Justifying the Bethe ansatz method as in Chapter 3, the vector Ψ will be described by the collection of functions

$$\psi_{mn}(x_1, x_2) = \sum_{P \in \mathscr{W}_2} \mathcal{A}^P_{mn} e^{i(k_{P1}x_1 + k_{P2}x_2)} \tag{4.1.36}$$

on rectangles D_{mn}, for $m \neq n$, and

$$\psi^{\pm}_{mm}(x_1, x_2) = \sum_{P \in \mathscr{W}_2} \mathcal{A}^{(P,\pm)}_{mm} e^{i(k_{P1}x_1 + k_{P2}x_2)} \tag{4.1.37}$$

on squares $D^{\pm}_{mm}$. The eigenvalue equation (4.1.7) is then satisfied with Laplace eigenvalues

$$E = k_1^2 + k_2^2. \tag{4.1.38}$$

Non-interacting particles

We begin by considering the example of two non-interacting particles on Γ. Such particles obey δ-type boundary conditions (4.1.14)–(4.1.15) with $\alpha = 0$. Together with the form specified by (4.1.37), these conditions imply the relations

$$\mathcal{A}_{mm}^{(P,+)} = \mathcal{A}_{mm}^{(P,-)} = \mathcal{A}_{mm}^{P} \tag{4.1.39}$$

for all $P \in \mathscr{W}_2$, where the final equality is made so that we can drop labels ±. Let us define the vector of amplitudes

$$\mathcal{A}^P = (\mathcal{A}_{mn}^P)_{m,n=1}^{|\mathcal{I}|} \tag{4.1.40}$$

and also the $d^2 \times d^2$ permutation matrix

$$\mathbb{T}_{d^2} = \begin{pmatrix} \mathbb{I}_d \otimes m_1 \\ \vdots \\ \mathbb{I}_d \otimes m_d \end{pmatrix}, \tag{4.1.41}$$

with row vectors

$$m_j = \begin{pmatrix} \underbrace{0 \dots 0}_{j-1} & 1 & \underbrace{0 \dots 0}_{d-j} \end{pmatrix}. \tag{4.1.42}$$

It is convenient to note the properties

$$\mathbb{T}_{d^2} \left(\mathcal{A}_{mn}\right)_{m,n=1}^{d} = \left(\mathcal{A}_{mn}\right)_{n,m=1}^{d} \tag{4.1.43}$$

for d^2-dimensional column vectors $\mathcal{A}$ and

$$\mathbb{T}_{d^2}(M \otimes N)\mathbb{T}_{d^2} = N \otimes M \tag{4.1.44}$$

for any $d \times d$ matrices M and N. Using the form (4.1.36), the boundary conditions (4.1.13) then imply the relations

$$\begin{aligned} &(A \otimes \mathbb{I}_{|\mathcal{I}|}) \begin{pmatrix} \mathcal{A}^P + \mathcal{A}^{PR} \\ (e^{ik_{P1}\boldsymbol{l}} \otimes \mathbb{I}_{|\mathcal{I}|})\mathcal{A}^P + (e^{-ik_{P1}\boldsymbol{l}} \otimes \mathbb{I}_{|\mathcal{I}|})\mathcal{A}^{PR} \end{pmatrix} \\ +ik_{P1}&(B \otimes \mathbb{I}_{|\mathcal{I}|}) \begin{pmatrix} \mathcal{A}^P - \mathcal{A}^{PR} \\ -(e^{ik_{P1}\boldsymbol{l}} \otimes \mathbb{I}_{|\mathcal{I}|})\mathcal{A}^P + (e^{-ik_{P1}\boldsymbol{l}} \otimes \mathbb{I}_{|\mathcal{I}|})\mathcal{A}^{PR} \end{pmatrix} = 0 \end{aligned} \tag{4.1.45}$$

and

$$\begin{aligned}(A\otimes\mathbb{I}_{|\mathcal{I}|})&\begin{pmatrix}\mathbb{T}_{|\mathcal{I}|^2}\mathcal{A}^{PT}+\mathbb{T}_{|\mathcal{I}|^2}\mathcal{A}^{PRT}\\(e^{ik_{P1}\boldsymbol{l}}\otimes\mathbb{I}_{|\mathcal{I}|})\mathbb{T}_{|\mathcal{I}|^2}\mathcal{A}^{PT}+(e^{-ik_{P1}\boldsymbol{l}}\otimes\mathbb{I}_{|\mathcal{I}|})\mathbb{T}_{|\mathcal{I}|^2}\mathcal{A}^{PRT}\end{pmatrix}\\+ik_{P1}(B\otimes\mathbb{I}_{|\mathcal{I}|})&\begin{pmatrix}\mathbb{T}_{|\mathcal{I}|^2}\mathcal{A}^{PT}-\mathbb{T}_{|\mathcal{I}|^2}\mathcal{A}^{PRT}\\-(e^{ik_{P1}\boldsymbol{l}}\otimes\mathbb{I}_{|\mathcal{I}|})\mathbb{T}_{|\mathcal{I}|^2}\mathcal{A}^{PT}+(e^{-ik_{P1}\boldsymbol{l}}\otimes\mathbb{I}_{|\mathcal{I}|})\mathbb{T}_{|\mathcal{I}|^2}\mathcal{A}^{PRT}\end{pmatrix}=0\end{aligned} \tag{4.1.46}$$

for all $P\in\mathscr{W}_2$, with $e^{ik\boldsymbol{l}}$ defined in (2.3.6). Using the properties of Kronecker products, each of these can be shown to imply the condition

$$\det\left[AX(k_{P1},\boldsymbol{l})+ikBY(k_{P1},\boldsymbol{l})\right]=0, \tag{4.1.47}$$

with $X(k,\boldsymbol{l})$ and $Y(k,\boldsymbol{l})$ defined in (2.3.5). Following the proof of Theorem 2.3.1 we arrive at the condition

$$Z(k_{P1})=0 \tag{4.1.48}$$

for all $P\in\mathscr{W}_2$, where

$$Z(k)=\det\left[\mathbb{I}_{2|\mathcal{I}|}-S_v(k)T(k,\boldsymbol{l})\right]. \tag{4.1.49}$$

Here $S_v(k)$ is the one-particle scattering matrix defined in (2.3.8) and $T(k,\boldsymbol{l})$ is defined in (2.3.7). We note that the forms of $S_v(k)$ and $T(k,\boldsymbol{l})$ are such that, if (4.1.48) is satisfied for some $P\in\mathscr{W}_2$, then it is necessarily satisfied for elements

$$PR \text{ and } PTRT \tag{4.1.50}$$

in $\mathscr{W}_2$. Thus we have two independent conditions,

$$Z(k_j)=0 \tag{4.1.51}$$

for $j\in\{1,2\}$, each corresponding to a single particle. As one might expect, these conditions are exactly those prescribed in Theorem 2.3.1 derived in the context of one-particle quantum graphs. Two-particle energies (4.1.38) in the non-interacting setting are simply the sums of the energies $E_j=k_j^2$ given by the one-particle quantisation conditions.

Two δ-interacting particles on an interval

One can establish agreement with the example of two δ-interacting particles in a box discussed in Section 3.2.3 by considering the simplest metric graph, an interval $[0, l]$, with Dirichlet boundary conditions

$$A = \mathbb{I}_2 \text{ and } B = 0. \tag{4.1.52}$$

Applying boundary conditions (4.1.13)–(4.1.15), and following the approach in Section 3.2.3, we arrive at the two-particle spectra prescribed by (3.2.48).

4.2 Equilateral stars with $\tilde{\delta}$-interactions

In the previous section we established appropriate boundary conditions on general two-particle quantum graphs with singular contact interactions by characterising self-adjoint extensions of the two-particle Laplacian. Such systems are, in general, not exactly solvable. The aim of this chapter is to calculate spectra for general two-particle graphs. The task is then to establish boundary conditions on general quantum graphs which are compatible with the Bethe ansatz method. In Section 3.3.2, systems of particles on two-edge star graphs, with non-local $\tilde{\delta}$-type particle interactions, were shown to be exactly solvable. Then prescribing length l to the edges and imposing certain coupling conditions, their spectra were deduced. In the remainder of this chapter, we extend this approach to systems of two particles on general graphs. Furthermore, we show that the corresponding boundary conditions provide a self-adjoint realisation of the two-particle Laplacian.

Before discussing general graphs it is convenient to consider a subset of graphs called equilateral stars. These graphs exhibit most of the essential features of the general case and thus act as a convenient way to introduce some key concepts. They also exhibit some distinguishing features associated with their spectral statistics. We revisit this point at the end of the chapter.

Let us define the equilateral star Γ_e as the graph $\Gamma(\mathcal{V}, \mathcal{I}, f)$ with the restrictions that

1. $l_j = l$;

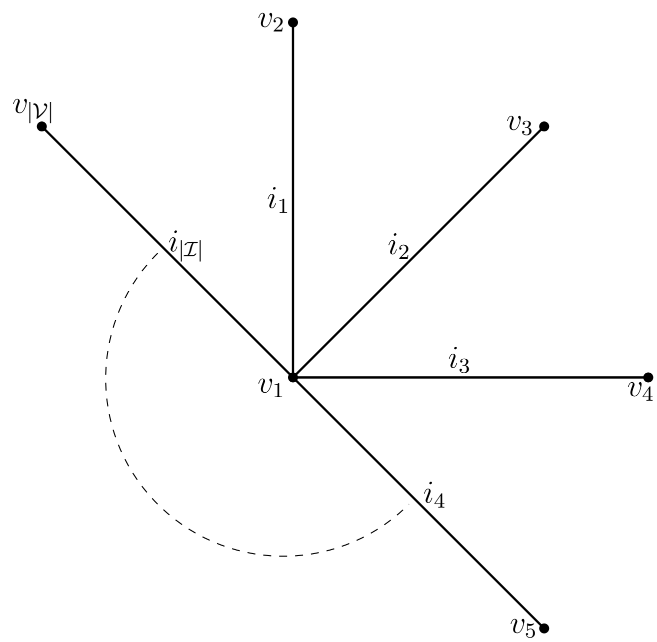

Figure 4.1: Equilateral star with $|\mathcal{I}|$ edges each of length l.

2. $f_0(i_j) = v_1$;
3. $f_l(i_j) = v_{j+1}$,

for all $i_j \in \mathcal{I}$, as depicted in Figure 4.1. Vectors $\Psi \in \mathscr{H}_2$ are then lists of two-particle functions $\psi_{mn} : D_{mn} \to \mathbb{C}$ in $L^2(D_{mn})$ with square subdomains

$$D_{mn} = (0,l) \times (0,l). \tag{4.2.1}$$

The total configuration space for two particles on Γ_e is the union

$$D_{\Gamma_e} = \dot{\bigcup_{m,n=1}^{|\mathcal{I}|}} D_{mn} \tag{4.2.2}$$

of these squares. The two-particle Hilbert space can then be written $\mathscr{H}_2 = L^2(D_{\Gamma_e})$.

Appropriate interactions will be analogous of the $\tilde{\delta}$-interactions imposed in Section 3.3.2 and take place when particles are situated on neighbouring edges, the same distance from the common vertex of the edges. We reiterate here that the set

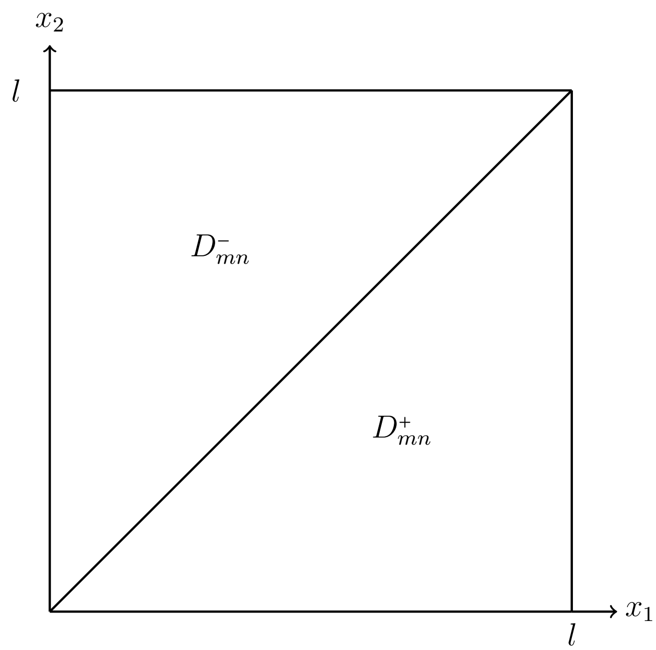

Figure 4.2: Dissected configuration space D^*_{mn} for an edge pair (i_m, i_n) on an equilateral star.

$\mathcal{N}$ of neighbouring edges includes pairs (i_m, i_m). On the equilateral star, such interactions will be implemented by dissecting all squares D_{mn} along the lines $x_1 = x_2$ and imposing suitable boundary conditions on functions ψ_{mn}. We thus have the dissected configuration space

$$D^*_{\Gamma_e} = \dot{\bigcup_{m,n=1}^{|\mathcal{I}|}} (D^+_{mn} \cup D^-_{mn}), \tag{4.2.3}$$

with subdomains of dissected squares D^*_{mn} defined as

$$D^+_{mn} = \{(x_1, x_2) \in D_{mn};\ x_1 > x_2\} \tag{4.2.4}$$

and

$$D^-_{mn} = \{(x_1, x_2) \in D_{mn};\ x_1 < x_2\} \tag{4.2.5}$$

as depicted in Figure 4.2. The total dissected two-particle Hilbert space is then

$\mathscr{H}_2^* = L^2(D^*_{\Gamma_e})$. Thus vectors $\Psi \in \mathscr{H}_2^*$ are lists

$$\Psi = \begin{pmatrix} (\psi^+_{mn})^{|\mathcal{I}|}_{m,n=1} \\ (\psi^-_{mn})^{|\mathcal{I}|}_{m,n=1} \end{pmatrix} \tag{4.2.6}$$

of square-integrable functions $\psi^\pm_{mn} : D^\pm_{mn} \to \mathbb{C}$. The corresponding Sobolev space $H^2(D^*_{\Gamma_e})$ is the set of $\Psi \in \mathscr{H}_2^*$ consisting of functions $\psi^\pm_{mn} \in H^2(D^\pm_{mn})$.

We are now in a position to be explicit about the types of interactions we would like to impose. They will appear as conditions on functions $\Psi \in H^2(D^*_{\Gamma_e})$ along the boundaries of $D^*_{\Gamma_e}$.

Interactions at the vertices will again be described by simple two-particle lifts of those imposed in the corresponding one-particle quantum graph. These will be given by conditions (4.1.13), with all edge lengths in boundary vectors (4.1.12) equal to l.

Interactions between particles will be analogues of the $\tilde{\delta}$-type interactions introduced in Section 3.3.2 and are characterised by the conditions

$$\psi^+_{mn}(x_1,x_2)|_{x_1=x_2^+} = \psi^-_{nm}(x_1,x_2)|_{x_1=x_2^-}; \tag{4.2.7}$$

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2\alpha\right)\psi^+_{mn}(x_1,x_2)|_{x_1=x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi^-_{nm}(x_1,x_2)|_{x_1=x_2^-}, \tag{4.2.8}$$

for all edge couples $(i_m, i_n) \in \mathcal{I} \otimes \mathcal{I}$. We reiterate here that $\tilde{\delta}$-type interactions can take place when particles are located on different edges and are therefore rather less physical than the familiar δ-interactions. Such interactions are chosen since, as we will show, they permit exact solutions via the Bethe ansatz method.

4.2.1 Self-adjoint realisation

The task is now to show that the interactions we would like to impose correspond to a self-adjoint Laplacian. To this end, adapting the method in [BK13c] (see Section 4.1.1), we deduce self-adjoint realisations of $-\Delta_2$ with domains characterised by conditions on boundary values of functions $\Psi \in H^2(D^*_{\Gamma_e})$ and their derivatives. We then show that from these conditions we can recover (4.1.13) and (4.2.7)–(4.2.8).

Let us define the boundary vectors

$$\Psi_{\text{bv}}(x) = \left(\psi_{mn,\text{bv}}(x)\right)_{m,n=1}^{|\mathcal{I}|} \text{ and } \Psi'_{\text{bv}}(x) = \left(\psi'_{mn,\text{bv}}(x)\right)_{m,n=1}^{|\mathcal{I}|} \tag{4.2.9}$$

for all $x \in (0,l)$, where each $\psi_{mn,\text{bv}}$ and $\psi'_{mn,\text{bv}}$ list values at the boundaries of D^*_{mn} so that

$$\psi_{mn,\text{bv}}(x) = \begin{pmatrix} \psi^-_{mn}(0,x) \\ \psi^+_{mn}(l,x) \\ \psi^+_{mn}(x,0) \\ \psi^-_{mn}(x,l) \\ \psi^+_{mn}(x,x) \\ \psi^-_{mn}(x,x) \end{pmatrix} \text{ and } \psi'_{mn,\text{bv}}(x) = \begin{pmatrix} \psi^-_{mn,1}(0,x) \\ \psi^+_{mn,1}(l,x) \\ \psi^+_{mn,2}(x,0) \\ \psi^-_{mn,2}(x,l) \\ \psi^+_{mn,d}(x,x) \\ \psi^-_{mn,d}(x,x) \end{pmatrix}. \tag{4.2.10}$$

Clearly we have that $\Psi_{\text{bv}}(x), \Psi'_{\text{bv}}(x) \in \mathbb{C}^{n(\mathcal{I})}$ with $n(\mathcal{I}) = 6|\mathcal{I}|^2$.

Carrying over the approach from the previous section we present the following theorem.

Theorem 4.2.1. Let bounded and measurable maps $P, L : [0,l] \to M(n(\mathcal{I}), \mathbb{C})$ be such that

1. $P(x) = \mathbb{I}_{n(\mathcal{I})} - Q(x)$ is an orthogonal projector of class C^1;
2. $L(x)$ a self-adjoint endomorphism on $\ker P(x)$,

for almost every $x \in [0,l]$. Additionally let bounded and self-adjoint operators Π and Λ on $L^2(0,l) \otimes \mathbb{C}^{n(\mathcal{I})}$ act according to $\Pi\chi(x) := P(x)\chi(x)$ and $\Lambda\chi(x) := L(x)\chi(x)$ on $\chi \in L^2(0,l) \otimes \mathbb{C}^{n(\mathcal{I})}$. Finally let us define the domain $D_2(P,L)$ as the set of $\Psi \in H^2(D^*_{\Gamma_e})$ such that

$$P(x)\Psi_{\text{bv}}(x) = 0 \text{ and } Q(x)\Psi'_{\text{bv}}(x) + L(x)Q(x)\Psi_{\text{bv}}(x) = 0. \tag{4.2.11}$$

The two-particle Laplacian $-\Delta_2$ with domain $D_2(P,L)$ is self-adjoint.

Now we have established the domain $D_2(P,L)$ of a self-adjoint Laplacian $-\Delta_2$ on Γ_e, we would like to recover boundary conditions (4.1.13) and (4.2.7)–(4.2.8) by choosing P and L appropriately. Firstly, to distinguish boundary values relating to vertex interactions from those relating to particle interactions, we assume the decomposition (4.1.30) where W_v and W_p now have dimension $4|\mathcal{I}|^2$ and $2|\mathcal{I}|^2$. Subspaces W_v and W_p are then composed of the top four and bottom two components in vectors (4.2.10) respectively. Choosing block diagonal forms (4.1.31) with respect to this decomposition, we impose that vertex and particle interactions are independent of each other.

For vertex interactions we again recover boundary conditions (4.1.13) by defining P_v and L_v as in Section 4.1.1.

For $\tilde{\delta}$-type particle interactions, we would first like to impose the further decomposition

$$W_p = \bigoplus_{m,n=1}^{|\mathcal{I}|} W_{p,mn}, \tag{4.2.12}$$

where in the case of Ψ_{bv}, each $W_{p,mn}$ is composed of the second bottom component of $\psi_{mn,\mathrm{bv}}$ and the bottom component of $\psi_{nm,\mathrm{bv}}$. Fixing the block diagonal forms

$$P_p = \bigoplus_{m,n=1}^{|\mathcal{I}|} P_{p,mn} \text{ and } L_p = \bigoplus_{m,n=1}^{|\mathcal{I}|} L_{p,mn} \tag{4.2.13}$$

with respect to this decomposition and setting

$$P_{p,mn}(x) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } L_{p,mn}(x) = -\alpha \mathbb{I}_2, \tag{4.2.14}$$

we recover the $\tilde{\delta}$-type boundary conditions (4.2.7)–(4.2.8).

4.2.2 Spectra

We have seen how to establish boundary conditions which correspond to systems of two particles on equilateral stars with $\tilde{\delta}$-interactions in the context of self-adjoint realisations of the two-particle Laplacian. In this section we show that these systems are exactly solvable and calculate their spectra.

The task is to specify eigenvectors $\Psi \in H^2(D^*_{\Gamma_e})$ of $-\Delta_2$ which satisfy vertex conditions (4.1.13) as well as $\tilde{\delta}$-type conditions (4.2.7)–(4.2.8). Justifying the Bethe ansatz approach as previously, the vector Ψ will be described by the collection of functions

$$\psi^{\pm}_{mn}(x_1,x_2) = \sum_{P\in\mathscr{W}_2} \mathcal{A}^{(P,\pm)}_{mn} e^{i(k_{P1}x_1+k_{P2}x_2)} \tag{4.2.15}$$

on squares $D^{\pm}_{mn}$. The eigenvalue equation (4.1.7) is then satisfied with eigenvalues (4.1.38).

We would first like to show that the system is indeed exactly solvable, that is, boundary conditions (4.1.13) and (4.2.7)–(4.2.8) imposed on Ψ are compatible with the properties of $\mathscr{W}_2$. Defining the $2|\mathcal{I}|^2$-dimensional vector

$$\mathcal{A}^P = \begin{pmatrix} \left(\mathcal{A}^{(P,-)}_{mn}\right)^{|\mathcal{I}|}_{m,n=1} \\ \mathbb{T}_{|\mathcal{I}|^2}\left(\mathcal{A}^{(PT,+)}_{mn}\right)^{|\mathcal{I}|}_{m,n=1} \end{pmatrix}, \tag{4.2.16}$$

the vertex condition (4.1.13), together with the form specified by (4.2.15), implies

$$\begin{aligned} &\left(\mathbb{I}_2\otimes A\otimes\mathbb{I}_{|\mathcal{I}|}\right)\mathbb{Q}\begin{pmatrix} \mathcal{A}^P+\mathcal{A}^{PR} \\ \mathcal{A}^{PT}e^{ik_{P1}l}+\mathcal{A}^{PRT}e^{-ik_{P1}l} \end{pmatrix} \\ &+ik_{P1}\left(\mathbb{I}_2\otimes B\otimes\mathbb{I}_{|\mathcal{I}|}\right)\mathbb{Q}\begin{pmatrix} \mathcal{A}^P-\mathcal{A}^{PR} \\ -\mathcal{A}^{PT}e^{ik_{P1}l}+\mathcal{A}^{PRT}e^{-ik_{P1}l} \end{pmatrix} = 0 \end{aligned} \tag{4.2.17}$$

for all $P\in\mathscr{W}_2$ where

$$\mathbb{Q} = \begin{pmatrix} \mathbb{I}_{|\mathcal{I}|^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{T}_{|\mathcal{I}|^2} \\ 0 & \mathbb{I}_{|\mathcal{I}|^2} & 0 & 0 \\ 0 & 0 & \mathbb{T}_{|\mathcal{I}|^2} & 0 \end{pmatrix}. \tag{4.2.18}$$

Equilateral stars have Dirichlet conditions at external vertices v_j for $j \geq 2$. We thus have the decomposition

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \tag{4.2.19}$$

with

$$A_2 = \mathbb{I}_{|\mathcal{I}|} \text{ and } B_2 = 0. \tag{4.2.20}$$

By using the properties of $\mathbb{T}_{|\mathcal{I}|^2}$, we then have that

$$\mathbb{Q}^{-1}\left(\mathbb{I}_2 \otimes S_v(k) \otimes \mathbb{I}_{|\mathcal{I}|}\right)\mathbb{Q} = \begin{pmatrix} \mathbb{I}_2 \otimes S_v^{(1)}(k) \otimes \mathbb{I}_{|\mathcal{I}|} & 0 \\ 0 & -\mathbb{I}_{2|\mathcal{I}|^2} \end{pmatrix}. \tag{4.2.21}$$

Rearranging (4.2.17), we can then extract the relation

$$\mathcal{A}^{PR} = \left(\mathbb{I}_2 \otimes S_v^{(1)}(-k_{P1}) \otimes \mathbb{I}_{|\mathcal{I}|}\right)\mathcal{A}^{P}. \tag{4.2.22}$$

The $\tilde{\delta}$-type conditions (4.2.7) and (4.2.8) imply

$$\mathcal{A}^{PT} = \left(S_p(k_{P1} - k_{P2}) \otimes \mathbb{I}_{|\mathcal{I}|^2}\right)\mathcal{A}^{P} \tag{4.2.23}$$

with $S_p(k)$ defined in (3.2.42). To prove exact solvability we need only show that relations (4.2.22) and (4.2.23) are consistent with the properties of $\mathscr{W}_2$. This amounts to the requirements

1. $S_v^{(1)}(u)S_v^{(1)}(-u) = \mathbb{I}_{|\mathcal{I}|}$;
2. $S_p(u)S_p(-u) = \mathbb{I}_2$;
3. $\left(\mathbb{I}_2 \otimes S_v^{(1)}(u) \otimes \mathbb{I}_{|\mathcal{I}|}\right)\left(S_p(u+v) \otimes \mathbb{I}_{|\mathcal{I}|^2}\right)\left(\mathbb{I}_2 \otimes S_v^{(1)}(v) \otimes \mathbb{I}_{|\mathcal{I}|}\right)\left(S_p(v-u) \otimes \mathbb{I}_{|\mathcal{I}|^2}\right)$ $= \left(S_p(v-u) \otimes \mathbb{I}_{|\mathcal{I}|^2}\right)\left(\mathbb{I}_2 \otimes S_v^{(1)}(v) \otimes \mathbb{I}_{|\mathcal{I}|}\right)\left(S_p(u+v) \otimes \mathbb{I}_{|\mathcal{I}|^2}\right)\left(\mathbb{I}_2 \otimes S_v^{(1)}(u) \otimes \mathbb{I}_{|\mathcal{I}|}\right)$.

The first two conditions are easily verified by the explicit forms of $S_v^{(1)}(u)$ and $S_p(u)$. The third follows from the commutation relation (3.3.28).

Now we have established that the system is exactly solvable, we would like to deduce the spectrum. This can be done in a number of ways. The method we choose here generalises that used for the one-particle case in [KS06b] which we presented in Section 2.3. Substituting (4.2.23) into (4.2.17) we have that

$$\det\Big[\left(\mathbb{I}_2\otimes(A+ik_{P1}B)\otimes\mathbb{I}_{|\mathcal{I}|}\right)\mathbb{Q}\frac{X(k_{P1},k_{P2},l)+Y(k_{P1},k_{P2},l)}{2} + \left(\mathbb{I}_2\otimes(A-ik_{P1}B)\otimes\mathbb{I}_{|\mathcal{I}|}\right)\mathbb{Q}\frac{X(k_{P1},k_{P2},l)-Y(k_{P1},k_{P2},l)}{2}\Big]=0, \tag{4.2.24}$$

with

$$X(k_1,k_2,l)=\begin{pmatrix}\mathbb{I}_2 & \mathbb{I}_2\\ S_p(k_1-k_2)e^{ik_1l} & S_p(-k_1-k_2)e^{-ik_1l}\end{pmatrix}\otimes\mathbb{I}_{|\mathcal{I}|^2} \tag{4.2.25}$$

and

$$Y(k_1,k_2,l)=\begin{pmatrix}\mathbb{I}_2 & -\mathbb{I}_2\\ -S_p(k_1-k_2)e^{ik_1l} & S_p(-k_1-k_2)e^{-ik_1l}\end{pmatrix}\otimes\mathbb{I}_{|\mathcal{I}|^2}. \tag{4.2.26}$$

Then, using the invertibility of $A\pm ikB$, and multiplying on the left by

$$\det\left[\left(\mathbb{I}_2\otimes(A+ik_{P1}B)\otimes\mathbb{I}_{|\mathcal{I}|}\right)\mathbb{Q}\right]^{-1} \tag{4.2.27}$$

and on the right by

$$\det\left[\frac{X(k_1,k_2,l)+Y(k_1,k_2,l)}{2}\right]^{-1}, \tag{4.2.28}$$

we arrive at the condition that

$$Z_e(k_{P1},k_{P2})=0, \tag{4.2.29}$$

with

$$Z_e(k_1,k_2)=\det\left[\mathbb{I}_{2|\mathcal{I}|}+e^{2ik_1l}\left(S_p(k_1-k_2)S_p(k_1+k_2)\otimes S_v^{(1)}(k_1)\right)\right], \tag{4.2.30}$$

is satisfied for all $P\in\mathscr{W}_2$. By using properties of determinants and the explicit forms of $S_p(k)$ and $S_v^{(1)}(k)$, it is easy to see that if (4.2.29) is satisfied for some $P\in\mathscr{W}_2$, then it is necessarily satisfied for elements $PR, PTRT\in\mathscr{W}_2$. With this in mind, we can state the main result of this section.

Theorem 4.2.2. Non-zero eigenvalues of a self-adjoint two-particle Laplacian $-\Delta_2$ defined on an equilateral star Γ_e with interactions at the central vertex specified through A_1, B_1 and $\tilde{\delta}$-type particle interactions, are the values $E = k_1^2 + k_2^2 \neq 0$ with multiplicity m, where (k_1, k_2), such that $0 \leq k_1 \leq k_2$, are solutions to the secular equations

$$Z_e(k_i, k_j) = 0 \tag{4.2.31}$$

for $j, i \neq j \in \{1, 2\}$, with multiplicity m.

4.2.3 Spectra from star representation

We have seen how to calculate the spectra of equilateral stars by imposing boundary conditions (4.1.13) and (4.2.7)–(4.2.8) on functions $\Psi \in H^2(D^*_{\Gamma_e})$. Before we move on, it is instructive to discuss an alternative method for calculating the spectra since we will follow a related method in the subsequent section where we extend the scope of our discussion to general graphs. The basic premise is to first consider Γ_e in its star representation $\Gamma_e^{(s)}$ as prescribed in Definition 2.4.1. This is the collection $\Gamma_e^{(s)}(\mathcal{V}, \mathcal{E}, f)$ of the single infinite star $\Gamma_1(v_1, \mathcal{E}_1, f)$ along with $|\mathcal{V}| - 1$ infinite one-edge stars $\Gamma_j(v_j, e_{|\mathcal{I}|+j-1}, f)$, for $j \in \{2, \ldots, |\mathcal{V}|\}$, as depicted in Figure 4.3. Deducing appropriate boundary conditions in this setting and constructing Laplace eigenfunctions using the Bethe ansatz, we show that we recover the spectrum prescribed in Theorem 4.2.2.

Consider the equilateral star Γ_e in its star representation $\Gamma_e^{(s)}$. The appropriate two-particle Hilbert space

$$\mathscr{H}_2^{(s)} = \left(\bigoplus_{j=1}^{|\mathcal{E}|} L^2(0, \infty) \right) \otimes \left(\bigoplus_{j=1}^{|\mathcal{E}|} L^2(0, \infty) \right) \tag{4.2.32}$$

on $\Gamma_e^{(s)}$ is the direct sum of constituent Hilbert spaces on each external edge couple $(e_m, e_n) \in \mathcal{E} \otimes \mathcal{E}$. Vectors

$$\Psi = \left(\psi_{mn}^{(s)} \right)_{m,n=1}^{|\mathcal{E}|} \tag{4.2.33}$$

in $\mathscr{H}_2^{(s)}$ are then lists of two-particle functions $\psi_{mn}^{(s)} : D_{mn}^{(s)} \to \mathbb{C}$ in $L^2(D_{mn}^{(s)})$ with

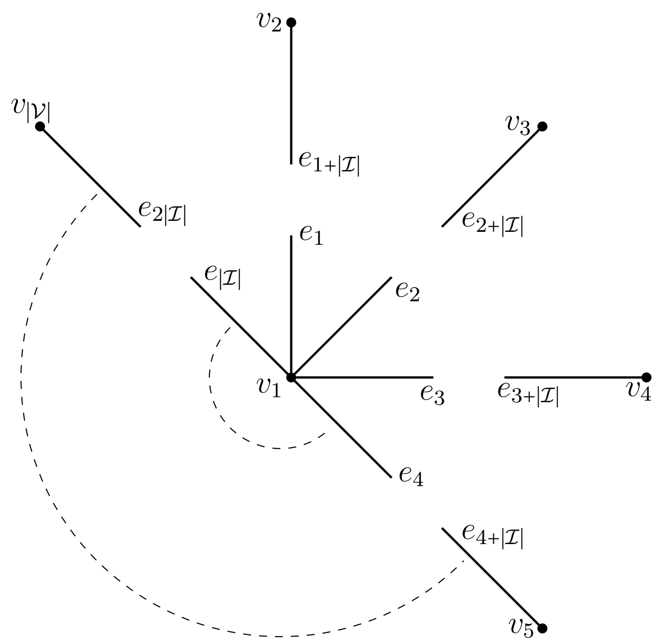

Figure 4.3: Star representation $\Gamma_e^{(s)}$ of equilateral star Γ_e with $|\mathcal{I}|$ edges.

infinite subdomains defined

$$D_{mn}^{(s)} = (0,\infty) \times (0,\infty). \tag{4.2.34}$$

The total configuration space for two particles on $\Gamma_e^{(s)}$ is the union

$$D_{\Gamma_e}^{(s)} = \dot{\bigcup_{m,n=1}^{|\mathcal{E}|}} D_{mn}^{(s)} \tag{4.2.35}$$

of these subdomains. The two-particle Hilbert space can then be written $\mathscr{H}_2^{(s)} = L^2(D_{\Gamma_e}^{(s)})$.

The task is now to specify appropriate boundary conditions which correspond to (4.1.13) and (4.2.7)–(4.2.8) imposed in the compact setting. To this end, let us make the definition

$$\begin{aligned} \mathcal{N}_e = \{&(e_m, e_n) \in \mathcal{E} \otimes \mathcal{E}; \\ &f(e_m) = f(e_n) = v_1 \text{ or } f(e_m), f(e_n) \in \{v_2, \dots, v_{|\mathcal{E}|}\}. \end{aligned} \tag{4.2.36}$$

On Γ_e, $\tilde{\delta}$-interactions require us to define dissections along the lines $x_1 = x_2$ of squares D_{mn}. On $\Gamma_e^{(s)}$, this corresponds to defining dissections of $D_{mn}^{(s)}$ with $(e_m, e_n) \in \mathcal{N}_e$, according to

$$D_{mn}^{(s,+)} = \{(x_1, x_2) \in D_{mn}^{(s)};\ x_1 > x_2\} \tag{4.2.37}$$

and

$$D_{mn}^{(s,-)} = \{(x_1, x_2) \in D_{mn}^{(s)};\ x_1 < x_2\}. \tag{4.2.38}$$

It is convenient, however, to extend these dissections to all edge pairs, as depicted in Figure 4.4, so that the total dissected configuration space is given by

$$D_{\Gamma_e}^{(s,*)} = \dot{\bigcup_{m,n=1}^{|\mathcal{E}|}} \left(D_{mn}^{(s,+)} \cup D_{mn}^{(s,-)} \right). \tag{4.2.39}$$

We note that later in the formalism we correct for this by imposing conditions of continuity across dissections of $D_{mn}^{(s)}$, with $(e_m, e_n) \notin \mathcal{N}_e$. The total dissected two-particle Hilbert space is then $\mathscr{H}_2^{(s,*)} = L^2(D_{\Gamma_e}^{(s,*)})$ with vectors

$$\Psi = \begin{pmatrix} \left(\psi_{mn}^{(s,+)}\right)_{m,n=1}^{|\mathcal{E}|} \\ \left(\psi_{mn}^{(s,-)}\right)_{m,n=1}^{|\mathcal{E}|} \end{pmatrix} \tag{4.2.40}$$

in $\mathscr{H}_2^{(s,*)}$, lists of square-integrable functions $\psi_{mn}^{(s,\pm)} : D_{mn}^{(s,\pm)} \to \mathbb{C}$. The corresponding Sobolev space $H^2(D_{\Gamma_e}^{(s,*)})$ is the set of $\Psi \in \mathscr{H}_2^{(s,*)}$ consisting of functions $\psi_{mn}^{(s,\pm)} \in H^2(D_{mn}^{(s,\pm)})$.

Interactions at the vertices will be described by simple two-particle lifts of those (2.4.4) imposed in the star representation of the corresponding one-particle quantum graph. Defining boundary vectors

$$\Psi_{\mathrm{bv}}^{(s,v)}(x) = \begin{pmatrix} \left(\psi_{mn}^{(s,-)}(0,x)\right)_{m,n=1}^{|\mathcal{E}|} \\ \left(\psi_{mn}^{(s,+)}(x,0)\right)_{n,m=1}^{|\mathcal{E}|} \end{pmatrix} \quad \text{and} \quad \Psi_{\mathrm{bv}}^{(s,v)\prime}(x) = \begin{pmatrix} \left(\psi_{mn,1}^{(s,-)}(0,x)\right)_{m,n=1}^{|\mathcal{E}|} \\ \left(\psi_{mn,2}^{(s,+)}(x,0)\right)_{n,m=1}^{|\mathcal{E}|} \end{pmatrix} \tag{4.2.41}$$

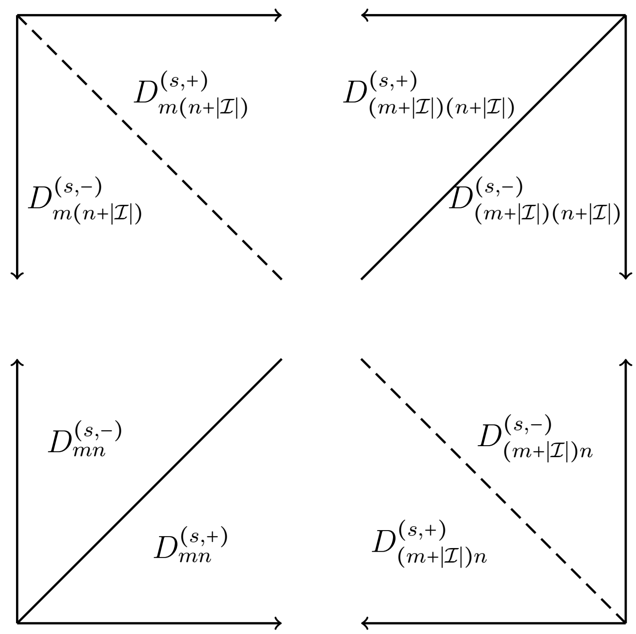

Figure 4.4: Four dissected infinite subdomains associated with internal edge couple (i_m, i_n). $\tilde{\delta}$-interactions imposed along solid diagonals and continuity imposed across dashed diagonals.

for all $x \in (0, \infty)$, boundary conditions at the vertices are given by

$$\left(\mathbb{I}_2 \otimes A \otimes \mathbb{I}_{|\mathcal{E}|}\right) \Psi^{(s,v)}_{\text{bv}} + \left(\mathbb{I}_2 \otimes B \otimes \mathbb{I}_{|\mathcal{E}|}\right) \Psi^{(s,v)\prime}_{\text{bv}} = 0. \tag{4.2.42}$$

The $\tilde{\delta}$-type interactions are implemented through the conditions

$$\psi^{(s,+)}_{mn}(x_1, x_2)|_{x_1 = x_2^+} = \psi^{(s,-)}_{nm}(x_1, x_2)|_{x_1 = x_2^-}; \tag{4.2.43}$$

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2\alpha\right) \psi^{(s,+)}_{mn}(x_1, x_2)|_{x_1 = x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) \psi^{(s,-)}_{nm}(x_1, x_2)|_{x_1 = x_2^-} \tag{4.2.44}$$

for edge couples $(e_m, e_n) \in \mathcal{N}_e$. Finally, we reestablish continuity across the dissections relating to edge couples $(e_m, e_n) \notin \mathcal{N}_e$ by imposing the conditions

$$\psi^{(s,+)}_{mn}(x_1, x_2)_{x_1 = x_2} = \psi^{(s,-)}_{mn}(x_1, x_2)_{x_1 = x_2^-}; \tag{4.2.45}$$

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) \psi^{(s,+)}_{mn}(x_1, x_2)|_{x_1 = x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) \psi^{(s,-)}_{mn}(x_1, x_2)|_{x_1 = x_2^-}. \tag{4.2.46}$$

The task is now to specify eigenvectors $\Psi \in H^2(D^{(s,*)}_{\Gamma_e})$ of $-\Delta_2$ which satisfy boundary conditions (4.2.42)–(4.2.46). Using the Bethe ansatz method, the vector Ψ will be described by the collection of functions

$$\psi_{mn}^{(s,\pm)}(x_1,x_2) = \sum_{P\in\mathscr{W}_2} \mathcal{A}_{mn}^{(P,\pm)} e^{i(k_{P1}x_1+k_{P2}x_2)} \tag{4.2.47}$$

on $D_{mn}^{(s,\pm)}$. Let us define the $2|\mathcal{E}|^2$-dimensional vector

$$\mathcal{A}^P = \begin{pmatrix} \left(\mathcal{A}_{mn}^{(P,-)}\right)_{m,n=1}^{|\mathcal{E}|} \\ \mathbb{T}_{|\mathcal{E}|^2}\left(\mathcal{A}_{mn}^{(PT,+)}\right)_{m,n=1}^{|\mathcal{E}|} \end{pmatrix}. \tag{4.2.48}$$

The vertex condition (4.2.42) then implies

$$\mathcal{A}^{PR} = \left(\mathbb{I}_2 \otimes S_v(-k_{P1}) \otimes \mathbb{I}_{|\mathcal{E}|}\right)\mathcal{A}^P \tag{4.2.49}$$

for all $P\in\mathscr{W}_2$. At this point, it is convenient to define the matrix $\boldsymbol{c}_e = \operatorname{diag}(c_{mn})_{m,n=1}^{|\mathcal{E}|}$ where

$$c_{mn} = \begin{cases} 1 & \text{if } (e_m,e_n)\in\mathcal{N}_e; \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.50}$$

which distinguishes domains with $\tilde{\delta}$-type interactions from those which are continuous across dissections. Conditions (4.2.43)–(4.2.46) then imply

$$\mathcal{A}^{PT} = Y_e(k_{P1}-k_{P2})\mathcal{A}^P, \tag{4.2.51}$$

with

$$Y_e(k) = S_p(k)\otimes\boldsymbol{c}_e + \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\otimes(\mathbb{I}_{|\mathcal{E}|^2}-\boldsymbol{c}_e)\mathbb{T}_{|\mathcal{E}|^2}. \tag{4.2.52}$$

To prove exact solvability we need only show that relations (4.2.49) and (4.2.51) are consistent with the properties of $\mathscr{W}_2$. This amounts to the requirements

1. $S_v(u)S_v(-u) = \mathbb{I}_{|\mathcal{E}|}$;

2. $Y_e(u)Y_e(-u) = \mathbb{I}_{2|\mathcal{E}|}$;

3. $\left(\mathbb{I}_2\otimes S_v(u)\otimes\mathbb{I}_{|\mathcal{E}|}\right)Y_e(u+v)\left(\mathbb{I}_2\otimes S_v(v)\otimes\mathbb{I}_{|\mathcal{E}|}\right)Y_e(v-u)$
$= Y_e(v-u)\left(\mathbb{I}_2\otimes S_v(v)\otimes\mathbb{I}_{|\mathcal{E}|}\right)Y_e(u+v)\left(\mathbb{I}_2\otimes S_v(u)\otimes\mathbb{I}_{|\mathcal{E}|}\right)$.

The first two conditions are easily verified by the explicit forms of $S_v(u)$ and $Y_e(u)$, noting that, since $c_{mn} = c_{nm}$, the properties of $\mathbb{T}_{|\mathcal{E}|^2}$ are such that

$$[\boldsymbol{c}_e, \mathbb{T}_{|\mathcal{E}|^2}] = 0. \tag{4.2.53}$$

Prescribing the connectivity of the star graph by choosing the block form (4.2.19) and Dirichlet conditions (4.2.20) at the outer vertices, the relation

$$[S_v(u) \otimes \mathbb{I}_{|\mathcal{E}|}, \boldsymbol{c}_e] = 0 \tag{4.2.54}$$

holds, and the third condition is easily verified.

Let us bring our attention back to the original equilateral star Γ_e. In order to turn the eigenfunctions in the star representation into eigenfunctions on the compact graph, it is sufficient to impose the relations

$$\psi^+_{mn}(x_1, x_2) = \psi^+_{(m+|\mathcal{I}|)n}(l - x_1, x_2) \text{ and} \tag{4.2.55}$$

$$\psi^-_{mn}(x_1, x_2) = \psi^-_{m(n+|\mathcal{I}|)}(x_1, l - x_2) \tag{4.2.56}$$

for all $m, n \in \{1, .., |\mathcal{I}|\}$ which imply

$$\mathcal{A}^{(P,+)}_{mn} = \mathcal{A}^{(PR,+)}_{(m+|\mathcal{I}|)n} e^{-ik_{P1}l} \text{ and} \tag{4.2.57}$$

$$\mathcal{A}^{(P,-)}_{mn} = \mathcal{A}^{(PTRT,-)}_{m(n+|\mathcal{I}|)} e^{-ik_{P2}l}. \tag{4.2.58}$$

These conditions then yield the relation

$$\mathcal{A}^P = E(-k_{P2})\mathcal{A}^{PTRT}, \tag{4.2.59}$$

with

$$E(k) = \mathbb{I}_{4|\mathcal{I}|} \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes \mathbb{I}_{|\mathcal{I}|} e^{ikl}. \tag{4.2.60}$$

Applying (4.2.49), (4.2.51) and (4.2.59) successively we arrive at the secular equation

$$\det\left[\mathbb{I}_{8|\mathcal{I}|^2} - E(k_2)Y_e(k_2 - k_1)\left(\mathbb{I}_2 \otimes S_v(k_2) \otimes \mathbb{I}_{2|\mathcal{I}|}\right) Y_e(k_1 + k_2)\right]. \tag{4.2.61}$$

Finally, defining the permutation matrix

$$\mathbb{V} = \left(\mathbb{I}_2 \otimes \mathbb{T}_4 \otimes \mathbb{I}_{|\mathcal{I}|^2}\right) \begin{pmatrix} \mathbb{I}_{4|\mathcal{I}|} \otimes \left(\mathbb{I}_{|\mathcal{I}|}, \mathbf{0}_{|\mathcal{I}|}\right) \\ \mathbb{I}_{4|\mathcal{I}|} \otimes \left(\mathbf{0}_{|\mathcal{I}|}, \mathbb{I}_{|\mathcal{I}|}\right) \end{pmatrix}, \tag{4.2.62}$$

where $\mathbf{0}_d$ is the $d \times d$ matrix of zeros, the spectrum prescribed in Theorem 4.2.2 is recovered by inserting the identity $\mathbb{I} = \mathbb{V}^{-1}\mathbb{V}$ between each element of (4.2.61), multiplying on the left and right by $\det[\mathbb{V}]$ and $\det[\mathbb{V}^{-1}]$ respectively, and using the properties of determinants.

4.2.4 Recovering specific results

Before we move on to general graphs, let us establish agreement between the spectra of equilateral stars presented in Theorem 4.2.2 and results derived in previous chapters.

Two-edge stars

Let us first explain how to recover the results of [CC07] presented in Section 3.3.3. Simply by substituting $|\mathcal{I}| = 2$ into (4.2.31), we immediately recover the appropriate quantisation conditions (3.3.44) for systems of two particles in a box with a central impurity. Furthermore, rather than choosing Dirichlet vertex conditions (4.2.20), which specify the connectivity of an equilateral star, and instead choosing standard boundary conditions

$$A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \tag{4.2.63}$$

to establish continuity at the outer vertices, we recover the quantisation conditions (3.3.36) which specify the spectra of systems of two particles on a circle with an impurity.

Non-interacting particles

Throughout this thesis we have used α to parameterise the strength of particle interactions. It is reasonable to expect then, that by setting $\alpha = 0$, one should arrive at separable quantisation conditions given by (2.3.9) for one-particle quantum graphs. Indeed, by substituting $\alpha = 0$ into the form (4.2.30), we recover

$$\det\left[\mathbb{I}_{|\mathcal{I}|} + e^{2ikl} S_v^{(1)}(k)\right] = 0 \tag{4.2.64}$$

which is exactly the corresponding one-particle condition. It is important to point out however, that $\tilde{\delta}$-type interactions with $\alpha = 0$ result in coupling between domains D_{mn} and D_{nm} and are thus clearly distinct from the truly non-interacting situation. For this reason we refer to such systems as *pseudo-non-interacting*. The fact the one-particle condition (4.2.64) is recovered in this case is a result of the specific geometry of the equilateral star. We will see in the subsequent section that this agreement does not hold for general graphs. We revisit this point in the final section of the chapter when discussing spectral statistics.

Bosons on equilateral stars

Later in this chapter, we would like to analyse examples of bosons on graphs. Computationally speaking, such examples are useful as the dimension of the matrix inside the determinant $Z_e(k_1, k_2)$ is halved. Imposing bosonic symmetry

$$\psi^-_{mn}(x_1, x_2) = \psi^+_{nm}(x_2, x_1) \tag{4.2.65}$$

we have that

$$\mathcal{A}^{(P,-)}_{mn} = \mathcal{A}^{(PT,+)}_{nm} \tag{4.2.66}$$

for all $P \in \mathscr{W}_2$. The matrix $S_p(k)$ then reduces to the scalar form $s_p(k)\mathbb{I}_2$ as defined in (3.2.17) so that from (4.2.30) we recover

$$Z_{e,b}(k_1, k_2) = \det\left[\mathbb{I}_{|\mathcal{I}|} + e^{2ik_1 l} s_p(k_1 - k_2) s_p(k_1 + k_2) S_v^{(1)}(k_1)\right]. \tag{4.2.67}$$

4.3 General graphs with $\tilde{\delta}$-interactions

We have seen, in the previous section, how to construct systems of two $\tilde{\delta}$-interacting particles on equilateral stars, defining appropriate boundary conditions in the context of a self-adjoint Laplacian, proving exact solvability and calculating spectra. The majority of quantum graphs literature, however, is concerned with the dynamics of single particles on graphs with, in general, different edge lengths. Indeed, in [KS97], rationally independent edge lengths are required to avoid degenerate energy levels and ensure spectral statistics following random matrix predictions. This section is concerned with extending the scope of our discussion to two particles on general compact graphs.

As imposing $\tilde{\delta}$-type interactions between particles on equilateral stars leads to exact solutions, it seems reasonable to assume that a suitable variant of such interactions in the general setting will also lead to exact solutions. However, general graphs bring added complications associated with edges of different length and distant vertices. The problem is then to choose an appropriate way to impose $\tilde{\delta}$-type interactions in the general setting which preserves compatibility with the Bethe ansatz method. To address this, let us consider a pair of particles on Γ viewed in its star representation $\Gamma^{(s)}$. At any one time, the particles will be located on some pair of infinite stars $(\Gamma_\gamma, \Gamma_\lambda)$ with $\gamma, \lambda \in \{1, \dots, |\mathcal{V}|\}$. We impose that, when particles are located on different stars ($\gamma \neq \lambda$), they will be independent of each other; there are no particle interactions. When, however, the particles are located on the same star ($\gamma = \lambda$), they will be allowed to interact. We postulate here that exact solvability is assured if these interactions are of $\tilde{\delta}$-type. In this setting, by following the method in Section 4.2.3, we show these systems are exactly solvable and calculate their spectra. Before we do this, however, we would like to establish that corresponding boundary conditions provide self-adjoint realisations of $-\Delta_2$ on the compact graph. To see this, let us first consider a pair of particles, with coordinates $x_1 \in [0, l_m]$ and $x_2 \in [0, l_n]$ respectively, on a neighbouring edge couple $(i_m, i_n) \in \mathcal{N}$ with common vertex v_η. Additionally, let us assume that the edges are orientated such that $f_0(i_m) = f_0(i_n) = v_\eta$. The $\tilde{\delta}$-interactions prescribed above become effective when $x_1 = x_2$, with the interaction cut off at the smaller of the two edge lengths involved. Thus the appropriate dissection of D_{mn} is given by

$$D^*_{mn} = D^+_{mn} \cup D^-_{mn}, \tag{4.3.1}$$

with subdomains defined as

$$D^+_{mn} = \{(x_1, x_2) \in D_{mn};\ x_1 > x_2\} \tag{4.3.2}$$

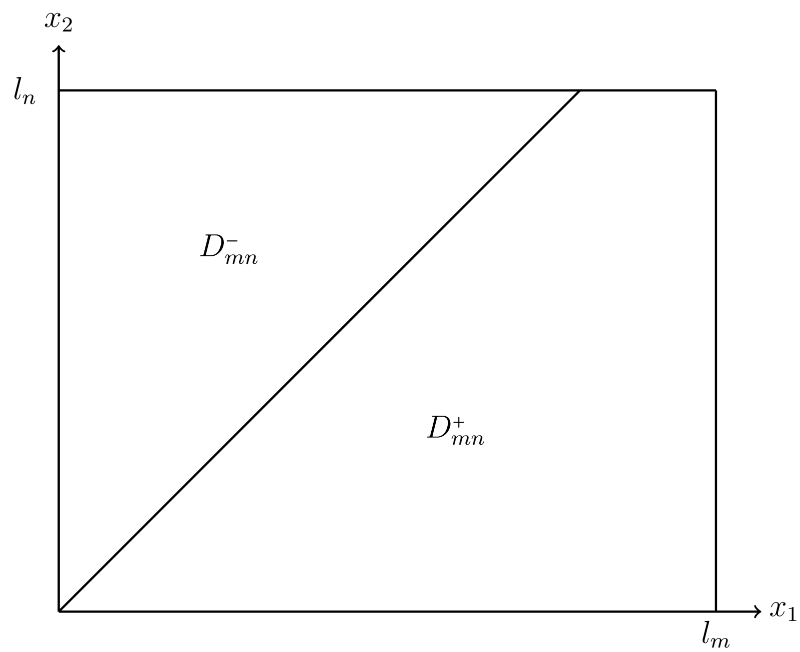

Figure 4.5: Dissected finite configuration space D^*_{mn} associated with internal edge couple (i_m, i_n).

and

$$D^-_{mn} = \{(x_1, x_2) \in D_{mn};\ x_1 < x_2\} \tag{4.3.3}$$

(see Figure 4.5). Accordingly, appropriate $\tilde{\delta}$-interactions will be imposed along these dissections.

Let us now extend this discussion from the pair of internal edges (i_m, i_n) to general graphs Γ. Before we continue, it is convenient to impose the further requirement that any two edges have at most one common vertex. This caveat facilitates the formalism that follows, namely by enforcing that any rectangle D_{mn} has at most one dissection. However, we argue that, dropping these restrictions, the method is easily generalised by imposing additional appropriate dissections.

For the example discussed above, where we assumed $f_0(i_m) = f_0(i_n)$, appropriate dissections were made along lines $x_1 = x_2$ of rectangles D_{mn}. In the general setting, we must, however, pay attention to additional complexities which arise from the possible orientations of neighbouring edge couples. Appropriate lines of dissection should be chosen with this in mind. Specifically, for possible orientations of $(i_m, i_n) \in \mathcal{N}$, with

$$f_0(i_m) = f_0(i_n),\ f_0(i_m) = f_l(i_n),\ f_l(i_m) = f_0(i_n) \text{ and } f_l(i_m) = f_l(i_n), \tag{4.3.4}$$

we require dissection along the lines

$$x_1 = x_2,\ x_1 = l_n - x_2,\ l_m - x_1 = x_2 \text{ and } l_m - x_1 = l_n - x_2, \tag{4.3.5}$$

respectively. These four types of dissection are shown in Figure 4.6. Of course for distant edge couples $(i_m, i_n) \in \mathcal{D}$, no dissection is required. By extending the arguments made in the example above, the appropriate dissected configuration space is given by

$$D_\Gamma^* = \left(\dot{\bigcup_{(i_m,i_n)\in\mathcal{D}}} D_{mn} \right) \dot{\bigcup} \left(\dot{\bigcup_{(i_m,i_n)\in\mathcal{N}}} (D_{mn}^+ \,\dot{\cup}\, D_{mn}^-) \right), \tag{4.3.6}$$

with subdomains of rectangles D_{mn}, for $(i_m, i_n) \in \mathcal{N}$, defined as

$$D_{mn}^+ = \begin{cases} \{(x_1,x_2) \in D_{mn};\ x_1 > x_2\} & \text{if } f_0(i_m) = f_0(i_n); \\ \{(x_1,x_2) \in D_{mn};\ x_1 > l_n - x_2\} & \text{if } f_0(i_m) = f_l(i_n); \\ \{(x_1,x_2) \in D_{mn};\ l_m - x_1 > x_2\} & \text{if } f_l(i_m) = f_0(i_n); \\ \{(x_1,x_2) \in D_{mn};\ l_m - x_1 > l_n - x_2\} & \text{if } f_l(i_m) = f_l(i_n), \end{cases} \tag{4.3.7}$$

and

$$D_{mn}^- = \begin{cases} \{(x_1,x_2) \in D_{mn};\ x_1 < x_2\} & \text{if } f_0(i_m) = f_0(i_n); \\ \{(x_1,x_2) \in D_{mn};\ x_1 < l_n - x_2\} & \text{if } f_0(i_m) = f_l(i_n); \\ \{(x_1,x_2) \in D_{mn};\ l_m - x_1 < x_2\} & \text{if } f_l(i_m) = f_0(i_n); \\ \{(x_1,x_2) \in D_{mn};\ l_m - x_1 < l_n - x_2\} & \text{if } f_l(i_m) = f_l(i_n). \end{cases} \tag{4.3.8}$$

The total dissected Hilbert space is then $\mathscr{H}_2^* = L^2(D_\Gamma^*)$. Thus vectors $\Psi \in \mathscr{H}_2^*$ are

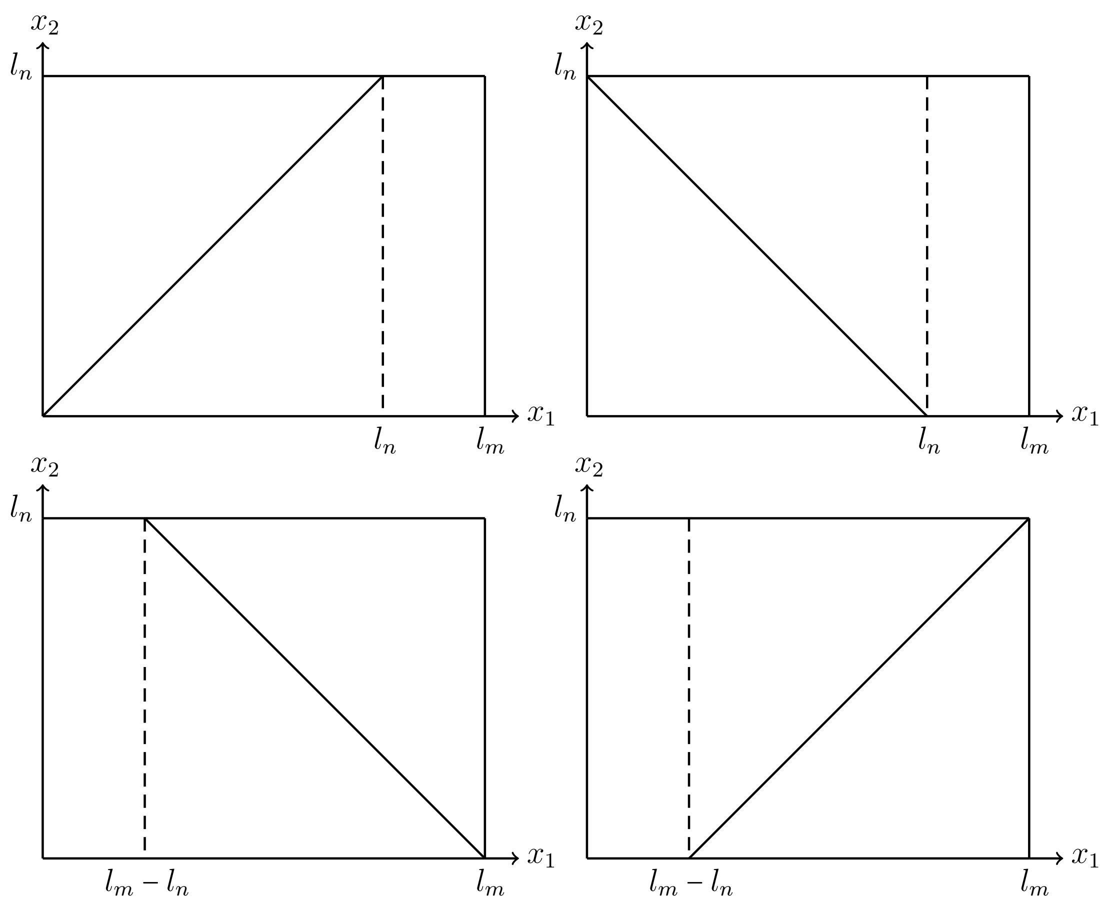

Figure 4.6: Dissected domains D^*_{mn} such that $l_m > l_n$ with four possible orientations. Top left: $f_0(i_m) = f_0(i_n)$, top right: $f_0(i_m) = f_l(i_n)$, bottom left: $f_l(i_m) = f_0(i_n)$, bottom right: $f_l(i_m) = f_l(i_n)$.

lists

$$\Psi = \begin{pmatrix} (\psi_{mn})_{(i_m,i_n)\in\mathcal{D}} \\ (\psi^+_{mn})_{(i_m,i_n)\in\mathcal{N}} \\ (\psi^-_{mn})_{(i_m,i_n)\in\mathcal{N}} \end{pmatrix} \tag{4.3.9}$$

of square-integrable functions $\psi_{mn} : D_{mn} \to \mathbb{C}$, for $(i_m, i_n) \in \mathcal{D}$, and $\psi^{\pm}_{mn} : D^{\pm}_{mn} \to \mathbb{C}$, for $(i_m, i_n) \in \mathcal{N}$. The corresponding Sobolev space $H^2(D^*_\Gamma)$ is the set of $\Psi \in \mathscr{H}^*_2$ consisting of functions $\psi_{mn} \in H^2(D_{mn})$, for $(i_m, i_n) \in \mathcal{D}$, and $\psi^{\pm}_{mn} \in H^2(D^{\pm}_{mn})$, for $(i_m, i_n) \in \mathcal{N}$.

Generalising the argument made earlier, we would like to impose $\tilde{\tilde{\delta}}$-type conditions along all dissections of D^*_Γ. In principle we can define such interactions on the dissected configuration space (4.3.6). However this presentation becomes rather convoluted. To this end, it is convenient to define an equivalent configuration space which simplifies this process. Let

$$(\tilde{x}_1, \tilde{x}_2) = \begin{cases} (x_1, x_2) & \text{if } f_0(i_m) = f_0(i_n); \\ (x_1, l_n - x_2) & \text{if } f_0(i_m) = f_l(i_n); \\ (l_m - x_1, x_2) & \text{if } f_l(i_m) = f_0(i_n); \\ (l_m - x_1, l_n - x_2) & \text{if } f_l(i_m) = f_l(i_n) \end{cases} \tag{4.3.10}$$

so that correct dissections are along the diagonals $\tilde{x}_1 = \tilde{x}_2$. The total dissected configuration space can then be written

$$D^*_\Gamma = \left(\dot{\bigcup_{(i_m,i_n)\in\mathcal{D}}} D_{mn} \right) \dot{\bigcup} \left(\dot{\bigcup_{(i_m,i_n)\in\mathcal{N}}} (\tilde{D}^+_{mn} \cup \tilde{D}^-_{mn}) \right), \tag{4.3.11}$$

with subdomains defined as

$$\tilde{D}^+_{mn} = \{(\tilde{x}_1, \tilde{x}_2) \in D_{mn};\ \tilde{x}_1 > \tilde{x}_2\} \tag{4.3.12}$$

and

$$\tilde{D}^-_{mn} = \{(\tilde{x}_1, \tilde{x}_2) \in D_{mn};\ \tilde{x}_1 < \tilde{x}_2\}. \tag{4.3.13}$$

We are now in a position to be explicit about the types of interactions we would like to impose. They will appear as conditions on functions $\Psi \in H^2(D^*_\Gamma)$ along the boundaries of D^*_Γ.

As in the previous two sections, single-particle interactions at the vertices are given by conditions (4.1.13).

Defining functions $\phi^\pm_{mn} : \tilde{D}^\pm_{mn} \to \mathbb{C}$ in $H^2(D^\pm_{mn})$, $\tilde{\delta}$-type boundary conditions are prescribed by the conditions

$$\phi^+_{mn}(\tilde{x}_1, \tilde{x}_2)|_{\tilde{x}_1 = \tilde{x}_2^+} = \phi^-_{nm}(\tilde{x}_1, \tilde{x}_2)|_{\tilde{x}_1 = \tilde{x}_2^-}; \tag{4.3.14}$$

$$\left(\frac{\partial}{\partial \tilde{x}_1} - \frac{\partial}{\partial \tilde{x}_2} - 2\alpha\right)\phi^+_{mn}(\tilde{x}_1, \tilde{x}_2)|_{\tilde{x}_1 = \tilde{x}_2^+} = \left(\frac{\partial}{\partial \tilde{x}_1} - \frac{\partial}{\partial \tilde{x}_2}\right)\phi^-_{nm}(\tilde{x}_1, \tilde{x}_2)|_{\tilde{x}_1 = \tilde{x}_2^-} \tag{4.3.15}$$

for neighbouring edge couples $(i_m, i_n) \in \mathcal{N}$, where

$$l^-_{mn} = \min(l_m, l_n). \tag{4.3.16}$$

The task is now to show that these conditions correspond to a valid self-adjoint two-particle Laplacian $-\Delta_2$.

4.3.1 Self-adjoint realisation

Following the formalism in Section 4.2.1, we deduce self-adjoint realisations of $-\Delta_2$ with domains characterised by conditions on boundary values of functions $\Psi \in H^2(D^*_\Gamma)$ and their derivatives. We then show that, from these conditions, we can recover (4.1.13) and (4.3.14)–(4.3.15).

Let us define the boundary vectors

$$\Psi_{\text{bv}}(y) = (\psi_{mn,\text{bv}}(y))^{|\mathcal{I}|}_{m,n=1} \text{ and } \Psi'_{\text{bv}}(y) = \left(\psi'_{mn,\text{bv}}(y)\right)^{|\mathcal{I}|}_{m,n=1} \tag{4.3.17}$$

for all $y \in (0,1)$, where $\psi_{mn,\text{bv}}$ and $\psi'_{mn,\text{bv}}$ list values at the boundaries of D^*_{mn}, for $(i_m, i_n) \in \mathcal{N}$, and D_{mn}, for $(i_m, i_n) \in \mathcal{D}$. Specifically, for $(i_m, i_n) \in \mathcal{D}$, there are no interactions between particles and we set

$$\psi_{mn,\mathrm{bv}}(y) = \begin{pmatrix} \psi_{mn}(0, l_n y) \\ \psi_{mn}(l_m, l_n y) \\ \psi_{mn}(l_m y, 0) \\ \psi_{mn}(l_m y, l_n) \end{pmatrix} \text{ and } \psi'_{mn,\mathrm{bv}}(y) = \begin{pmatrix} \psi_{mn,1}(0, l_n y) \\ \psi_{mn,1}(l_m, l_n y) \\ \psi_{mn,2}(l_m y, 0) \\ \psi_{mn,2}(l_m y, l_n) \end{pmatrix}. \tag{4.3.18}$$

For $(i_m, i_n) \in \mathcal{N}$, we must include boundary values along the diagonals $\tilde{x}_1 = \tilde{x}_2$ to accommodate $\tilde{\delta}$-interactions. We thus set

$$\psi_{mn,\mathrm{bv}}(y) = \begin{pmatrix} \psi_{mn}(0, l_n y) \\ \psi_{mn}(l_m, l_n y) \\ \psi_{mn}(l_m y, 0) \\ \psi_{mn}(l_m y, l_n) \\ \phi^+_{mn}(l^-_{mn} y, l^-_{mn} y) \\ \phi^-_{mn}(l^-_{mn} y, l^-_{mn} y) \end{pmatrix} \text{ and } \psi'_{mn,\mathrm{bv}}(y) = \begin{pmatrix} \psi_{mn,1}(0, l_n y) \\ \psi_{mn,1}(l_m, l_n y) \\ \psi_{mn,2}(l_m y, 0) \\ \psi_{mn,2}(l_m y, l_n) \\ \phi^+_{mn,d}(l^-_{mn} y, l^-_{mn} y) \\ \phi^-_{mn,d}(l^-_{mn} y, l^-_{mn} y) \end{pmatrix}, \tag{4.3.19}$$

where functions $\phi^{\pm}_{mn,d}$ are inward derivatives normal to the lines $\tilde{x}_1 = \tilde{x}_2$. We note that we have dropped the labels ± denoting the appropriate subdomains of D_{mn} in the first four components of (4.3.19) in order to keep the presentation compact. Clearly we have that $\Psi_{\mathrm{bv}}(y), \Psi'_{\mathrm{bv}}(y) \in \mathbb{C}^{n(\mathcal{I},\mathcal{N})}$ with $n(\mathcal{I},\mathcal{N}) = 4|\mathcal{I}|^2 + 2|\mathcal{N}|$.

Carrying over the approach in [BK13c], used for equilateral stars in Section 4.2.1, we have the following theorem.

Theorem 4.3.1. Let bounded and measurable maps $P, L : [0,1] \to M(n(\mathcal{I},\mathcal{N}), \mathbb{C})$ be such that

1. $P(y) = \mathbb{I}_{n(\mathcal{I},\mathcal{N})} - Q(y)$ is an orthogonal projector of class C^1;
2. $L(y)$ a self-adjoint endomorphism on $\ker P(y)$,

for almost every $y \in [0,1]$. Additionally let bounded and self-adjoint operators Π and Λ on $L^2(0,1) \otimes \mathbb{C}^{n(\mathcal{I},\mathcal{N})}$ act according to $\Pi\chi(y) := P(y)\chi(y)$ and $\Lambda\chi(y) := L(y)\chi(y)$ on $\chi \in L^2(0,1) \otimes \mathbb{C}^{n(\mathcal{I},\mathcal{N})}$. Finally let us define the domain $D_2(P, L)$ the set of $\Psi \in H^2(D^*_\Gamma)$ such that

$$P(y)\Psi_{\mathrm{bv}}(y) = 0 \text{ and } Q(y)\Psi'_{\mathrm{bv}}(y) + L(y)Q(y)\Psi_{\mathrm{bv}}(y) = 0. \tag{4.3.20}$$

The two-particle Laplacian $-\Delta_2$ with domain $D_2(P,L)$ is self-adjoint.

Now we have established the domain $D_2(P,L)$ of a self-adjoint Laplacian $-\Delta_2$ on Γ, we would like to recover boundary conditions (4.1.13) and (4.3.14)–(4.3.15) by choosing P and L appropriately. To distinguish boundary values relating to vertex interactions from those relating to particle interactions, we assume the decomposition

$$\mathbb{C}^{n(\mathcal{I},\mathcal{N})} = W_v \oplus W_p, \tag{4.3.21}$$

where W_v and W_p have dimension $4|\mathcal{I}|^2$ and $2|\mathcal{N}|$ respectively. Here W_v is composed of all components in vectors (4.3.18) as well as the top four components in vectors (4.3.19) with W_p composed of the remaining components. Choosing block diagonal forms as in (4.1.31) with respect to this decomposition, we impose that vertex and particle interactions are independent of each other.

For vertex interactions, we again recover boundary conditions (4.1.13) by defining P_v and L_v as in Section 4.1.1. For $\tilde{\delta}$-type particle interactions we impose the further decomposition

$$W_p = \bigoplus_{(i_m,i_n)\in\mathcal{N}} W_{p,mn}, \tag{4.3.22}$$

where in the case of Ψ_{bv}, each $W_{p,mn}$ is composed of the fifth component of $\psi_{mn,\mathrm{bv}}$ and the sixth component of $\psi_{nm,\mathrm{bv}}$ in (4.3.19). Fixing the block diagonal forms

$$P_p = \bigoplus_{(i_m,i_n)\in\mathcal{N}} P_{p,mn} \text{ and } L_p = \bigoplus_{(i_m,i_n)\in\mathcal{N}} L_{p,mn} \tag{4.3.23}$$

with respect to the decomposition (4.3.22) and setting

$$P_{p,mn}(y) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } L_{p,mn}(x) = -\alpha \mathbb{I}_2, \tag{4.3.24}$$

we arrive at the $\tilde{\delta}$-type conditions (4.3.14)–(4.3.15).

4.3.2 Spectra

We have seen how to establish boundary conditions which correspond to two-particle quantum graphs with $\tilde{\delta}$-type interactions by means of self-adjoint extension. We would now like to show that such systems are exactly solvable and calculate their spectra.

For equilateral stars Γ_e considered in Section 4.2, exact solvability was shown by substituting the ansatz (4.2.15) directly into boundary conditions (4.1.13) and (4.2.7)–(4.2.8) defined on $D^*_{\Gamma_e}$. The spectra then followed by generalising the approach in [KS06b] to two particles. While, in principle, we can use the same method in the general graph case, the extra complexity brought about by different edge lengths and distant edge couples makes the presentation rather convoluted. To this end we will use the method presented in Section 4.2.3 which utilises the star representation $\Gamma^{(s)}$ of the compact graph Γ.

The appropriate two-particle Hilbert space

$$\mathscr{H}_2^{(s)} = \left(\bigoplus_{j=1}^{|\mathcal{E}|} L^2(0,\infty) \right) \otimes \left(\bigoplus_{j=1}^{|\mathcal{E}|} L^2(0,\infty) \right) \tag{4.3.25}$$

on $\Gamma^{(s)}$ is the direct sum of constituent Hilbert spaces on each external edge couple $(e_m, e_n) \in \mathcal{E} \otimes \mathcal{E}$. Vectors

$$\Psi = \left(\psi_{mn}^{(s)} \right)_{m,n=1}^{|\mathcal{E}|} \tag{4.3.26}$$

in $\mathscr{H}_2^{(s)}$ are then lists of two-particle functions $\psi_{mn}^{(s)} : D_{mn}^{(s)} \to \mathbb{C}$ in $L^2(D_{mn}^{(s)})$ with infinite subdomains defined

$$D_{mn}^{(s)} = (0,\infty) \times (0,\infty). \tag{4.3.27}$$

The total configuration space for two particles on $\Gamma^{(s)}$ is the union

$$D_\Gamma^{(s)} = \dot{\bigcup_{m,n=1}^{|\mathcal{E}|}} D_{mn}^{(s)} \tag{4.3.28}$$

of these subdomains. The two-particle Hilbert space can then be written $\mathscr{H}_2^{(s)} =$

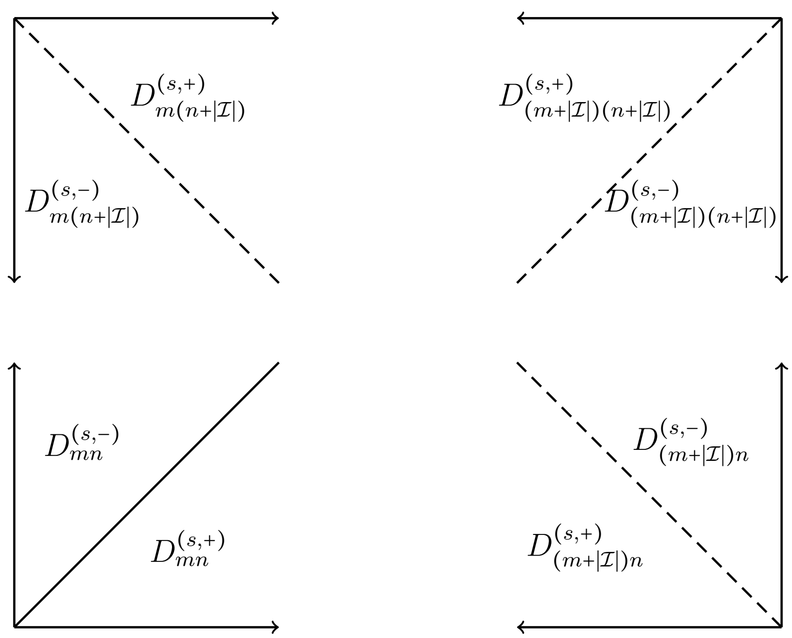

Figure 4.7: Four dissected infinite subdomains associated with internal edge couple (i_m, i_n) with $f_0(i_m) = f_0(i_n)$. $\tilde{\delta}$-interactions imposed along solid diagonal and continuity imposed across dashed diagonals.

$L^2(D^{(s)}_\Gamma)$.

In the compact setting, $\tilde{\delta}$-interactions require us to define dissections along the lines $\tilde{x}_1 = \tilde{x}_2$ of the domains D_{mn} which relate to neighbouring edge pairs $(i_m, i_n) \in \mathcal{N}$. On $\Gamma^{(s)}$, this corresponds to defining dissections of $D^{(s)}_{mn}$, with $f(e_m) = f(e_n)$, according to

$$D^{(s,+)}_{mn} = \{(x_1, x_2) \in D^{(s)}_{mn};\ x_1 > x_2\} \tag{4.3.29}$$

and

$$D^{(s,-)}_{mn} = \{(x_1, x_2) \in D^{(s)}_{mn};\ x_1 < x_2\}. \tag{4.3.30}$$

As in Section 4.2.3, it is convenient to extend these dissections to all edge pairs, as depicted in Figure 4.7, so that the total configuration space is given by

$$D^{(s,*)}_\Gamma = \dot{\bigcup_{m,n=1}^{|\mathcal{E}|}} \left(D^{(s,+)}_{mn} \cup D^{(s,-)}_{mn}\right). \tag{4.3.31}$$

The total dissected two-particle Hilbert space is then $\mathscr{H}_2^{(s,*)} = L^2(D_\Gamma^{(s,*)})$ with vectors

$$\Psi = \begin{pmatrix} \left(\psi_{mn}^{(s,+)}\right)_{m,n=1}^{|\mathcal{E}|} \\ \left(\psi_{mn}^{(s,-)}\right)_{m,n=1}^{|\mathcal{E}|} \end{pmatrix} \tag{4.3.32}$$

in $\mathscr{H}_2^{(s,*)}$, lists of square-integrable functions $\psi_{mn}^{(s,\pm)}(x_1,x_2) : D_{mn}^{(s,\pm)} \to \mathbb{C}$. The corresponding Sobolev space $H^2(D_\Gamma^{(s,*)})$ is the set of $\Psi \in \mathscr{H}_2^{(s,*)}$ consisting of functions $\psi_{mn}^{(s,\pm)} \in H^2(D_{mn}^{(s,\pm)})$.

Interactions at the vertices in this setting will again be described by simple two-particle lifts of the corresponding one-particle conditions. Defining boundary vectors

$$\Psi_{\text{bv}}^{(s,v)}(x) = \begin{pmatrix} \left(\psi_{mn}^{(s,-)}(0,x)\right)_{m,n=1}^{|\mathcal{E}|} \\ \left(\psi_{mn}^{(s,+)}(x,0)\right)_{n,m=1}^{|\mathcal{E}|} \end{pmatrix} \text{ and } {\Psi_{\text{bv}}^{(s,v)}}'(x) = \begin{pmatrix} \left(\psi_{mn,1}^{(s,-)}(0,x)\right)_{m,n=1}^{|\mathcal{E}|} \\ \left(\psi_{mn,2}^{(s,+)}(x,0)\right)_{n,m=1}^{|\mathcal{E}|} \end{pmatrix}, \tag{4.3.33}$$

for all $x \in (0,\infty)$, the appropriate boundary conditions at the vertices are given by

$$\left(\mathbb{I}_2 \otimes A \otimes \mathbb{I}_{|\mathcal{E}|}\right) \Psi_{\text{bv}}^{(s,v)} + \left(\mathbb{I}_2 \otimes B \otimes \mathbb{I}_{|\mathcal{E}|}\right) {\Psi_{\text{bv}}^{(s,v)}}' = 0 \tag{4.3.34}$$

with $|\mathcal{E}| \times |\mathcal{E}|$ matrices A and B defined as in (2.4.6). As discussed previously we would like to impose $\tilde{\delta}$-type interactions between particles located on the same infinite star. Such interactions are prescribed by the boundary conditions

$$\psi_{mn}^{(s,+)}(x_1,x_2)|_{x_1=x_2^+} = \psi_{nm}^{(s,-)}(x_1,x_2)|_{x_1=x_2^-}; \tag{4.3.35}$$

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2\alpha\right)\psi_{mn}^{(s,+)}(x_1,x_2)|_{x_1=x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi_{nm}^{(s,-)}(x_1,x_2)|_{x_1=x_2^-}, \tag{4.3.36}$$

with $f(e_m) = f(e_n)$. Since in $D_\Gamma^{(s,*)}$, we defined dissections in D_{mn} for all edge pairs, we must reestablish continuity where there are no interactions, that is for

$f(e_m) \neq f(e_n)$, according to

$$\psi_{mn}^{(s,+)}(x_1,x_2)|_{x_1=x_2^+} = \psi_{mn}^{(s,-)}(x_1,x_2)|_{x_1=x_2^-}; \tag{4.3.37}$$

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi_{mn}^{(s,+)}(x_1,x_2)|_{x_1=x_2^+} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi_{mn}^{(s,-)}(x_1,x_2)|_{x_1=x_2^-}. \tag{4.3.38}$$

The task is now to specify eigenvectors $\Psi \in H^2(D_\Gamma^{(s,*)})$ of $-\Delta_2$ which satisfy boundary conditions (4.3.34)–(4.3.38). Using the Bethe ansatz method, the vector Ψ will be described by the collection of functions

$$\psi_{mn}^{(s,\pm)}(x_1,x_2) = \sum_{P\in\mathscr{W}_2} \mathcal{A}_{mn}^{(P,\pm)} e^{i(k_{P1}x_1+k_{P2}x_2)} \tag{4.3.39}$$

on $D_{mn}^{(s,\pm)}$. Let us define the $2|\mathcal{E}|^2$-dimensional vector

$$\mathcal{A}^P = \begin{pmatrix} \left(\mathcal{A}_{mn}^{(P,-)}\right)_{m,n=1}^{|\mathcal{E}|} \\ \mathbb{T}_{|\mathcal{E}|^2}\left(\mathcal{A}_{mn}^{(PT,+)}\right)_{m,n=1}^{|\mathcal{E}|} \end{pmatrix}. \tag{4.3.40}$$

The vertex conditions (4.3.34) then imply

$$\mathcal{A}^{PR} = \left(\mathbb{I}_2 \otimes S_v(-k_{P1}) \otimes \mathbb{I}_{|\mathcal{E}|}\right)\mathcal{A}^P \tag{4.3.41}$$

for all $P \in \mathscr{W}_2$. At this point, it is convenient to define the diagonal matrix $\boldsymbol{c} = \mathrm{diag}(c_{mn})_{m,n=1}^{|\mathcal{E}|}$ where

$$c_{mn} = \begin{cases} 1 & \text{if } f(e_m) = f(e_n); \\ 0 & \text{otherwise,} \end{cases} \tag{4.3.42}$$

which distinguishes domains with $\tilde{\delta}$-type interactions from those which are continuous across dissections. The $\tilde{\delta}$-type conditions (4.3.35)–(4.3.38) then imply

$$\mathcal{A}^{PT} = Y(k_{P1} - k_{P2})\mathcal{A}^P, \tag{4.3.43}$$

with

$$Y(k) = S_p(k) \otimes \boldsymbol{c} + \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes (\mathbb{I}_{|\mathcal{E}|^2} - \boldsymbol{c})\mathbb{T}_{|\mathcal{E}|^2}. \tag{4.3.44}$$

To prove exact solvability we need only show that relations (4.3.41) and (4.3.43) are consistent with the properties of $\mathscr{W}_2$. This amounts to the requirements

1. $S_v(u)S_v(-u) = \mathbb{I}_{|\mathcal{E}|}$;
2. $Y(k)Y(-k) = \mathbb{I}_{2|\mathcal{E}|}$;
3. $\left(\mathbb{I}_2 \otimes S_v(u) \otimes \mathbb{I}_{|\mathcal{E}|}\right) Y(u+v) \left(\mathbb{I}_2 \otimes S_v(v) \otimes \mathbb{I}_{|\mathcal{E}|}\right) Y(v-u)$
$= Y(v-u) \left(\mathbb{I}_2 \otimes S_v(v) \otimes \mathbb{I}_{|\mathcal{E}|}\right) Y(u+v) \left(\mathbb{I}_2 \otimes S_v(u) \otimes \mathbb{I}_{|\mathcal{E}|}\right)$.

The first two conditions are easily verified by the explicit forms of $S_v(u)$ and $Y(u)$, noting that, since $c_{mn} = c_{nm}$, the properties of $\mathbb{T}_{|\mathcal{E}|^2}$ are such that

$$[\boldsymbol{c}, \mathbb{T}_{|\mathcal{E}|^2}] = 0. \tag{4.3.45}$$

Noting then that the relation

$$[S_v(u) \otimes \mathbb{I}_{|\mathcal{E}|}, \boldsymbol{c}] = 0 \tag{4.3.46}$$

holds if boundary conditions at the vertices are local, that is matrices A, B are subject to the conditions (2.4.6), and also the relation (3.3.28), the third condition is easily verified.

Let us bring our attention back to the original compact graph Γ. In order to turn the eigenfunctions in the star representation into eigenfunctions on the compact graph, it is sufficient to impose the relations

$$\psi_{mn}^{(s,+)}(x_1, x_2) = \psi_{(m+|\mathcal{I}|)n}^{(s,+)}(l_m - x_1, x_2) \text{ and} \tag{4.3.47}$$

$$\psi_{mn}^{(s,-)}(x_1, x_2) = \psi_{m(n+|\mathcal{I}|)}^{(s,-)}(x_1, l_n - x_2) \tag{4.3.48}$$

for all $m, n \in \{1, .., |\mathcal{I}|\}$ which imply

$$\mathcal{A}_{mn}^{(P,+)} = \mathcal{A}_{(m+|\mathcal{I}|)n}^{(PR,+)} e^{-ik_{P1}l_m} \text{ and} \tag{4.3.49}$$

$$\mathcal{A}_{mn}^{(P,-)} = \mathcal{A}_{m(n+|\mathcal{I}|)}^{(PTRT,-)} e^{-ik_{P2}l_n}. \tag{4.3.50}$$

These conditions then yield the relation

$$\mathcal{A}^P = E(-k_{P2})\mathcal{A}^{PTRT}, \tag{4.3.51}$$

where

$$E(k) = \mathbb{I}_{4|\mathcal{I}|} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes e^{i k \boldsymbol{l}}, \tag{4.3.52}$$

with $e^{ik\boldsymbol{l}}$ defined as in (2.3.6). Applying (4.3.41), (4.3.43) and (4.3.51) successively we have the condition that

$$Z(k_{P1}, k_{P2}) = 0, \tag{4.3.53}$$

with

$$Z(k_1, k_2) = \det\left[\mathbb{I}_{8|\mathcal{I}|^2} - E(k_2) Y(k_2 - k_1)\left(\mathbb{I}_2 \otimes S_v(k_2) \otimes \mathbb{I}_{2|\mathcal{I}|}\right) Y(k_1 + k_2)\right], \tag{4.3.54}$$

is satisfied for all $P \in \mathscr{W}_2$. By using properties of determinants, the commutation relations established above, and the explicit forms of $Y(k)$, $S_v(k)$ and $E(k)$, it is easy to see that if (4.3.53) is satisfied for some $P \in \mathscr{W}_2$, then it is necessarily satisfied for elements $PR, PTRT \in \mathscr{W}_2$. With this in mind we can state the main result of this section.

Theorem 4.3.2. Non-zero eigenvalues of a self-adjoint two-particle Laplacian $-\Delta_2$ defined on Γ with local vertex interactions specified through A, B and $\tilde{\delta}$-type interactions between particles when they are located on neighbouring edges, are the values $E = k_1^2 + k_2^2 \neq 0$ with multiplicity m, where (k_1, k_2), such that $0 \leq k_1 \leq k_2$, are solutions to the secular equations

$$Z(k_i, k_j) = 0, \tag{4.3.55}$$

for $j, i \neq j \in \{1, 2\}$, with multiplicity m.

4.3.3 Recovering specific results

To finish the section, let us establish agreement between the spectra of general two-particle quantum graphs presented in Theorem 4.3.2 and results derived and discussed earlier in the thesis.

Equilateral stars

We would like to show that the quantisation condition prescribed in Theorem 4.3.2 for general graphs Γ reduces to that prescribed in Theorem 4.2.2 for equilateral stars Γ_e when appropriate parameters are imposed. Firstly, the appropriate vertex conditions are given by (4.2.19) and (4.2.20) so that interactions at a central vertex are prescribed by A_1, B_1 with Dirichlet conditions at outer vertices. We also impose equal lengths $l_j = l$ for all $j \in \{1, \dots, |\mathcal{I}|\}$. We recall that on equilateral stars, $\tilde{\delta}$-type interactions are imposed along the diagonals $x_1 = x_2$ of all square domains D_{mn}. Viewed in the star representation, this corresponds to $\tilde{\delta}$-type interactions along dissections of $D_{mn}^{(s,*)}$ with $(e_m, e_n) \in \mathcal{N}_e$ and continuity otherwise. To this end, we replace the diagonal matrix $\boldsymbol{c}$ with $\boldsymbol{c}_e$ as prescribed by (4.2.50). Substituting these parameters into $Z(k_i, k_j)$ and using the properties of determinants we recover the form $Z_e(k_i, k_j)$ as required.

Pseudo-non-interacting particles

In Section 4.2.4, we introduced the notion of pseudo-non-interacting particles and showed that in the equilateral star setting, the corresponding quantisation condition is indeed that of the truly non-interacting case. However, in the general setting, this agreement does not hold. The spectra of such systems is calculated first by identifying the matrix

$$\lim_{\alpha \to 0} Y(k) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes \left(\boldsymbol{c} + (\mathbb{I}_{|\mathcal{E}|^2} - \boldsymbol{c})\mathbb{T}_{|\mathcal{E}|^2}\right). \tag{4.3.56}$$

Substitution into (4.3.54) then yields the quantisation condition

$$\begin{aligned} Z(k) = \det\Big[\mathbb{I}_{4|\mathcal{I}|^2} - &\left(S_v(k) \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes e^{ik\boldsymbol{l}}\right)\boldsymbol{c} \\ &- \left(\mathbb{I}_{2|\mathcal{I}|} \otimes \left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes e^{ik\boldsymbol{l}}\right) S_v(k)\right)\left(\mathbb{I}_{4|\mathcal{I}|^2} - \boldsymbol{c}\right)\Big] \end{aligned} \tag{4.3.57}$$

which we notice is dependent on the single momentum k.

Non-interacting particles

Truly non-interacting systems are recovered by turning off all coupling between domains D_{mn} and D_{nm}. This is achieved by setting $\boldsymbol{c} = \mathbf{0}$. We then have that

$$Y(k)|_{\boldsymbol{c}=\mathbf{0}} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes \mathbb{T}_{|\mathcal{E}|^2}. \tag{4.3.58}$$

By substituting into (4.3.54) we recover the secular equation (2.3.9) for the one-particle quantum graph.

Bosons on a general graph

In the subsequent section we would like to analyse examples of bosons on a graph. Imposing bosonic symmetry

$$\psi_{mn}^{(s,-)}(x_1, x_2) = \psi_{nm}^{(s,+)}(x_2, x_1) \tag{4.3.59}$$

we have that

$$\mathcal{A}_{mn}^{(P,-)} = \mathcal{A}_{nm}^{(PT,+)} \tag{4.3.60}$$

for all $P \in \mathscr{W}_2$. The matrix $Y(k)$ then reduces to the form

$$Y(k) = \mathbb{I}_2 \otimes Y_b(k) \tag{4.3.61}$$

with

$$Y_b(k) = s_p(k)\boldsymbol{c} + (\mathbb{I}_{|\mathcal{E}|^2} - \boldsymbol{c})\mathbb{T}_{|\mathcal{E}|^2}, \tag{4.3.62}$$

so that from (4.3.54), we recover

$$Z_b(k_1, k_2) = \det\left[\mathbb{I}_{4|\mathcal{I}|^2} - E_b(k_2)Y_b(k_2 - k_1)\left(S_v(k_2) \otimes \mathbb{I}_{2|\mathcal{I}|}\right)Y_b(k_1 + k_2)\right], \tag{4.3.63}$$

with $E(k) = \mathbb{I}_2 \otimes E_b(k)$.

4.4 Spectral statistics

One of the main motivations for the study of quantum graphs is to analyse their spectral statistics. In doing so, we can investigate the chaotic nature of their classical counterparts. Again we pay particular attention to the nearest neighbour energy level distribution (2.6.13). We also compare the counting function $N(E)$ as defined in (2.5.1) to the Weyl law (3.4.1) which is proven for singular contact interactions. Of course, the majority of this thesis is not concerned with δ-type interactions, but with $\tilde{\delta}$-interactions. Nonetheless, for each example, we again assign a line of best fit (3.4.2) to $N(E)$ and compare the leading term to (3.4.1) by calculating the value (3.4.3).

In [KS97], nearest neighbour energy level distributions of one-particle quantum tetrahedra were shown to exhibit GOE spectral statistics and thus imply chaotic classical counterparts (see Section 2.6). In this section we analyse the spectra of two-particle quantum graphs calculated in the previous sections, looking for a potential dependence of spectral correlations on the interaction strength. We refer to the result by Srivastava et al. [STL+16] who analysed the spectral properties of interacting kicked rotors which individually show GOE statistics. For the combined spectra, they found a transition from Poissonian to GOE statistics as the strength of the interaction was increased.

4.4.1 The tetrahedron

Let us take, as a first example, a system of two $\tilde{\delta}$-interacting bosons on a tetrahedron (see Figure 2.3) with local boundary conditions (2.6.1), such that scattering matrices $S_v^{(\eta)}(k)$ corresponding to each vertex v_η are identical. The appropriate spectra are calculated according to Theorem 4.3.2 by finding solutions to the pair of secular equations

$$Z_b(k_1, k_2) = 0 \text{ and } Z_b(k_2, k_1) = 0 \tag{4.4.1}$$

given by (4.3.63). In order to reduce the computational expense of this problem let us define the permutation matrix

$$\mathbb{V} = \mathbb{I}_4 \otimes \left(\mathbb{I}_{12} \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \mathbb{I}_{12} \otimes \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^{\mathrm{T}}, \mathbb{I}_{12} \otimes \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}} \right). \tag{4.4.2}$$

Multiplying on the left and right of (4.3.63) by

$$\det[\mathbb{V}(\mathbb{P} \otimes \mathbb{P})] \text{ and } \det[(\mathbb{P}^{-1} \otimes \mathbb{P}^{-1})\mathbb{V}^{-1}], \tag{4.4.3}$$

we arrive at the block form

$$Z_b(k_1, k_2) = \det[\mathbb{I}_{144} - \bigoplus_{m=1}^{4} M_m(k_1, k_2)], \tag{4.4.4}$$

with

$$\begin{aligned} M_m(k_1, k_2) = &\left(\mathbb{P}\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \otimes e^{ik_2 \boldsymbol{l}} \right) \mathbb{P}^{-1} \otimes \mathbb{I}_3 \right) \\ &\left(\operatorname{diag}(\delta_{mn})_{n=1}^4 \otimes s_p(k_2 - k_1) s_p(k_1 + k_2) \left(\mathbb{I}_3 \otimes S_v^{(\eta)}(k_2) \right) \right. \\ &\left. + \operatorname{diag}(1 - \delta_{mn})_{n=1}^4 \otimes S_v^{(\eta)}(k_2) \otimes \mathbb{I}_3 \right). \end{aligned} \tag{4.4.5}$$

Laplace eigenvalues $E = k_1^2 + k_2^2$ are then given by solutions (k_1, k_2) of the pairs of secular equations

$$\det\left[\mathbb{I}_{36} - M_u(k_1, k_2)\right] = 0 \text{ and } \det\left[\mathbb{I}_{36} - M_v(k_2, k_1)\right] = 0, \tag{4.4.6}$$

with $u, v \in \{1, \dots, 4\}$.

Boundary conditions at the vertices are determined by choosing Discrete Fourier Transform (DFT) scattering matrices $S_v^{(\eta, DFT)}$ as defined in (2.4.17). For the tetrahedron, appropriate DFT scattering matrices at each vertex v_η are

$$S_v^{(\eta, DFT)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{3}} & e^{\frac{4i\pi}{3}} \\ 1 & e^{\frac{4i\pi}{3}} & e^{\frac{8i\pi}{3}} \end{pmatrix}, \tag{4.4.7}$$

with distinct eigenvalues $\{-1, 1, i\}$. With this choice, the spectrum of the two-particle Laplacian with $\tilde{\delta}$-interactions is non-degenerate.

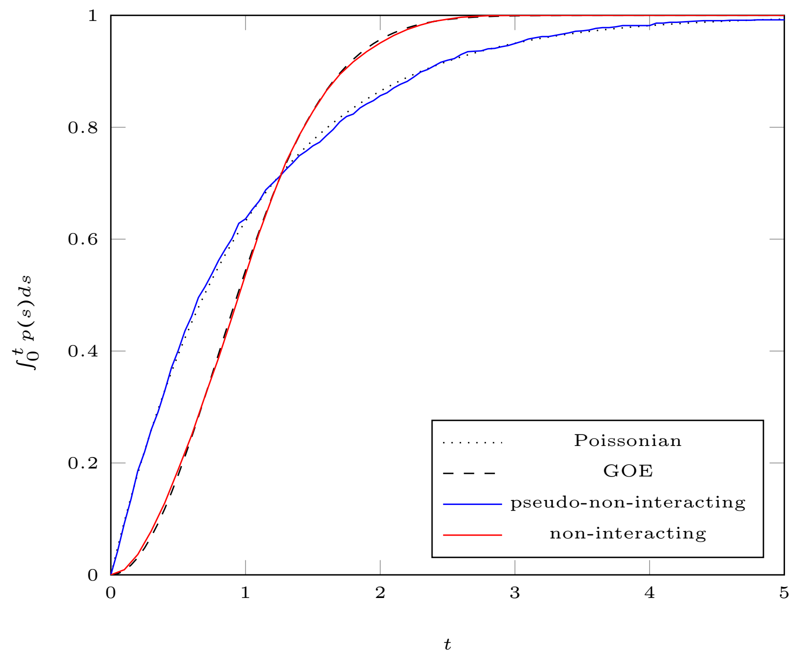

Figure 4.8: Integrated level spacings distributions for the single-particle spectra associated with non-interacting and pseudo-non-interacting systems on a tetrahedron. First 50,000 eigenvalues.

Before analysing two-particle spectra, let us consider the spectra of non-interacting systems. Figure 4.8 plots the nearest neighbour distributions for the single-particle spectra associated with truly non-interacting ($\boldsymbol{c} = \mathbf{0}$) and pseudo-non-interacting ($\alpha = 0$) particles on the tetrahedron with DFT scattering matrices. As is well known [KS97] and confirmed in Figure 4.8, the one-particle spectrum follows GOE statistics. The pseudo-non-interacting system, however, shows Poissonian statistics. The crucial point here is that two-particle systems prescribed in Theorem 4.3.2 in fact couple systems of pseudo-non-interacting particles which individually possess spectra with Poissonian statistics, not systems of truly non-interacting particles which individually follow GOE statistics. Thus we cannot expect a transition to GOE statistics as in [STL+16]. Figure 4.9 plots the α-dependency of the lowest energy levels of a system of $\tilde{\delta}$-interacting bosons on a tetrahedron with DFT vertex scattering matrices. There is no obvious transition to a regime of energy level repulsion as we increase α. Indeed, plots of nearest neighbour distributions reveal Poissonian statistics for all interaction strengths. Figure 4.10 shows these

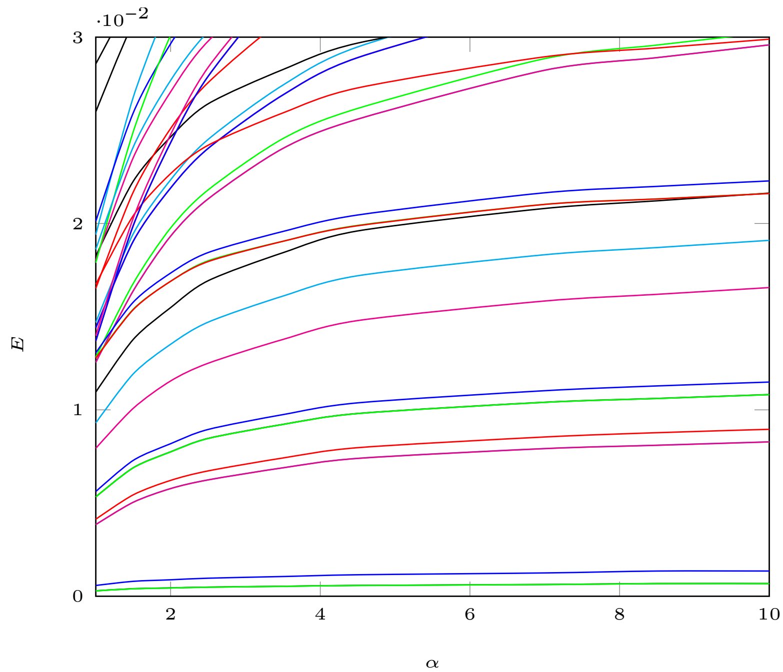

Figure 4.9: Dependency on interaction strength of small eigenvalues of a system of two bosons on a tetrahedron.

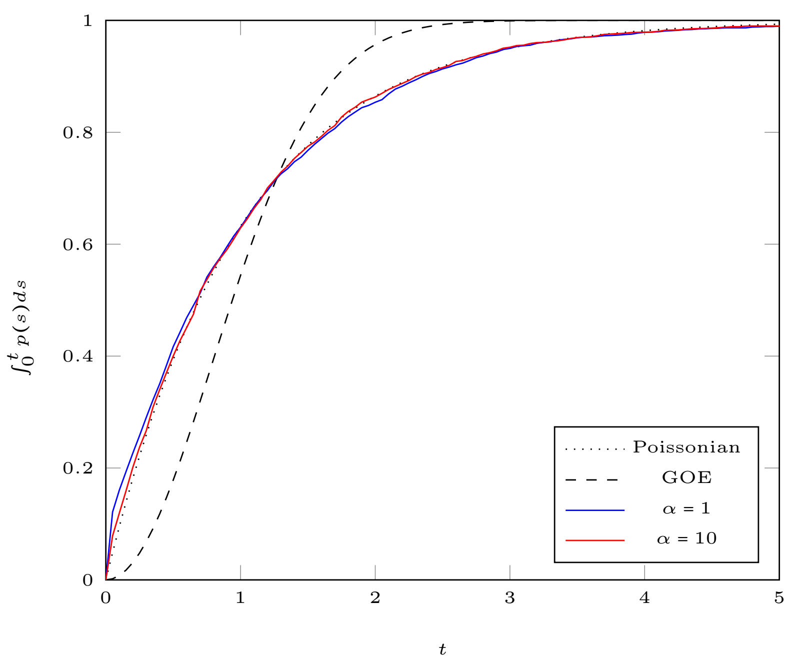

Figure 4.10: Integrated level spacings distributions for systems of two bosons on a tetrahedron. First 3000 eigenvalues.

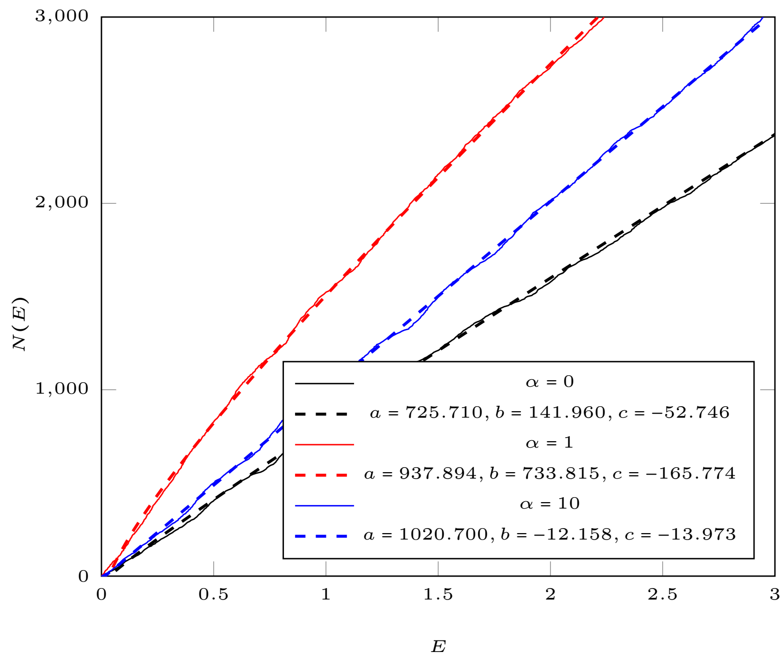

Figure 4.11: Counting functions $N(E)$ (solid line) with lines of best fit $\overline{N}(E)$ (dashed line) for systems of two bosons on a tetrahedron.

plots for interaction strengths $\alpha = 1$ and $\alpha = 10$.

Figure 4.11 plots counting functions $N(E)$ for strengths $\alpha \in \{0, 1, 10\}$ together with quadratic lines of best fit $\bar{N}(E)$ given by (3.4.2). In each case, the leading term does not agree with the Weyl law (3.4.1) predicted for contact interactions; the values $\frac{\mathcal{L}}{8\pi a}$ are

$$8.36 \times 10^{-3},\ 6.47 \times 10^{-3} \text{ and } 5.94 \times 10^{-3} \tag{4.4.8}$$

for α equal to 0, 1 and 10 respectively.

4.4.2 Equilateral stars

To examine the spectral statistics of coupled chaotic systems we must look for two-particle quantum graphs for which the one-particle spectra recovered when setting $\alpha = 0$ are chaotic. We have seen that two-particle tetrahedra with $\tilde{\delta}$-interactions do not fulfil this requirement. Let us then focus our attention on equilateral stars which we discussed in Section 4.2. Therein, we showed that true

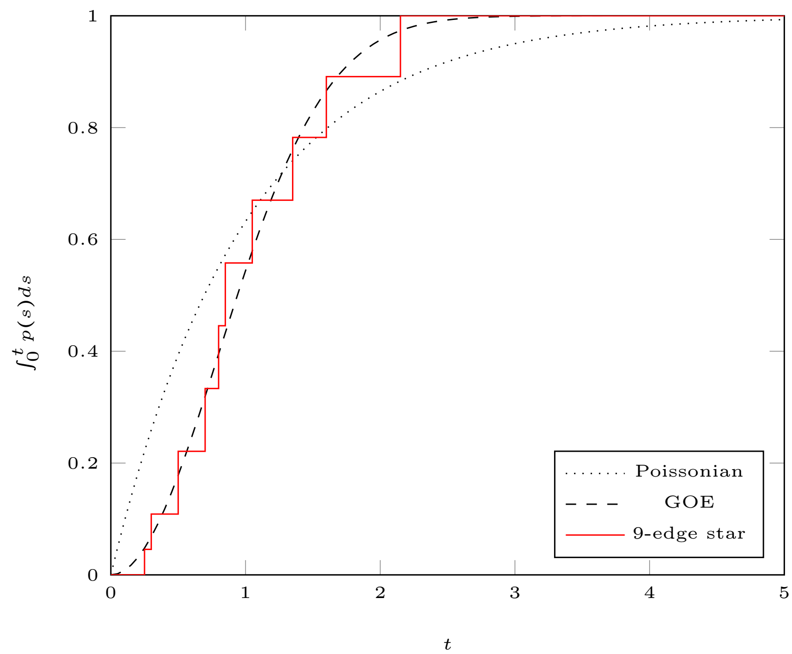

Figure 4.12: Integrated level spacings distributions for a single particle on a 9-edge equilateral star. First 50,000 eigenvalues.

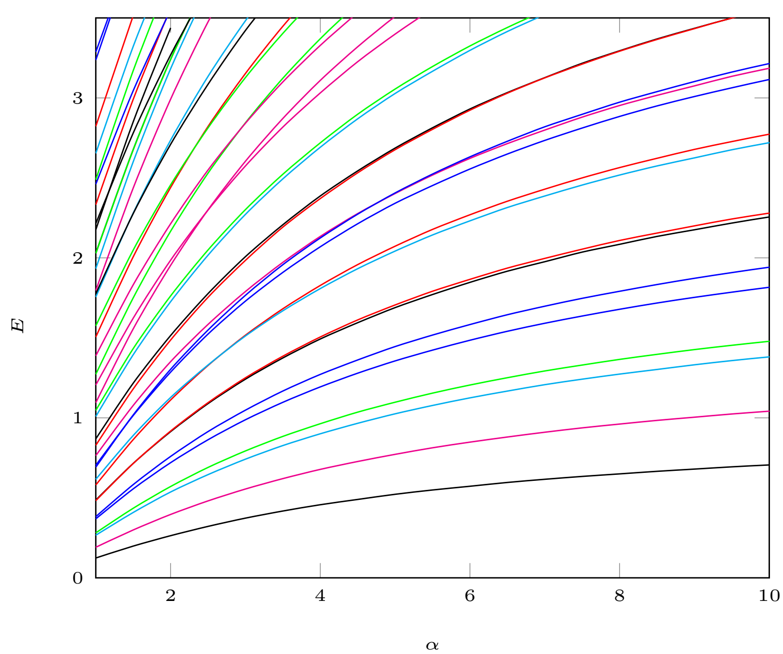

Figure 4.13: Dependency on interaction strength of small eigenvalues of a system of two bosons on a 9-edge equilateral star.

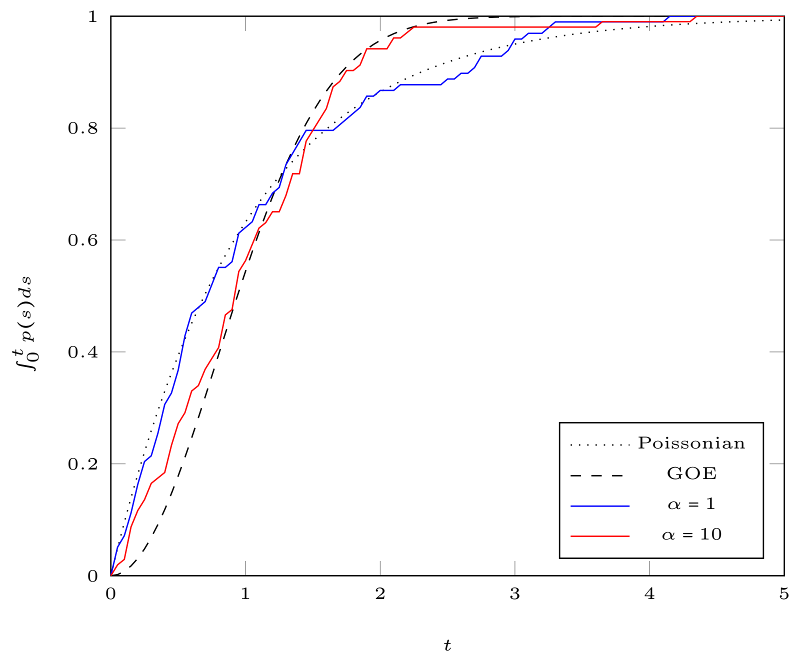

Figure 4.14: Integrated level spacings distributions for systems of two bosons on a 9-edge equilateral star. First 100 eigenvalues.

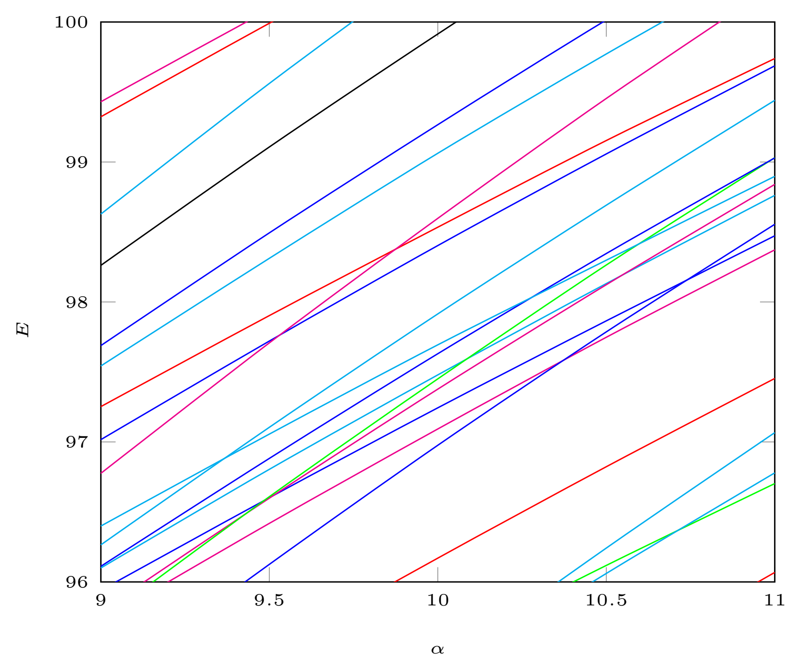

Figure 4.15: Dependency on interaction strength of large eigenvalues of a system of two bosons on a 9-edge equilateral star.

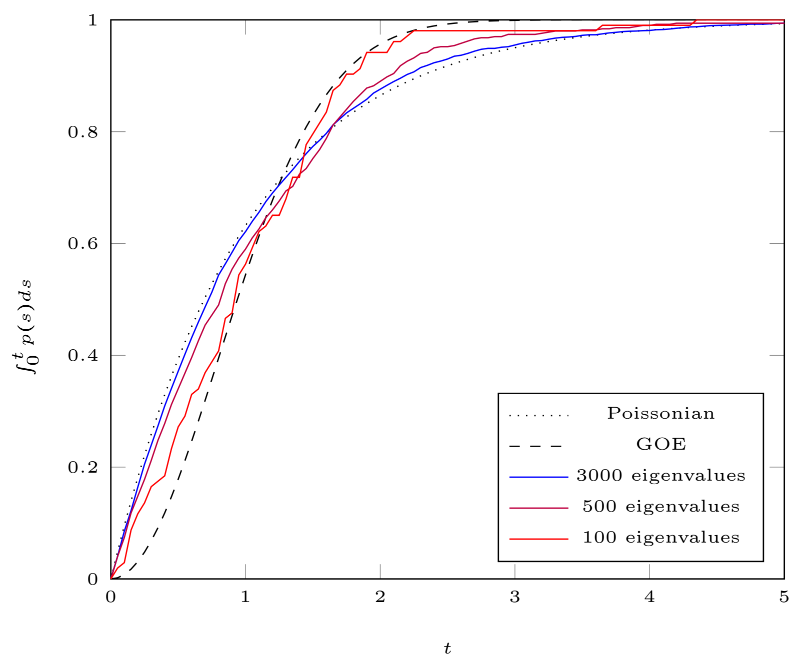

Figure 4.16: Integrated level spacings distributions for systems of two bosons on a 9-edge equilateral star with $\alpha = 10$.

one-particle spectra are recovered when setting $\alpha = 0$. Thus we can discuss coupled chaotic systems in the spirit of [STL+16] if we can find one-particle equilateral stars which exhibit GOE statistics. Such systems are characterised by the quantisation condition (4.2.64) which can be written

$$e^{-2ikl} = -\mu(k), \tag{4.4.9}$$

where $\mu(k)$ is an eigenvalue of $S_v^{(1)}(k)$. Clearly, the multiplicity of solutions k are equal to the multiplicity these eigenvalues. For example, equilateral stars with boundary conditions characterised by the DFT scattering matrix (2.4.17) at the central vertex yield solutions corresponding to $\mu = \{1, -1, i, -i\}$, with degenerate values arising for $d > 3$. Clearly degenerate energy levels would obscure conclusions made in the context of spectral statistics. To navigate this issue, we must choose a scattering matrix with non-degenerate eigenvalues. In what follows, we choose a randomly generated $d_{|\mathcal{I}|} \times d_{|\mathcal{I}|}$ unitary matrix. Figure 4.12 plots the nearest neighbour distribution for a single particle on such an equilateral star with 9 edges. The degenerate energy level spacings arise from the imposition of equal lengths.

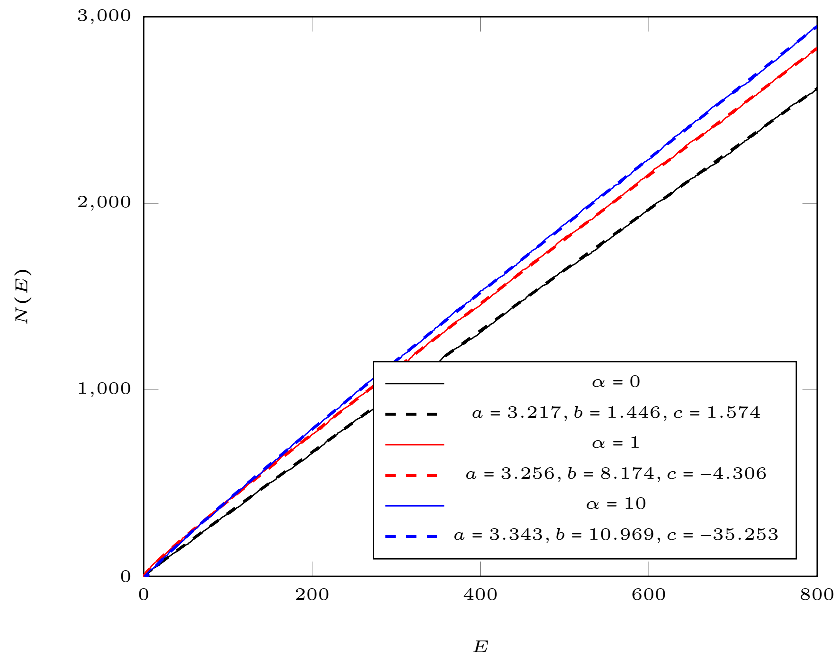

Figure 4.17: Counting functions $N(E)$ (solid line) with lines of best fit $\overline{N}(E)$ (dashed line) for systems of two bosons on a 9-edge equilateral star.

Indeed in studies of one-particle quantum graph spectra, rationally independent lengths are chosen to avoid degenerate level spacings. We do however see approximate agreement with GOE statistics. In this setting we can thus investigate the coupling of two chaotic spectra by increasing α from 0.

Figure 4.13 plots the α-dependency of the lowest energy levels of a system of two bosons on a 9-edge equilateral star with a random unitary central scattering matrix. We clearly see a transition to level repulsion as α is increased. Figure 4.14 plots nearest neighbour distributions for the first 100 energy levels. There is a clear shift from Poissonian, for $\alpha = 1$, towards GOE statistics, for $\alpha = 10$. We note, however, that this level repulsion becomes less apparent as we include larger energy levels; Figure 4.15 shows level crossing at higher energies and Figure 4.16 shows how the spectral statistics for the $\alpha = 10$ case tend to Poissonian as we include higher energies.

Figure 4.17 plots counting functions for $\alpha \in \{0, 1, 10\}$ together with quadratic lines of best fit $\bar{N}(E)$ given by (3.4.2). The leading term is consistent with the Weyl law (3.4.1) for all interactions strengths; the values $\frac{\mathcal{L}}{8\pi a}$ are

$$1.002,\ 0.990 \text{ and } 0.964 \tag{4.4.10}$$

for α equal to 0, 1 and 10 respectively. Indeed, for the non-interacting ($\alpha = 0$) case, this agreement is almost exact. As the interaction strength increases, the counting function diverges from this exact agreement.

Chapter 5

Many-particle quantum graphs

In the previous chapter we constructed exactly solvable two-particle quantum graphs with $\tilde{\delta}$-interactions and calculated their spectra using the Bethe ansatz. In this chapter we generalise this approach to n-particle graphs.

5.1 Preliminaries

Before we proceed it is useful to define the symmetric group S_n, and the Weyl group $\mathscr{W}_n$ of the root system C_n, which we will use to characterise the symmetries of exactly solvable n-particle systems. Material, taken from [Hum72, AMP81, Bou68], generalises that in Section 3.1.

Definition 5.1.1. Elements Q in the symmetric group S_n acting on the set $\{1,\dots,n\}$ will be written in term of generators

$$T_1,\dots,T_{n-1} \tag{5.1.1}$$

which act according to

$$T_i(1,\dots,i-1,i,i+1,i+2,\dots,n) = (1,\dots,i-1,i+1,i,i+2,\dots,n) \tag{5.1.2}$$

and satisfy the conditions

1. $T_iT_i = I$;
2. $T_iT_j = T_jT_i$ for $|i-j|>1$;

3. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

Consider the Euclidean space $\mathbb{R}^n$ with standard basis consisting of vectors

$$\epsilon_i = (0, \dots, 1, \dots, 0)^{\mathrm{T}}, \tag{5.1.3}$$

with 1, in the i^{th} position, the only non-zero entry. The root system C_n, for $n > 2$, is the set of $2n^2$ vectors $\{\pm\epsilon_i \pm \epsilon_j\}$, with $1 \le i < j \le n$, and $\{\pm 2\epsilon_i\}$, with $1 \le i \le n$. The Weyl group $\mathscr{W}_n$ is the group of isometries generated by the reflections through hyperplanes perpendicular to the roots of C_n.

Definition 5.1.2. Elements P in the Weyl group

$$\mathscr{W}_n := (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n, \tag{5.1.4}$$

of order $2^n n!$, acting on the set $\{\pm 1, \dots, \pm n\}$ will be written in terms of generators

$$T_1, \dots, T_n, R_1 \tag{5.1.5}$$

which act according to

1. $T_i(1, \dots, i-1, i, i+1, i+2, \dots, n) = (1, \dots, i-1, i+1, i, i+2, \dots, n)$;

2. $R_1(1, 2, \dots, n) = (-1, 2, \dots, n)$,

and satisfy the conditions

1. $R_1 R_1 = I$;

2. $T_i T_i = I$;

3. $T_i T_j = T_j T_i$ for $|i - j| > 1$;

4. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$;

5. $R_1 T_1 R_1 T_1 = T_1 R_1 T_1 R_1$;

6. $R_1 T_i = T_i R_1$ for $i > 1$.

It will be convenient to define elements

$$R_i = T_{i-1} \dots T_1 R_1 T_1 \dots T_{i-1} \tag{5.1.6}$$

so that

$$R_i(1,\ldots,i,\ldots,n) = (1,\ldots,-i,\ldots,n). \tag{5.1.7}$$

We note then that, with S_n and $\mathscr{W}_n$ defined as above, the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ in $\mathscr{W}_n$ can be written in terms of generators $R_1,\ldots,R_n$. With this in mind, the conditions in Definition 3.1.4, which validate the semidirect product are easily verified.

Finally it will be useful to relate the Weyl groups $\mathscr{W}_n$ and $\mathscr{W}_{n-1}$. To this end, let us define the cyclic permutation

$$C_n = T_{n-1}T_{n-2}\ldots T_1 \tag{5.1.8}$$

so that

$$C_n(1,2,\ldots,n-1,n) = (n,1,\ldots,n-2,n-1), \tag{5.1.9}$$

where we note the relation

$$R_n = C_n R_1 C_n^{-1}. \tag{5.1.10}$$

The Weyl group $\mathscr{W}_n$ can then be written in terms of $\mathscr{W}_{n-1}$ according to

$$\mathscr{W}_n = \left\{\{C_n^d X, C_n^d R_n X\}_{X\in\mathscr{W}_{n-1}}\right\}_{d=0}^{n-1}. \tag{5.1.11}$$

5.2 Bosons in a box

In order to establish some key concepts in the n-particle setting, we begin by presenting the model of n δ-interacting bosons in a box solved by Gaudin [Gau71] and presented for $n = 2$ in Section 3.2.2. The problem is formulated as a search for n-particle solutions

$$\psi = \psi(x_1,\ldots,x_n) \tag{5.2.1}$$

of the Schrödinger equation

$$\left(-\Delta_n + 2\alpha \sum_{i \neq j} \delta(x_i - x_j)\right)\psi = E\psi \tag{5.2.2}$$

with particle positions $x_1, \ldots, x_n$ defined on the half-line $\mathbb{R}_+ = (0, \infty)$. Here the n-particle Laplacian acts according to

$$-\Delta_n \psi = -\sum_{j=1}^{n} \frac{\partial^2 \psi}{\partial {x_j}^2}. \tag{5.2.3}$$

By requiring that $-\Delta_n$ is self-adjoint and imposing bosonic symmetry

$$\begin{aligned} &\psi(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \\ =&\psi(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n), \end{aligned} \tag{5.2.4}$$

equation (5.2.2) can be shown to decompose into the eigenvalue equation

$$-\Delta_n \psi = E\psi \tag{5.2.5}$$

alongside the set of $n-1$ jump conditions in the derivatives

$$\left(\frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j}\right)\psi|_{x_{j+1}=x_j^+} = \alpha\psi|_{x_{j+1}=x_j^+}, \tag{5.2.6}$$

for $j \in \{1, \ldots, n-1\}$, and the Dirichlet condition

$$\psi|_{x_1=0} = 0, \tag{5.2.7}$$

with ψ restricted to the subspace

$$d^I = \{\mathbb{R}_+^n;\ x_1 < \cdots < x_n\}. \tag{5.2.8}$$

Together with the imposition of bosonic symmetry, the problem is then also defined in all subspaces

$$d^Q = \{\mathbb{R}_+^n;\ x_{Q1} < \cdots < x_{Qn}\} \tag{5.2.9}$$

with $Q \in S_n$ and thus all of $\mathbb{R}^n_+$. We recall that, in the two-particle setting, interactions between particles meant $\mathbb{R}^2_+$ was naturally dissected into two subspaces $d^\pm$. Here, in the n-particle case, appropriate dissections result in $n!$ subspaces labeled by elements $Q \in S_n$. Bosonic symmetry establishes equivalence between each of these subspaces so that we need only consider one.

The task is then to construct explicit Laplace eigenfunctions ψ in d^I which satisfy conditions (5.2.6) and (5.2.7). The justification for the appropriate ansatz is a straightforward generalisation of that given in the two-particle setting. Let us consider an n-particle plane wave state

$$\psi_I = e^{i(k_1 x_1 + \cdots + k_n x_n)} \tag{5.2.10}$$

defined with momenta

$$k_n \le \cdots \le k_1 \le 0 \text{ and } (k_1, \ldots, k_n) \neq 0, \tag{5.2.11}$$

so that the system is approaching one of the n boundaries, $x_j = x_{j+1}$, for $j \in \{1, \ldots, n-1\}$, and $x_1 = 0$, of d^I. The possible results of δ-type conditions at each boundary $x_j = x_{j+1}$ are the momenta of each participating particle being swapped

$$(k_j, k_{j+1}) \to (k_{j+1}, k_j), \tag{5.2.12}$$

or else remaining as they were

$$(k_j, k_{j+1}) \to (k_j, k_{j+1}). \tag{5.2.13}$$

Dirichlet boundary conditions at the latter boundary result in momentum reversal

$$(k_1, k_2, \ldots, k_n) \to (-k_1, k_2, \ldots, k_n). \tag{5.2.14}$$

Taking into account all possible interactions, we expect that any resulting n-particle state must be one of $n!2^n$ n-particle plane waves

$$\psi_P = e^{i(k_{P1} x_1 + \cdots + k_{Pn} x_n)}, \tag{5.2.15}$$

with elements $P \in \mathscr{W}_n$ as prescribed in Definition 5.1.2. We can then think of each $P \in \mathscr{W}_n$ as corresponding to some configuration of momenta

$$\boldsymbol{k} = (k_{P1}, \ldots, k_{Pn}). \tag{5.2.16}$$

The Bethe ansatz method in this context is the assumption that the appropriate ansatz is the sum of possible constituent plane wave states

$$\psi = \sum_{P \in \mathscr{W}_n} \mathcal{A}^P e^{i(k_{P1}x_1 + \cdots + k_{Pn}x_n)} \tag{5.2.17}$$

with $\mathcal{A}^P$ the amplitudes of constituent states ψ_P.

Using the form (5.2.17), equation (5.2.6) is satisfied with Laplace eigenvalues

$$E = \sum_{j=1}^{n} k_j^2. \tag{5.2.18}$$

Boundary conditions (5.2.6) and (5.2.7) then imply

$$\mathcal{A}^{PT_i} = s_p(k_{Pi} - k_{P(i+1)})\mathcal{A}^P, \tag{5.2.19}$$

for $i \in \{1, \ldots, n-1\}$, and

$$\mathcal{A}^{PR} = -\mathcal{A}^P \tag{5.2.20}$$

for all $P \in \mathscr{W}_n$. Exact solvability is assured if relations (5.2.19) and (5.2.20) are compatible with the properties of $\mathscr{W}_n$. This amounts only to the requirement $s_p(u)s_p(-u) = 1$ which is easily verified.

Let us bring our attention back to the compact setting. Enclosing the particles in a box length l is enforcing the Dirichlet condition

$$\psi(x_1, \ldots, x_{n-1}, l) = 0 \tag{5.2.21}$$

which implies the relation

$$\begin{aligned}\mathcal{A}^P &= -\, e^{2ik_{Pn}l}\mathcal{A}^{PR_n} \\ &= -\, e^{2ik_{Pn}l}\mathcal{A}^{PC_nR_1C_n^{-1}},\end{aligned} \tag{5.2.22}$$

where for the latter equality we have used the relation (5.1.10). Finally, applying (5.2.19), (5.2.20) and (5.2.22) successively, we arrive at the condition

$$e^{-2ik_{Pn}l} = \prod_{i=1}^{n-1} s_p(k_{Pn} + k_{Pi})s_p(k_{Pn} - k_{Pi}) \tag{5.2.23}$$

for all $P \in \mathscr{W}_n$. We note here that the form of $s_p(k)$ is such that, if (5.2.23) is satisfied for some $P \in \mathscr{W}_n$ then it is necessarily satisfied for elements

$$PT_1, \dots, PT_{n-2}, PR_1 \text{ and } PR_n \tag{5.2.24}$$

in $\mathscr{W}_n$ and thus for every

$$PX \text{ and } PR_nX \tag{5.2.25}$$

in $\mathscr{W}_n$ with $X \in \mathscr{W}_{n-1}$. Using (5.1.11), we thus have the n quantisation conditions

$$e^{-2ik_jl} = \prod_{i\neq j} s_p(k_j + k_i)s_p(k_j - k_i), \tag{5.2.26}$$

with $j \in \{1, \dots, n\}$. Solutions $(k_1, \dots, k_n) \neq (0, \dots, 0)$, such that $0 \leq k_1 \cdots \leq k_n$, then constitute energies (5.2.18).

5.3 General graphs with $\tilde{\delta}$-interactions

Now we have established how to construct exactly solvable n-particle systems on an interval, we would like to extend the approach to general graphs. As in the two-particle setting, we will impose $\tilde{\delta}$-interactions to ensure compatibility with the Bethe ansatz method.

In Section 4.3, we calculated the spectra of two-particle quantum graphs by first viewing a general graph Γ in its star representation $\Gamma^{(s)}$ and then imposing $\tilde{\delta}$-type interactions between particles located on the same star. In this section we extend this notion to n-particle quantum graphs. Appropriate boundary conditions will be n-particle analogues of (4.3.34)–(4.3.38). Defining an appropriate n-particle Bethe ansatz, we show exact solvability and calculate quantisation conditions which provide the exact spectra. In previous chapters, boundary conditions were shown to provide self-adjoint realisations of the appropriate Laplacian. For compactness, we claim that the arguments made in this context can be carried over to the n-particle setting in the obvious way (see [BK13c]).

Consider the compact graph Γ viewed in its star representation $\Gamma^{(s)}$. The appropriate n-particle Hilbert space on $\Gamma^{(s)}$ is

$$\mathscr{H}_n^{(s)} = \bigotimes_{i=1}^{n}\left(\bigoplus_{j=1}^{|\mathcal{E}|} L^2(0,\infty)\right). \tag{5.3.1}$$

Vectors

$$\Psi = \left(\psi^{(s)}_{j_1\cdots j_n}\right)_{j_1,\dots,j_n=1}^{|\mathcal{E}|} \tag{5.3.2}$$

in $\mathscr{H}_n^{(s)}$ are then lists of n-particle functions

$$\psi^{(s)}_{j_1\cdots j_n} : D^{(s)}_{j_1\cdots j_n} \to \mathbb{C} \tag{5.3.3}$$

in $L^2(D^{(s)}_{j_1\cdots j_n})$ with infinite subdomains defined as

$$D^{(s)}_{j_1\cdots j_n} = (0,\infty)^n. \tag{5.3.4}$$

The total configuration space for n particles on $\Gamma^{(s)}$ is the disjoint union

$$D^{(s)}_{\Gamma} = \dot{\bigcup_{j_1,\dots,j_n=1}^{|\mathcal{E}|}} D^{(s)}_{j_1\cdots j_n} \tag{5.3.5}$$

of these subdomains. The n-particle Hilbert space can then be written $\mathscr{H}_n^{(s)} = L^2(D^{(s)}_{\Gamma})$.

In the two-particle setting, interactions take place along the diagonals $x_1 = x_2$ of two-dimensional configuration spaces $D^{(s)}_{mn}$. In the n particle case, we wish to impose interactions at the boundaries of subdomains

$$D^{(s,Q)}_{j_1\cdots j_n} = \{(x_1,\dots,x_n) \in D^{(s)}_{j_1\cdots j_n};\ x_{Q1} < \cdots < x_{Qn}\}, \tag{5.3.6}$$

with $Q \in S_n$. The appropriate total dissected configuration space is then

$$D^{(s,*)}_\Gamma = \dot{\bigcup_{j_1,\dots,j_n=1}^{|\mathcal{E}|}} \left(\dot{\bigcup_{Q\in S_n}} D^{(s,Q)}_{j_1\cdots j_n} \right), \tag{5.3.7}$$

with the total dissected two-particle Hilbert space $\mathscr{H}_n^{(s,*)} = L^2(D^{(s,*)}_\Gamma)$. Thus vectors

$$\Psi = \left(\left(\psi^{(s,Q)}_{j_1\cdots j_n} \right)^{|\mathcal{E}|}_{j_1,\dots,j_n=1} \right)_{Q\in S_n} \tag{5.3.8}$$

in $\mathscr{H}_n^{(s,*)}$ are lists of square-integrable functions $\psi^{(s,Q)}_{j_1\cdots j_n}(x_1,\dots,x_n) : D^{(s,Q)}_{j_1\cdots j_n} \to \mathbb{C}$. The corresponding Sobolev space $H^2(D^{(s,*)}_\Gamma)$ is the set of $\Psi \in \mathscr{H}_n^{(s,*)}$ consisting of functions $\psi^{(s,Q)}_{j_1\cdots j_n} \in H^2(D^{(s,Q)}_{j_1\cdots j_n})$.

Boundary conditions will be imposed on eigenfunctions $\Psi \in H^2(D^{(s,*)}_\Gamma)$ of the n-particle Laplacian $-\Delta_n$. We reiterate here that these will be n-particle analogues of the boundary conditions (4.3.34)–(4.3.38) imposed in the two-particle setting. Before we proceed with establishing these conditions, it is convenient to define the permutations $\mathbb{Q}$ as representations of $Q \in S_n$ on

$$\bigotimes_{j=1}^{n} \mathbb{C}^{|\mathcal{E}|} \tag{5.3.9}$$

according to

1. $\mathbb{I} = \mathbb{I}_{|\mathcal{E}|^n}$ is the representation of I;

2. $\mathbb{T}^{(i)} = \mathbb{I}_{|\mathcal{E}|^{i-1}} \otimes \mathbb{T}_{|\mathcal{E}|^2} \otimes \mathbb{I}_{|\mathcal{E}|^{n-i-1}}$ is the representation of T_i.

We note the properties

$$\mathbb{T}^{(i)} \left(\mathcal{A}_{j_1\cdots j_n} \right)^{|\mathcal{E}|}_{j_1,\dots,j_{i-1},j_i,j_{i+1},j_{i+2}\cdots,j_n=1} \tag{5.3.10}$$

$$= \left(\mathcal{A}_{j_1\cdots j_n} \right)^{|\mathcal{E}|}_{j_1,\dots,j_{i-1},j_{i+1},j_i,j_{i+2},\dots,j_n=1} \tag{5.3.11}$$

for $|\mathcal{E}|^n$-dimensional column vectors $\mathcal{A}$ and that

$$\begin{aligned}&\mathbb{T}^{(i)}(M_1 \otimes \cdots \otimes M_{i-1} \otimes M_i \otimes M_{i+1} \otimes M_{i+2} \otimes \cdots \otimes M_n)\mathbb{T}^{(i)} \\ &= M_1 \otimes \cdots \otimes M_{i-1} \otimes M_{i+1} \otimes M_i \otimes M_{i+2} \otimes \cdots \otimes M_n\end{aligned} \tag{5.3.12}$$

for any $|\mathcal{E}| \times |\mathcal{E}|$ matrices M_j.

Let us define boundary vectors

$$\begin{aligned}\Psi_{\mathrm{bv}}^{(v)} &= \left(\mathbb{Q}^{-1}(\psi^{Q}_{j_1 \dots j_n}(x_1, \dots, x_n)|_{x_{Q1}=0})^{|\mathcal{E}|}_{j_1,\dots,j_n=1}\right)_{Q \in S_n}; \\ \Psi_{\mathrm{bv}}^{(v)\prime} &= \left(\mathbb{Q}^{-1}(\psi^{Q}_{j_1 \dots j_n, Q1}(x_1, \dots, x_n)|_{x_{Q1}=0})^{|\mathcal{E}|}_{j_1,\dots,j_n=1}\right)_{Q \in S_n},\end{aligned} \tag{5.3.13}$$

where $\psi^{Q}_{j_1 \dots j_n, Q1}(x_1, \dots, x_n)$ are inward derivatives normal to the line $x_{Q1} = 0$. Single-particle interactions with the vertices are then prescribed by local boundary conditions

$$(\mathbb{I}_{n!} \otimes A \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}})\Psi_{\mathrm{bv}}^{(v)} + (\mathbb{I}_{n!} \otimes B \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}})\Psi_{\mathrm{bv}}^{(v)\prime} = 0 \tag{5.3.14}$$

with $|\mathcal{E}| \times |\mathcal{E}|$ matrices A and B defined as in (2.4.6).

We would like to impose $\tilde{\delta}$-type interactions between particles located on the same infinite star and impose continuity across dissections otherwise. We then have the conditions

$$\begin{aligned}&\psi^{Q}_{j_{Q^{-1}1} \dots j_{Q^{-1}n}}(x_1, \dots, x_n)|_{x_{Qi}=x_{Q(i+1)}} \\ &= \psi^{QT_i}_{j_{T_i Q^{-1}1} \dots j_{T_i Q^{-1}n}}(x_1, \dots, x_n)|_{x_{Qi}=x_{Q(i+1)}}; \\ &\left(\frac{\partial}{\partial x_{Q(i+1)}} - \frac{\partial}{\partial x_{Qi}} - 2\alpha\right)\psi^{Q}_{j_{Q^{-1}1} \dots j_{Q^{-1}n}}(x_1, \dots, x_n)|_{x_{Qi}=x_{Q(i+1)}} \\ &= \left(\frac{\partial}{\partial x_{Q(i+1)}} - \frac{\partial}{\partial x_{Qi}}\right)\psi^{QT_i}_{j_{T_i Q^{-1}1} \dots j_{T_i Q^{-1}n}}(x_1, \dots, x_n)|_{x_{Qi}=x_{Q(i+1)}},\end{aligned} \tag{5.3.15}$$

for $f(e_{j_i}) = f(e_{j_{i+1}})$ and

$$
\begin{aligned}
&\psi^{Q}_{j_{Q^{-1}1}\cdots j_{Q^{-1}n}}(x_1,\dots,x_n)|_{x_{Qi}=x_{Q(i+1)}} \\
=&\psi^{QT_i}_{j_{Q^{-1}1}\cdots j_{Q^{-1}n}}(x_1,\dots,x_n)|_{x_{Qi}=x_{Q(i+1)}}; \\
&\left(\frac{\partial}{\partial x_{Q(i+1)}}-\frac{\partial}{\partial x_{Qi}}\right)\psi^{Q}_{j_{Q^{-1}1}\cdots j_{Q^{-1}n}}(x_1,\dots,x_n)|_{x_{Qi}=x_{Q(i+1)}} \\
=&\left(\frac{\partial}{\partial x_{Q(i+1)}}-\frac{\partial}{\partial x_{Qi}}\right)\psi^{QT_i}_{j_{Q^{-1}1}\cdots j_{Q^{-1}n}}(x_1,\dots,x_n)|_{x_{Qi}=x_{Q(i+1)}},
\end{aligned}
\tag{5.3.16}
$$

for $f(e_{j_i}) \neq f(e_{j_{i+1}})$.

The task is now to specify eigenvectors $\Psi \in H^2(D_\Gamma^{(s,*)})$ which satisfy boundary conditions (5.3.14)-(5.3.16). Taking care to distinguish between subdomains $D^{(s,Q)}_{j_1\cdots j_n}$, the vector Ψ will be described by the collection of functions

$$\psi^{Q}_{j_1\cdots j_n} = \sum_{P\in\mathscr{W}_n} \mathcal{A}^{(P,Q)}_{j_1\cdots j_n} e^{i(k_{P1}x_1+\cdots+k_{Pn}x_n)}. \tag{5.3.17}$$

This form obviously leads to eigenfunctions of $-\Delta_n$ with Laplace eigenvalues (5.2.18).

Let us define the $|\mathcal{E}|^n$-dimensional vectors

$$\mathcal{A}^{(P,Q)} = \left(\mathcal{A}^{(P,Q)}_{j_1\cdots j_n}\right)^{|\mathcal{E}|}_{j_1,\dots,j_n=1} \tag{5.3.18}$$

and then the $n!|\mathcal{E}|^n$-dimensional vectors

$$\mathcal{A}^{P} = \left(\mathbb{Q}^{-1}\mathcal{A}^{(PQ^{-1},Q)}\right)_{Q\in S_n}. \tag{5.3.19}$$

It is convenient at this point to impose an ordering on (5.3.19) by associating with each element Q the number $[Q] \in (1,\dots,n!)$ so that

$$\mathbb{Q}^{-1}\mathcal{A}^{(PQ^{-1},Q)} \tag{5.3.20}$$

is the $[Q]^{\text{th}}$ block in the list $\mathcal{A}^P$.

Boundary conditions at the vertices (5.3.14) imply the relations

$$\mathbb{Q}^{-1}\mathcal{A}^{(PR_{Q1},Q)} = \left(S_v(-k_{PQ1}) \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}}\right)\mathbb{Q}^{-1}\mathcal{A}^{(P,Q)}. \tag{5.3.21}$$

Noting then, that the properties of $\mathscr{W}_n$ imply

$$R_{Q1} = QR_1Q^{-1}, \tag{5.3.22}$$

we have that

$$\mathcal{A}^{PR_1} = \mathbb{I}_{n!} \otimes S_v(-k_{P1}) \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}}\mathcal{A}^P. \tag{5.3.23}$$

At this point, it is convenient to define the diagonal matrices

$$\boldsymbol{c}_i = \mathrm{diag}(c^{(i)}_{j_1\dots j_n})^{|\mathcal{E}|}_{j_1,\dots,j_n=1}, \tag{5.3.24}$$

where

$$c^{(i)}_{j_1\dots j_n} = \begin{cases} 1 & \text{if } f(e_{j_i}) = f(e_{j_{i+1}}); \\ 0 & \text{otherwise,} \end{cases} \tag{5.3.25}$$

which distinguish domains with $\tilde{\delta}$-type interactions from those which are continuous across dissections. We notice here the relations

$$\boldsymbol{c}_i = \mathbb{I}_{|\mathcal{E}|^{i-1}} \otimes \boldsymbol{c} \otimes \mathbb{I}_{|\mathcal{E}|^{n-i-1}}. \tag{5.3.26}$$

The $\tilde{\delta}$-type conditions (5.3.15) and continuity conditions (5.3.16) imply the relations

$$\begin{aligned}(\mathbb{I}_2 \otimes \boldsymbol{c}_i)&\begin{pmatrix}\mathbb{Q}^{-1}\mathcal{A}^{(PT_iQ^{-1},Q)} \\ \mathbb{T}^{(i)}\mathbb{Q}^{-1}\mathcal{A}^{(PQ^{-1},QT_i)}\end{pmatrix} \\ &= \left(S_p(k_{Pi} - k_{P(i+1)}) \otimes \mathbb{I}_{|\mathcal{E}|^n}\right)(\mathbb{I}_2 \otimes \boldsymbol{c}_i)\begin{pmatrix}\mathbb{Q}^{-1}\mathcal{A}^{(PQ^{-1},Q)} \\ \mathbb{T}^{(i)}\mathbb{Q}^{-1}\mathcal{A}^{(PT_iQ^{-1},QT_i)}\end{pmatrix}\end{aligned} \tag{5.3.27}$$

and

$$(\mathbb{I}_{|\mathcal{E}|^n} - \boldsymbol{c}_i)\mathbb{Q}^{-1}\mathcal{A}^{(PQ^{-1},Q)} = (\mathbb{I}_{|\mathcal{E}|^n} - \boldsymbol{c}_i)\mathbb{Q}^{-1}\mathcal{A}^{(PQ^{-1},QT_i)} \tag{5.3.28}$$

respectively. We then have that

$$\mathcal{A}^{PT_i} = Y_i(k_{Pi} - k_{P(i+1)})\mathcal{A}^P, \tag{5.3.29}$$

where

$$(Y_i(k))_{[Q][Q']} = \frac{-i\alpha}{k + i\alpha}\boldsymbol{c}_i\delta_{[Q][Q']} + \left(\frac{k}{k + i\alpha}\boldsymbol{c}_i + \mathbb{T}^{(i)}\left(\mathbb{I}_{|\mathcal{E}|^n} - \boldsymbol{c}_i\right)\right)\delta_{[QT_i][Q']}. \tag{5.3.30}$$

Exact solvability is assured if relations (5.3.23) and (5.3.29) are compatible with the properties of $\mathscr{W}_n$. This amounts to the consistency relations

1. $S_v(u)S_v(-u) = \mathbb{I}_{|\mathcal{E}|}$;
2. $Y_i(u)Y_i(-u) = \mathbb{I}_{n!|\mathcal{E}|^n}$;
3. $Y_i(u)Y_j(v) = Y_j(v)Y_i(u)$ for $|i - j| > 1$;
4. $Y_{i+1}(u)Y_i(u + v)Y_{i+1}(v) = Y_i(v)Y_{i+1}(u + v)Y_i(u)$;
5. $\left(\mathbb{I}_{n!} \otimes S_v(u) \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}}\right) Y_1(u + v) \left(\mathbb{I}_{n!} \otimes S_v(v) \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}}\right) Y_1(v - u)$
$= Y_1(v - u) \left(\mathbb{I}_{n!} \otimes S_v(v) \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}}\right) Y_1(u + v) \left(\mathbb{I}_{n!} \otimes S_v(u) \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}}\right)$;
6. $Y_i(u)\left(\mathbb{I}_{n!} \otimes S_v(v) \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}}\right) = \left(\mathbb{I}_{n!} \otimes S_v(v) \otimes \mathbb{I}_{|\mathcal{E}|^{n-1}}\right) Y_i(u)$ for $i > 1$.

These conditions can be verified by the explicit forms of $S_v(k)$ (for local boundary conditions) and $Y_i(k)$ using the commutation relations (4.3.46), (4.3.45) and (3.3.28) alongside the decompositions of $\boldsymbol{c}_i$ and $\mathbb{T}^{(i)}$ in terms of $\boldsymbol{c}$ and $\mathbb{T}_{|\mathcal{E}|}$ prescribed above.

In order to turn the eigenfunctions in the star representation into eigenfunctions on the compact graph, we must impose appropriate joining conditions. These are analogues of (4.3.47)–(4.3.48) and are written

$$\psi^Q_{j_1\cdots j_n}(x_1, \ldots, x_n) = \psi^Q_{j'_1\cdots j'_n}(x'_1, \ldots, x'_n) \tag{5.3.31}$$

for all $Q \in S_n$, where

$$(x'_{Q1}, \ldots, x'_{Qn}) = (x_{Q1}, \ldots, x_{Q(n-1)}, l_{j_{Qn}} - x_{Qn}) \tag{5.3.32}$$

and

$$(j'_{Q1},\dots,j'_{Qn}) = (j_{Q1},\dots,j_{Q(n-1)},j_{Qn}+|\mathcal{E}|). \tag{5.3.33}$$

We then have

$$\begin{aligned}\mathcal{A}^{P} &= E(-k_{Pn})\mathcal{A}^{PR_n}\\ &= E(-k_{Pn})\mathcal{A}^{PC_nR_1C_n^{-1}},\end{aligned} \tag{5.3.34}$$

where

$$E(k) = \mathbb{I}_{n!|\mathcal{E}|^{n-1}} \otimes \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \otimes e^{ik\boldsymbol{l}}. \tag{5.3.35}$$

Applying (5.3.23), (5.3.29) and (5.3.34) successively we have that the relation

$$Z(k_{P1},\dots,k_{Pn}) = 0, \tag{5.3.36}$$

with

$$\begin{aligned}Z(k_1,\dots,k_n) = \det\Big[&\mathbb{I}_{n!|\mathcal{E}|^n} - E(k_n)Y_{n-1}(k_n-k_{n-1})\dots Y_1(k_n-k_1)\\ &\big(\mathbb{I}_{n!}\otimes S_v(k_n)\otimes \mathbb{I}_{|\mathcal{E}|^{n-1}}\big)Y_1(k_1+k_n)\dots Y_{n-1}(k_{n-1}+k_n)\Big],\end{aligned} \tag{5.3.37}$$

is satisfied for all $P \in \mathscr{W}_n$. By using properties of determinants, it can be shown that the explicit forms of $Y_i(k)$, $S_v(k)$ and $E(k)$ are such that if (5.3.36) is satisfied for some $P \in \mathscr{W}_n$, then it is necessarily satisfied for elements

$$PT_1,\dots,PT_{n-2}, R_1 \text{ and } R_n \tag{5.3.38}$$

in $\mathscr{W}_n$ and thus for every

$$PX \text{ and } PR_nX \tag{5.3.39}$$

in $\mathscr{W}_n$ with $X \in \mathscr{W}_{n-1}$. Using (5.1.11), we can state the main result of this section.

Theorem 5.3.1. Non-zero eigenvalues of a self-adjoint n-particle Laplacian $-\Delta_n$ defined on Γ with local vertex interactions specified through A, B and $\tilde{\delta}$-type interactions between particles when they are located on neighbouring edges, are the values $E = k_1^2 + \dots + k_n^2 \neq 0$ with multiplicity m, where $(k_1,\dots,k_n)$, such that

$0 \le k_1 \le \cdots \le k_n$, are solutions to the n secular equations

$$Z(k_{i_1}, \ldots, k_{i_n}) = 0, \tag{5.3.40}$$

for $(i_1, \ldots, i_n) \in \{C_n^d(1, \ldots, n)\}_{d=0}^{n-1}$, with multiplicity m.

5.3.1 Recovering specific results

In this final section, by choosing particular parameters, we show how to recover established results from the general n-particle quantisation condition prescribed by Theorem 5.3.1.

Equilateral stars

Let us recover the spectra of n-particle equilateral stars Γ_e. We begin by imposing vertex conditions (4.2.19) and (4.2.20) and equal lengths $l_j = l$ for all $j \in \{1, \ldots, |\mathcal{I}|\}$. We recall that in the star representation of two-particle equilateral stars, $\tilde{\delta}$-type interactions were imposed along the diagonals $x_1 = x_2$ of domains $D_{mn}^{(s)}$ with $(e_m, e_n) \in \mathcal{N}_e$. Extending this notion to n particles, the diagonal matrices $\boldsymbol{c}$ in $Y_i(k)$, are replaced with $\boldsymbol{c}_e$ as prescribed by (4.2.50). Substituting these parameters into (5.3.37) we recover the spectra for n-particle equilateral stars.

Simply by choosing $|\mathcal{I}| = 2$, one immediately recovers the spectra of n particles in a box with a central impurity. By instead defining vertex conditions according to (4.2.63) to establish continuity at the outer vertices, we recover the spectra of systems of two particles on a circle with an impurity. These spectra are exactly those prescribed in [CC07] (see Proposition 3.1 therein).

Non-interacting particles

Non-interacting systems are recovered by turning off all coupling between domains D_{mn} and D_{nm}. This is achieved by setting $\boldsymbol{c} = \boldsymbol{0}$. Matrices

$$Y_i(k)|_{\boldsymbol{c}=\boldsymbol{0}} \tag{5.3.41}$$

are then composed of blocks

$$(Y_i(k)|_{\boldsymbol{c}=\boldsymbol{0}})_{[Q][Q']} = \mathbb{T}^{(i)}\delta_{[QT_i][Q']}. \tag{5.3.42}$$

By substituting into (5.3.37) we recover the secular equation (2.3.9) for the one-particle quantum graph.

Chapter 6

Summary and Outlook

In this thesis, we have constructed exactly solvable many-particle quantum graphs with boundary conditions which provide self-adjoint realisations of the Laplacian. Using the Bethe ansatz, we calculated and analysed their spectra.

We began by introducing basic concepts and ideas associated with one-particle graphs before introducing the Bethe ansatz in the context of simple, exactly solvable, two-particle systems. We then constructed general two-particle quantum graphs by establishing self-adjoint realisations of the two-particle Laplacian which prescribe single particle interactions with the vertices as well as δ-type particle interactions. Such systems are, in general, not exactly solvable. Adapting the approach so that self-adjoint Laplacians instead prescribe non-local $\tilde{\delta}$-type particle interactions, we constructed exactly solvable two-particle equilateral star graphs. In this setting we introduced two methods for calculating spectra using the Bethe ansatz. The latter was used in extending the approach to general two-particle quantum graphs, finally arriving at an exact expression for the spectra

$$E = k_1^2 + k_2^2, \tag{6.0.1}$$

with (k_1, k_2) given by solutions to a pair of secular equations

$$Z(k_1, k_2) = 0 \text{ and } Z(k_2, k_1) = 0. \tag{6.0.2}$$

For two examples, we performed numerical eigenvalue searches to obtain explicit spectra. We then compared the spectral counting function and level-spacings distribution with known results in quantum graphs and random matrix theory. Mostly, Poissonian statistics resulted, with some level repulsion detected in examples of equilateral stars. None of the examples, however, reproduced clear GOE statistics. Spectral counting functions, in general, did not agree with the two-particle Weyl law for contact interactions. Agreement, however, was observed in the equilateral star case. Finally, we extended the Bethe ansatz approach to n-particle quantum graphs with $\tilde{\delta}$-interactions, deducing an exact expression for the spectra

$$E = \sum_{j=1}^{n} E_j^2 \tag{6.0.3}$$

with $(k_1, \ldots, k_n)$ given by solutions to the collection of n secular equations

$$Z(k_{i_1}, \ldots, k_{i_n}) = 0, \tag{6.0.4}$$

for $(i_1, \ldots, i_n)$ equal to cyclic permutations of $(1, \ldots, n)$.

There are several directions for further research in this area. Firstly, quantisation conditions in the form of secular equations provide the possibility to establish a many-particle quantum graph trace formula analogous to (2.5.2) for the one-particle quantum graph. Such an expression would provide an analytical connection between the spectra of many-particle quantum graphs and the dynamics of their classical counterparts in terms of periodic orbits.

Using the trace formula, or otherwise, one might wish to deduce an appropriate Weyl law for quantum graphs with $\tilde{\delta}$-interactions. Indeed this would shed light on the apparent disagreement between the spectral counting functions for certain two-particle quantum graphs and the Weyl laws (1.0.26)–(1.0.27) which are valid for contact interactions.

In each of the examples studied in Sections 3.4 and 4.4, we referred to the assertion [BGS84] that systems which are chaotic in their classical limit exhibit spectral statistics which follow GOE predictions. It is well known [KS97] that this is the case for one-particle quantum graphs. Indeed we showed this for the one-particle tetrahedron. However, none of the two-particle quantum graphs we studied exhibited clear GOE statistics. In [STL+16], it was shown that, when coupling individually chaotic systems, the resulting spectra reproduce GOE statistics in the limit of a large number of systems. Of course, in this thesis, the n-particle spectra have been deduced exactly and thus, in principle, it is possible to test this argument for n-particle quantum graphs. However this amounts to eigenvalue searches in high dimensions which is computationally expensive. Revealing numerical results could be obtained either by increasing computational power or developing efficient root-finding methods. Additionally, it may be possible to reduce the dimension of certain secular equations by making use of inherent symmetries associated with specific examples.

Exact solvability on general many-particle quantum graphs was assured by imposing $\tilde{\delta}$-interactions between particles. In this way we constructed explicit Laplace eigenfunctions using the Bethe ansatz. Of course there may be other possible types of particle interaction which lead to exactly solvable models. It would then be interesting to compare the spectral statistics of such models.

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