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Order-Driven Markets are Almost Competitive

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Abstract

This paper studies a market game under uncertainty in which agents may submit multiple limit and market orders. When agents know their preferences at all states, the competitive equilibrium can be supported as a Nash equilibrium of the market game, that is, agents behave as if they were price takers. Therefore, if the associated competitive economy has a fully revealing rational expectations equilibrium, then so does the market game. This resolves the puzzle that agents behave as if prices were given, even though prices aggregate private information, at least for this "private values" case. Necessary conditions for Nash equilibrium show that the resulting allocation cannot deviate too far from a competitive equilibrium. When agents do not know their preferences at some states, though, a characterization result shows that the Nash equilibria of the market game tend to be far from competitive.

Keywords. Competitive equilibrium; market game; limit order book; rational expectations equilibrium.

JEL classification. C72; D47; D53; D82; G14.

1 Introduction

When it comes to the analysis of markets there are two basic paradigms: At a *competitive equilibrium* all agents take prices as given; *strategic agents*, on the other hand, take into account the impact of their trades on market prices. In the former approach prices are determined by an abstract market clearing condition; in the latter they result from the traders' interaction in a strategic (Nash) equilibrium.

The reconciliation of these two concepts has been a longstanding concern in economics and finance (see Subsection 1.1 for a literature review). This endeavor becomes even more challenging when uncertainty is involved and prices are supposed to reveal information. For, as Beja (1977) and Hellwig (1980) observe, at a competitive equilibrium with uncertainty traders appear to behave "schizophrenically," taking prices as given when trading, even though they infer information from them. Hence, private information must influence prices if it is to be reflected by them.

This paper tackles the strategic foundations of competitive equilibrium by a market game under uncertainty in which agents submit multiple limit and market orders. Limit and market orders are the two dominant order types on stock exchanges. A limit order is an ex-ante commitment to buy or sell up to a specified limit quantity not above or not below a specified limit price. A market order only specifies a limit quantity, but no limit price. Executable trades are determined by ranking orders. That is, buy orders that bid a higher price obtain priority over those that bid lower prices, and sell orders that ask a lower price are given priority over those asking higher prices. The resulting supply and demand schedules determine which orders are fulfilled.

The clearing mechanism requires that executed trades on the same side of the market pay resp. receive the same price—executed purchases pay the market bid price, and executed sales receive the market ask price. This is in deviation from the literature, which typically assumes that the execution prices are the traders' limit prices (see, e.g., Wilson, 1977; Dubey, 1982; Simon, 1984; Parlour, 1998; Glosten, 1994, calls this a *discriminatory* limit order book). Empirically, on the other hand, many markets have the obligation to treat orders symmetrically, ruling out such discriminatory practices.[1] In fact, the clearing mechanism is

[1] This is also the spirit of Regulations ATS and NMS, as implemented by the U.S. Security and Exchange Commission in 1998 and 2007, respectively (see Hendershott and Jones, 2005). Roughly, those require that customers at one market platform are offered the best quote at any other market nationwide.

inspired by the electronic limit order books run on electronic communication networks, like Island, BATS, Direct Edge, Instinet, IEX, Chi-X Europe, or Archipelago.[2] Those are the market platforms to which the bulk of trading in stocks and exchange traded funds has migrated during the past two decades. These financial markets are thus quantitatively important. Therefore, unlike the literature, this paper studies an *undiscriminatory* limit order book (in the terminology of Glosten, 1994).

The second novelty concerns information. While most of the literature has focused on the case of certainty, this paper introduces *uncertainty* about the state of the world. This is why *multiple* orders are allowed. Agents, who are uncertain about which state obtains, may hedge by submitting several orders that will execute in different events.

The results about the undiscriminatory limit order book with uncertainty are as follows. When agents privately know their preferences, they justify the concept of competitive equilibrium with uncertainty (or *rational expectations equilibrium*), even though agents act strategically. First, there is always an equilibrium that generates precisely the competitive allocation, irrespective of how many agents there are (Theorem 1 below). Consequently, if the competitive economy has a fully revealing rational expectations equilibrium, then the market game has a Nash equilibrium that induces the same fully revealing prices. Second, for all Nash equilibria the associated equilibrium prices stay in a vicinity of the competitive prices in the following sense: They cannot deviate further from competitive prices than the latter would (from their original values) if one agent were removed from the economy (Theorem 2 below).

The undiscriminatory limit order book thus achieves two goals. First, it provides a trading rule which allows for a Nash equilibrium that corresponds to a competitive allocation, even if the number of traders is finite. A limit results for the number of agents going to infinity (as, e.g., in Rustichini, Satterthwaite, and Williams, 1994, or Forges and Minelli, 1997) is, therefore, not needed. Instead, the impact of an individual agent on prices in a Nash equilibrium is captured by how much her removal would change competitive prices. Second, the limit order book resolves the puzzle that traders behave as if they were taking

[2] Archipelago was acquired by the New York Stock Exchange (NYSE). A part of Instinet was spun off, merged with Island into Inet, and acquired by Nasdaq. The remainder of Instinet launched the electronic trading platform Chi-X Europe. Most Euronext markets, which includes the NYSE as well as the exchanges in Amsterdam, Brussels, Lisbon, and Paris, and the futures market in London, also operate electronic limit order books in addition to floor trading.

prices as given, yet take into account the information that is incorporated in prices. That is, agents are aware of their influence on prices when planning their trades, yet they end up behaving as if they were price takers. This also holds under uncertainty, provided the agents know their preferences and can place sufficiently many orders.

The intuition for this result is as follows. The very nature of limit orders induces constraints on an individual trader that look like step functions (in quantity-price space), because residual supply and demand are step functions. (A price-taking agent would face a degenerate step function given by a single fixed price for all quantities.) Hence, at most feasible allocations a trader has to take a price as given—given by the limit price that some other market participant chooses. Therefore, locally she optimally behaves *as if* she were a price-taker who faces a globally infinite price elasticity.[3]

The only exception occurs when the trader has no competition on the same side of the market at the price under scrutiny. This corresponds to a "market corner"—and a feasible corner of the opportunity set. If each agent is small compared to the aggregate, every trader will face competition on her side of the market at the realized price and market corners cannot occur.

This intuition carries over to the case of uncertainty, provided multiple orders are allowed. By placing an order for each event that she regards possible a trader can hedge against all contingencies. This allows a trader to incorporate into her orders the information that she anticipates for the various events. That is, agents do not infer information from the price *before* they place an order, as in the competitive model. Instead, they foresee for all possible contingencies what the market will yield, inclusive of the prices in these events, and place optimal orders for all event. After trades execute, traders learn which event has realized (and the associated price) by observing which orders were executed. In this sense prices are informative ex-post, but not ex-ante. Still, they do aggregate private information.

Thus, whether or not uncertainty is involved, there are market organizations that induce finitely many traders to act as if they were price takers, when in fact they behave strategically. Hence, price taking behavior may be a reasonable approximation to these market outcomes, provided agents are privately informed

[3] That agents behave locally as if they were price-takers contrasts with models of competition in supply functions. If one *assumes* that multiple orders give rise to a continuous function, the slope of this residual demand function facing an individual trader is necessarily bounded away from zero; hence, the key effect is lost.

about their preferences. Further, under uncertainty private information can get revealed by prices even though the equilibrium allocation is as if agents did not recognize their influence on prices.

Yet, the results are comforting for the concept of competitive equilibrium with uncertainty only to a certain extent. In particular, that each agent always knows her preferences is (almost) necessary for these results. For, without this condition monotonicity of demand functions is lost, i.e., an agent may demand more at a higher price or less at a lower price without being able to distinguish between these two events based on her private information. Because the limit order book ranks orders according to price priority, monotonicity of excess demand schedules is crucial for its operation. In fact, this condition *characterizes* when the Nash equilibria of the market game are close to competitive outcomes (Theorem 3 below). Not surprisingly (see, e.g., Schmeidler and Postlewaite, 1986; Palfrey and Srivastava, 1989; Blume and Easley, 1990), therefore, asymmetric information that concerns a "common value" component may be inconsistent with outcomes that resemble competitive behavior.

1.1 Relations to the Literature

There is an extensive literature on the strategic foundations of competitive equilibrium á la Arrow and Debreu (1954), e.g., Shapley and Shubik (1977), Shubik (1977), Wilson (1977, 1978, 1987), Schmeidler (1980), Dubey (1982, 1994), Simon (1984), Sorin (1996), Dubey and Geanakoplos (2003), or Mertens (2003); Giraud (2003) provides an overview. These papers study the case of certainty in order to focus on the properties of the market mechanism.

Uncertainty and the ability of prices to aggregate information has been addressed by a related but distinct literature. For instance, within the framework of competitive markets Hellwig (1982) and Blume and Easley (1984) study dynamic economies where traders condition on past information only. Kyle (1985) and Glosten and Milgrom (1985) develop models of over-the-counter markets where a market maker quotes prices and infers information from order flows. Another literature, starting from Wilson (1977) and Milgrom (1981), studies Vickrey-type auctions and how those aggregate information (see, e.g., Pesendorfer and Swinkels, 1997; Satterthwaite and Williams, 2002; Perry and Reny, 2006). Yet another approach considers competition in supply functions (see, e.g., Grossman, 1981b; Kyle, 1989; Klemperer and Meyer, 1989; Biais, Martimort, and Rochet, 2000). In finance the literature on market microstructure studies models where risk neutral traders arrive sequentially and repeatedly at the market and trade indivisible units under limit prices (see, e.g., Parlour, 1998; Foucault, 1999; Goettler, Parlour, and Rajan, 2005).

The narrower branch of literature concerned with *strategic market games* has by and large studied "competition in quantities"—á la Cournot, as it were—as formalized by the trading-post model (Shapley and Shubik, 1977). In that, buyers simultaneously deposit money and then receive quantities of the commodity in proportion to their shares in aggregate deposits. This literature has largely focused on the case of certainty. The few exception that allow uncertainty include Dubey, Geanakoplos, and Shubik (1987), who extend the trading-post model to a multi-period setting in which information is revealed from one period to the next, yet being better informed is still profitable. Forges and Minelli (1997) consider both a one-shot and a repeated version of the ("sell-all") trading-post model, amended by a pre-play communication stage in the spirit of correlated equilibrium (Aumann, 1974). Codognato and Ghosal (2003) extend this analysis to Shapley's "windows model" (see Sahi and Yao, 1989), also with an atomless continuum of traders. Peck (2014) uses a trading-post model to study price manipulation by informed "bulls" and "bears."

Dubey (1982) and Simon (1984) are the seminal contributions that introduce market games with price competition by allowing limit orders—á la Bertrand, as it were. Both obtain a competitive allocation as the outcome of a Nash equilibrium. Yet, once again these models are stated under certainty; and they assume that the execution prices are the traders' limit prices. The model by Mertens (2003) also assumes certainty, but is set up in such a way that all trades are executed at the same price. Yet, limit orders in Mertens' model are the supply functions of artificial agents with linear utility functions. Hence, when such an agent sells at a price strictly above her limit price, the order must be fully executed. Hence, no rationing can occur, while in the present model rationing is possible (even though it does not occur in equilibrium).

As far as I can tell, there is no paper that applies market games with price competition to the case of uncertainty. Similarly, an undiscriminatory limit order book has not been studied so far. In particular, that a competitive equilibrium under uncertainty is the outcome of a strategic market game with finitely many players is a new result.

The remainder of the paper is organized as follows. Section 2 presents the model, the benchmark competitive equilibrium, and the market game. Section 3 identifies necessary and sufficient conditions for the Nash equilibria of the Bayesian market game with private values. Section 4 discusses economies, where the agents' utility functions are not necessarily measurable with respect to private information, and characterizes when the market game has equilibria close to a competitive allocation for this case. Section 5 concludes. All proofs are relegated to the Appendix.

2 The Model

Consider an economy with two goods $j = 1, 2$ and a finite number of agents $i \in I = \{1, ..., n\}$ for $n > 1$. Two commodities are assumed for simplicity.[4] The agents' preferences over their final holdings of the two goods are represented by utility functions $u : \mathbb{R}^2_+ \to \mathbb{R}$ that are continuously differentiable, strictly increasing in both arguments, strictly quasi-concave, and are such that the (excess) demand functions for good 2 are strictly decreasing in the price of good 2. (A sufficient condition for this is that the two goods are gross substitutes.) Let $\mathcal{U}$ denote the set of all utility functions with these properties. Differentiability is assumed for convenience, and monotonicity means that both goods are desirable. That demand is downward sloping in the price is assuming the "law of demand," which will be important in Section 3. Under expected utility strict quasi-concavity will imply (strict) risk aversion. Without expected utility (strict) risk aversion is assumed, i.e., for each non-degenerate lottery an agent strictly prefers the associated expected allocation for sure over the lottery.

The economy can be in a number of *states* $\varpi \in \Omega$ that determine the agents' characteristics. The latter consist of a utility function, an endowment vector, and an information partition for each agent. That is, for each agent $i \in I$ there is a finite partition T_i of the state space Ω that summarizes player i's private information.[5] The functions (random variables) $f = (f_i)_{i \in I} : \Omega \to \left(\mathcal{U} \times \mathbb{R}^2_{++}\right)^n$ assign to each agent $i \in I$ a utility function $f_{i1}(\varpi) = u_i(\cdot|\varpi) \in \mathcal{U}$ and an endowment vector $f_{i2}(\varpi) = \mathbf{w}_i(\varpi) = (w_{i1}(\varpi), w_{i2}(\varpi)) \gg 0$ for each $\varpi \in \Omega$.

[4] The online appendix to this paper contains the game specification for the case of more than two goods and a generalization of Theorem 1 to an arbitrary but finite number of commodities. This is feasible because more goods will not affect what happens on the equilibrium path. Difficulties would only arise *off* the equilibrium path. For, there the effects of and penalties for bankruptcy (see below) render the system open; that is, outside of equilibrium resources may leak out of the economy—but this is also the case in general equilibrium theory.

[5] Whether the state space Ω is finite or not is immaterial. What counts is that the partition T_i is finite for each $i \in I$. If Ω is infinite, then utility functions are to be interpreted as expected utility functions with respect to a (possibly subjective) prior.

(Clearly, that $f_{i1} \in \mathcal{U}$ is equivalent to state-dependent utility.) To specify a prior probability measure on the state space Ω will not be necessary. For each agent $i \in I$ both coordinates of $f_i : \Omega \to \mathcal{U} \times \mathbb{R}^2_{++}$ are assumed measurable with respect to the partition T_i, that is, $f_i^{-1}(u, \mathbf{w}) \in T_i$ for all $(u, \mathbf{w}) \in f_i(\Omega)$. In line with the standard terminology in game theory a cell $t_i \in T_i$ of player i's information partition is referred to as a *type* of player $i \in I$, since each cell is the preimage of a utility-endowment pair.

That $f_{i2} = \mathbf{w}_i$ is measurable is merely the statement that agents observe their endowments in each state. That $f_{i1} = u_i$ is measurable is a popular assumption in the literature on economies with differential information. Still, this is a strong assumption. It rules out asymmetric information in the sense that an agent holds private information that is relevant to the preferences of someone else, e.g., superior information about the return of an asset. The assumption that f_i is measurable with respect to T_i will be referred to as "private values."[6] It captures a case, where the agents know the utility that they get from each possible trade. Intuitively, the measurability assumption says that agents always (for each trade) know what they want and what they own, but not necessarily in which environment they live. Section 4 will discuss what happens if measurability of $f_{i1} = u_i$ with respect to T_i is violated—the case of "common values."

With the measurability assumption it is justified to write $\mathbf{w}_i(\varpi) = \mathbf{w}_i(t_i)$ for all $\varpi \in t_i$, all $t_i \in T_i$, and all $i \in I$. Since utility functions $u_i(\cdot\,|\varpi) = f_{i1}(\varpi)$ are also measurable with respect to T_i, the notation $u_i(\cdot\,|t_i)$ refers to the utility of type $t_i \in T_i$ of agent i. In the definition of competitive equilibrium (below) utility functions are *not* assumed measurable; in that case it is assumed that agents are expected utility maximizers, and $u_i(\cdot\,|\pi)$ refers to the (conditional) expected utility of agent i (with respect to some prior distribution on Ω) at an event π from an information partition Π_i that refines T_i. For instance, if type t_i of agent i also observes the market price and infers information from it, her information partition Π_i may be finer than T_i.

Denote by $\mathcal{S} = \bigwedge_{i \in I} T_i = \left\{ \bigcap_{i \in I} t_i \mid t_i \in T_i, \forall i \in I \right\}$ the coarsest common refinement of the partitions T_i. Then the partition $\mathcal{S}$ represents all the information that is available in the economy. Since $\mathcal{S}$ incorporates all the information there is, one may assume without loss of generality that $\mathcal{S} = \{\{\varpi\} \mid \varpi \in \Omega\}$.

[6] This terminology is borrowed from Forges and Minelli (1997, p. 401). Admittedly, this is somewhat inappropriate. For, the term "private values" is used in the auction literature for an independence assumption—that valuations are i.i.d.

Still, the elements of $\mathcal{S}$ are called *events*. For each $s \in \mathcal{S}$ and every agent $i \in I$ let $\tau_i(s) \in T_i$ denote the unique type that satisfies $s \subseteq \tau_i(s)$, i.e., the type of i that occurs at event $s \in \mathcal{S}$. And, for each agent $i \in I$ denote by $\tau_i^{-1}(t_i) = \{s \in \mathcal{S} \mid s \subseteq t_i\}$ the events that type $t_i \in T_i$ regards possible.

2.1 Competitive Equilibrium

The benchmark for the economy are its competitive equilibria, or more precisely its competitive rational expectations equilibria (Radner, 1968, 1972; Lucas, 1972; Grossman, 1977, 1981a). For those it is not assumed that utility functions $f_{i1} = u_i$ are measurable, but the assumption that endowments $f_{i2} = w_i$ are measurable is maintained.

To develop competitive equilibrium, normalize relative prices such that the price of good $j = 1$ is 1, and start with a hypothetical economy in which all agents have access to the pooled information (the partition $\mathcal{S}$). A *full communication equilibrium*[7] for the economy is an allocation function $\mathbf{x} = (\mathbf{x}_i)_{i \in I} : \mathcal{S} \to \mathbb{R}_+^{2n}$ together with a price function $p^* : \mathcal{S} \to \mathbb{R}_+$ such that, for each agent $i \in I$, all events $s \in \mathcal{S}$, and all consumption vectors $\hat{\mathbf{x}} \in \mathbb{R}_+^2$,

$$\begin{gathered}(1, p^*(s)) \cdot \mathbf{w}_i(\tau_i(s)) \geq (1, p^*(s)) \cdot \hat{\mathbf{x}} \Rightarrow u_i(\mathbf{x}_i(s) \,|s) \geq u_i(\hat{\mathbf{x}} \,|s), \text{ and} \\ \sum_{i \in I} \mathbf{x}_i(s) \leq \sum_{i \in I} \mathbf{w}_i(\tau_i(s)). \end{gathered} \tag{1}$$

On the market, though, agents do not necessarily have access to the pooled information. Yet, they may infer information from the price. In particular, for a given price function $\hat{p} : \mathcal{S} \to \mathbb{R}_+$ let $\Pi_i(\hat{p}) = \left\{\tau_i(s) \cap \hat{p}^{-1}(\hat{p}(s)) \,| s \in \mathcal{S}\right\}$ be the partition generated by the types and (the preimages of) the price observations for agent $i \in I$.

A *competitive (rational expectations) equilibrium* $(\mathbf{x}, p)$ is again an allocation function $\mathbf{x} = (x_i)_{i \in I} : \mathcal{S} \to \mathbb{R}_+^{2n}$ together with a price function $p : \mathcal{S} \to \mathbb{R}_+$ such that, for each agent $i \in I$, all events $s \in \mathcal{S}$, and all consumption vectors $\hat{\mathbf{x}} \in \mathbb{R}_+^2$,

$$\begin{gathered}(1, p(s)) \cdot \mathbf{w}_i(\tau_i(s)) \geq (1, p(s)) \cdot \hat{\mathbf{x}} \Rightarrow \\ u_i\left(\mathbf{x}_i(s) \,\middle|\, \tau_i(s) \cap p^{-1}(p(s))\right) \geq u_i\left(\hat{\mathbf{x}} \,\middle|\, \tau_i(s) \cap p^{-1}(p(s))\right), \end{gathered} \tag{2}$$

each function $\mathbf{x}_i : \mathcal{S} \to \mathbb{R}_+^2$ is measurable with respect to the partition $\Pi_i(p)$,

[7] The term "full communication equilibrium" was coined by Radner (1979). Today it appears somewhat inappropriate, because communication is not explicitly modeled. A way to introduce explicit communication was proposed later by Forges and Minelli (1997).

and the market clearing condition (1) holds for all $s \in \mathcal{S}$.

The information incorporated in $\Pi_i(p)$ may or may not be coarser than $\mathcal{S}$ (resp. finer than T_i). To distinguish, a competitive equilibrium $(\mathbf{x}, p)$ is called *fully revealing* if the price function p is one-to-one, i.e., if $p(s) = p(s')$ implies $s = s'$ for all $s, s' \in \mathcal{S}$. That is, the price function in a competitive equilibrium is fully revealing, if it distinguishes the occurrence of any two events that can be distinguished by some agent. In that case $\Pi_i(p) = \mathcal{S}$ for all $i \in I$. If the competitive equilibrium is not fully revealing, then it is assumed that agents use some (possibly subjective) prior distribution on Ω to evaluate their utility functions. The above definition makes no statement about whether or not the competitive equilibrium is fully revealing.

Radner (1979) and Allen (1981) have established *generic* existence of fully revealing competitive equilibria. Universal existence of competitive equilibria would require that utility functions are measurable with respect to private information (private values) and strictly concave (for details see de Castro, Pesce, and Yannelis, 2011, Theorem 4.1 and Remark 4.2).[8] If utility functions are not measurable with respect to private information, Kreps (1977) gives a counterexample to existence with state-dependent and unobserved expected utility. A full communication equilibrium, on the other hand, always exists (see Hart, 1974, Theorem 3.3; and Jordan, 1983, Proposition 2.8). Clearly, a full communication equilibrium with an injective price function constitutes also a rational expectations equilibrium.

It is assumed throughout that the competitive equilibrium, if it exists, involves trade in at least some events. Furthermore, if a competitive equilibrium exists, with prices $p(s) = p_s$ for all $s \in \mathcal{S}$, then it involves no loss of generality to assume that $\mathcal{S} = \{1, ..., |\mathcal{S}|\}$ and $p_1 \leq p_2 \leq ... \leq p_{|\mathcal{S}|}$. If it is fully revealing, then the latter inequalities are all strict.

Because preferences are strictly increasing in both goods, the budget constraint (see (2)) must bind at a competitive equilibrium for all events and all agents. Therefore, for any price $p \in \mathbb{R}_{++}$ and any information partition Π_i that (weakly) refines T_i, agent i's excess demand function $\xi_i : \mathbb{R}_{++} \times \Pi_i \to \mathbb{R}_+$ (for good $j = 2$) can be defined by

$$\xi_i(p, \pi) = \arg \max_{-pw_{i2}(t_i) \leq px \leq w_{i1}(t_i)} u_i(w_{i1}(t_i) - px, w_{i2}(t_i) + x \,|\pi). \qquad (3)$$

[8] Existence of a rational expectations equilibrium is not an issue in this paper. This is so, because for the existence of a Nash equilibrium of the market game it is sufficient that a full communication equilibrium exists.

for all $\pi \in \Pi_i$ with $\pi \subseteq t_i \in T_i$. By standard arguments this is a continuous function of p that is strictly decreasing in p by the assumption of the law of demand, for any fixed $\pi \in \Pi_i$. As long as $-w_{i2}(t_i) < \xi_i(p, t_i) < w_{i1}(t_i)/p$, the first-order condition,

$$\frac{\partial u_i(x_1, x_2 \,|\pi)}{\partial x_2} = p\frac{\partial u_i(x_1, x_2 \,|\pi)}{\partial x_1}, \tag{4}$$

must hold at $x_1 = w_{i1}(t_i) - p\xi_i(p, t_i)$ and $x_2 = w_{i2}(t_i) + \xi_i(p, t_i)$.

Competitive equilibria only serve as a benchmark. The focus in this paper is on the market game that is described in the following subsection.

2.2 Market Game

The market game is an idealized version of an electronic limit order book. The idealization concerns three points. First, at real-world markets price increments are finite, so-called "ticks," and quantities are traded in discrete "lots" (see Hasbrouck, Sofianos, and Sosebee, 1993). By contrast, here price increments may be infinitesimal and quantity is perfectly divisible, i.e., prices and quantities are real numbers. The reason for this assumption is that without it a number of results about competitive equilibrium, which are used in this paper, would not apply. Second, in practice many platforms charge small proportional fees that are assumed away in the model.[9] Third, this paper considers a one-shot (static) model. This rules out dynamic effects that are often considered important on limit order markets. On the one hand the assumptions on utility functions are general enough to interpret them as value functions of a dynamic optimization problem. On the other hand such an interpretation would have to assume that after each round of trading all unexecuted orders are canceled. Otherwise priority rules based on timing would kick in. Most electronic order books operate a "first in-first out" principle that grants priority to limit orders (with the same limit price) that arrived earlier. In high-frequency markets this may generate queuing uncertainty (see Yueshen, 2014). Such effects cannot be captured by a static model.

Formally, the game starts with a chance move that determines the state $\varpi \in \Omega$. After chance has determined the state, agents privately learn their types and then the market opens. At the market all trades are made as exchanges of good $j = 1$, which serves as the numeraire, against commodity $j = 2$. Thus,

[9] These fees, that range from 0.1 to 2 cents per share, act like a distortive tax.

every trade can be summarized by the quantity of good $j = 2$ traded and its price in terms of good $j = 1$.

Trades are done by orders that all agents submit simultaneously. An order may be a limit order or a market order. A *limit buy order* by agent $i \in I$ is a pair $(p_i, x_i) \in \mathbb{R}_+ \times \mathbb{R}_{++}$. It constitutes a commitment on the part of agent i to buy at any price below or equal to the bid price $p_i \geq 0$ any quantity smaller or equal to $x_i > 0$ of good $j = 2$. A *limit sell order* by i is a pair $(p_i, x_i) \in \mathbb{R}_{++} \times \mathbb{R}_{-}$. It constitutes a commitment on the part of i to sell at any price above or equal to the ask price $p_i > 0$ any quantity not exceeding $|x_i| > 0$ of good $j = 2$. A *market buy order* by agent i, (∞, x_i) with $x_i > 0$, is a commitment to buy at any non-negative price any quantity not exceeding $x_i > 0$. A *market sell order* by i, $(0, x_i)$ with $x_i < 0$, is a commitment to sell at any non-negative price any quantity smaller than or equal to $|x_i| > 0$. That is, in contrast to Mertens (2003), an order, as formalized here, does not force full execution of the quantity if the relevant market price is strictly below or above the limit price.

Each agent $i \in I$ may place up to $m \geq |\mathcal{S}|$ orders. Under certainty agents would only place a single order ($x_i = 0$ for at least $m - 1$ orders). With uncertainty it will be seen that agents have an incentive to place multiple orders.

When placing orders, agents must respect a budget and a short selling constraint. In particular, if agent $i \in I$ is of type $t_i \in T_i$, then all orders in her order vector $\mathbf{y}_i(t_i) = ((p_{ik}, x_{ik}))_{k=1}^m \in (\mathbb{R}_+ \times \mathbb{R})^m$ must satisfy, for all prices $p \in \mathbb{R}_+$,

$$w_{i2}(t_i) + \sum_{p_{ik} \geq p} \max\{0, x_{ik}\} \geq -\sum_{p_{ik} \leq p} \min\{0, x_{ik}\} \text{ and} \tag{5}$$

$$w_{i1}(t_i) - p \sum_{p_{ik} \leq p} \min\{0, x_{ik}\} \geq p \sum_{p_{ik} \geq p} \max\{0, x_{ik}\}. \tag{6}$$

This is to be read as follows. If the market price p is below the smallest limit price among t_i's sell orders, $p < \min_{x_{ik}<0} p_{ik}$, but does not exceed the largest limit price among t_i's buy orders, $p \leq \max_{x_{ik}>0} p_{ik}$, then t_i only buys. In that case (5) is void, and (6) reduces to $p \sum_{p_{ik} \geq p} \max\{0, x_{ik}\} \leq w_{i1}(t_i)$, a budget constraint. If the price p is not below the smallest limit price among t_i's sell orders, i.e. $p \geq \min_{x_{ik}<0} p_{ik}$, but exceeds the largest limit price among t_i's buy orders, i.e. $p > \max_{x_{ik}>0} p_{ik}$, then t_i only sells. In that case (6) is void, and (5) reduces to $w_{i2}(t_i) + \sum_{p_{ik} \leq p} \min\{0, x_{ik}\} \geq 0$, a short selling constraint.

If $\min_{x_{ik}<0} p_{ik} \leq p \leq \max_{x_{ik}>0} p_{ik}$, then t_i may trade on both sides of the market. In that case (5) demands that t_i's endowment of good $j = 2$ plus what she buys at p must be sufficient to cover all of t_i's sales at p; and (6) demands that t_i's endowment of good $j = 1$ plus what she earns from sales must finance her purchases. Clearly, if t_i submits a market sell order, $(0, x_{ik})$ with $x_{ik} < 0$, then the smallest limit price among t_i's sell orders is zero; and if she submits a market buy order, (∞, x_{ik}) with $x_{ik} > 0$, no price $p \in \mathbb{R}_+$ exceeds the largest limit price among t_i's buy orders.

Hence, (5) restricts short selling, as there is only one round of trading. And (6) ensures that t_i does not go bankrupt at a single price p by placing limit buy orders. For all types $t_i \in T_i$ of agent $i \in I$ denote by $B_i(t_i)$ the set of all order vectors that satisfy (5) and (6), where $p_{ik} \in \mathbb{R}_+ \cup \{\infty\}$ so as to allow for market buy orders. The set $B_i(t_i)$ is non-empty, because $0 \in B_i(t_i) \subseteq (\mathbb{R}_+ \times \mathbb{R})^m$ for all $t_i \in T_i$, and it does not depend on market prices, because (5) and (6) must hold for all $p \in \mathbb{R}_+$.

The formulation of constraints (5) and (6) stipulate a single price p at which sales and purchases are executed. If the market ask price a and the market bid price b can be different—as they may be in the current model—an agent may still go bankrupt. For instance, if type t_i of agent i places one market buy order (∞, x) and one market sell order $(0, -x)$ with $x > 0$ (and $x_{ik} = 0$ for all $k = 3, ..., m$), then the analogue of (6) is $w_{i1}(t_i) + ax \geq bx$ with $a < b$. For sufficiently large x this inequality must fail and t_i is bankrupt. More generally, say that type t_i of agent i is *bankrupt at* the market ask price a and the market bid price b if either

$$w_{i2}(t_i) + \sum_{p_{ik} \geq b} \max\{0, x_{ik}\} < -\sum_{p_{ik} \leq a} \min\{0, x_{ik}\} \text{ or} \tag{7}$$

$$w_{i1}(t_i) - a \sum_{p_{ik} \leq a} \min\{0, x_{ik}\} < b \sum_{p_{ik} \geq b} \max\{0, x_{ik}\} \tag{8}$$

or both hold at (a, b). Note that, in contrast to (5) and (6), the inequalities (7) and (8) refer to a market bid price b and a market ask price a (that may be different, $a < b$), rather than to a single price p. If t_i is bankrupt, her endowments are confiscated, but her feasible trades are carried out by an external agency. Any other bankruptcy penalty would also do, as long as agents have an incentive to avoid bankruptcy.

Transactions are executed mechanically by a pricing rule. Given all agents' orders, the clearing rule is as follows. First, the trades that maximize turnover in terms of good $j = 2$ are determined. Then a market bid price and a market ask price for these trades are set such that their difference, the *spread*, is maximal. These two steps fully specify the price determination, as will be shown below.

If the quantity demanded at the market bid price does not match the quantity supplied at the market ask price, a random rationing mechanism becomes effective. If there is excess demand, this mechanism rations buy orders that bid the lowest price among executable buy orders such that each of those has a nonzero chance of being rationed. Likewise, for excess supply all sell orders that ask the highest price among executable orders are randomly rationed. Inframarginal orders are not rationed.

More precisely, call a profile of order vectors $\mathbf{y} = \left(\left((p_{ik}, x_{ik})\right)_{k=1}^{m}\right)_{i\in I}$, one order vector for each agent, an *order book*. Say that the market is *active* at the order book $\mathbf{y}$ if there are $i, j \in I$ and $k, h \in M \equiv \{1, ..., m\}$ such that $x_{ik} > 0$, $x_{jh} < 0$, and $p_{ik} \geq p_{jh}$. If the market is active at y, define the functions

$$D_{\mathbf{y}}(p) = \sum_{(j,k)\in\{(i,h)\in I\times M \mid p_{ih}\geq p\}} \max\{0, x_{jk}\} \text{ and} \tag{9}$$

$$S_{\mathbf{y}}(p) = -\sum_{(j,k)\in\{(i,h)\in I\times M \mid p_{ih}\leq p\}} \min\{0, x_{jk}\}. \tag{10}$$

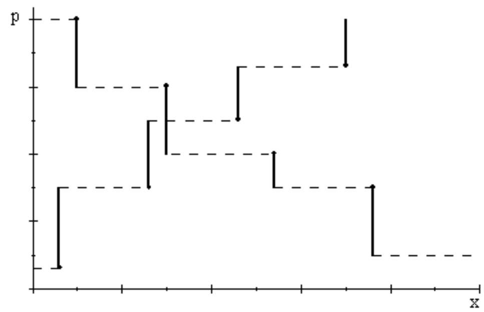
By definition the aggregate demand function $D_{\mathbf{y}}$ in (9) is a nonincreasing step function in $p \in \mathbb{R}_+$ that is continuous from below; and the aggregate supply function $S_{\mathbf{y}}$ in (10) is a nondecreasing step function in $p \in \mathbb{R}_+$ that is continuous from above. Next, define the auxiliary prices $\underline{b}(\mathbf{y}) = \inf\{p \in \mathbb{R}_+ \mid D_{\mathbf{y}}(p) < S_{\mathbf{y}}(p)\}$ and $\overline{a}(\mathbf{y}) = \sup\{p \in \mathbb{R}_+ \mid D_{\mathbf{y}}(p) > S_{\mathbf{y}}(p)\}$. The bid price $\underline{b}(\mathbf{y})$ is the smallest bid price among buy orders that will be executed under turnover maximization. The ask price $\overline{a}(\mathbf{y})$ is the largest ask price among sell orders that will be executed at a turnover maximum. Finally, define

$$\overline{b}(\mathbf{y}) = \max\{p \in \mathbb{R}_+ \mid D_{\mathbf{y}}(p) = D_{\mathbf{y}}(\underline{b}(\mathbf{y}))\} \text{ and} \tag{11}$$

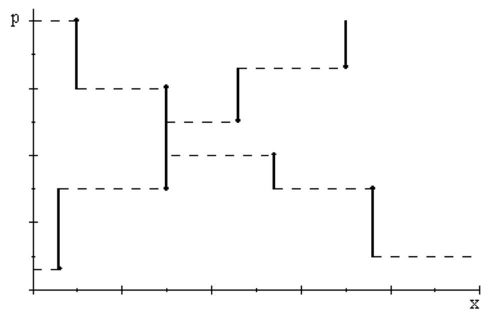
$$\underline{a}(\mathbf{y}) = \min\{p \in \mathbb{R}_+ \mid S_{\mathbf{y}}(p) = S_{\mathbf{y}}(\overline{a}(\mathbf{y}))\}. \tag{12}$$

The prices $\overline{b}(\mathbf{y})$ and $\underline{a}(\mathbf{y})$ in (11) and (12) are the *market bid* and *ask prices* that maximize the spread, keeping turnover at its maximum.

Remark 1 *All results in this paper continue to hold if the market bid and ask prices as in (11) and (12) are replaced by* $\lambda_b \underline{b}(\mathbf{y}) + (1-\lambda_b)\overline{b}(\mathbf{y})$ *and* $\lambda_a \overline{a}(\mathbf{y}) + (1-\lambda_a)\underline{a}(\mathbf{y})$ *for some constants* $\lambda_a, \lambda_b \in [0, 1]$. *This is a consequence of Lemma*



(a) $\underline{a} = \overline{a} = \underline{b} < \overline{b}$



(b) $\underline{a} < \overline{a} < \underline{b} < \overline{b}$

Figure 1: Two (active) order books.

2 below. Anecdotal evidence suggests that in practice $\lambda_a = \lambda_b = 0$ is actually used (except for so-called "single-price auctions;" see Hendershott, 2003, p. 10). More importantly, since $\lambda_a = \lambda_b = 0$ maximizes the spread, this specification may be taken as a reduced form model of brokers or high-frequency traders front-running their clients' orders. Since front-running is a major concern for electronic limit order books (see The New York Times, April 6, 2014, p. MM27), simplifying notation by assuming $\lambda_a = \lambda_b = 0$ seems worthwhile.

The mechanics of the definitions is encapsulated in the first auxiliary result that applies to a fixed order book. (Under uncertainty the order book that realizes depends on the event that obtains, of course.)

Lemma 1 *For all order books* $\mathbf{y}$*:*
(a) $\overline{b}(\mathbf{y}) \geq \underline{b}(\mathbf{y}) \geq \overline{a}(\mathbf{y}) \geq \underline{a}(\mathbf{y})$*;*
(b) if the market is active at $\mathbf{y}$*, then* $\underline{b}(\mathbf{y}) > \overline{a}(\mathbf{y})$ *implies* $D_{\mathbf{y}}\left(\overline{b}(\mathbf{y})\right) = D_{\mathbf{y}}\left(\underline{b}(\mathbf{y})\right) = D_{\mathbf{y}}(p) = S_{\mathbf{y}}(p) = S_{\mathbf{y}}(\overline{a}(\mathbf{y})) = S_{\mathbf{y}}(\underline{a}(\mathbf{y}))$ *for all* $p \in (\overline{a}(\mathbf{y}), \underline{b}(\mathbf{y}))$*;*

(c) if the market is active at $\mathbf{y}$*, then*

$$\begin{aligned} \underline{a}(\mathbf{y}) &< \overline{a}(\mathbf{y}) \Rightarrow D_{\mathbf{y}}(\overline{a}(\mathbf{y})) > S_{\mathbf{y}}(\overline{a}(\mathbf{y})) \text{ and} \\ \overline{b}(\mathbf{y}) &> \underline{b}(\mathbf{y}) \Rightarrow D_{\mathbf{y}}(\underline{b}(\mathbf{y})) < S_{\mathbf{y}}(\underline{b}(\mathbf{y})). \end{aligned}$$

Lemma 1(a) states that ask prices never exceed bid prices. Part (b) says, first, that if there is no price p such that $D_{\mathbf{y}}(p) = S_{\mathbf{y}}(p)$, then $\underline{b}(\mathbf{y}) = \overline{a}(\mathbf{y})$; second, if $\underline{b}(\mathbf{y}) > \overline{a}(\mathbf{y})$, then there is a whole interval of prices for which demand equals supply. Part (c) says that if $D_{\mathbf{y}}(\overline{a}(\mathbf{y})) \leq S_{\mathbf{y}}(\overline{a}(\mathbf{y}))$, then $\overline{a}(\mathbf{y}) = \underline{a}(\mathbf{y})$, and if $D_{\mathbf{y}}(\underline{b}(\mathbf{y})) \geq S_{\mathbf{y}}(\underline{b}(\mathbf{y}))$, then $\overline{b}(\mathbf{y}) = \underline{b}(\mathbf{y})$. Figure 1 illustrates two possible order book configurations.

With these definitions an *allocation rule* can now be defined for each order book $\mathbf{y}$. First, if the market is not active at $\mathbf{y}$, then no transactions take place. If the market is active at $\mathbf{y}$, then:

1. If $D_{\mathbf{y}}(\underline{b}(\mathbf{y})) = S_{\mathbf{y}}(\overline{a}(\mathbf{y}))$, then each order $y_{ik} = (p_{ik}, x_{ik})$ with $p_{ik} \geq \underline{b}(\mathbf{y})$ and $x_{ik} > 0$, or with $p_{ik} \leq \overline{a}(\mathbf{y})$ and $x_{ik} < 0$, will fully be carried out at the market bid price $\overline{b}(\mathbf{y})$ if $x_{ik} > 0$ or at the market ask price $\underline{a}(\mathbf{y})$ if $x_{ik} < 0$. All other orders are canceled. Accordingly, the *net trades* $\theta_i(\mathbf{y}) \in \mathbb{R}^2$ of agent i at the order book $\mathbf{y}$ are given by

$$\theta_{i1}(\mathbf{y}) = -\overline{b}(\mathbf{y}) \sum_{p_{ik} \geq \overline{b}(\mathbf{y})} \max\{0, x_{ik}\} - \underline{a}(\mathbf{y}) \sum_{p_{ik} \leq \underline{a}(\mathbf{y})} \min\{0, x_{ik}\}, \quad (13)$$

$$\theta_{i2}(\mathbf{y}) = \sum_{p_{ik} \leq \underline{a}(\mathbf{y})} \min\{0, x_{ik}\} + \sum_{p_{ik} \geq \overline{b}(\mathbf{y})} \max\{0, x_{ik}\}. \quad (14)$$

2. If $\underline{b}(\mathbf{y}) = \overline{a}(\mathbf{y})$, then either the former is applicable, or one side of the market is longer than the other at this price, but trades are possible (see Lemma 1). For concreteness suppose that $D_{\mathbf{y}}(\underline{b}(\mathbf{y})) > S_{\mathbf{y}}(\overline{a}(\mathbf{y}))$. (The rule for the reverse strict inequality is analogous.) Then $\overline{b}(\mathbf{y}) = \underline{b}(\mathbf{y})$ by Lemma 1(c) and all buy orders (p_{ik}, x_{ik}), for which $p_{ik} = \underline{b}(\mathbf{y})$ and $x_{ik} > 0$, will be rationed with positive probability so as to balance the quantity demanded with the quantity supplied $S_{\mathbf{y}}(\overline{a}(\mathbf{y}))$. (If only one order satisfies that, this order is rationed with certainty.) The exact nature of the random rationing mechanism is immaterial, because agents will avoid being rationed by risk aversion. Buy orders ($x_{ik} > 0$) are executed at the market bid price $\overline{b}(\mathbf{y})$ and sell orders ($x_{ik} < 0$) at the market ask price $\underline{a}(\mathbf{y})$.

3. If an active trader is bankrupt at $\left(\underline{a}\left(\mathbf{y}\right), \overline{b}\left(\mathbf{y}\right)\right)$, as defined by (7) and (8), her endowment is confiscated (so that she ends up with zero holdings of both goods) and an external agency trades her executable orders.[10]

According to this allocation rule agents do not necessarily pay their limit prices, but buyers pay the market bid price $\overline{b}\left(\mathbf{y}\right)$ from (11) and sellers receive the market ask price $\underline{a}\left(\mathbf{y}\right)$ from (12) per unit traded. The idea is that (unmodeled) brokers maximizes their profits by front-running, but they cannot not price-discriminate among executable orders on the same side of the market. If they did, on real-world markets this would cause litigation.

The allocation rule above translates each order book $\mathbf{y}$ into payoffs for all agents and all types. Therefore, restricting agents' choices to order vectors in $B_i\left(t_i\right)$ for all $t_i \in T_i$ and all $i \in I$, this defines a Bayesian game in which agent $i \in I$, when she is of type $t_i \in T_i$, has the strategy set $B_i\left(t_i\right)$. More precisely, the players in the game are the types $t_i \in T_i$ of the agents $i \in I$ and their strategy sets are $B_i\left(t_i\right)$. Thus, technically speaking the Bayesian game is analyzed in Harsanyi form (Harsanyi, 1967-8). The solution concept is Nash equilibrium (Nash, 1950, 1951) in pure strategies.

3 Equilibria of the Market Game

Existence of a Nash equilibrium for the market game is trivially established, because inactivity is always an equilibrium. But autarky is not an interesting equilibrium. The focus here is on *active equilibria*, that is, on equilibria that involve trade.

The key to understanding active equilibria is the observation that agents may use multiple orders to hedge against all contingencies that they regard possible. If agent $i \in I$ is of type $t_i \in T_i$, she regards possible the events in $\tau_i^{-1}\left(t_i\right) = \{s \in \mathcal{S} \mid s \subseteq t_i\}$. Since she can place multiple orders (and $m \geq |\mathcal{S}|$), she can perfectly hedge by placing a separate order for each $s \in \tau_i^{-1}\left(t_i\right)$. By risk aversion it is optimal to do so. Therefore, in equilibrium type t_i of agent i submits an order vector $\mathbf{y}_i\left(t_i\right) = \left(\left(p_{is}, x_{is}\right)\right)_{s \subseteq t_i}$ with one (nonzero) order for each event that she regards possible. (The remaining orders have $x_{ik} = 0$.

[10] This is the point where a generalization of the model to more than two goods could pose difficulties. If a trader is bankrupt, this will affect all markets on which she placed orders and the ensuing repercussions would have to be modeled (for a discussion see Dubey, 1982). The assumption that the feasible trades of a bankrupt trader are carried out by an external agency sterilizes this effect—at the cost that outside of equilibrium the system may not be closed anymore, of course.

Note that this kind of hedging would be impossible in a trading-post model á la Shapley and Shubik, 1977.)

This is vaguely reminiscent of general equilibrium theory with complete markets. In that theory also all possible contingencies can be insured against, because there is a complete set of Arrow-Debreu securities. The difference here is that the hedging is done on the *same* market, so that transactions are dependent. In particular, all buy orders with bid prices not below the market bid will be cleared, and all sell orders with asks not above the market ask also will.

If event $s \in \mathcal{S}$ materializes, then for each $i \in I$ there is a unique type $\tau_i(s) \in T_i$ such that $s \subseteq \tau_i(s)$. Therefore, the order book at event $s \in \mathcal{S}$ is uniquely determined by $\mathbf{y}(s) = (\mathbf{y}_i(\tau_i(s)))_{i \in I}$. And all agents rationally foresee that the order book $\mathbf{y}(s)$ will obtain at event s. The ability to perfectly hedge implies that the spread is zero in all active equilibria.

Lemma 2 *In any active equilibrium of the market game* $\underline{a}(\mathbf{y}(s)) = \overline{a}(\mathbf{y}(s)) = \underline{b}(\mathbf{y}(s)) = \overline{b}(\mathbf{y}(s))$ *for all* $s \in \mathcal{S}$.

Lemma 2 states that in an active equilibrium front-runners cannot profit from a positive spread. Therefore, in equilibrium the system is closed in the sense that no resources leak out. More importantly, the lemma implies that at any active equilibrium each event $s \in S$ is associated with an equilibrium price $p_s = \underline{a}(\mathbf{y}(s)) = \overline{b}(\mathbf{y}(s))$ that is both the market bid and the market ask price at the event s. (This implies that using other market bid and ask prices, as discussed in Remark 1, does not change the results.) Thus, henceforth, p_s denotes both the equilibrium market bid and market ask price.

Lemma 2 also implies that bankruptcy does not occur in equilibrium. For, if the market bid and ask price both equal p_s, then that agent i at event $s \in \mathcal{S}$ has to choose from $B_i(\tau_i(s))$ guarantees solvency by (5) and (6).

3.1 Sufficient Condition

This section considers existence of active Nash equilibria. Assuming that a full communication equilibrium involves trade in at least some states, its existence (Hart, 1974, Theorem 3.3; Jordan, 1983, Proposition 2.8) implies existence of an active Nash equilibrium for the market game.

Theorem 1 *An active pure strategy Nash equilibrium for the market game always exists. This Nash equilibrium induces precisely the same allocation as the full communication equilibrium.*

Theorem 1 establishes that there is always an active equilibrium for the market game at which agents behave *as if* they were price takers. Its proof is constructive, showing that competitive behavior indeed constitutes an equilibrium of the market game. For, assume for a moment that there is no uncertainty, $|\mathcal{S}| = 1$, and let p^* denote the market clearing price. If all buyers bid p^*, no seller who asks a price above p^* will be able to trade; likewise, if all sellers ask p^*, no buyer who bids below p^* will trade. Given that everybody else insists on p^*, the optimal trade for an individual agent is precisely her competitive excess demand at the price p^*.

Now add uncertainty; say, there are three elementary events, $\mathcal{S} = \{1, 2, 3\}$. For instance, one group of agents knows whether or not $s = 1$ obtains, $T_1 = \{\{1\}, \{2, 3\}\}$, the other whether or not $s = 3$ obtains, $T_2 = \{\{1, 2\}, \{3\}\}$. Suppose that at the full communication equilibrium $p_1^* < p_2^* < p_3^*$. Types $t_1 = \{1\}$ in the first group bid resp. ask p_1^*, and types $t_2 = \{3\}$ in the second group bid resp. ask p_3^*. Types $t_1 = \{2, 3\}$ in the first group all place two limit orders, one at p_2^* and one at p_3^*; types $t_2 = \{1, 2\}$ in the second group also place two orders, one at p_1^* and one at p_2^*. All quantities in the orders correspond to the associated competitive excess demands.[11] If, say, $s = 2$ realizes, then the orders at p_2^* precisely clear the market. Given that $s = 2$, there is excess demand at p_1^*, because the orders at that price lack their counterparts from types $t_1 = \{1\}$; and at p_3^* there is excess supply, because the orders at p_3^* lack their counterparts from $t_2 = \{3\}$. The other events are similar. The rest of the proof establishes individual optimality.

Theorem 1 has a further important implication. It claims that private information gets aggregated by market prices.

Corollary 1 *Suppose that the economy has a fully revealing competitive equilibrium. Then the market game has a (pure strategy) Nash equilibrium which induces the same allocation and the same prices.*

For the case of private values this result resolves the problem raised by Beja (1977) and Hellwig (1980), that traders act rationally with respect to information, yet fail to recognize their influence on the price. In the present model agents *do* recognize their influence on the price, and this is *why* the price reflects private information. Still they behave *as if* they were price takers, because of the nature of order-driven markets. These markets generate incentives that

[11] More precisely, they are the increments of competitive excess demands. For instance, $t_1 = \{2, 3\}$ orders $\xi_1\left(p_3^*, \{2, 3\}\right)$ at p_3^* and $\xi_1\left(p_2^*, \{2, 3\}\right) - \xi_1\left(p_3^*, \{2, 3\}\right)$ at p_2^*.

locally act like given prices, because traders take as given the limit prices of others.

Theorem 1 contrasts sharply with trading-post models (Shapley and Shubik, 1977), whose equilibria with finitely many agents do *not* include a competitive allocation. This is because (very much like models of competition in supply functions) the residual demand functions facing an individual have non-zero slope. Hence, the first-order condition (4) fails for trading-post models, as it contains a (non-zero) term that depends on the slope of residual demand. Unlike trading-posts, the undiscriminatory limit order book implements the full communication equilibrium as a Nash equilibrium of the associated Bayesian game even if there is only a finite number of agents, irrespective of how large or small these agents are. If the full communication equilibrium is a rational expectations equilibrium, the same holds for the latter.

The driving forces of Theorem 1 are the law of demand and the "private values" assumption. Theorem 3 below will show that private values are in fact necessary. The reason is that with "common values" the law of demand is likely to break down. For, if an agent wishes to buy at a high price and, say, sell at a low price, in events that she cannot distinguish based on her private information, her two orders may execute against each other. This make it impossible to hedge and destroys the limit order book's ability to implement a competitive allocation. And this is likely to be the case without the measurability assumption, as Theorem 3 below will show.

3.2 Necessary Condition

Even with private values the competitive allocation may be but one equilibrium of the market game. Indeed the market game may have other active equilibria. Yet, these will now be shown to stay in a vicinity of the competitive equilibrium, provided agents are small compared to the market.

Since agents are risk averse, they will avoid being rationed: At a given price they prefer trading the expected quantity for sure over a lottery that results from rationing. Therefore, they will plan their orders for the events $s \in \tau_i^{-1}(t_i)$ by considering which quantities can be bought or sold at the equilibrium prices p_s (from Lemma 2) without risking rationing. What these quantities are, given an order book $\mathbf{y}$, can be summarized by a correspondence $F_{\mathbf{y}} : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$. This is constructed as follows. For an order book $\mathbf{y}$ let

$$\begin{aligned} A(\mathbf{y}) &= \{p \in \mathbb{R}_+ \mid \exists (i,k) \in I \times M : x_{ik} < 0,\, p_{ik} = p\} \text{ and} \\ B(\mathbf{y}) &= \{p \in \mathbb{R}_+ \mid \exists (i,k) \in I \times M : x_{ik} > 0,\, p_{ik} = p\} \end{aligned}$$

be the sets of ask and bid prices that occur in the order book $\mathbf{y}$. Now consider the opportunity to place one additional order. Which additional orders will be executed without rationing? This is described by the correspondence $F_{\mathbf{y}}$ defined by

$$F_{\mathbf{y}}(p) = \begin{cases} [S_{\mathbf{y}}(p) - D_{\mathbf{y}}(p)\,,\, 0] & \text{if} \quad \overline{a}(\mathbf{y}) \geq p \notin A(\mathbf{y})\,, \\ (\lim_{\varepsilon \searrow 0} S_{\mathbf{y}}(p-\varepsilon) - D_{\mathbf{y}}(p)\,,\, 0] & \text{if} \quad \overline{a}(\mathbf{y}) \geq p \in A(\mathbf{y})\,, \\ \{0\} & \text{if} \quad \overline{a}(\mathbf{y}) < p < \underline{b}(\mathbf{y})\,, \\ [0,\, S_{\mathbf{y}}(p) - \lim_{\varepsilon \searrow 0} D_{\mathbf{y}}(p+\varepsilon)) & \text{if} \quad \underline{b}(\mathbf{y}) \leq p \in B(\mathbf{y})\,, \\ [0,\, S_{\mathbf{y}}(p) - D_{\mathbf{y}}(p)] & \text{if} \quad \underline{b}(\mathbf{y}) \leq p \notin B(\mathbf{y})\,. \end{cases} \tag{15}$$

That is, by asking a price p that does not exceed $\overline{a}(\mathbf{y})$ and is not asked in any existing sell order, $p \notin A(\mathbf{y})$, the quantity $S_{\mathbf{y}}(p) - D_{\mathbf{y}}(p)$ can be sold at the price p, which will be the new market ask price $\underline{a}$. If the price p does not exceed $\overline{a}(\mathbf{y})$, but is asked in some existing sell order, $p \in A(\mathbf{y})$, then by asking a price $p - \varepsilon$ for some small $\varepsilon > 0$ the existing sell order is undercut, and any quantity strictly larger than $\lim_{\varepsilon \searrow 0} [S_{\mathbf{y}}(p-\varepsilon) - D_{\mathbf{y}}(p-\varepsilon)] = \lim_{\varepsilon \searrow 0} S_{\mathbf{y}}(p-\varepsilon) - D_{\mathbf{y}}(p) < 0$ can be sold at the price p, because the existing sell order determines the market ask price. (Note that $\lim_{\varepsilon \searrow 0} S_{\mathbf{y}}(p-\varepsilon) < S_{\mathbf{y}}(p)$, because $S_{\mathbf{y}}$ is continuous from above and at p an existing sell order becomes executable.)

At prices strictly between $\overline{a}(\mathbf{y})$ and $\underline{b}(\mathbf{y})$ nothing can be sold or bought. By bidding a price $p + \varepsilon > \underline{b}(\mathbf{y})$, when p is bid in some existing buy order, i.e. $p \in B(\mathbf{y})$, any quantity strictly less than $\lim_{\varepsilon \searrow 0} [S_{\mathbf{y}}(p+\varepsilon) - D_{\mathbf{y}}(p+\varepsilon)] = S_{\mathbf{y}}(p) - \lim_{\varepsilon \searrow 0} D_{\mathbf{y}}(p+\varepsilon) > 0$ can be bought at the price p, because the existing buy order determines the market bid price (where $\lim_{\varepsilon \searrow 0} D_{\mathbf{y}}(p+\varepsilon) < D_{\mathbf{y}}(p)$, since $D_{\mathbf{y}}$ is continuous from below and at p an existing buy order becomes effective). At a price $p \geq \underline{b}(\mathbf{y})$ that is not bid in any existing buy order, $p \notin B(\mathbf{y})$, any quantity not exceeding $S_{\mathbf{y}}(p) - D_{\mathbf{y}}(p)$ can be bought, because p will be the new market bid price $\overline{b}$. Thus, every pair (x, p) in the intersection of the graph of $F_{\mathbf{y}}$ with the budget set $B_i(t_i)$ of type t_i of agent i is a feasible trade with no risk of rationing.

The correspondence $F_{\mathbf{y}}$ satisfies that $0 > x \in F_{\mathbf{y}}(p)$ and $p \leq \overline{a}(\mathbf{y})$ imply

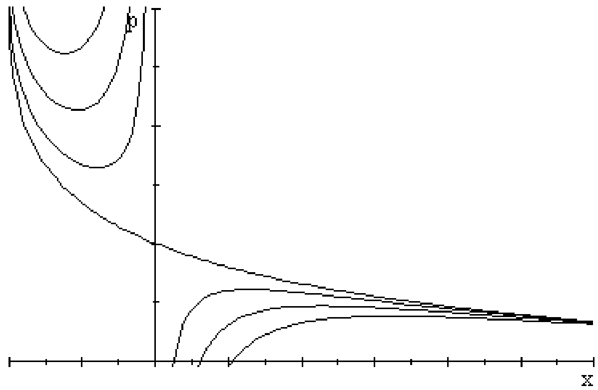


Figure 2: Indifference curves in order space.

$x \in F_{\mathbf{y}}(q)$ for all $q \leq p$, and $0 < x \in F_{\mathbf{y}}(p)$ and $p \geq \underline{b}(\mathbf{y})$ imply $x \in F_{\mathbf{y}}(q)$ for all $q \geq p$. Its boundary is like a step function, except that occasionally the corner points are missing, i.e., the horizontal pieces may be closed or half-open intervals. They are half-open when there is a competing order on the same side of the market; and they are closed when the constraint comes from the other side of the market. Whenever a trader's optimal choice is located at a horizontal segment of (the boundary of the graph of) $F_{\mathbf{y}}$, then she locally faces an infinite price elasticity—as if she were a price taker.

For what follows abbreviate $\xi_i(p_s, \tau_i(s))$ for an event $s \in \mathcal{S}$ by $\xi_i(p_s, s)$ for all $i \in I$. Combining the correspondence $F_{\mathbf{y}}$ with the preferences of traders (as illustrated in Figure 2) yields the following.

Proposition 1 *In any active equilibrium of the market game, for all $s \in \mathcal{S}$:*
(a) the quantity traded by type $\tau_i(s) \in T_i$ of agent $i \in I$ at event $s \in \mathcal{S}$ does not exceed in absolute value her excess demand $\xi_i(p_s, s)$ at the equilibrium price p_s;
(b) on each side of the market there is at most one agent for whom the quantity traded is strictly smaller in absolute value than her excess demand $\xi_i(p_s, s)$ at the equilibrium price p_s;
(c) at most one order is rationed.

The essence of Proposition 1 is encapsulated in Figure 3.[12] Because excess demand is a step function, the opportunity set graph $(F_{\mathbf{y}}) \cap B_i(t_i)$ of type t_i of agent i at an equilibrium is an area bounded by a nondecreasing step function,

[12] The better-direction for preferences is down-right and the opportunity set is the area to the upper left of the step function.

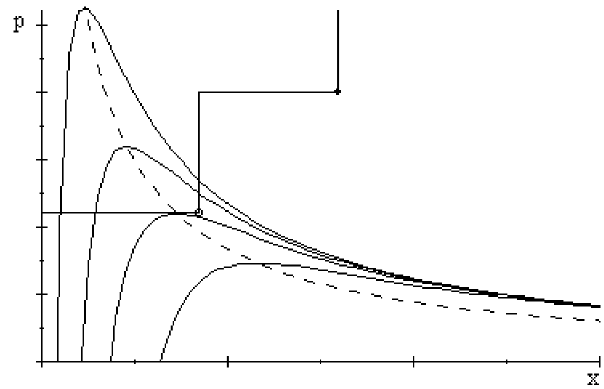


Figure 3: Opportunity set and indifference curves in order space.

that is, the boundary consists of vertical and horizontal pieces. The optimal choice cannot lie on a vertical piece, as there the same quantity can be bought at a lower price resp. sold at a higher price. Hence, it must be located at a horizontal segment—or at a corner point. At a horizontal segment the agent is precisely at her excess demand function, as if she were a price taker (see (4)). The only subtlety arises at corner points, where the agent may be below her competitive excess demand. But corner points can only be reached when the bidder has no competitors on the same side of the market and the constraint comes purely from the other side of the market. Therefore, this can apply to at most one bidder. (The arguments on the supply side are analogous.) These insights identify necessary conditions for active equilibria of the market game for the private values case.

Theorem 2 *If a pure strategy combination for the market game constitutes an active equilibrium, then for each event $s \in \mathcal{S}$ there are an equilibrium price p_s, two agents $i(s), j(s) \in I$ with $i(s) \neq j(s)$, and numbers $\theta_{i(s)}, \theta_{j(s)} \in \mathbb{R}$ such that*

$$\begin{aligned} \xi_{i(s)}(p_s, s) &\leq \theta_{i(s)} \leq 0 \leq \theta_{j(s)} \leq \xi_{j(s)}(p_s, s) \text{ and} \\ \theta_{i(s)} + \theta_{j(s)} &+ \sum_{k \in I \setminus \{i(s), j(s)\}} \xi_k(p_s, s) = 0, \end{aligned}$$

where $\theta_{i(s)}$ and $\theta_{j(s)}$ denote the net trades of agents $i(s)$ and $j(s)$, and all other agents' net trades are equal to their excess demands $\xi_k(p_s, s)$ at p_s.

There is an alternative formulation of Theorem 2 that looks at how far equilibrium prices can deviate from their competitive levels. In particular, for each agent $j \in I$ let $P_s(j)$ denote the market clearing price at event $s \in \mathcal{S}$ when agent j has been excluded from the economy, i.e., the price that solves $\sum_{i \in I \setminus \{j\}} \xi_i (P_s(j), s) = 0$. Then, the prices p_s at an equilibrium of the market game satisfy

$$\min_{j \in I} P_s(j) \leq p_s \leq \max_{j \in I} P_s(j) \tag{16}$$

for all events $s \in \mathcal{S}$. For, by Proposition 1(b) on each side of the market at most one agent is short of her competitive (excess) demand. No agent will trade more than her competitive (excess) demand either. Hence, the thought experiment of removing one agent on each side of the market, fully or partially, gives the possible range of equilibrium prices.

If the rational expectations equilibrium is fully revealing and the intervals from (16) are disjoint across events $s \in \mathcal{S}$, then all Nash equilibria of the market game are also fully revealing. That, of course, requires that removing one agent from the economy has very little effect on the full communication equilibrium. Still, this observation yields a notion when all (of finitely many) agents are small: If the intervals from (16) are small. (An example in the next subsection illustrates what may happen if these intervals are large.)

Such a condition gives substance to the claim that small agents in a large market have almost no influence on the price, even when the number of agents is finite. While it is conceivable that an equilibrium of the market game may not precisely result in a competitive allocation, the deviation from the latter is bounded—by how much a single agent can move the market clearing price. In particular, no seller can drive the price further up than what would result if she withheld her whole supply; and no buyer can drive the price further down than what would result if she withheld her whole demand.

There are other condition that would eliminate non-competitive Nash equilibria, of course. Replicating the economy, for instance, would eliminate non-competitive equilibria in the limit. As an alternative, Simon (1984, p. 223) proposes the condition that markets are "thick." Translated to the case of uncertainty and an undiscriminatory order book, markets are *thick* at a particular strategy combination if at all events $s \in \mathcal{S}$ at least two buyers bid the market bid price and at least two sellers ask the market ask price. This condition eliminates the corner points of the opportunity set (15) at the realized equilibrium prices. Therefore, it immediately implies that *every* Nash equilibrium of the market game induces a competitive allocation. The *no-surplus* condition, as introduced by Ostroy (1980) and Makowski (1980), has clearly the same effect, by (16).

3.3 Discussion

On the other hand, if a seller, say, controls a large fraction of aggregate supply, then by withholding supply she can drive the market price up significantly—markets are "thin." And in equilibrium buyers will shade their demands accordingly. Such a constellation is often called a *market corner*.[13] The following example illustrates that in the presence of large traders there may be Nash equilibria—besides the fully revealing one from Theorem 1—at which prices do not reveal information.

Example 1 *Suppose there are three states $\Omega = \{1,2,3\}$ and four agents with $f_1 = f_2$ and $f_3 = f_4$, $T_1 = T_2 = \{\{1\},\{2,3\}\}$, and $T_3 = T_4 = \{\{1,2\},\{3\}\}$.*[14] *That is, agents 1 and 2 are identical, and so are agents 3 and 4. Assume also that $\xi_1(p,\{1\}) = \xi_2(p,\{1\}) > \xi_1(p,\{2,3\}) = \xi_2(p,\{2,3\}) \geq 0$ and $0 \geq \xi_3(p,\{1,2\}) = \xi_4(p,\{1,2\}) > \xi_3(p,\{3\}) = \xi_4(p,\{3\})$ for all $p > 0$. That is, agents 1 and 2 are always buyers, but wish to buy more at $s = 1$ than at event $\{2,3\}$; agents 3 and 4 are always sellers, but supply less at event $\{1,2\}$ than at $s = 3$, for all prices p. Therefore, if this economy has a fully revealing competitive equilibrium with prices $(p_s)_{s=1,2,3}$, then $p_1 > p_2 > p_3$. And this can be supported by the same strategies as in the proof of Theorem 2. Observe that by construction $2\xi_1(p_2,\{2,3\}) + 2\xi_3(p_2,\{1,2\}) = 0$, but $2\xi_1(p_2,\{1\}) + 2\xi_3(p_2,\{1,2\}) > 0$ and $2\xi_1(p_2,\{2,3\}) + 2\xi_3(p_2,\{3\}) < 0$.*

Assume that type $\{1\}$ of agent 1 and type $\{3\}$ of agent 3 are individually large enough so that, by withholding their excess demands, they can turn the sign of aggregate excess demand at the "middle" price p_2, i.e., $\xi_1(p_2,\{1\}) + 2\xi_3(p_2,\{1,2\}) < 0$ and $2\xi_1(p_2,\{2,3\}) + \xi_3(p_2,\{3\}) > 0$. (Hence, the intervals from (16) are overlapping.) For further reference define the quantities $z_1 =$

[13] Historical examples include Cornelius Vanderbilt accumulating shares of Harlem Railroad in 1863, James Fisk and Jay Gould cornering the gold market in 1869, Thomas F. Ryan cornering stocks of Stutz Motor company in 1920, or Nelson B. and William H. Hunt's attempt to corner the silver market in the late 1970ties.

[14] If $\varpi = 1$, then agents 1 and 2 know that, but agents 3 and 4 do not. At $\varpi = 2$ no agent knows the true state. At state $\varpi = 3$ agents 3 and 4 know it, but 1 and 2 do not. The event that 1 and 2 know the true state, therefore, is $\{1\}$, and the event that 3 and 4 know the true state is $\{3\}$. Consequently, at $\varpi = 1$ agents 3 and 4 do not know that 1 and 2 know the true state, and at $\varpi = 3$ agents 1 and 2 do not know that 3 and 4 know the true state.

$-\xi_1(p_2,\{1\}) - 2\xi_3(p_2,\{1,2\})$ and $z_3 = -2\xi_1(p_2,\{2,3\}) - \xi_3(p_2,\{3\})$, where $0 < z_1 < \xi_1(p_2,\{1\})$ and $\xi_3(p_2,\{3\}) < z_3 < 0$.

Then consider the following strategy combination, which is such that each type of every agent places only one nonzero order. Type $\{1\}$ of agent 1 submits a limit buy order (p_2, z_1) and type $\{2,3\}$ of agent 1 submits a limit buy order $(p_2, \xi_1(p_2,\{2,3\}))$; type $\{1\}$ of agent 2 places a market buy order $(\infty, \xi_1(p_2,\{1\}))$ and type $\{2,3\}$ of 2 submits a limit buy order $(p_2, \xi_1(p_2,\{2,3\}))$. Type $\{1,2\}$ of agent 3 submits a limit sell order $(p_2, \xi_3(p_2,\{1,2\}))$ and type $\{3\}$ of agent 3 submits a limit sell order (p_2, z_3); type $\{1,2\}$ of agent 4 submits a limit sell order $(p_2, \xi_3(p_2,\{1,2\}))$ and type $\{3\}$ of agent 4 submits a market sell order $(0, \xi_3(p_2,\{3\}))$.

It is easily verified that this constitutes an equilibrium: At $\varpi = 2$ the types of agents 1 and 2 is $\{2,3\}$ and the type of agents 3 and 4 is $\{1,2\}$, so that two limit orders with limit price p_2 on each side of the market clear against each other; all market participants are at their excess demand functions. At $\varpi = 1$ the type of agents 1 and 2 is $\{1\}$ and the type of agents 3 and 4 is $\{1,2\}$; at the market two limit sell orders with limit price p_2 clear against one market buy order and one limit buy order with limit price p_2. Only the buyer type $\{1\}$ of agent 1 is not at her demand function, but she is constrained purely from the sell side of the market. At $\varpi = 3$ the situation is similar, where two limit buy orders with limit price p_2 clear against one market sell order and one limit sell order with limit price p_2. Only the seller type $\{3\}$ of agent 3 is not at her supply function, but she is constrained from the other side of the market. In this equilibrium prices reveal no information at all.

The example shows that in the presence of large traders an equilibrium may not only deviate from a competitive allocation, but information revelation may be distorted too. In the example type $\{1\}$ of agent 1 and type $\{3\}$ of agent 3 (partially) withhold their demand/supply so as to conceal information. And this is possible, because they are large in the sense that the intervals from (16) are overlapping.

4 Common Values

For this subsection the assumption, that preferences $f_{i1} = u_i$ are measurable with respect to private information T_i, is dropped. Then an economy with "common values" is obtained. There are common values cases, where the above market mechanism can also support a competitive allocation, at least approximately. Ironically this is the case, for instance, in Kreps' (1977) example.

Example 2 *Let there be two events,* $\mathcal{S} = \{1, 2\}$, *and two agents with utility functions* $u_1(x\,|s) = s\ln(x_1) + x_2$ *and* $u_2(x\,|s) = (3 - s)\ln(x_2) + x_1$. *Endowments are* $\mathbf{w}_1 = \mathbf{w}_2 = (2, 3/2)$ *for all* $s \in \mathcal{S}$. *Types are* $T_1 = \{\{1\}, \{2\}\}$ *and* $T_2 = \mathcal{S}$, *that is, agent 1 is fully informed and agent 2 not at all. It is not difficult to see that prices in the full communication equilibrium are* $p^*(1) = p^*(2) = 1$, *i.e., they are uninformative.*

While no rational expectations equilibrium exists, the market game has a Nash equilibrium that is arbitrarily close to the full communication equilibrium. Let $\varepsilon > 0$ *be small and suppose that type* $\{1\}$ *of agent 1 places the single limit sell order* $\mathbf{y}_1(\{1\}) = (1 - \varepsilon, (3\varepsilon - 1)/(2 - 2\varepsilon)) \to_{\varepsilon\searrow 0} (1, -1/2)$ *and type* $\{2\}$ *of agent 1 the limit buy order* $\mathbf{y}_1(\{2\}) = (1 + \varepsilon, (1 - 3\varepsilon)/(2 + 2\varepsilon)) \to_{\varepsilon\searrow 0} (1, 1/2)$.

If agent 2 puts herself into the shoes of event $s = 1$, *her opportunity set is determined by the single sell order* $\mathbf{y}_1(\{1\})$. *Since the slope of 2's indifference curve at* $s = 1$ *through the point* $(p, \xi) = (1 - \varepsilon, (1 - 3\varepsilon)(2 - 2\varepsilon)$ *is* $(1 - \varepsilon)^2 (4 + 6\varepsilon) / \left(4 - 18\varepsilon + 18\varepsilon^2\right) > 0$, *it is optimal for agent 2 to place a limit buy order* $y_{21} = (1 - \varepsilon, (1 - 3\varepsilon)/(2 - 2\varepsilon))$ *for event* $s = 1$. *If she puts herself into the shoes of* $s = 2$, *she faces the single buy order* $\mathbf{y}_1(\{2\})$. *Since the slope of 2's indifference curve at* $s = 2$ *through the point* $(p, \xi) = (1 + \varepsilon, (3\varepsilon - 1)/(2 + 2\varepsilon))$ *is* $6\varepsilon(1 + \varepsilon)^2 / \left(1 - 9\varepsilon^2\right) > 0$, *it is optimal to place a limit sell order* $y_{22} = (1 + \varepsilon, (3\varepsilon - 3)/(2 + 2\varepsilon))$ *for event* $s = 2$.

Then both types of agent 1 will be faced with the two orders, the buy order $(1 - \varepsilon, (1 - 3\varepsilon)/(2 - 2\varepsilon))$ *and the sell order* $(1 + \varepsilon, (3\varepsilon - 3)/(2 + 2\varepsilon))$. *Since* $(1 - \varepsilon, (3\varepsilon - 1)/(2 - 2\varepsilon))$ *lies on the excess demand function for type* $\{1\}$ *of agent 1, and* $(1 + \varepsilon, (3 - 3\varepsilon)/(2 + 2\varepsilon))$ *lies on the excess demand function for type* $\{2\}$ *of agent 1, agent 1's orders are also optimal in both events. It follows that an equilibrium has been constructed.*

In this Nash equilibrium the uninformed agent 2 is slightly short of her excess demands in both events, while both types of agent 1 are on their excess demand functions. That is, in both events agent 2 is constrained from the opposite side of the market. Since $\varepsilon > 0$ *can be arbitrarily small, the Nash equilibrium approximates the full communication equilibrium. That is, while this is a precise Nash equilibrium (not an approximate* ε*-Nash equilibrium), it mimics an* approximate *rational expectations equilibrium.*

The special feature of this example is the tie between prices in the full communication equilibrium, $p^*(1) = p^*(2)$. Even though the demand functions of both agents vary with the event $s \in \mathcal{S}$, this is inconsequential, as prices do not vary at all. Therefore, Nash equilibria of the market game *approximate* a competitive allocation, i.e., for every $\varepsilon > 0$ there is a Nash equilibrium within ε from the full communication equilibrium.

4.1 A Characterization

The key property that makes the proof of Theorem 1 work is that for each type $t_i \in T_i$ a strict inequality between full communication equilibrium prices, $p^*(s) > p^*(s')$, for events that type t_i cannot distinguish, $s, s' \subseteq t_i$, translates into a reversed weak inequality for i's excess demands at s and s', or equivalently,

$$\text{if } p^*(s) > p^*(s') \text{ and } \xi_i(p^*(s), s) > \xi_i(p^*(s'), s'), \text{ then } \tau_i(s) \neq \tau_i(s'), \quad (17)$$

for all $s, s' \in \mathcal{S}$ and all $i \in I$.[15] In words, (17) says that any two states $s, s' \in \tau_i^{-1}(t_i)$ between which i's (excess) demand is *increasing* in the (full communication equilibrium) price must be distinguishable based on i's private information. In Kreps' (1977) example (17) holds trivially, because the hypothesis is void. It turns out, though, that condition (17) is not only sufficient for a Nash equilibrium of the market game to support a competitive allocation—it is also necessary.

Theorem 3 *If utility functions $f_{i1} = u_i$ are not measurable with respect to private information T_i, then the market game has a Nash equilibrium that approximates a full communication equilibrium if and only if condition (17) holds.*

The striking part of Theorem 3 is the "only if." For, this states that when condition (17) fails, then every Nash equilibrium of the market game must be bounded away from a full communication equilibrium. That is, without (17) Nash equilibria of the market game are necessarily far from competitive.

The reason is that the limit order book ranks orders according to price priority. If an agent submits two orders that violate (17), say, a sell order at a low price and a buy order at a high price, then the two orders will execute against

[15] Condition (17) corresponds to the assumption of *convex* transfer schedules, offered by market makers, as adopted by Biais, Martimort, and Rochet (2000). Hence, what these authors refer to as "common values" comprises the special case characterized in Theorem 3 below and illustrated by Example 2.

each other (at least partially). Therefore, the market organization dictates that agent i's orders satisfy the constraints

$$(x_{ik} - x_{ih})(p_{ik} - p_{ih}) \leq 0 \text{ for all } k, h \in M \tag{18}$$

for all $i \in I$. When (17) fails, the competitive excess demands at the full communication equilibrium prices do not fulfill these conditions (18). Hence, the Nash equilibria of the market game are bounded away from the competitive equilibria.

Despite Example 2 condition (17) is, in fact, very restrictive. This is so, because under common values excess demand functions will vary in the event $s \in \mathcal{S}$ in general. And if they do, the only possibility for (17) to hold is that each agent can distinguish all events between which her equilibrium excess demand is increasing in the (full communication) equilibrium prices.

This can be seen as follows. Suppose that there are two events $s, s' \in \mathcal{S}$ such that for the full communication equilibrium $p^*(s) > p^*(s')$ and $\xi_\iota(p^*(s), s) \neq \xi_\iota(p^*(s'), s')$ for some agent $\iota \in I$. Then there must be an agent $j \in I$ such that $\xi_j(p^*(s), s) > \xi_j(p^*(s'), s')$. For, if $\xi_i(p^*(s), s) \leq \xi_i(p^*(s'), s')$ for all $i \in I$, then $\sum_{i \in I} \xi_i(p^*(s), s) = 0 = \sum_{i \in I} \xi_i(p^*(s'), s')$ implies $\xi_i(p^*(s), s) = \xi_i(p^*(s'), s')$ for all $i \in I$, contradicting $\xi_\iota(p^*(s), s) \neq \xi_\iota(p^*(s'), s')$. Hence, condition (17) can only be true if $\tau_j(s) \neq \tau_j(s')$, that is, if agent j can distinguish between s and s' based on her private information.

4.2 Discussion

It seems conceivable that the constraint (18) could be finessed by allowing agents to "sterilize" orders at specific prices. This may seem achievable by introducing *stop orders*. A stop order is like a reverse limit order. More precisely, a simple *buy stop order*, $((q_i, \infty]; x_i)$ with $x_i > 0$, is a commitment of agent $i \in I$ to buy up to the specified quantity $x_i > 0$ if the market price is *above* the stop price $q_i \geq 0$. That is, a simple buy stop order becomes a market buy order, once the market price is above the stop price q_i. A simple *sell stop order*, $([0, q_i); x_i)$ with $x_i < 0$, is a commitment of agent i to sell up to the quantity $|x_i| > 0$ if the market price is *below* the stop price $q_i \geq 0$. That is, a simple sell stop order becomes a market sell order, once the market price is below the stop price q_i.

Stop orders may also be combined with price limits. A *stop limit buy order*, $((q_i, p_i]; x_i)$ with $x_i > 0$, is the automatic placement of the limit buy order

(p_i, x_i) in the event that the market price is above the stop price $q_i \geq 0$, where $x_i > 0 \Rightarrow q_i < p_i$. A *stop limit sell order*, $([p_i, q_i); x_i)$ with $x_i < 0$, is the automatic placement of the limit sell order (p_i, x_i) in the event that the market price is below the stop price $q_i \geq 0$, where $x_i < 0 \Rightarrow q_i > p_i$.

Stop orders allow agents to "undo" other orders. Yet, the presence of stop orders destroys the monotonicity of demand and supply functions derived from the order book. They also destroy the continuity properties (continuity from below for demand and continuity from above for supply) of the demand and supply functions as derived from an order book. Consequently, the market may not work properly in the presence of stop orders. This is illustrated by an example that can be found in the online appendix.

5 Conclusions

This paper proposes a strategic market game under uncertainty in which traders may place multiple limit and market orders. The pricing rule is as in electronic limit order books. When utility functions are measurable with respect to private information and individual demand functions are downward sloping in the price, this mechanism generates incentives for traders to behave as if they were price takers: Locally they face constraints with infinite price elasticity. Therefore, there is always an equilibrium of the market game that supports a competitive allocation. In fact, if agents are small relative to the market, the allocations resulting from Nash equilibria of the market game cannot deviate too much from competitive allocations.

Since agents may place multiple orders, under uncertainty they can hedge against all possible contingencies. Consequently, if the competitive equilibrium has fully revealing prices, then so does the corresponding Nash equilibrium of the market game. Yet, if agents are large, Nash equilibria may not be competitive and information revelation may be distorted, as shown by an example.

The paper thus resolves two issues. First, why finitely many agents behave as if they were price takers, even though they are involved in strategic interaction, provided each agent is small in the sense that her withdrawing from the economy does not move market prices too much. Second, how private information gets incorporated in prices even though the allocation is as if agents did not recognize their influence on prices.

The caveat to these results is that they take private values, in the sense that preferences are measurable with respect to private information, and the law of demand. With common values (when measurability fails), the picture changes. Even though there are cases, where the market game has approximately competitive Nash equilibria, these are rare. In many instances of asymmetric information economies the Nash equilibria of order-driven markets are far from competitive, as shown by a characterization result. The reason is that with common values individual demand functions are typically not all downward sloping in the price—and this conflicts with the limit order book's price-priority rule.

A Appendix: Proofs

Proof of Lemma 1. (a) The first and the last inequality follow from the definitions (11) and (12). To see the second inequality, note that by definition $D_{\mathbf{y}}(p) < S_{\mathbf{y}}(p)$ for all $p > \underline{b}(\mathbf{y})$ and $D_{\mathbf{y}}(p) > S_{\mathbf{y}}(p)$ for all $p < \overline{a}(\mathbf{y})$. Therefore, $\underline{b}(\mathbf{y}) < \overline{a}(\mathbf{y})$ would imply that there is some $p \in (\underline{b}(\mathbf{y}), \overline{a}(\mathbf{y}))$ such that $D_{\mathbf{y}}(p) < S_{\mathbf{y}}(p)$ and $D_{\mathbf{y}}(p) > S_{\mathbf{y}}(p)$, which is clearly impossible.

(b) If $\underline{b}(\mathbf{y}) > \overline{a}(\mathbf{y})$, then from $D_{\mathbf{y}}(p) \leq S_{\mathbf{y}}(p)$ for all $p > \overline{a}(\mathbf{y})$ and $D_{\mathbf{y}}(p) \geq S_{\mathbf{y}}(p)$ for all $p < \underline{b}(\mathbf{y})$ it follows that $D_{\mathbf{y}}(p) = S_{\mathbf{y}}(p)$ for all $p \in (\underline{b}(\mathbf{y}), \overline{a}(\mathbf{y}))$. Since $D_{\mathbf{y}}$ is continuous from below and $S_{\mathbf{y}}$ is continuous from above, and both are step functions, also $D_{\mathbf{y}}(\underline{b}(\mathbf{y})) = D_{\mathbf{y}}(p)$ and $S_{\mathbf{y}}(\overline{a}(\mathbf{y})) = S_{\mathbf{y}}(p)$ for all $p \in (\overline{a}(\mathbf{y}), \underline{b}(\mathbf{y}))$. The remaining equalities follow from the definitions (11) and (12).

(c) Suppose first that $\overline{a}(\mathbf{y}) > \underline{a}(\mathbf{y})$. Then the definition of $\overline{a}(\mathbf{y})$ implies that $D_{\mathbf{y}}(p) > S_{\mathbf{y}}(p)$ for all $p < \overline{a}(\mathbf{y})$. The definition of $\underline{a}(\mathbf{y})$ implies that $S_{\mathbf{y}}(p) = S_{\mathbf{y}}(\overline{a}(\mathbf{y}))$ for all $p \in [\underline{a}(\mathbf{y}), \overline{a}(\mathbf{y})]$. Thus, the hypothesis, continuity of $D_{\mathbf{y}}$ from below, and the definition of $\overline{a}(\mathbf{y})$ imply $D_{\mathbf{y}}(\overline{a}(\mathbf{y})) > S_{\mathbf{y}}(\overline{a}(\mathbf{y}))$. Similarly, if $\overline{b}(\mathbf{y}) > \underline{b}(\mathbf{y})$, then $D_{\mathbf{y}}(p) < S_{\mathbf{y}}(p)$ for all $p > \underline{b}(\mathbf{y})$, $D_{\mathbf{y}}(p) = D_{\mathbf{y}}(\underline{b}(\mathbf{y}))$ for all $p \in \left[\underline{b}(\mathbf{y}), \overline{b}(\mathbf{y})\right]$, continuity of $S_{\mathbf{y}}$ from above, and the definition of $\underline{b}(\mathbf{y})$ imply that $D_{\mathbf{y}}(\underline{b}(\mathbf{y})) < S_{\mathbf{y}}(\underline{b}(\mathbf{y}))$. □

Proof of Lemma 2. Suppose that at some event $s \in \mathcal{S}$ the market bid price $\overline{b}(\mathbf{y}(s))$ is strictly larger than $\underline{b}(\mathbf{y}(s))$. Then there must be an agent $i \in I$ and a type $\tau_i(s) \in T_i$ such that type $\tau_i(s)$ has placed an order with a bid price equal to $\overline{b}(\mathbf{y}(s))$. This type of agent i can profitably deviate to bidding $\underline{b}(\mathbf{y}(s))$ without the risk of being rationed, because by Lemma 1(c) $D_{\mathbf{y}}(\underline{b}(\mathbf{y})) < S_{\mathbf{y}}(\underline{b}(\mathbf{y}))$. Since this contradicts optimality, it must be the case that in any equilibrium $\overline{b}(\mathbf{y}(s)) = \underline{b}(\mathbf{y}(s))$ for all events $s \in \mathcal{S}$. An analogous argument on the supply side establishes that in equilibrium $\underline{a}(\mathbf{y}(s)) = \overline{a}(\mathbf{y}(s))$.

Next, suppose that at some event $s \in S$ it is the case that $\overline{a}(\mathbf{y}(s)) < \underline{b}(\mathbf{y}(s))$. Then by Lemma 1(b) supply equals demand at all prices between $\overline{a}(\mathbf{y}(s))$ and $\underline{b}(\mathbf{y}(s))$. Therefore, the agent, who bids $\overline{b}(\mathbf{y}(s))$ (asks $\underline{a}(\mathbf{y}(s))$), can reduce her bid price to $\overline{a}(\mathbf{y}(s))$ (increase her ask price to $\underline{b}(\mathbf{y}(s))$) without risking rationing. Since this contradicts optimality, in any equilibrium $\overline{a}(\mathbf{y}(s)) = \underline{b}(\mathbf{y}(s))$ must hold for all $s \in \mathcal{S}$. $\square$

Proof of Theorem 1. As mentioned, a full communication equilibrium always exists under the current assumptions (Hart, 1974, Theorem 3.3; Jordan, 1983, Proposition 2.8). That is, for each $s \in \mathcal{S}$ there is $p_s = p^*(s)$ such that $\sum_{i \in I} \xi_i(p_s, s) = 0$. The proof now proceeds by constructing strategies that support the full communication equilibrium, hence a rational expectations equilibrium (if the latter exists).

If agent $i \in I$ is of type $t_i \in T_i$, she regards possible the events in $\tau_i^{-1}(t_i) = \{s \in \mathcal{S} \mid s \subseteq t_i\}$. Because the excess demand function ξ_i is strictly decreasing in the price and continuous, and $\xi_i(p, s) = \xi_i(p, s')$ for all $s, s' \in \tau_i^{-1}(t_i)$ and all $p \in \mathbb{R}_+$ by measurability of utility functions, there is a unique $p_0(t_i) \in \mathbb{R}_{++}$ such that $\xi_i(p_0(t_i), s) = 0$ for all $s \in \tau_i^{-1}(t_i)$. For all $i \in I$ and each $t_i \in T_i$ let

$$\begin{aligned} \alpha(t_i) &= \min\left\{p_s \mid p_s \geq p_0(t_i),\ s \in \tau_i^{-1}(t_i)\right\} \text{ and} \\ \beta(t_i) &= \max\left\{p_s \mid p_s \leq p_0(t_i),\ s \in \tau_i^{-1}(t_i)\right\}, \end{aligned}$$

when both sets are non-empty; otherwise only one of them is relevant. For each event $s \in \tau_i^{-1}(t_i)$ such that the associated price p_s satisfies $p_s \leq p_0(t_i)$ type t_i of agent i places a limit buy order

$$(p_{is}, x_{is}) = \left(p_s, \xi_i(p_s, s) - \xi_i\left(\min\left\{p_r \mid p_s < p_r \leq p_0(t_i),\ r \in \tau_i^{-1}(t_i)\right\}, s\right)\right),$$

if there is $r \in \tau_i^{-1}(t_i)$ with $p_s < p_r \leq p_0(t_i)$, and $(p_{is}, x_{is}) = (\beta(t_i), \xi_i(\beta(t_i), t_i))$ otherwise. For each event $s \in \tau_i^{-1}(t_i)$ such that the associated price p_s satisfies $p_s \geq p_0(t_i)$ type t_i of agent i places a limit sell order

$$(p_{is}, x_{is}) = \left(p_s, \xi_i(p_s, s) - \xi_i\left(\max\left\{p_r \mid p_s > p_r \geq p_0(t_i),\ r \in \tau_i^{-1}(t_i)\right\}, s\right)\right),$$

if there is $r \in \tau_i^{-1}(t_i)$ with $p_s > p_r \geq p_0(t_i)$, and $(p_{is}, x_{is}) = (\alpha(t_i), \xi_i(\alpha(t_i), t_i))$ otherwise. (All remaining orders are with $x_{ik} = 0$.) Intuitively, strategies are constructed by ordering the possible equilibrium prices in $\tau_i^{-1}(t_i)$ and bidding (or asking) the equilibrium prices together with limit quantities that correspond to the increment of the excess demand (or supply) function over inframarginal bids (or asks). Price ties, $p_s = p_{s+1}$, do not matter, because by measurability of preferences the excess demand functions do not vary in the events $s \in \tau_1^{-1}(t_i)$.

Suppose that event $s \in \mathcal{S}$ realizes, so that agent $i \in I$ is of type $\tau_i(s) \in T_i$. Letting $\mathbf{y}(s)$ denote the associated order book under the above strategies, the construction of strategies implies that

$$D_{\mathbf{y}(s)}(p_s) = \sum_{i \in I} \max\{0, \xi_i(p_s, s)\} = S_{\mathbf{y}(s)}(p_s) = -\sum_{i \in I} \min\{0, \xi_i(p_s, s)\}.$$

Therefore, $\underline{a}(\mathbf{y}(s)) = \overline{a}(\mathbf{y}(s)) = \underline{b}(\mathbf{y}(s)) = \overline{b}(\mathbf{y}(s)) = p_s$ by Lemma 1, because excess demand functions are strictly decreasing in the price, so that at any price $p > p_s$ we have $D_{\mathbf{y}(s)}(p) < S_{\mathbf{y}(s)}(p)$ and at any price $p < p_s$ we have $D_{\mathbf{y}(s)}(p) > S_{\mathbf{y}(s)}(p)$. Hence, no active trader is rationed.

It remains to show that the above strategies are optimal. The opportunity set at event $s \in S$ and the equilibrium price p_s for a buyer at her order is given by $F_{\mathbf{y}(s)}(p_s) = \left[0, S_{\mathbf{y}(s)}(p_s) - \lim_{\varepsilon \searrow 0} D_{\mathbf{y}(s)}(p_s + \varepsilon)\right)$, and for a seller by $F_{\mathbf{y}(s)}(p_s) = \left(\lim_{\varepsilon \searrow 0} S_{\mathbf{y}(s)}(p_s - \varepsilon) - D_{\mathbf{y}(s)}(p_s), 0\right]$. Because the limit price of all (active) traders at $s \in \mathcal{S}$ equals p_s, both these intervals are non-empty, since $\lim_{\varepsilon \searrow 0} D_{\mathbf{y}}(p_s + \varepsilon) < D_{\mathbf{y}(s)}(p_s) = S_{\mathbf{y}(s)}(p_s)$ and $\lim_{\varepsilon \searrow 0} S_{\mathbf{y}(s)}(p - \varepsilon) < S_{\mathbf{y}(s)}(p_s) = D_{\mathbf{y}(s)}(p_s)$. In fact, the upper bound for a buyer's interval is at least as large as the sum of all buy orders with limit prices p_s, hence larger than the increment of the buyer's excess demand at p_s over the sum of her inframarginal orders. Therefore, the buyer obtains precisely her excess demand at p_s and her order is located at a horizontal piece of her opportunity set; hence it is optimal. An analogous argument establishes that a seller's order is optimal. Thus, an equilibrium has been constructed. □

PROOF OF PROPOSITION 1. (a) To see the first statement takes the correspondence $F_{\mathbf{y}}$, as defined by (15), the budget set $B_i(\tau_i(s))$, as defined by (5) and (6), and the shape of indifference curves in order space (p, x). The opportunity set $F_{\mathbf{y}}(p)$ has been explained in the text. The budget set,

$$-w_{i2}(t_i) \leq \sum_{p_{ik} \geq p} \max\{0, x_{ik}\} + \sum_{p_{ik} \leq p} \min\{0, x_{ik}\} \leq w_{i1}(t_i)/p,$$

corresponds to a fixed (negative) lower bound on supplies, and a downward sloping hyperbola for demands. To understand indifference curves in the space of price and net trades, implicitly differentiate the equation

$$u_i\left(w_{i1}\left(t_i\right)-px, w_{i2}\left(t_i\right)+x\,|s\right)=c$$

for a constant $c>u_i\left(w_{i1}\left(t_i\right), w_{i2}\left(t_i\right)|s\right)$ at some $x\in\left(-w_{i2}\left(t_i\right), w_{i1}\left(t_i\right)/p\right)$. This yields

$$\frac{dp}{dx}=\frac{1}{x}\left(\frac{\partial u_i/\partial x_2}{\partial u_i/\partial x_1}-p\right).$$

That is, for a net buyer ($x>0$) that p is above the marginal rate of substitution (MRS), $\left(\partial u_i/\partial x_2\right)/\left(\partial u_i/\partial x_1\right)$, implies that the indifference curve (in (p,x)-space) is downward sloping, and that p is below the MRS implies that the indifference curve is upward sloping; for a net seller ($x<0$) that p is above the MRS implies that the indifference curve is upward sloping, and that p is below the MRS implies that it is downward sloping. Clearly, utility is increasing in the price p for a net seller and decreasing in the price for a net buyer. Thus, indifference curves (for utility levels above the utility of inactivity) in (p,x)-space are downward opening curves with a unique maximum at $x=\xi_i\left(p,s\right)$ for buyers, and upward opening curves with a unique minimum at $x=\xi_i\left(p,s\right)$ for sellers (see Figure 2). Quasi-concavity of utility functions implies that the preferred sets for a fixed price p are convex (intervals).

For concreteness, consider an agent, who has placed a buy order $\left(p_{is}, x_{is}\right)\gg 0$ for event $s\in\mathcal{S}$ in some equilibrium. (The arguments for sell orders are analogous.) If $\mathbf{y}$ denotes for the moment the order book exclusive of i's buy order, her opportunity set is given by the intersection of $B_i\left(\tau_i\left(s\right)\right)$ with the graph of $F_{\mathbf{y}}$. If the optimal choice is located at the boundary of $B_i\left(\tau_i\left(s\right)\right)$, then the indifference curve through it must be (weakly) upward sloping and the claim follows directly. Otherwise the optimal choice is interior and must be located on the boundary of (the graph of) $F_{\mathbf{y}}$, because any interior point allows for an improvement by the shape of indifference curves. Lemma 2 implies that there is $p_s\in\mathbb{R}_+$ such that $\overline{a}\left(\mathbf{y}\left(s\right)\right)=p_s=\underline{b}\left(\mathbf{y}\left(s\right)\right)$. Therefore, the optimal order must either be on a corner point, where $x_{is}=S_{\mathbf{y}}\left(p_s\right)-D_{\mathbf{y}}\left(p_s\right)$ and $p_{is}\geq\underline{b}\left(\mathbf{y}\left(s\right)\right)=p_s$, or it must satisfy

$$\frac{\partial u_i\left(w_{i1}-p_sx_{is}, w_{i2}+x_{is}\,|s\right)}{\partial x_2}=p_s\frac{\partial u_i\left(w_{i1}-p_sx_{is}, w_{i2}+x_{is}\,|s\right)}{\partial x_1},$$

at $w_{ij}=w_{ij}\left(\tau_i\left(s\right)\right)$ for $j=1,2$ (see (4)). In the first case the indifference curve through $\left(p_s, x_{is}\right)$ must be upward sloping and, therefore, $x_{is}\leq\xi_i\left(p_s,s\right)$.

In the second case clearly $x_{is} = \xi_i(p_s, s)$; see Figure 3 for an illustration. This demonstrates statement (a).

(b) Suppose that for some equilibrium there is an event $s \in \mathcal{S}$ at which there are several bidders for whom the quantity bought is strictly smaller than $\xi_i(p_s, s)$. (Again, the arguments on the supply side are analogous.) This can only occur if the optimal buy order for each of these bidders sits on a corner point of $F_{\mathbf{y}(s)}$ at p_s. But then each of these bidders must bid precisely p_s, which contradicts the hypothesis that there is more than one bidder for whom the quantity bought is strictly smaller than $\xi_i(p_s, s)$, as corner points are only feasible if the constraint comes from the other side of the market. Hence, this can hold only for at most one bidder.

(c) By the specification of the trading mechanism only traders on the same side of the market can be rationed simultaneously. Suppose at some event $s \in \mathcal{S}$ rationing occurs at the demand side, $D_{\mathbf{y}(s)}(p_s) > S_{\mathbf{y}(s)}(p_s)$. (The arguments on the supply side are analogous.) Since by Lemma 2 $p_s = \underline{b}(\mathbf{y}(s))$, at any price $p > p_s$ it must be the case that $D_{\mathbf{y}(s)}(p) < S_{\mathbf{y}(s)}(p)$. Yet, if more than one buyer is rationed, then several buyers must bid $p_s = \underline{a}(\mathbf{y}(s)) = \overline{a}(\mathbf{y}(s)) = \underline{b}(\mathbf{y}(s)) = \overline{b}(\mathbf{y}(s))$, because only marginal bidders are rationed. But then each of those marginal bidders can, by slightly increasing her bid, obtain with certainty every non-negative quantity strictly smaller than the excess supply at p_s over inframarginal demands, $S_{\mathbf{y}(s)}(p_s) - \lim_{\varepsilon \searrow 0} D_{\mathbf{y}(s)}(p_s + \varepsilon)$ (which corresponds to the fourth line in the definition of $F_{\mathbf{y}(s)}$, when $\mathbf{y}(s)$ excludes the order under scrutiny). Since in equilibrium it cannot be profitable to raise the market bid price above $p_s = \underline{b}(\mathbf{y}(s))$, the optimal quantity demanded at p_s by a marginal bidder must be strictly less than $z \equiv S_{\mathbf{y}(s)}(p_s) - \lim_{\varepsilon \searrow 0} D_{\mathbf{y}(s)}(p_s + \varepsilon) = \lim_{\varepsilon \searrow 0} \left[S_{\mathbf{y}(s)}(p_s + \varepsilon) - D_{\mathbf{y}(s)}(p_s + \varepsilon) \right] > 0$. Since z is precisely the quantity that will be distributed by the random rationing mechanism, by risk aversion each marginal bidder prefers to demand the expected value of her share in z over the lottery induced by the rationing mechanism. Since the expected values of the marginal (active) bidders' shares in z must add up to z, this contradicts equilibrium. $\square$

Proof of Theorem 2. For each $s \in \mathcal{S}$ there is p_s that constitutes both the market bid and the market ask price by Lemma 2. By Proposition 1(a) and (b) there are at most two agents, $i(s), j(s) \in I$, on different sides of the market, $i(s) \neq j(s)$, who are not on their excess demand functions, $\xi_{i(s)}(p_s, s)$ and $\xi_{j(s)}(p_s, s)$, at p_s. Both of these must bid resp. ask the market price p_s, because their optimal choice must sit on a corner point of $F_{\mathbf{y}}$ (see (15)), with (the sums of executable) limit quantities not exceeding in absolute value their excess demands (by Proposition 1(a)). Assume without loss of generality that at p_s agent $i(s)$ is a net seller and $j(s)$ a net buyer, let

$$z_{i(s)} = \sum_{p_{i(s)k} \leq p_s} \min\left\{0, x_{i(s)k}\right\} + \sum_{p_{i(s)k} \geq p_s} \max\left\{0, x_{i(s)k}\right\} \leq 0 \text{ and}$$
$$z_{j(s)} = \sum_{p_{j(s)k} \geq p_s} \max\left\{0, x_{j(s)k}\right\} + \sum_{p_{j(s)k} \leq p_s} \min\left\{0, x_{j(s)k}\right\} \geq 0,$$

denote the "notional" net trades of $i(s)$ and $j(s)$ at p_s and define

$$\theta_{i(s)} = \max\left\{z_{i(s)}, -\sum_{k \in I \setminus \{i(s)\}} \xi_k(p_s, s)\right\} \text{ and}$$
$$\theta_{j(s)} = \min\left\{z_{j(s)}, -\sum_{k \in I \setminus \{j(s)\}} \xi_k(p_s, s)\right\}.$$

Then the equation claimed in the statement must hold. For, if $D_{\mathbf{y}(s)}(p_s) < S_{\mathbf{y}(s)}(p_s)$, then the seller $i(s)$ is rationed. If $\theta_{j(s)} < \xi_{j(s)}(p_s, s)$ would hold, the buyer $j(s)$ could profitably deviate to a higher limit quantity (at the same bid price), taking up quantity supplied that the rationing cuts away from $i(s)$'s limit quantity. Thus, $\theta_{j(s)} = \xi_{j(s)}(p_s, s)$ must hold and $\theta_{i(s)} = -\sum_{k \in I \setminus \{i(s)\}} \xi_k(p_s, s)$. Similarly, if $D_{\mathbf{y}}(p_s) > S_{\mathbf{y}}(p_s)$ and the buyer is rationed, then $\theta_{i(s)} = \xi_{i(s)}(p_s, s)$ and $\theta_{j(s)} = -\sum_{k \in I \setminus \{j(s)\}} \xi_k(p_s, s)$. If $D_{\mathbf{y}}(p_s) = S_{\mathbf{y}}(p_s)$, then no rationing occurs, so that $\theta_{i(s)} = z_{i(s)}$ and $\theta_{j(s)} = z_{j(s)}$ verify the equation. □

Proof of Theorem 3. "if:" When condition (17) holds and the full communication equilibrium, with prices $p^*(s) = p_s$ for all $s \in \mathcal{S} = \{1, ..., |\mathcal{S}|\}$, is fully revealing, i.e. $0 < p_1 < ... < p_{|\mathcal{S}|}$, the proof of Theorem 1 carries over to the case of common values. When condition (17) holds, but $0 < p_{s-1} < p_s = p_{s+1} \leq p_{|\mathcal{S}|}$, say, then the tie needs to be broken.[16] This is done by starting with the lowest full communication equilibrium price p_s for which a tie (with p_{s+1}) occurs and working upwards, defining new candidate equilibrium prices $\widetilde{p}_s$ with $\widetilde{p}_s \leq p_s$ for all $s \in \mathcal{S}$.

More precisely, if $\xi_i(p_s, s) = \xi_i(p_{s+1}, s+1)$ for all $i \in I$, then the same limit orders for s and for $s+1$ will do. (In fact, then s and $s+1$ can be

[16] This is not needed with private values, because excess demand functions are constant in the events $s \in \tau_i^{-1}(t_i)$ for all $i \in I$.

identified to a single event.) Therefore, assume without loss of generality that $\xi_j(p_s, s) \neq \xi_j(p_{s+1}, s+1)$ for some $j \in I$. Then, in particular, at least one of the events s or $s+1$ must involve trade in equilibrium. Assume, again without loss of generality, that s involves trade in the full communication equilibrium. Then there is $\iota \in I$ such that $\sum_{i \in I} \xi_i(p_s, s) = 0 < \xi_\iota(p_s, s)$. By continuity and strict monotonicity of excess demand there is ε_s, with $0 < \varepsilon_s < p_s - p_{s-1}$ if $s > 1$ and $0 < \varepsilon_s < p_1$ otherwise, such that

$$0 < \sum_{i \in I} \xi_i(p_s - \varepsilon_s, s) < \xi_\iota(p_s - \varepsilon_s, s). \tag{19}$$

(For all events $s' \in \mathcal{S}$ for which $p_{s'} < p_{s'+1}$ set $\varepsilon_{s'} = 0$.) Construct strategies as follows. For all $i \in I$ and $s \in \mathcal{S} \setminus \{1, |\mathcal{S}|\}$ define

$$\Sigma_i^-(s) = \{s' \subseteq \tau_i(s) \setminus \{s\} \,|p_{s'} \leq p_s\} \text{ and } \sigma_i^-(s) = \arg\max_{s' \in \Sigma_i^-(s)} p_{s'}$$

$$\Sigma_i^+(s) = \{s' \subseteq \tau_i(s) \setminus \{s\} \,|p_{s'} \geq p_s\} \text{ and } \sigma_i^+(s) = \arg\min_{s' \in \Sigma_i^+(s)} p_{s'}$$

so that by condition (17) $p_s > p_{\sigma_i^-(s)} \Rightarrow \xi_i(p_s, s) \leq \xi_i\left(p_{\sigma_i^-(s)}, \sigma_i^-(s)\right)$ and $p_{\sigma_i^+(s)} > p_s \Rightarrow \xi_i\left(p_{\sigma_i^+(s)}, \sigma_i^+(s)\right) \leq \xi_i(p_s, s)$. For all $j \in I$ with $\xi_j(p_s - \varepsilon_s, s) \leq 0$ let

$$(p_{js}, x_{js}) = \left(p_s - \varepsilon_s, \xi_j(p_s - \varepsilon_s, s) - \xi_j\left(p_{\sigma_j^-(s)}, \sigma_j^-(s)\right)\right)$$

if $\Sigma_j^-(s) \neq \emptyset$, and $(p_{js}, x_{js}) = (p_s - \varepsilon_s, \xi_j(p_s - \varepsilon_s, s))$ if $\Sigma_j^-(s) = \emptyset$, be the limit sell order that type $\tau_j(s)$ places for event s. For all $j \in I \setminus \{\iota\}$ for whom $\xi_j(p_s - \varepsilon_s, s) > 0$ let

$$(p_{js}, x_{js}) = \left(p_s - \frac{\varepsilon_s}{2}, \xi_j(p_s - \varepsilon_s, s) - \xi_j\left(p_{\sigma_j^+(s)}, \sigma_j^+(s)\right)\right)$$

if $\Sigma_j^+(s) \neq \emptyset$, and $(p_{js}, x_{js}) = (p_s - \varepsilon_s/2, \xi_j(p_s - \varepsilon_s, s))$ if $\Sigma_j^+(s) = \emptyset$, be the limit buy order that type $\tau_j(s)$ places for event s. Finally, let

$$(p_{\iota s}, x_{\iota s}) = \left(p_s - \varepsilon_s, \left(-\sum_{i \in I \setminus \{\iota\}} \xi_i(p_s - \varepsilon_s, s)\right) - \xi_\iota\left(p_{\sigma_\iota^+(s)}, \sigma_\iota^+(s)\right)\right)$$

if $\Sigma_\iota^+(s) \neq \emptyset$, and $(p_{\iota s}, x_{\iota s}) = \left(p_s - \varepsilon_s, -\sum_{i \in I \setminus \{\iota\}} \xi_i(p_s - \varepsilon_s, s)\right)$ if $\Sigma_\iota^+(s) = \emptyset$, be the limit sell order that type $\tau_\iota(s)$ of agent ι places for event s. Note that ι bids a lower price than all other potential buyers, and $\sum_{i \in I \setminus \{\iota\}} \xi_i(p_s - \varepsilon_s, s) < 0$

by (19).

Set $\widetilde{p}_s = p_s - \varepsilon_s$, and repeat this construction for all other price ties. This gives new candidate equilibrium prices $0 < \widetilde{p}_1 < \widetilde{p}_2 < ... < \widetilde{p}_{|S|}$, that are ε_s-close to the full communication equilibrium prices, and associated strategies with the following property: If no price tie (for full communication equilibrium prices) occurred, all agents bid and ask the same price ($\widetilde{p}_s = p_s$); if a price tie occurred ($\widetilde{p}_s = p_s - \varepsilon_s$), all potential sellers and the buyer ι ask resp. bid the price $\widetilde{p}_s$, but all other potential buyers overbid (by $\varepsilon_s/2$).

If $\widetilde{p}_s = p_s$, the proof that $D_{\mathbf{y}(s)}(\widetilde{p}_s) = S_{\mathbf{y}(s)}(\widetilde{p}_s)$, $\underline{a}(\mathbf{y}(s)) = \overline{b}(\mathbf{y}(s)) = \widetilde{p}_s$, and that strategies are optimal works as in the proof of Theorem 1. If $\widetilde{p}_s = p_s - \varepsilon_s$, then by construction $D_{\mathbf{y}(s)}(\widetilde{p}_s) = S_{\mathbf{y}(s)}(\widetilde{p}_s)$, because agent ι is the only buyer, who bids the price $\widetilde{p}_s$ and she is constrained from the other side of the market. By (19) ι's limit quantity is below her excess demand at $\widetilde{p}_s$, thus, her limit order is optimal. All other market participants are on their excess demand functions for event $s \in \mathcal{S}$, so that their limit orders are also optimal. It follows that a Nash equilibrium has been constructed. Since the ε_ss can be chosen arbitrarily small, this Nash equilibrium approximates the full communication equilibrium.

"only if:" Suppose that condition (17) fails, but that there is a Nash equilibrium for the market game that is arbitrarily close to a full communication equilibrium (with prices $p^*(s) = p_s$ for all $s \in \mathcal{S}$). Then there are a type $t_i \in T_i$ for some agent $i \in I$ and two events $s, s' \in \tau_i^{-1}(t_i)$ such that $p_s > p_{s'}$ and $\xi_i(p_s, s) > \xi_i(p_{s'}, s')$. Suppose first that $\xi_i(p_s, s) > 0$. If the market game has a Nash equilibrium close to the full communication equilibrium, with market bid and ask prices equal to $\widetilde{p}_s \approx p_s$ for all $s \in \mathcal{S}$, then type t_i must bid at least $\widetilde{p}_s$ for event s; yet, as $\widetilde{p}_s > \widetilde{p}_{s'}$, an order with a limit price of at least $\widetilde{p}_s$ will also execute at event s' and cannot be rationed at s'. Therefore, at event s' type t_i cannot be close to her excess demand at $\widetilde{p}_{s'}$, a contradiction. If $\xi_i(p_s, s) \leq 0$, then $\xi_i(p_{s'}, s') < 0$, too. Then type t_i cannot ask more than $\widetilde{p}_{s'} \approx p_{s'}$ for event s'; yet, as $\widetilde{p}_s > \widetilde{p}_{s'}$, a sell order with a limit price not exceeding $\widetilde{p}_{s'}$ will also execute at event s, where it cannot be rationed, since it is inframarginal. It follows that at event s type t_i cannot be close to her excess demand at $\widetilde{p}_s$, again a contradiction. $\square$

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