The Davenport constant of finite abelian groups

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- A sequence S over G is called *zero-sum* if $|S| = 0_G$.
- The *Davenport constant* $D(G)$ of G is the smallest $d \in \mathbb{N}$ such that every sequence over G of length d has a non-empty zero sum subsequence.

The Davenport constant

Question (Rogers, 1963, and Davenport, 1966)

What is the value of $D(G)$ for an arbitrary finite abelian group G?

Trivial bounds for $D(G)$

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Structure theorem

For any non-trivial finite abelian group G there exists unique parameters $1 < n_1 | \cdots | n_r \in \mathbb{N}$ such that $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$.

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For a finite abelian group $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ with $1 < n_1 | \cdots | n_r$ define $d^*(G) := \sum_{i=1}^{r}(n_i - 1)$.

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$$1 + d^*(G) \leq D(G)$$

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For any non-trivial finite abelian group G:

$$1 + d^*(G) \leq D(G) \leq |G|$$

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Every sequence of $2n-1$ elements of $\mathbb{Z}_n$ contains a non-empty zero-sum subsequence of length n.

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Davenport's problem: Find the smallest x such that every sequence of x elements of G contains a non-empty zero-sum subsequence, i.e find $D(G)$

Applications

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- Applications in graph theory:
  N. Alon, S. Friedland, and G. Kalai. "Regular subgraphs of almost regular graphs". In: *J. Combin. Theory Ser. B* 37.1 (1984), pp. 79–91

The lower bound

Question

Which groups G satisfy $D(G) = 1 + d^*(G)$ and which satisfy $D(G) > 1 + d^*(G)$?

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$\boldsymbol{D(G) = 1 + d^*(G)}$	$\boldsymbol{D(G) > 1 + d^*(G)}$
p-groups	$\mathbb{Z}_2^{4k} \oplus \mathbb{Z}_{4k+2}$, $k \in \mathbb{N}$
groups of rank ≤ 2	$\mathbb{Z}_m \oplus \mathbb{Z}_n^2 \oplus \mathbb{Z}_{2n}$, m, $n \geq 3$ are odd and $m\|n$
$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$, $d \geq 1$	$\mathbb{Z}_n^n \oplus \mathbb{Z}_{nm}$, m, $n \geq 2$ and $m\|n-1$
many others...	some others...

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$D(G) = 1 + d^*(G)$	$D(G) > 1 + d^*(G)$
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groups of rank ≤ 2	$\mathbb{Z}_m \oplus \mathbb{Z}_n^2 \oplus \mathbb{Z}_{2n},\ m,\ n \geq 3$ are odd and $m\vert n$
$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d},\ d \geq 1$	$\mathbb{Z}_n^n \oplus \mathbb{Z}_{nm},\ m,\ n \geq 2$ and $m\vert n-1$
many others...	some others...

Question

Does the equality $D(G) = 1 + d^*(G)$ hold for all finite abelian groups G of rank three?

$D(\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10})$

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Theorem (A.S. 2015)

The equality $D(G) = 1 + d^*(G)$ *holds for any finite abelian group* $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$

Open problems

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- Does the equality $D(G) = 1 + d^*(G)$ hold for $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{15}$?

Thank you for listening!