The Davenport constant of finite abelian groups of rank three

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- The *Davenport constant* $D(G)$ of G is the smallest $m \in \mathbb{N}$ such that every sequence S over G with $l(S) \geq m$ has a non-empty zero sum subsequence.

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Question (Rogers, 1963, and Davenport, 1966)

What is the value of $D(G)$ for an arbitrary finite abelian group G?

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Structure theorem

For any non-trivial finite abelian group G there exists unique parameters $1 < d_1 | \cdots | d_r \in \mathbb{N}$ such that $G \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r}$.

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$$1 + d^*(G) \leq D(G) \leq |G|$$

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Every sequence of $2n-1$ *elements of* $\mathbb{Z}_n$ *contains a non-empty zero-sum subsequence of length* n.

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Davenport's problem: Find the smallest x such that every sequence of x elements of G contains a non-empty zero-sum subsequence, i.e find $D(G)$

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- Applications in graph theory: N Alon, S Friedland, and G Kalai. "Regular subgraphs of almost regular graphs". In: *Journal of Combinatorial Theory, Series B* 37 (1984), pp. 79–91

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Question

Which groups G satisfy $D(G) = 1 + d^*(G)$ and which satisfy $D(G) > 1 + d^*(G)$?

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$\boldsymbol{D(G) = 1 + d^*(G)}$	$\boldsymbol{D(G) > 1 + d^*(G)}$
p-groups	$\mathbb{Z}_2^{4k} \oplus \mathbb{Z}_{4k+2}$, $k \in \mathbb{N}$
groups of rank ≤ 2	$\mathbb{Z}_m \oplus \mathbb{Z}_n^2 \oplus \mathbb{Z}_{2n}$, m, $n \geq 3$ are odd and $m \mid n$
$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$, $d \geq 1$	$\mathbb{Z}_n^n \oplus \mathbb{Z}_{nm}$, m, $n \geq 2$ and $m \mid n-1$
many others...	some others...

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Question

Does the equality $D(G) = 1 + d^*(G)$ hold for all finite abelian groups G of rank three?

$D(\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10})$

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Theorem (A.S. 2015)

The equality $D(G) = 1 + d^*(G)$ *holds for any finite abelian group* $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$

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- Does the equality $D(G) = 1 + d^*(G)$ hold for $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{15}$?

Thank you for listening!