

Complete Bredon cohomology and its applications to hierarchically defined groups

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Abstract

By considering the Bredon analogue of complete cohomology of a group, we show that every group in the class $\mathbf{LH}^{\mathfrak{F}}$ of type Bredon-FP $_{\infty}$ admits a finite dimensional model for $E_{\mathfrak{F}}G$. We also show that abelian-by-infinite cyclic groups admit a 3-dimensional model for the classifying space for the family of virtually nilpotent subgroups. This allows us to prove that for \mathfrak{F} , the class of virtually cyclic groups, the class of $\mathbf{LH}^{\mathfrak{F}}$ -groups contains all locally virtually soluble groups and all linear groups over \mathbb{C} of integral characteristic.

1. Introduction

Classifying spaces with isotropy in a family have been the subject of intensive research, with a large proportion focussing on \underline{EG} , the classifying space with finite isotropy [19–21]. Classes of groups admitting a finite dimensional model for \underline{EG} abound, such as elementary amenable groups of finite Hirsch length [9, 14], hyperbolic groups [28], mapping class groups [21] and $\text{Out}(F_n)$ [31]. Finding manageable models for \underline{EG} , the classifying space for virtually cyclic isotropy, has been shown to be much more elusive. So far manageable models have been found for crystallographic groups [17], polycyclic-by-finite groups [24], hyperbolic groups [12], certain HNN-extensions [10], elementary amenable groups of finite Hirsch length [5, 6, 11] and groups acting isometrically with discrete orbits on separable complete CAT(0)-spaces [7, 22].

Let \mathfrak{F} be a family of subgroups of a given group and denote by $E_{\mathfrak{F}}G$ the classifying space with isotropy in \mathfrak{F} . In this note we propose a method to decide whether a group has a finite dimensional model for $E_{\mathfrak{F}}G$ without actually providing a bound. This is closely related to Kropholler’s Theorem that a torsion-free group in $\mathbf{LH}\mathfrak{F}$ of type FP $_{\infty}$ has finite integral cohomological dimension [15]. To do this we consider groups belonging to the class $\mathbf{LH}^{\mathfrak{F}}$, a class recently considered in [8]:

Let \mathfrak{F} be a class of groups closed under taking subgroups. Let G be a group and set $\mathfrak{F} \cap G = \{H \leq G \mid H \text{ is isomorphic to a subgroup in } \mathfrak{F}\}$. Let \mathfrak{X} be a class of groups. Then $\mathbf{H}^{\mathfrak{F}}\mathfrak{X}$ is defined as the smallest class of groups containing the class \mathfrak{X} with the property that if a group G acts cellularly on a finite dimensional CW-complex X with all isotropy subgroups in $\mathbf{H}^{\mathfrak{F}}\mathfrak{X}$, and such that for each subgroup $F \in \mathfrak{F} \cap G$ the fixed point set X^F is contractible, then G is in $\mathbf{H}^{\mathfrak{F}}\mathfrak{X}$. The class $\mathbf{LH}^{\mathfrak{F}}\mathfrak{X}$ is

defined to be the class of groups that are locally $\mathbf{H}^{\mathfrak{F}}\mathfrak{X}$ -groups.

In this definition and throughout the paper, we always assume that a cellular action of group on a CW-complex is admissible. That is, if an element of group stabilises a cell, then it fixes it pointwise.

We generalise complete cohomology of a group to the Bredon setting and verify that some of the main results hold in this new context. This allows us to establish:

Theorem A. *Let G be group in $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ of type Bredon-FP $_{\infty}$. Then G admits a finite dimensional model for $E_{\mathfrak{F}}G$.*

We consider the class $\mathbf{LH}^{\mathfrak{F}}\mathfrak{X}$, especially when $\mathfrak{F} = \mathfrak{X}$ is either the class of all finite groups or the class of all virtually cyclic groups. Note, that if \mathfrak{F} contains the trivial group only, then $\mathbf{LH}^{\mathfrak{F}}\mathfrak{X}$ is exactly Kropholler's class $\mathbf{LH}\mathfrak{X}$. If \mathfrak{F} is the class of all finite groups, then $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ also turns out to be quite large. It contains all elementary amenable groups and all linear groups over a field of arbitrary characteristic (see [7], [8]). It is also closed under extensions, taking subgroups, amalgamated products, HNN-extensions, and countable directed unions. Here we show that similar closure operations hold when \mathfrak{F} is the class of virtually cyclic groups.

In [8], it was shown that when \mathfrak{F} is the class of finite groups, then $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ contains all elementary amenable groups. We show that when $\mathfrak{F} = \mathfrak{F}_{\text{vc}}$, the class of virtually cyclic groups, then $\mathbf{LH}^{\mathfrak{F}_{\text{vc}}}\mathfrak{F}_{\text{vc}}$ contains all locally virtually soluble groups. We also show that any countable subgroup of a general linear group $\text{GL}_n(\mathbb{C})$ of integral characteristic lies in $\mathbf{H}^{\mathfrak{F}_{\text{vc}}}\mathfrak{F}_{\text{vc}}$. Both of these results rely on the following:

Theorem B. *Let G be a semi-direct product $A \rtimes \mathbb{Z}$ where A is a countable abelian group. Define \mathfrak{H} to be the family of all virtually nilpotent subgroups of G . Then there exists a 3-dimensional model for $E_{\mathfrak{H}}G$.*

Another consequence of Theorem B is that any semi-direct product $A \rtimes \mathbb{Z}$ where A is a countable abelian group lies in $\mathbf{H}_3^{\mathfrak{F}_{\text{vc}}}\mathfrak{F}_{\text{vc}}$.

2. Background on Bredon cohomology

In this note, a family \mathfrak{F} of subgroups of a group G is closed under conjugation and taking subgroups. The families most frequently considered are the family $\mathfrak{F}_{\text{fin}}(G)$ of all finite subgroups of G and the family $\mathfrak{F}_{\text{vc}}(G)$ of all virtually cyclic subgroups of G .

For a subgroup $K \leq G$ we consider:

$$\mathfrak{F} \cap K = \{H \cap K \mid H \in \mathfrak{F}\}.$$

Bredon cohomology has been introduced for finite groups by Bredon [3] and later generalised to arbitrary groups by Lück [19].

The orbit category $\mathcal{O}_{\mathfrak{F}}G$ is defined as follows: objects are the transitive G -sets G/H with $H \leq G$ and $H \in \mathfrak{F}$; morphisms of $\mathcal{O}_{\mathfrak{F}}G$ are all G -maps $G/H \rightarrow G/K$, where $H, K \in \mathfrak{F}$.

An $\mathcal{O}_{\mathfrak{F}}G$ -module, or Bredon module, is a contravariant functor $M: \mathcal{O}_{\mathfrak{F}}G \rightarrow \mathfrak{Ab}$ from the orbit category to the category of abelian groups. A natural transformation $f: M \rightarrow N$ between two $\mathcal{O}_{\mathfrak{F}}G$ -modules is called a morphism of $\mathcal{O}_{\mathfrak{F}}G$ -modules.

The trivial $\mathcal{O}_{\mathfrak{F}}G$ -module is denoted by $\mathbb{Z}_{\mathfrak{F}}$. It is given by $\mathbb{Z}_{\mathfrak{F}}(G/H) = \mathbb{Z}$ and $\mathbb{Z}_{\mathfrak{F}}(\varphi) = \text{id}$ for all objects and morphisms φ of $\mathcal{O}_{\mathfrak{F}}G$.

The category of $\mathcal{O}_{\mathfrak{F}}G$ -modules, denoted $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$, is a functor category and therefore inherits

properties from the category \mathfrak{Ab} . For example, a sequence $L \rightarrow M \rightarrow N$ of Bredon modules is exact if and only if, when evaluated at every $G/H \in \mathcal{O}_{\mathfrak{F}}G$, we obtain an exact sequence $L(G/H) \rightarrow M(G/H) \rightarrow N(G/H)$ of abelian groups.

Since \mathfrak{Ab} has enough projectives, so does $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$, and we can define homology functors in $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ analogously to ordinary cohomology, using projective resolutions.

There now follow the basic properties of free and projective $\mathcal{O}_{\mathfrak{F}}G$ -modules as described in [19, 9.16, 9.17]. An \mathfrak{F} -set Δ is a collection of sets $\{\Delta_K \mid K \in \mathfrak{F}\}$. For any two \mathfrak{F} -sets Δ and Ω , an \mathfrak{F} -map is a family of maps $\{\Delta_K \rightarrow \Omega_K \mid K \in \mathfrak{F}\}$. Hence we have a forgetful functor from the category of $\mathcal{O}_{\mathfrak{F}}G$ -modules to the category of \mathfrak{F} -sets. One defines the free functor as the left adjoint to this forgetful functor. This satisfies the usual universal property.

There is a more constructive description of free Bredon-modules as follows: Consider the right Bredon-module: $\mathbb{Z}[-, G/K]_{\mathfrak{F}}$ with $K \in \mathfrak{F}$. When evaluated at G/H we obtain the free abelian group $\mathbb{Z}[G/H, G/K]_{\mathfrak{F}}$ on the set $[G/H, G/K]_{\mathfrak{F}}$ of G -maps $G/H \rightarrow G/K$. These modules are free, cf. [19, p. 167], and can be viewed as the building blocks of the free right Bredon-modules. Generally, a free module is one of the form $\mathbb{Z}[-, \Delta]_{\mathfrak{F}}$, where Δ is a G -set with isotropy in \mathfrak{F} . Projectives are now defined to be direct summands of frees.

Given a covariant functor $F: \mathcal{O}_{\mathfrak{F}_1}G_1 \rightarrow \mathcal{O}_{\mathfrak{F}_2}G_2$ between orbit categories, one can now define induction and restriction functors along F , see [19, p. 166]:

$$\begin{aligned} \text{Ind}_F: \mathcal{O}_{\mathfrak{F}_1}G_1 &\rightarrow \mathcal{O}_{\mathfrak{F}_2}G_2 \\ M(-) &\mapsto M(-) \otimes_{\mathfrak{F}_1} [-, F(-)]_{\mathfrak{F}_2} \end{aligned}$$

and

$$\begin{aligned} \text{Res}_F: \mathcal{O}_{\mathfrak{F}_2}G_2 &\rightarrow \mathcal{O}_{\mathfrak{F}_1}G_1 \\ M(-) &\mapsto M \circ F(-) \end{aligned}$$

Since these functors are adjoint to each others, Ind_F commutes with arbitrary colimits [26, pp. 118f.] and preserves free and projective Bredon modules [19, p. 169]. The case of particular interest is when F is given by inclusion of a subgroup of G .

For subgroup K of G we consider the following functor

$$\begin{aligned} \iota_K^G: \mathcal{O}_{\mathfrak{F} \cap K}K &\rightarrow \mathcal{O}_{\mathfrak{F}}G \\ K/H &\mapsto G/H. \end{aligned}$$

and denote the corresponding induction and restriction functors by Ind_K^G and Res_K^G respectively.

LEMMA 2.1. [30, Lemma 2.9] *Let K be a subgroup of G . Then Ind_K^G is an exact functor.*

Symmond's [30] methods also yields; for a short account see also the proof of Lemma 3.5 in [14]:

LEMMA 2.2. *Let $K \leq H \leq G$ be subgroups. Then*

$$\text{Ind}_K^G \mathbb{Z}_{\mathfrak{F}} \cong \mathbb{Z}[-, G/K]_{\mathfrak{F}},$$

and

$$\text{Ind}_H^G \mathbb{Z}[-, H/K]_{\mathfrak{F} \cap H} \cong \mathbb{Z}[-, G/K]_{\mathfrak{F}}.$$

The Bredon cohomological dimension $\text{cd}_{\mathfrak{F}}G$ of a group G with respect to the family \mathfrak{F} of subgroups is the projective dimension $\text{pd}_{\mathfrak{F}}\mathbb{Z}_{\mathfrak{F}}$ of the trivial $\mathcal{O}_{\mathfrak{F}}G$ -module $\mathbb{Z}_{\mathfrak{F}}$. The cellular chain complex of a model for $E_{\mathfrak{F}}G$ yields a free resolution of the trivial $\mathcal{O}_{\mathfrak{F}}G$ -module $\mathbb{Z}_{\mathfrak{F}}$ [19, pp. 151f.].

In particular, this implies that for the Bredon geometric dimension $\text{gd}_{\mathfrak{F}} G$, the minimal dimension of a model for $E_{\mathfrak{F}} G$, we have

$$\text{cd}_{\mathfrak{F}} G \leq \text{gd}_{\mathfrak{F}} G.$$

Furthermore, one always has:

PROPOSITION 2.3. [23, Theorem 0.1 (i)] *Let G be a group. Then*

$$\text{gd}_{\mathfrak{F}} G \leq \max(3, \text{cd}_{\mathfrak{F}} G).$$

Next, suppose \mathfrak{T} and \mathfrak{H} are families of subgroups of a group G where $\mathfrak{T} \subseteq \mathfrak{H}$. In Section 5, we will need to adapt a model for $E_{\mathfrak{T}} G$ to obtain a model for $E_{\mathfrak{H}} G$. For this we will use a general construction of Lück and Weiermann (see [24, §2]). We recall the basics of this construction:

Suppose that there exists an equivalence relation \sim on the set $\mathcal{S} = \mathfrak{H} \setminus \mathfrak{T}$ that satisfies the following properties:

- $\forall H, K \in \mathcal{S} : H \subseteq K \Rightarrow H \sim K$;
- $\forall H, K \in \mathcal{S}, \forall x \in G : H \sim K \Leftrightarrow H^x \sim K^x$.

An equivalence relation that satisfies these properties is called a *strong equivalence relation*. Let $[H]$ be an equivalence class represented by $H \in \mathcal{S}$ and denote the set of equivalence classes by $[\mathcal{S}]$. The group G acts on $[\mathcal{S}]$ via conjugation, and the stabiliser group of an equivalence class $[H]$ is

$$\text{N}_G[H] = \{x \in G \mid H^x \sim H\}.$$

Note that $\text{N}_G[H]$ contains H as a subgroup. Let \mathcal{S} be a complete set of representatives $[H]$ of the orbits of the conjugation action of G on $[\mathcal{S}]$. Define for each $[H] \in \mathcal{S}$ the family

$$\mathfrak{T}[H] = \{K \leq \text{N}_G[H] \mid K \in \mathcal{S}, K \sim H\} \cup (\text{N}_G[H] \cap \mathfrak{T})$$

of subgroups of $\text{N}_G[H]$.

PROPOSITION 2.4 (Lück-Weiermann, [24, 2.5]). *Let $\mathfrak{T} \subseteq \mathfrak{H}$ be two families of subgroups of a group G such that $\mathcal{S} = \mathfrak{H} \setminus \mathfrak{T}$ is equipped with a strong equivalence relation. Denote the set of equivalence classes by $[\mathcal{S}]$ and let \mathcal{S} be a complete set of representatives $[H]$ of the orbits of the conjugation action of G on $[\mathcal{S}]$. If there exists a natural number d such that $\text{gd}_{\mathfrak{T} \cap \text{N}_G[H]}(\text{N}_G[H]) \leq d - 1$ and $\text{gd}_{\mathfrak{T}[H]}(\text{N}_G[H]) \leq d$ for each $[H] \in \mathcal{S}$, and such that $\text{gd}_{\mathfrak{T}}(G) \leq d$, then $\text{gd}_{\mathfrak{H}}(G) \leq d$.*

3. Complete Bredon cohomology

Since $\text{Mod-}\mathcal{O}_{\mathfrak{F}} G$ is an abelian category, we can just follow the approaches of Mislin [29] and Benson-Carlson [2]. We will, however, include the main steps of the construction. We will begin by describing the Satellite construction due to Mislin [29]. The methods used there can be carried over to the Bredon-setting by applying [25, XII.7-8.].

Let M be an $\mathcal{O}_{\mathfrak{F}} G$ -module and denote by FM the free $\mathcal{O}_{\mathfrak{F}} G$ -module on the underlying \mathfrak{F} -set of M . Let $\Omega M = \ker(FM \rightarrow M)$, and inductively $\Omega^n M = \Omega(\Omega^{n-1} M)$. Let T be an additive functor from $\text{Mod-}\mathcal{O}_{\mathfrak{F}} G$ to the category of abelian groups. Then the left satellite of T is defined as

$$S^{-1}T(M) = \ker(T(\Omega M) \rightarrow T(FM)).$$

Furthermore, $S^{-n}T(M) = S^{-1}(S^{-n+1}T(M))$, and the family $\{S^{-n} \mid n \geq 0\}$ forms a connected sequence of functors where $S^{-n}T(P) = 0$ for all projective $\mathcal{O}_{\mathfrak{F}} G$ -modules P and $n \geq 1$. Following

the approach in [29] further, we call a connected sequence of additive functors $T^* = \{T^n \mid n \in \mathbb{Z}\}$ from $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ to the category of abelian groups a $(-\infty, +\infty)$ -Bredon-cohomological functor, if for every short exact sequence $M' \rightarrow M \rightarrow M''$ of $\mathcal{O}_{\mathfrak{F}}G$ -modules the associated sequence

$$\cdots \rightarrow T^n M' \rightarrow T^n M \rightarrow T^n M'' \rightarrow T^{n+1} M' \rightarrow \cdots$$

is exact. Obviously, Bredon-cohomology $H_{\mathfrak{F}}^*(G, -)$ is such a functor with the convention that $H_{\mathfrak{F}}^n(G, -) = 0$ whenever $n < 0$.

DEFINITION 3.1. A $(-\infty, +\infty)$ -Bredon-cohomological functor $T^* = \{T^n \mid n \in \mathbb{Z}\}$ is called P -complete if $T^n(P) = 0$ for all $n \in \mathbb{Z}$ and every projective $\mathcal{O}_{\mathfrak{F}}G$ -module P .

A morphism $\varphi^* : U^* \rightarrow V^*$ of $(-\infty, +\infty)$ -Bredon-cohomological functors is called a P -completion, if V^* is P -complete and if every morphism $U^* \rightarrow T^*$ into a P -complete $(-\infty, +\infty)$ -Bredon-cohomological functor T^* factors uniquely through $\varphi^* : U^* \rightarrow V^*$.

The following theorem is now the exact analogue to [29, Theorem 2.2].

THEOREM 3.2. Every $(-\infty, +\infty)$ -Bredon-cohomological functor T^* admits a unique P -completion \widehat{T}^* given by

$$\widehat{T}^j(M) = \varinjlim_{k \geq 0} S^{-k} T^{j+k}(M)$$

for any $M \in \text{Mod-}\mathcal{O}_{\mathfrak{F}}G$. \square

In particular, we have, for every $\mathcal{O}_{\mathfrak{F}}G$ -module M that

$$\widehat{\text{Ext}}_{\mathfrak{F}}^j(M, -) = \varinjlim_{k \geq 0} S^{-k} \text{Ext}_{\mathfrak{F}}^{j+k}(M, -).$$

We have immediately:

LEMMA 3.3. Let M and N be $\mathcal{O}_{\mathfrak{F}}G$ -modules. If either of these has finite projective dimension, then

$$\widehat{\text{Ext}}_{\mathfrak{F}}^*(M, N) = 0.$$

We can also mimic Benson and Carlson's approach [2]. For any two $\mathcal{O}_{\mathfrak{F}}G$ -modules we denote by $[M, N]_{\mathfrak{F}}$ the quotient of $\text{Hom}_{\mathfrak{F}}(M, N)$ by the subgroup of those homomorphisms factoring through a projective module. Then it follows that there is a homomorphism $[M, N]_{\mathfrak{F}} \rightarrow [\Omega M, \Omega N]_{\mathfrak{F}}$ and it can be shown analogously to [29, Theorem 4.4] that

$$\widehat{\text{Ext}}_{\mathfrak{F}}^n(M, M) = \varinjlim_{k, k+n \geq 0} [\Omega^{k+n} M, \Omega^k N]_{\mathfrak{F}}.$$

This now allows us to deduce the following Lemma, which is an analogue to [15, 4.2].

LEMMA 3.4. $\widehat{\text{Ext}}_{\mathfrak{F}}^0(M, M) = 0$ if and only if M has finite projective dimension. In particular,

$$\widehat{H}_{\mathfrak{F}}^0(G, \mathbb{Z}_{\mathfrak{F}}) = 0 \iff \text{cd}_{\mathfrak{F}} G < \infty.$$

4. Proof of Theorem A

The proof of Theorem A is analogous to the proof of the main result in [15]. We begin by recording two easy lemmas, which have their analogues in [15, 3.1] and [15, 4.1] respectively.

LEMMA 4.1. *Let*

$$0 \rightarrow M_n \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_1 \rightarrow M_0 \rightarrow L \rightarrow 0$$

be an exact sequence of $\mathcal{O}_{\mathfrak{F}}G$ -modules and i be an integer such that $\mathbf{H}_{\mathfrak{F}}^i(G, L) \neq 0$. Then there exists an integer $0 \leq j \leq n-1$ such that $\mathbf{H}_{\mathfrak{F}}^{j+i}(G, M_j) \neq 0$.

Proof. This is an easy dimension shifting argument. \square

LEMMA 4.2. *Let G be a group such that $\mathbf{H}_{\mathfrak{F}}^k(G, -)$ commutes with direct limits for infinitely many k , then $\widehat{\mathbf{H}}_{\mathfrak{F}}^k(G, -)$ commutes for all $k \in \mathbb{Z}$.*

Proof. This follows from the fact that direct limits commute with each other. \square

The proof of Theorem A now relies on the fact that one can hierarchically decompose the class $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ in exactly the same way as Kropholler's decomposition, see [8, 15]:

- $\mathbf{H}_0^{\mathfrak{F}}\mathfrak{F} = \mathfrak{F}$;
- For an ordinal $\alpha > 0$, we let $\mathbf{H}_{\alpha}^{\mathfrak{F}}\mathfrak{F}$ be the class of groups acting cellularly on a finite dimensional complex X such that each stabiliser subgroup lies in $\mathbf{H}_{\beta}^{\mathfrak{F}}\mathfrak{F}$ for some $\beta < \alpha$ and such that X^K is contractible for all $K \in \mathfrak{F}$.

A group G now lies in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ if and only if it lies in some $\mathbf{H}_{\alpha}^{\mathfrak{F}}\mathfrak{F}$ for some ordinal α .

In particular, $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ is subgroup closed.

LEMMA 4.3. *Let G be a group and G_{λ} , $\lambda \in \Lambda$ its finitely generated subgroups. Then we have the following isomorphism:*

$$\varinjlim_{\lambda \in \Lambda} \mathbb{Z}[-, G/G_{\lambda}]_{\mathfrak{F} \cap G_{\lambda}} \cong \mathbb{Z}_{\mathfrak{F}}.$$

Proof. This follows directly from Lemma 2.2. \square

THEOREM 4.4. *Let G be a group in $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ and suppose that $\widehat{\mathbf{H}}_{\mathfrak{F}}^*(G, -)$ commutes with direct limits. Then $\text{cd}_{\mathfrak{F}} G < \infty$.*

Proof. We prove this by contradiction and suppose that $\text{cd}_{\mathfrak{F}} G = \infty$. Hence, by Lemma 3.4, we have that $\widehat{\mathbf{H}}_{\mathfrak{F}}^0(G, \mathbb{Z}_{\mathfrak{F}}) \neq 0$. We claim that then there exists a group $H \in \mathfrak{F}$ and an integer $i \geq 0$ such that $\widehat{\mathbf{H}}_{\mathfrak{F}}^i(G, \text{Ind}_H^G \mathbb{Z}_{\mathfrak{F} \cap H}) \neq 0$. By Lemma 2.2, we have $\text{Ind}_H^G \mathbb{Z}_{\mathfrak{F} \cap H} \cong \mathbb{Z}[-, G/H]$, which is projective, giving us the desired contradiction.

It now remains to prove the claim: Let \mathcal{S} be the set of ordinals β such there exists a $i \geq 0$ and $H \leq G$ lying in $\mathbf{H}_{\beta}^{\mathfrak{F}}\mathfrak{F}$ and such that $\widehat{\mathbf{H}}_{\mathfrak{F}}^i(G, \text{Ind}_H^G \mathbb{Z}_{\mathfrak{F} \cap H}) \neq 0$. If we can prove that $0 \in \mathcal{S}$, we are done.

(1) We show that \mathcal{S} is not empty: Let $\{G_{\lambda} \mid \lambda \in \Lambda\}$ be the family of all finitely generated subgroups of G . Hence, applying Lemma 4.3 and the fact that $\widehat{\mathbf{H}}_{\mathfrak{F}}^*(G, -)$ commutes with direct limits, we get

$$\widehat{\mathbf{H}}_{\mathfrak{F}}^0(G, \mathbb{Z}_{\mathfrak{F}}) \cong \widehat{\mathbf{H}}_{\mathfrak{F}}^0(G, \varinjlim_{\lambda \in \Lambda} \mathbb{Z}[-, G/G_{\lambda}]_{\mathfrak{F} \cap G_{\lambda}}) \cong \varinjlim_{\lambda \in \Lambda} \widehat{\mathbf{H}}_{\mathfrak{F}}^0(G, \mathbb{Z}[-, G/G_{\lambda}]_{\mathfrak{F} \cap G_{\lambda}}).$$

Since $\widehat{\mathbf{H}}_{\mathfrak{F}}^0(G, \mathbb{Z}_{\mathfrak{F}}) \neq 0$, there exists a finitely generated subgroup G_{λ} such that, see also Lemma 2.2,

$$\widehat{\mathbf{H}}_{\mathfrak{F}}^0(G, \mathbb{Z}[-, G/G_{\lambda}]_{\mathfrak{F} \cap G_{\lambda}}) \cong \widehat{\mathbf{H}}_{\mathfrak{F}}^0(G, \text{Ind}_{G_{\lambda}}^G \mathbb{Z}_{\mathfrak{F} \cap G_{\lambda}}) \neq 0.$$

Since $G \in \mathbf{LH}^{\mathfrak{F}} \mathfrak{F}$, and $\mathbf{H}^{\mathfrak{F}} \mathfrak{F}$ is subgroup closed, $G_\lambda \in \mathbf{H}^{\mathfrak{F}} \mathfrak{F}$ and in particular, there is an ordinal β such that $G_\lambda \in \mathbf{H}_\beta^{\mathfrak{F}} \mathfrak{F}$. Hence $\beta \in \mathcal{S}$.

(2) We now show that, if $0 \neq \beta \in \mathcal{S}$, then there is an ordinal $\gamma < \beta$ such that $\gamma \in \mathcal{S}$: Let $0 \neq \beta \in \mathcal{S}$. Then there is a $H \in G$ and $i \geq 0$ such that $H \in \mathbf{H}_\beta^{\mathfrak{F}} \mathfrak{F}$ and

$$\widehat{\mathbf{H}}_{\mathfrak{F}}^i(G, \text{Ind}_H^G \mathbb{Z}_{\mathfrak{F} \cap H}) \neq 0.$$

Hence H acts cellularly on a finite dimensional contractible space X such that each isotropy group lies in some $\mathbf{H}_\gamma^{\mathfrak{F}} \mathfrak{F}$ for $\gamma < \beta$ and such that X^K is contractible if $K \in \mathfrak{F}$. Hence we have an exact sequence of free $\mathcal{O}_{\mathfrak{F}} H$ -modules:

$$0 \rightarrow C_n(X^{(-)}) \rightarrow C_{n-1}(X^{(-)}) \rightarrow \dots \rightarrow C_1(X^{(-)}) \rightarrow C_0(X^{(-)}) \rightarrow \mathbb{Z}_{\mathfrak{F} \cap H} \rightarrow 0.$$

Each

$$C_k(X^{(-)}) \cong \mathbb{Z}[-, \bigoplus_{\sigma_k \in \Delta_k} H/H\sigma_k],$$

where Δ_k is the set of orbit representatives for the k -cells of X . Furthermore, by Lemma 2.2, upon induction, we obtain an exact sequence of $\mathcal{O}_{\mathfrak{F}} G$ -modules as follows:

$$0 \rightarrow \bigoplus_{\sigma_n \in \Delta_n} \text{Ind}_{H\sigma_n}^G \mathbb{Z}_{\mathfrak{F} \cap H\sigma_n} \rightarrow \dots \rightarrow \bigoplus_{\sigma_1 \in \Delta_1} \text{Ind}_{H\sigma_1}^G \mathbb{Z}_{\mathfrak{F} \cap H\sigma_1} \rightarrow \bigoplus_{\sigma_0 \in \Delta_0} \text{Ind}_{H\sigma_0}^G \mathbb{Z}_{\mathfrak{F} \cap H\sigma_0} \rightarrow \text{Ind}_H^G \mathbb{Z}_{\mathfrak{F} \cap H} \rightarrow 0.$$

Now, by Lemma 4.1, there is a $k \geq 0$ such that

$$\widehat{\mathbf{H}}_{\mathfrak{F}}^{j+k}(G, \bigoplus_{\sigma_k \in \Delta_k} \text{Ind}_{H\sigma_k}^G \mathbb{Z}_{\mathfrak{F} \cap H\sigma_k}) \neq 0.$$

Since $\widehat{\mathbf{H}}_{\mathfrak{F}}^{j+k}(G, -)$ commutes, in particular, with direct sums, there is a $\sigma_k \in \Delta_k$ such that

$$\widehat{\mathbf{H}}_{\mathfrak{F}}^{j+k}(G, \text{Ind}_{H\sigma_k}^G \mathbb{Z}_{\mathfrak{F} \cap H\sigma_k}) \neq 0,$$

thus proving the claim.

□

COROLLARY 4.5. *Let G be a group in $\mathbf{H}^{\mathfrak{F}} \mathfrak{F}$ and suppose that $\widehat{\mathbf{H}}_{\mathfrak{F}}^*(G, -)$ commutes with direct sums. Then $\text{cd}_{\mathfrak{F}} G < \infty$.*

Proof. The proof is analogous to the proof of Theorem 4.4. To show that \mathcal{S} is not empty, we can use the fact that $G \in \mathbf{H}_\beta^{\mathfrak{F}} \mathfrak{F}$ for some β . Then follow step (2) as above. □

Theorem A now follows directly from Theorem 4.4, as, for groups of type Bredon-FP $_\infty$ it follows that $\widehat{\mathbf{H}}_{\mathfrak{F}}^*(G, -)$ commutes with direct limits, see Lemma 4.2 and [27, Theorem 5.3].

5. Some properties of $\mathbf{LH}^{\mathfrak{F}} \mathfrak{F}$

We consider containment and closure properties of the class $\mathbf{LH}^{\mathfrak{F}} \mathfrak{F}$ especially when \mathfrak{F} either the class of finite groups or the class of virtually cyclic groups.

Let A be an abelian group and $\mathbb{Z} = \langle t \rangle$. Consider the semi-direct product $G = A \rtimes \mathbb{Z}$ with t acting on A by conjugation. To shorten the notation, wherever necessary, we will identify A with its image in G . Fix an arbitrary integer $k > 0$. For each integer $i \geq 0$, we define the subgroups P_i^k of A inductively as follows:

- $P_0^k = \langle 1 \rangle$,
- $P_{i+1}^k = \{x \in A \mid t^k(x)x^{-1} \in P_i^k\}$ for $i \geq 0$.

An easy induction on i shows that each P_i^k is a normal subgroup of G . We set $P^k = \cup_{i \geq 0} P_i^k$. Note that P^k is also a normal subgroup of G and it has the property that if $t^k(x)x^{-1} \in P^k$ and $x \in A$ then $x \in P^k$. In fact, P^k can be defined as the smallest subgroup of G with this property.

LEMMA 5.1. *Let $a \in A$. For each $i \geq 0$, consider the subgroup $G_i^k = \langle P_i^k, (a, t^k) \rangle$ of G . Then $P_i^k = G_i^k \cap A$ and G_i^k is nilpotent of nilpotency class at most $i + 1$.*

Proof. For the first part one only needs to check that $G_i^k \cap A$ is in P_i^k as the reverse inclusion is trivially satisfied. But this follows immediately from the fact that $G_i^k \cong P_i^k \rtimes \langle (a, t^k) \rangle$.

For the second claim, note that $[G_i^k, G_i^k]$ lies in A . Let $0 \leq m \leq i$. The only possibly nontrivial m -fold commutators starting with an element $x \in P_i^k$ are of the form

$$y_m = [(a_1, t^{kn_1}), [(a_2, t^{kn_2}), \dots [(a_m, t^{kn_m}), x] \dots]]$$

for $a_1, \dots, a_m \in P_i^k$ where we denote $y_0 = x$. We claim that y_m is in P_{i-m}^k . Assuming the claim, we have that y_i is trivial and hence G_i^k is nilpotent of nilpotency class at most $i + 1$.

To prove the claim we use induction on m . The case $m = 0$ is trivially satisfied. Now, suppose $m > 0$. Then, by induction, the $(m - 1)$ -fold commutator

$$z = [(a_2, t^{kn_2}), \dots [(a_m, t^{kn_m}), x] \dots] \in P_{i-m+1}^k.$$

But then

$$y_m = [(a_1, t^{kn_1}), z] = t^{kn_1}(z)z^{-1} \in P_{i-m}^k$$

because $z \in P_{i-m+1}^k$. This finishes the claim. \square

LEMMA 5.2. *For a given integer $i > 0$, let N be a nilpotent subgroup of G of nilpotency class i , which is not contained in A . Then $N = \langle B, (a, t^k) \rangle$ where $B = P_i^k \cap N$ for some $a \in A$ and $k > 0$. In particular, N is contained in $G_i^k = \langle P_i^k, (a, t^k) \rangle$.*

Proof. Clearly, $N = \langle B, (a, t^k) \rangle$ where $B = A \cap N$ for some $a \in A$ and $k > 0$. It is left to show that $B \leq P_i^k$. Let $0 \leq m \leq i$ and consider $(i - m)$ -fold commutator

$$y_{(i-m)} = [(a, t^k), [(a, t^k), \dots [(a, t^k), x] \dots]]$$

where we denote $y_0 = x \in B$. We will prove by induction that $y_{(i-m)} \in P_m^k$. Since N has nilpotency class i , $y_i = 1 \in P_0^k$. So, assume $m > 0$. Consider $z = [(a, t^k), y_{(i-m)}]$. By induction, $z \in P_{m-1}^k$. But $z = t^k(y_{(i-m)})y_{(i-m)}^{-1}$. So, by the definition of P_m^k , we have $y_{(i-m)} \in P_m^k$.

Now, taking $m = i$, gives us that each $x \in B$ lies in P_i^k . \square

PROPOSITION 5.3. *Define $P = \cup_{k > 0} P^k$ in A . Then*

- P is a normal subgroup of G .
- P is the smallest subgroup of G defined by the property that if $t^k(x)x^{-1} \in P$ for some $k > 0$ and $x \in A$, then $x \in P$.
- Let $N = \langle B, (a, t^l) \rangle$ where $B \leq P$, $a \in A$ and $l \geq 1$. Then N is locally virtually nilpotent.
- Let N be a locally nilpotent subgroup of G not contained in A . Then $N \cap A$ is contained in P .

Proof. (a). Given any integers $k_1, k_2 > 0$ such that k_1 divides k_2 , it follows that $P_{k_1} \subseteq P_{k_2}$. This shows that the set P is a subgroup of A . Since each P_k is a normal subgroup of G , their union P is also a normal in G .

(b). Let P' be the smallest subgroup of G defined by the property stated in (b); denote this property by (*). Note that $P = \cup_{i \geq 0} P_i$ where the subgroups P_i are defined inductively by:

- $P_0 = \langle 1 \rangle$,
- $P_{i+1} = \{x \in A \mid \exists k > 0, t^k(x)x^{-1} \in P_i\}$ for $i \geq 0$.

An easy induction on i shows that each P_i is a subgroup of P' . Hence, $P \leq P'$. But since P has the property (*) and P' is the smallest subgroup of G with the property (*), we deduce that $P = P'$.

(c). Let $H = \langle b_1, \dots, b_s, (a, t^l) \rangle$, for some $b_1, \dots, b_s \in P$, $a \in A$, and $l, s \geq 1$. It suffices to show that H is virtually nilpotent. Since $P = \cup_{i, k > 0} P_i^k$, we conclude that for each $j \in \{1, \dots, s\}$, we have $b_j \in P_{i_j}^{k_j}$ for some $i_j, k_j > 0$. Set $k = \prod_{j=1}^s k_j$ and $i = \sup\{i_j \mid 1 \leq j \leq s\}$. It follows that the group $H' = \langle b_1, \dots, b_s, (a, t^l)^k \rangle$ is a finite index subgroup of H , and $H' \leq \langle P_i^{kl}, (a, t^l)^k \rangle$. So, by Lemma 5.1, H' is nilpotent.

(d). This is a direct consequence of Lemma 5.2. \square

THEOREM 5.4. *Let G be a semi-direct product $A \rtimes \mathbb{Z}$ where A is a countable abelian group. Define \mathfrak{H} to be the family of all virtually nilpotent subgroups of G . Then there exists a 3-dimensional model for $E_{\mathfrak{H}}G$.*

Proof. Let \mathfrak{T} be the subfamily of \mathfrak{H} consisting of all countable subgroups of A . We will use the construction of Lück and Weiermann that adapts the model for $E_{\mathfrak{T}}G$ to a model for the larger family \mathfrak{H} .

First, we need a strong equivalence relation on the set

$$\mathcal{S} = \mathfrak{H} \setminus \mathfrak{T} = \{H \leq G \mid H \not\leq A \text{ and } H \text{ is virtually nilpotent}\}.$$

Let $\bar{\cdot} : G \rightarrow G/P$ denote the quotient homomorphism. By Proposition 5.3, we have that if $H \in \mathfrak{H}$, then \bar{H} is virtually cyclic.

Now, for $H, S \in \mathcal{S}$, we say that there is a relation $H \sim S$ if $|\bar{H} \cap \bar{S}| = \infty$. It is not difficult to show that this indeed defines a strong equivalence relation on the set \mathcal{S} . Our group G acts by conjugation on the set of equivalence classes $[\mathcal{S}]$ and the stabiliser of an equivalence class $[H]$ is

$$N_G[H] = \{x \in G \mid H^x \sim H\}.$$

Note that $H \sim Z$ if $Z = \langle h \rangle$, $h \in H$, $h \notin A$. Hence $N_G[H] = N_G[Z]$. Clearly, Z is a subgroup of $N_G[Z]$ and $N_G[Z] = \langle B, Z \rangle$ for some subgroup $B \leq A$. But for each $b \in B$, we have $Z^b \sim Z$. Writing $h = (a, t^k)$ for some $a \in A$ and $k > 0$, this implies that $\overline{b^{-1}(a, t^k)^n b} = \overline{(a, t^k)^n}$ in G/P for some nonzero integer n . A quick computation then shows that $t^{kn}(\overline{b}) = \overline{b}$ in G/P . This means that $t^{kn}(b)b^{-1} \in P$. Then, by Proposition 5.3(b), $b \in P$. Hence, by part (c) of Proposition 5.3, we have that every finitely generated subgroup K of $N_G[Z]$ that contains Z is virtually nilpotent. Thus $K \in \mathcal{S}$ and $K \sim Z$ and hence it is in the family

$$\mathfrak{T}[H] = \{K \leq N_G[H] \mid K \in \mathcal{S}, K \sim H\} \cup (N_G[H] \cap \mathfrak{T})$$

of subgroups of $N_G[H]$. It follows that $N_G[H]$ is a countable directed union of subgroups that are in $\mathfrak{T}[H]$ but are not in $N_G[H] \cap \mathfrak{T}$. Denote by T the tree on which $N_G[H]$ acts with stabilisers as such subgroups. Note that the action of G on \mathbb{R} via the natural projection of G onto \mathbb{Z} makes \mathbb{R} into a model for $E_{\mathfrak{T}}G$. Restricting this action to $N_G[H]$ and considering the induced action on the

join $T * \mathbb{R}$ gives us a 3-dimensional model for $E_{\mathfrak{T}[H]}N_G[H]$. Invoking Proposition 2.4 entails a 3-dimensional model for $E_{\mathfrak{H}}G$, as was required to prove. \square

REMARK 5.5. Since finitely generated nilpotent groups lie in $\mathbf{H}_1^{\mathfrak{F}_{\text{vc}}}$, it follows that countable virtually nilpotent groups are in $\mathbf{H}_2^{\mathfrak{F}_{\text{vc}}}$. We obtain that the group $G = A \rtimes \mathbb{Z} \in \mathbf{H}_3^{\mathfrak{F}_{\text{vc}}}$.

REMARK 5.6. In the statement of Theorem 5.4, one could enlarge \mathfrak{H} to be the family of all locally virtually nilpotent subgroups of G . Then its proof together with Proposition 5.3(c)-(d) would imply that $N_G[H]$ is $\mathfrak{T}[H]$. So, a point with the trivial action of $N_G[H]$ would then be a model for $E_{\mathfrak{T}[H]}N_G[H]$ for each $H \in \mathcal{S}$. Applying Proposition 2.4 would give us a 2-dimensional model for $E_{\mathfrak{H}}G$.

In the next example, we illustrate that the family \mathfrak{H} of all virtually nilpotent subgroups of G can contain nilpotent subgroups of G of arbitrarily high nilpotency class.

EXAMPLE 5.7. Consider the unrestricted wreath product $W = \mathbb{Z} \wr \mathbb{Z}$. Rewriting this group as a semi-direct product, we have that $W = A \rtimes \mathbb{Z}$ where $A = \prod_{i \in \mathbb{Z}} \mathbb{Z}$ and the standard infinite cyclic subgroup of W is generated by t and acts on A by translations. Define G to be the subgroup of W given by $G = P \rtimes \mathbb{Z}$. For each $k > 0$, note that P_1^k is the subgroup of A of all k -periodic sequences of integers and hence $P_1^k \cong \mathbb{Z}^k$. Since $P_1 = \cup_{k > 0} P_1^k$, it is countable of infinite rank. Similarly, one can argue that P_2/P_1 is countable of infinite rank and hence P_2 is also countable. Continuing in this manner, one obtains that P_i is countable for each $i > 0$ and since P is a countable union of these groups it is itself countable. This shows that the group G satisfies the hypothesis of Theorem 5.4.

Now, it is not difficult to see, that for each $i > 0$, the subgroup $P_i^1 \rtimes \mathbb{Z}$ of G is nilpotent of nilpotency class i .

THEOREM 5.8. *Let \mathfrak{F} be a class of subgroups of finitely generated groups. Then $\mathbf{H}^{\mathfrak{F}}$ is closed under countable directed unions. If \mathfrak{F} is the class of all virtually cyclic groups, then $\mathbf{H}^{\mathfrak{F}}$ is closed under finite extensions and under extensions with virtually soluble kernels. In particular, $\mathbf{LH}^{\mathfrak{F}}$ contains all locally virtually soluble groups.*

Proof. The proof of the first fact is the same as for the class of finite groups \mathfrak{F} given in Proposition 5.5 in [8]. That is, let G be a countable directed union of groups that are in $\mathbf{H}^{\mathfrak{F}}$. Then G acts on a tree with stabilisers exactly the subgroups that comprise this union. It is now easy to see that the action of G on the tree satisfies the stabiliser and the fixed-point set conditions of the definition of $\mathbf{H}^{\mathfrak{F}}$ -groups. This shows that G is in $\mathbf{H}^{\mathfrak{F}}$.

For the second part, first note that by the Serre's Construction, $\mathbf{H}^{\mathfrak{F}}$ is closed under finite extensions (see the proof of [20, 2.3(2)]).

Let G be a countable group that fits into an extension $K \twoheadrightarrow G \twoheadrightarrow Q$ such that K is virtually soluble and $Q \in \mathbf{H}^{\mathfrak{F}}$. Suppose K is finite. Then an easy transfinite induction on the ordinal associated to the class containing Q shows G lies in $\mathbf{H}^{\mathfrak{F}}$. In general, since K is virtually soluble, it contains a soluble characteristic subgroup of finite index, which must be normal in G . In view of these facts, without loss of generality, we can assume that K is soluble.

Next, we proceed by the induction on the derived length of K to prove that $G \in \mathbf{H}^{\mathfrak{F}}$. When K is the trivial group, then $G = Q \in \mathbf{H}^{\mathfrak{F}}$. Suppose K is nontrivial. Since $[K, K]$ is a characteristic subgroup of K , it is a normal subgroup of G . So, there are extensions

$$[K, K] \twoheadrightarrow G \twoheadrightarrow G/[K, K] \quad \text{and} \quad K/[K, K] \twoheadrightarrow G/[K, K] \twoheadrightarrow Q.$$

We claim that $G/[K, K] \in \mathbf{H}^{\delta} \mathfrak{F}$. Then by induction applied to the first extension $G \in \mathbf{H}^{\delta} \mathfrak{F}$. Let us now prove the claim.

In view of the second extension, it suffices to show that given an extension

$$A \twoheadrightarrow S \twoheadrightarrow Q$$

where A is abelian and $Q \in \mathbf{H}_{\alpha}^{\delta} \mathfrak{F}$, then $S \in \mathbf{H}^{\delta} \mathfrak{F}$. We use transfinite induction on the ordinal α . When $\alpha = 0$, then S is virtually a semi-direct product $A \rtimes \mathbb{Z}$. Hence, by Theorem 5.4, it is in $\mathbf{H}^{\delta} \mathfrak{F}$. Suppose $\alpha > 0$, then there is a finite dimensional Q -CW-complex X such that each stabiliser subgroup lies in $\mathbf{H}_{\beta}^{\delta} \mathfrak{F}$ for some $\beta < \alpha$ and such that X^H is contractible for all $H \in \mathfrak{F}$. The group S also acts on X via the projection onto Q . Each stabiliser of this action is abelian-by- $\mathbf{H}_{\beta}^{\delta} \mathfrak{F}$ and hence by transfinite induction is in $\mathbf{H}^{\delta} \mathfrak{F}$. Therefore, $S \in \mathbf{H}^{\delta} \mathfrak{F}$. This finishes the claim and the proof. \square

Recall that a subgroup G of $\mathrm{GL}_n(\mathbb{C})$ is said to be of *integral characteristic* if the coefficients of the characteristic polynomial of every element of G are algebraic integers. It follows that G has integral characteristic if and only if the characteristic roots of every element of G are algebraic integers (see [1, §2]).

THEOREM 5.9. *Let G be a countable subgroup of some $\mathrm{GL}_n(\mathbb{C})$ of integral characteristic. Then G lies in $\mathbf{H}^{\delta_{\mathrm{vc}}} \mathfrak{F}_{\mathrm{vc}}$.*

Proof. Since the class $\mathbf{H}^{\delta_{\mathrm{vc}}} \mathfrak{F}_{\mathrm{vc}}$ is closed under countable directed unions, it is enough to prove the claim when G is finitely generated. Note that under a standard embedding of $\mathrm{GL}_n(\mathbb{C})$ into $\mathrm{SL}_{n+1}(\mathbb{C})$ the image of G is still of integral characteristic. So, we can assume that G is a subgroup of $\mathrm{SL}_n(\mathbb{C})$ of integral characteristic. Let A be the finitely generated subring of \mathbb{C} generated by the matrix entries of a finite set of generators of G and their inverses. Then G is a subgroup of $\mathrm{SL}_n(A)$.

Let \mathbb{F} denote the quotient field of A . Proceeding as in the proof of Theorem 3.3 of [1], there is an epimorphism $\rho : G \rightarrow H_1 \times \cdots \times H_r$ such that the kernel U of ρ is a unipotent subgroup of G and for each $1 \leq i \leq r$, H_i is a subgroup of some $\mathrm{GL}_{n_i}(A)$ of integral characteristic where the canonical action of H_i on \mathbb{F}^{n_i} is irreducible and $\sum n_i = n$. So, by the proof of Theorem B in [4], each group H_i admits a finite dimensional model for $E_{\mathfrak{F}_{\mathrm{vc}} \cap H_i} H_i$. Applying [24, 5.6], one immediately sees that the product $Q = H_1 \times \cdots \times H_r$ admits a finite dimensional model for $E_{\mathfrak{F}_{\mathrm{vc}} \cap Q} Q$. So, Q is in $\mathbf{H}_1^{\delta_{\mathrm{vc}}} \mathfrak{F}_{\mathrm{vc}}$. By Theorem 5.8, it follows that G lies in $\mathbf{H}^{\delta_{\mathrm{vc}}} \mathfrak{F}_{\mathrm{vc}}$. \square

COROLLARY 5.10. *Let \mathfrak{F} be either the class of all finite groups or the class of all virtually cyclic groups and let G be a group such that $\widehat{\mathbf{H}}_{\mathfrak{F}}^*(G, -)$ commutes with direct limits. If G is a subgroup of some $\mathrm{GL}_n(\mathbb{C})$ of integral characteristic or if G is a subgroup of some $\mathrm{GL}_n(\mathbb{F})$ where \mathbb{F} is a field of positive characteristic, then $\mathrm{cd}_{\mathfrak{F}}(G) < \infty$.*

Proof. Suppose H is a finitely generated subgroup of G . If G is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ of integral characteristic, then by [1] when \mathfrak{F} is the class of finite groups or by the previous theorem when \mathfrak{F} is the class of virtually cyclic groups, we know that H in $\mathbf{H}^{\delta} \mathfrak{F}$. If G embeds into $\mathrm{GL}_n(\mathbb{F})$ for some field \mathbb{F} of positive characteristic, then by [7, Corollary 5], H has finite Bredon cohomological dimension and hence it is in $\mathbf{H}^{\delta} \mathfrak{F}$. This shows that G is in $\mathbf{LH}^{\delta} \mathfrak{F}$. The result now follows from Theorem 4.4. \square

6. Change of family

In this section we discuss the question when the functor $\widehat{\mathbf{H}}_{\mathfrak{F}}^*(G, -)$ commutes with direct limits. By the above, it is obvious that groups of finite Bredon cohomological dimension as well as

groups of Bredon-type FP_∞ satisfy this condition. It would be interesting to see whether there are groups a priori satisfying neither, that also have continuous $\widehat{H}_{\mathfrak{F}}^*(G, -)$.

Considering Lemma 4.2, we see that it is enough to require that $H_{\mathfrak{F}}^k(G, -)$ commutes with direct limits for infinitely many k . This, for example holds for groups, for which the trivial Bredon-module $\mathbb{Z}_{\mathfrak{F}}$ has a Bredon-projective resolution, which is finitely generated from a certain point onwards.

As mentioned in the introduction, the families of greatest interest are the families $\mathfrak{F}_{\text{fin}}$ of finite subgroups and \mathfrak{F}_{vc} of virtually finite subgroups. In light of Juan-Pineda and Leary's conjecture [12], which asserts that no non-virtually cyclic group is of type $\underline{\text{FP}}_\infty$, the question above is of particular interest for the family \mathfrak{F}_{vc} .

Let us begin with the following:

QUESTION 6.1. *Does $\widehat{H}^*(G, -)$ being continuous imply that $\widehat{H}_{\mathfrak{F}_{\text{fin}}}^*(G, -)$ is continuous?*

The converse of this question is obviously not true. Take any group G with $\text{cd}_{\mathfrak{F}_{\text{fin}}} G < \infty$, which is not of type FP_∞ and which has no bound on the orders of the finite subgroups. It follows from [15] that groups with $\text{cd}_{\mathbb{Q}} G < \infty$ and continuous $\widehat{H}^*(G, -)$ have a bound on the orders of their finite subgroups. Locally finite groups and Houghton's groups satisfy this condition. On the other hand [16, Theorem 2.7], any group G in $\text{LH}\mathfrak{F}$, for which $\widehat{H}^*(G, -)$ is continuous has finite $\text{cd}_{\mathfrak{F}_{\text{fin}}} G$, hence $\widehat{H}_{\mathfrak{F}_{\text{fin}}}^*(G, -)$ is continuous.

Also note that there are examples of groups of type FP_∞ , which are not of type Bredon- FP_∞ for the class of finite subgroups [18]. These groups, however, satisfy $\text{cd}_{\mathfrak{F}_{\text{fin}}} G < \infty$, hence have continuous $\widehat{H}_{\mathfrak{F}_{\text{fin}}}^*(G, -)$.

QUESTION 6.2. *Is $\widehat{H}_{\mathfrak{F}_{\text{fin}}}^*(G, -)$ being continuous equivalent to $\widehat{H}_{\mathfrak{F}_{\text{vc}}}^*(G, -)$ being continuous?*

Any group of type $\underline{\text{FP}}_\infty$ is of type FP_∞ (see [13]) and any group with $\text{cd}_{\mathfrak{F}_{\text{vc}}} G < \infty$ also has $\text{cd}_{\mathfrak{F}_{\text{fin}}} G < \infty$ (see [24]). Hence we may ask:

QUESTION 6.3. *Suppose $\text{cd}_{\mathfrak{F}_{\text{fin}}} G < \infty$. Does this imply that $\widehat{H}_{\mathfrak{F}_{\text{vc}}}^*(G, -)$ is continuous?*

If this question has a positive answer, Theorem A would imply that any group in $\text{LH}^{\mathfrak{F}_{\text{vc}}}\mathfrak{F}_{\text{vc}}$ with $\text{cd}_{\mathfrak{F}_{\text{fin}}} G < \infty$ satisfies $\text{cd}_{\mathfrak{F}_{\text{vc}}} G < \infty$.

We end with two questions on the family $\text{LH}^{\mathfrak{F}_{\text{vc}}}\mathfrak{F}_{\text{vc}}$.

QUESTION 6.4. *Is the class $\text{LH}^{\mathfrak{F}_{\text{vc}}}\mathfrak{F}_{\text{vc}}$ closed under extensions?*

This reduces to asking whether an infinite cyclic extension of group in $\text{LH}^{\mathfrak{F}_{\text{vc}}}\mathfrak{F}_{\text{vc}}$ is also in $\text{LH}^{\mathfrak{F}_{\text{vc}}}\mathfrak{F}_{\text{vc}}$.

QUESTION 6.5. *Does the class $\text{LH}^{\mathfrak{F}_{\text{vc}}}\mathfrak{F}_{\text{vc}}$ contain all elementary amenable groups?*

Note that a positive answer to Question 6.4 implies a positive answer to this question.

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