Density of Skew Brownian motion and its functionals with application in finance

Alexander Gairat* and Vadim Shcherbakov†

Abstract

We derive the joint density of a Skew Brownian motion, its last visit to the origin, its local and occupation times. The result allows to obtain explicit analytical formulas for pricing European options under both a two valued local volatility model and a displaced diffusion model with constrained volatility.

Key words: Skew Brownian motion, local volatility model, displaced diffusion, local time, occupation time, simple random walk, option pricing

1 Introduction

A Skew Brownian motion (SBM) with parameter p is a Markov process that evolves as a standard Brownian motion reflected at the origin so that the next excursion is chosen to be positive with probability p. SBM was introduced in Ito and McKean (1963) and has been studied extensively in probability since then. The process naturally appears in diverse applications, e.g. Appuhamillage et al. (2011) and Lejay (2006), and, in particular, in finance applications, e.g. Decamps, De Schepper and Goovaerts (2004), Decamps, Goovaerts and Schoutens (2006a,b) and Rossello (2012). In this paper, we derive the joint distribution of SBM and some of its functionals and apply this distribution to derivative pricing under both a local volatility model with discontinuity and a displaced diffusion model with constrained volatility.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and let $\{W_t, \mathcal{F}_t,\, t \geq 0\}$ be a standard Brownian motion (BM) with its natural filtration. As usual, denote by $\mathbb{R}$ and $\mathbb{R}_+$ sets of all real and all non-negative real numbers respectively. A local volatility model (LVM) for the underlying price S_t is given by the following equation

$$dS_t = \mu(t)S_t dt + \sigma(t, S_t)S_t dW_t, \tag{1}$$

*Gazprombank, Moscow. Email address: Alexander.Gairat@gazprombank.ru

†Department of Mathematics, Royal Holloway, University of London. Email address: vadim.shcherbakov@rhul.ac.uk

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where $\mu(t) \in \mathbb{R}$ and $\sigma(t, S_t) \in \mathbb{R}_+$. LVM is a natural extension of the famous Black-Scholes model. The latter is a particular case of (1) where both drift μ and volatility σ are constant. LVM is actively used in practice because it can be easily calibrated to the market. Furthermore, by Gyongy's lemma (Gyongy (1986)) a wider class of stochastic volatility models can be reduced to LVM.

A number of approximations to LVM have been developed for both calibration purposes and qualitative analysis (Guyon (2011)). We apply our probabilistic results primarily to a particular case of LVM that can be used as a benchmark model for analyzing the quality of such approximations. Namely, we consider a driftless LVM with a two-valued volatility (two-valued LVM)

$$\sigma(t, S) = \sigma_1 1_{\{S \geq S^*\}} + \sigma_2 1_{\{S < S^*\}}, \tag{2}$$

where $\sigma_i > 0, i = 1, 2$, $S^* > 0$, and 1_A is used to denote the indicator function of set A. Without loss of generality we assume that $S^* = 1$ in what follows.

In Section 3.1, we show that if S_t follows the two-valued LVM then process $X_t = \log(S_t)/\sigma(S_t)$ is a solution of a stochastic differential equation (SDE) of the following type

$$X_t = X_0 + \int_0^t m(X_s)ds + (2p-1)L_t^{(0)}(X) + W_t, \tag{3}$$

where $L_t^{(0)}(X)$ is the local time of process X_t at zero, $p \in (0, 1)$,

$$m(x) = m_1 1_{\{x \geq 0\}} + m_2 1_{\{x < 0\}},\ m_1, m_2 \in \mathbb{R}, \tag{4}$$

and both p and pair (m_1, m_2) are uniquely determined by σ_1 and σ_2 (Lemma 3.1). Notice that SDE (3) belongs to the following class of SDE with local time

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}} \nu(dx)dL_t^{(x)}(X), \tag{5}$$

where ν is a finite signed measure with atoms at the points, where both b and σ can be discontinuous, and $L_t^{(x)}(X)$ is the local time of process X at x. It is known that SDE (5) has a unique strong solution under certain general conditions which are satisfied in the case of equation (3) (e.g. Le Gall (1985), Lejay (2006) and references therein). In particular, if $m \equiv 0$ then a unique strong solution of equation (3) is a SBM with parameter p which we are going to denote by $W_t^{(p)}$ from now on. If $m_1 = m_2 = m$, then equation (3) takes the following form

$$X_t = X_0 + mt + (2p-1)L_t^{(0)} + W_t. \tag{6}$$

A diffusion process defined by equation (6) appears, for instance, in a study of dispersion across an interface in Appuhamillage et al. (2011) and is named there as a SBM with parameter p and drift m. By analogy, we refer to the solution of equation (3) with two-valued drift (4) as a SBM with a two-valued drift. A SBM with two-valued drift (4) is reflected at the origin in the same way as a driftless $W_t^{(p)}$ and evolves as a BM with drift m_1 when it is above zero and with drift m_2 when it is below zero.

We derive explicit formulas for the joint density of $W_t^{(p)}$, its last visit to the origin, local and occupation times in both the driftless and the two-valued drift cases. The joint density is then applied to option pricing under both LVM with volatility (2) and a displaced diffusion model with constrained volatility (defined in Section 6).

It turns out that in both cases European option prices can be expressed analytically in terms of both the univariate standard normal distribution and a bivariate normal distribution only. It should be noted that these models belong to a more general class of diffusion processes with discontinuity which has been used in financial applications. For example, semi-analytical results have been obtained in Decamps, Goovaerts and Schoutens (2006a) for LVM where $\sigma(t, S) = \sigma(S)$ is continuous at all but one point. In particular, they have shown how SBM naturally appears in LVM with such type of discontinuity. A similar model has been considered in Lipton and Sepp (2011), where semi-analytical results have been obtained for LVM with a so-called tiled local volatility. Another example is provided by Akahori and Imamuri (2014), where a model with discontinuity at a single point appeared in relation to pricing of barrier options.

Joint distributions of SBM and its functionals are of interest in their own right. For example, the joint density of SBM with a constant drift, its local and occupation times was obtained in Appuhamillage et al. (2011). The result of Appuhamillage et al. (2011) generalizes the classic result of Karatzas and Shreve (1984), where the same trivariate density was obtained for the standard BM. In Appuhamillage et al. (2011) the technique of Ito and McKean (1963) was modified to obtain a Feynman–Kac formula for SBM and this allowed them to adopt the method of Karatzas and Shreve (1984). In turn, the method of Karatzas and Shreve (1984) is based on the computation of the Laplace transform of the joint density. In contrast, we use a discrete approximation of SBM by a random walk and a key step of our approach consists in combining an intuitively clear path decomposition for the discrete process with some well known properties of the symmetric simple random walk. This allows us to derive analytically tractable expressions for the joint density of discrete analogues of quantities of interest and to compute the limit density.

A discrete approximation is a well known method for obtaining joint distribution of both BM and SBM and their functionals (e.g. Lulko (2012) or Takacs (1995)). We were inspired by the use of this method in Billingsley (1968) for the computation of the joint distribution of the standard BM, its occupation time and its last visit to the origin.

The paper is organized as follows. We formulate the results on the joint distribution of SBM and its functionals in Section 2. Section 3 describes the relationship between LVM with the two-valued volatility and SBM with the two-valued drift. Theorem 3.1 in Section 3.2 is an example of an option pricing theorem under the two-valued LVM. Proofs are given in Section 4. In Section 5, we also derive in a special case a simple closed form approximation for option prices based on the Black-Scholes formula. Effectiveness of the approximate result is tested in comparison with the exact result in Theorem 3.1 and another approximation derived in Lipton and Sepp (2011). Finally, we discuss in Section 6 how our results can be applied to derivative pricing under the displaced diffusion model with constrained volatility.

2 Density of Skew Brownian motion, its last visit to the origin, occupation and local times

Given a continuous semimartingale X_t, $t \in [0,T]$, define the following quantities

$$\tau = \max\{t \in (0,T] : X_t = 0\},\ V = \int_{\tau_0}^{\tau} 1_{\{X_t \geq 0\}} dt \tag{7}$$

and $\tau_0 = \min\{t : X_t = 0\}$. Let $L_t^{(x)}(X)$ be the symmetric local time of X_t at point x. For example, if X_t is a SBM with a two-valued drift, then $L_t^{(x)}(X) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{x-\varepsilon \leq X_u \leq x+\varepsilon\}} du$. In what follows we consider only symmetric local times.

Our principal result about joint density of SBM and its functionals is the following theorem.

Theorem 2.1. *Let $X_t = W_t^{(p)}$ and let (τ, V) be the quantities defined in (7). Given $X_0 = 0$ the joint density of $\left(\tau, V, X_T, L_T^{(0)}(X)\right)$ is*

$$\psi_{p,T}(t,v,x,l) = 2a(x)h(v,lp)h(t-v,lq)h(T-t,x),\ 0 \leq v \leq t \leq T, l \geq 0, \tag{8}$$

where $q = 1-p$, $a(x) = p1_{\{x \geq 0\}} + q1_{\{x<0\}}$ and $h(s,y) = \frac{|y|}{\sqrt{2\pi s^3}} e^{-\frac{y^2}{2s}}$, $y \in \mathbb{R}, s \in \mathbb{R}_+$, is the probability density function of the first passage time to zero of the standard BM starting at y.

Theorem 2.2. *Let X_t be a SBM with two-valued drift (4). Let $\left(\tau, V, L_T^{(0)}(X)\right)$ be as defined in (7). Given $X_0 = 0$, the joint density of $\left(\tau, V, X_T, L_T^{(0)}(X)\right)$ is given by the following function*

$$\phi_T(t,v,x,l) = \psi_{p,T}(t,v,x,l) e^{-\frac{m_1^2 v + m_2^2 (T-v)}{2} - l(m_1 p - q m_2) + x m(x)},\ 0 \leq v \leq t \leq T, l \geq 0, \tag{9}$$

where $\psi_{p,T}(t,v,z,l)$ is defined in (8).

Let us briefly comment on the relationship between Theorems 2.1 and 2.2 and some known results. First, we rewrite the joint density (9) in terms of the total occupation time. Given $T > 0$, define $U = \int_0^T 1_{\{X_t \geq 0\}} dt$ the total occupation time of the non-negative half-line during time period $[0,T]$ and notice that if $X_0 = 0$ then $U = (V + T - \tau)1_{\{X_T \geq 0\}} + V 1_{\{X_T < 0\}}$. If X is SBM with parameter p and drift $m(x)$, then this equation and Theorem 2.1 yield that the joint density of $(\tau, U, X_T, L_T^{(0)}(X))$ is given by the following equation

$$\varphi_T(t,u,x,l) = \begin{cases} 2ph(u+t-T, lp)h(T-u, lq)h(T-t, x)e^{-\frac{m_1^2 u + m_2^2(T-u)}{2} + xm_1 - l(pm_1 - qm_2)}, \\ \text{if } x \geq 0,\ l > 0,\ \text{and } t \leq T,\ T-t \leq u \leq T, \\ 2qh(u, lp)h(t-u, lq)h(T-t, x)e^{-\frac{m_1^2 u + m_2^2(T-u)}{2} + xm_2 - l(pm_1 - qm_2)}, \\ \text{if } x < 0,\ l > 0,\ \text{and } 0 \leq u \leq t \leq T. \end{cases} \tag{10}$$

If $m_1 = m_2 = m = const$, then we obtain the density of the quartet in the case of constant drift

$$\varphi_{T,m}(t,u,x,l) = \begin{cases} 2ph(u+t-T, lp)h(T-u, lq)h(T-t, x)e^{-\frac{m^2 T}{2} + xm - lm(p-q)}, \\ \text{if } x \geq 0,\ l > 0,\ \text{and } t \leq T,\ T-t \leq u \leq T, \\ 2qh(u, lp)h(t-u, lq)h(T-t, x)e^{-\frac{m^2 T}{2} + xm - lm(p-q)}, \\ \text{if } x < 0,\ l > 0,\ \text{and } 0 \leq u \leq t \leq T. \end{cases}$$

Further, setting $m = 0$ in the preceding display and integrating out variable t we get the joint density of SBM with parameter p, its (total) occupation and local time (Theorem 1.2 in Appuhamillage et al. (2011))

$$\begin{aligned} \rho(u,z,b) &= \begin{cases} \int_0^T 2ph(u+t-T, lp)h(T-u, lq)h(T-t, x)dt, & x \geq 0, \\ \int_u^T 2qh(u, lp)h(T-u, lq)h(T-t, x)dt, & x < 0, \end{cases} \\ &= \begin{cases} 2ph(T-u, bq)h(u, lp+x), & x \geq 0, \\ 2qh(u, lp)h(T-u, lq-x), & x < 0. \end{cases} \end{aligned} \tag{11}$$

In a particular case $p = 1/2$ density (11) is the trivariate density obtained in Karatzas and Shreve (1984) for the standard BM. It should be noticed that the local time in Karatzas and Shreve (1984) equals a half of the local time we consider in this paper.

3 Application in finance

3.1 Relationship between LVM with discontinuity and SBM

Fix $\sigma_1 > 0$ and $\sigma_2 > 0$ and consider the following LVM

$$dS_t = \sigma(S_t)S_t dW_t, \tag{12}$$

where $\sigma(S) = \sigma_1 1_{\{S \geq 1\}} + \sigma_2 1_{\{S<1\}}$. Lemma 3.1 below explains the relationship between SBM and LVM defined by (12). This lemma can be regarded as a particular case of Theorem 1 in Decamps, Goovaerts and Schoutens (2006a)) (see also an argument on p.687 in Decamps, De

Schepper and Goovaerts (2004)) and is based on application of the symmetric Tanaka-Meyer formula (e.g. see either formula (7.4) in Karatzas and Shreve (1991), or Exercise 1.25, Chapter VI in Revuz and Yor (1998), or formula (32) in Lejay (2006)). We provide the proof here for the sake of completeness and for the reader's convenience.

Lemma 3.1. *If S_t is a solution of equation (12), then a random process $X_t = \log(S_t)/\sigma(S_t)$ is a solution of the following SDE with local time*

$$dX_t = \mu(X_t)dt + dW_t + (p-q)dL_t^{(0)}(X), \tag{13}$$

where

$$\mu(x) = -\frac{\sigma\left(e^x\right)}{2} = \begin{cases} \mu_1 = -\sigma_1/2, & x \geq 0, \\ \mu_2 = -\sigma_2/2, & x < 0, \end{cases} \tag{14}$$

and $p = \frac{\sigma_2}{\sigma_1+\sigma_2}$, $q = 1-p = \frac{\sigma_1}{\sigma_1+\sigma_2}$. In other words, X_t is SBM with parameter $p = \frac{\sigma_2}{\sigma_1+\sigma_2}$ and discontinuous drift $\mu(x)$.

Proof. First, define $Y_t = \log(S_t)$ and notice that by Ito's formula

$$dY_t = -\frac{\sigma^2(S_t)}{2}dt + \sigma(S_t)dW_t = -\frac{\sigma^2\left(e^{Y_t}\right)}{2}dt + \sigma\left(e^{Y_t}\right)dW_t.$$

In terms of process Y_t, we have that $X_t = f(Y_t)$, where $f(y) = \frac{y}{\sigma_1}1_{\{y\geq 0\}} + \frac{y}{\sigma_2}1_{\{y<0\}}$. It is easy to see that f is a difference of two convex functions and, hence, $X_t = f(Y_t)$ is a semimartingale. Define $f'(y) = \frac{1}{2}\left(f_l'(y) + f_r'(y)\right)$, where $f_l'(y)$ and $f_r'(y)$ are the left and right derivatives of f. It is easy to see that $f^{'}(y) = \frac{1}{\sigma(y)}1_{\{y\neq 0\}} + \frac{\sigma_1+\sigma_2}{2\sigma_1\sigma_2}1_{\{y=0\}}$ and $f^{''}(y) = \delta(y)\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)$, where $\delta(x)$ is the delta function. Applying the symmetric Tanaka-Meyer formula to semimartingale $f(Y_t)$, we get that

$$\begin{aligned} X_t = f(Y_t) &= f(Y_0) + \int_0^t f'(Y_u)dY_u + \int_{\mathbb{R}} f^{''}(y)L_t^{(y)}(Y)dy, \\ &= f(Y_0) + \int_0^t \left(\frac{1}{\sigma(y)}1_{\{y\neq 0\}} + \frac{\sigma_1+\sigma_2}{2\sigma_1\sigma_2}1_{\{y=0\}}\right)dY_u + \frac{1}{2}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)L_t^{(0)}(Y) \qquad (15) \\ &= X_0 - \int_0^t \frac{\sigma\left(e^{X_u}\right)}{2}du + W_t + \frac{1}{2}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)L_t^{(0)}(Y), \qquad (16) \end{aligned}$$

where $L_t^{(0)}(Y)$ is the local time of Y_t at zero and where we also used that $\int_0^t 1_{\{Y_u=0\}}dY_u = 0$ and $\sigma\left(e^{Y_t}\right) = \sigma\left(e^{X_t}\right)$, in order to get (16) from (15).

It is left to express $L_t^{(0)}(Y)$ in terms of $L_t^{(0)}(X)$. Firstly, we apply symmetric Tanaka-Meyer formula to semimartingale X_t with convex function $|x|$ and get that

$$|X_t| = |X_0| + \int_0^t sgn(X_u)dX_u + L_t^{(0)}(X), \tag{17}$$

where $sgn(x) = 1$ if $x > 0$, $sgn(x) = -1$, if $x < 0$, and $sgn(0) = 0$. Secondly, consider $|X_t|$ as a result of applying convex function $g(y) = |f(y)| = \frac{y}{\sigma_1}1_{\{y\geq 0\}} - \frac{y}{\sigma_2}1_{\{y<0\}}$ to semimartingale Y_t. Let g' be the arithmetic mean of the right and the left derivatives of g. It is easy to see that

$$g'(y) = \frac{1}{2}\left(g_l'(y) + g_r'(y)\right) = sgn(y)\frac{1}{\sigma(y)} + \frac{1}{2}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)1_{\{y=0\}}.$$

The second generalised derivative g'' of g is $\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)\delta(y)$. Applying symmetric Tanaka-Meyer formula to $g(Y_t)$, we obtain that

$$|X_t| = |f(Y_t)| = |X_0| + \int_0^t g'(Y_u)dY_u + \frac{1}{2}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)L_t^{(0)}(Y). \tag{18}$$

Noticing that

$$\begin{aligned}\int_0^t sgn(X_u)dX_u &= \int_0^t sgn(Y_u)\frac{1}{\sigma(Y_u)}\left(-\frac{\sigma^2\left(e^{Y_u}\right)}{2}du + \sigma\left(e^{Y_u}\right)dW_u\right)\\ &= \int_0^t g'(Y_u)dY_u - g'(0)\int_0^t 1_{\{Y_u=0\}}dY_u = \int_0^t g'(Y_u)dY_u\end{aligned}$$

and comparing the right sides of (17) and (18) yields that $L_t^{(0)}(X) = \frac{\sigma_1+\sigma_2}{2\sigma_1\sigma_2}L_t^{(0)}(Y)$, and, hence, we get equation $dX_t = \mu(X_t)dt + dW_t + (p-q)dL_t^{(0)}(X)$ as claimed. Lemma 3.1 is proved. □

Remark 3.1. Denote by $\mathbb{Q}_T$ the probability distribution of SBM with parameter p and drift (4) on the time interval $[0,T]$ and by $\mathbb{P}_T$ the probability distribution of $W_t^{(p)}$, $t \in [0,T]$. By the Girsanov's theorem, we have that

$$\begin{aligned}\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}(X_\cdot) &= e^{\int_{X_0}^{X_T} m(u)\,du - \frac{1}{2}\int_0^T m^2(X_t)dt - (pm_1 - qm_2)L_T^{(0)}(X)}\\ &= e^{\int_{X_0}^{X_T} m(u)\,du - \frac{1}{2}\int_0^T m^2(X_t)dt - (pm_1 - qm_2)L_T^{(0)}(X)}\\ &= e^{\int_{x_0}^{x} m(u)\,du - \frac{1}{2}(m_1^2 w - m_2^2(T-w)) - (pm_1 - qm_2)l}\end{aligned} \tag{19}$$

for any trajectory $X_\cdot$ such that $X_0 = x_0, X_T = x$, $\int_0^T 1_{\{X_t\geq 0\}}dt = w$, $L_T^{(0)}(X) = l$.

3.2 Option pricing under the two-valued local volatility model

In this section we show how the results of Section 2 can be applied to option pricing under the two-valued LVM. We do it by example in the case of a European call option. Recall first some terminology and facts from option pricing theory. A European call option (call option) with strike price (strike) K and expiration date T is a derivative whose payoff is $(S_T - K)^+ = \max(S_T - K, 0)$, where S_T is the price of the underlying asset at expiration. A knock-in call option with barrier H is a regular call option that *comes into existence* only when the underlying reaches the barrier. A knock-out call option with barrier H is a regular call option that *ceases to exist* as soon as the underlying reaches the barrier.

Consider the two valued LVM defined by equations (12) with $\sigma(S) = \sigma_1 1_{\{S\geq 1\}} + \sigma_2 1_{\{S<1\}}$. Given value S_0 of the underlying at $t = 0$, strike K and expiry date T, denote by $\mathsf{C}(S_0, K, T)$ and $\mathsf{C}_{\text{in}}(S_0, K, T)$ the price of a corresponding call option and the price of a corresponding knock-in call option with the barrier level of 1 respectively, where both prices are computed under the two-valued LVM. Also, given the same parameters denote by $\mathsf{C}_{\text{out}}(S_0, K, T, \sigma_1, 1)$ the price of a knock-out call option with the barrier level of 1 computed under the log-normal model with volatility σ_1.

It is easy to see that if $K > 1$ then

$$\mathsf{C}(S_0, K, T) = \begin{cases} \mathsf{C}_{\text{in}}(S_0, K, T) + \mathsf{C}_{\text{out}}(S_0, K, T, \sigma_1, 1), & S_0 \geq 1, \\ \mathsf{C}_{\text{in}}(S_0, K, T), & S_0 < 1. \end{cases}$$

Prices of barrier options under the log-normal model are known (e.g., see ch.22 in Hull (2009)). Therefore, if $K > 1$, it is only left to find $\mathsf{C}_{\text{in}}(S_0, K, T)$ under the two-valued LVM in order to price a call option. A formula for the knock-in call option price $\mathsf{C}_{\text{in}}(S_0, K, T)$ is given by Theorem 3.1 below.

The price of a call option with strike $K < 1$ and prices of put options can be obtained in a similar way. Notice that in the case of a call (put) option with strike $K < 1$ ($K > 1$) it seems technically more convenient to start with computing the price of a put (call) option with the same parameters and then to use the put-call parity equation.

Let us introduce some functions that will appear in Theorem 3.1 and its proof. Let

$$\mathrm{n}(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},\ x \in \mathbb{R}, \quad \text{and} \quad \Phi(z) = \int_{-\infty}^{z} \mathrm{n}(y)dy,\ z \in \mathbb{R}, \tag{20}$$

be the probability density function and the cumulative distribution function respectively of the standard normal distribution. Let

$$\mathcal{N}(x, y, \rho) = \int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} \frac{e^{\frac{1}{1-\rho^2}\left(-\frac{z_1^2}{2} - \rho z_1 z_2 + \frac{z_2^2}{2}\right)}}{2\pi\sqrt{1-\rho^2}} dz_1 dz_2,\ x, y \in \mathbb{R}, \tag{21}$$

be the joint cumulative distribution function of the bivariate normal distribution with zero means, unit variances and correlation ρ. Also, denote

$$\phi(S) = \frac{\log(S)}{\sigma(S)} = \frac{\log(S)}{\sigma_1} 1_{\{S \geq 1\}} + \frac{\log(S)}{\sigma_2} 1_{\{S<1\}},$$

and

$$h(t, x, b) = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{(x+bt)^2}{2t}}, \ t \in \mathbb{R}_+, x, b \in \mathbb{R}, \tag{22}$$

Finally, to simplify notation, we assume in Theorem 3.1 that the risk-free interest rate is zero.

Theorem 3.1. *Let S_t be the random process that follows (12) with $\sigma(S) = \sigma_1 1_{\{S \geq 1\}} + \sigma_2 1_{\{S<1\}}$. Given $K > 0$ and $S_0 > 0$ denote $k = \phi(K)$ and $x_0 = \phi(S_0)$. Let $\mathsf{C}_{\text{in}} = \mathsf{C}_{\text{in}}(S_0, K, T)$ be the price of a knock-in European call option with strike K and expiration date T given the initial price S_0.*

1) If $S_0 \geq 1, K > 1$, then

$$\mathsf{C}_{\text{in}} = pe^{-\frac{\sigma_1 x_0}{2}} \left(F_{\text{call}} \left(\frac{\sigma_1}{2}, x_0 \right) - e^{\sigma_1 k} F_{\text{call}} \left(-\frac{\sigma_1}{2}, x_0 \right) \right),$$

where

$$F_{\text{call}}(a, x_0) = \int_0^T F_1(T-t) F_2(a, t, x_0, 1) e^{-\frac{t\sigma_1^2}{8}} dt, \tag{23}$$

and where, in turn,

$$F_1(s) = \frac{\sqrt{2} \left(\sigma_1 e^{-\frac{\sigma_2^2 s}{8}} - \sigma_2 e^{-\frac{\sigma_1^2 s}{8}} \right) + \sqrt{\pi s} \sigma_1 \sigma_2 \left(\Phi\left(\frac{\sqrt{s}\sigma_2}{2} \right) - \Phi\left(\frac{\sqrt{s}\sigma_1}{2} \right) \right)}{\sqrt{\pi s}(\sigma_1 - \sigma_2)}, \tag{24}$$

$$F_2(a, t, x_0) = \frac{1}{\sqrt{2\pi}\sqrt{t}} e^{ka - \frac{(|x_0|+|k|)^2}{2t}} + a e^{a|x_0| + \frac{ta^2}{2}} \left(1 - \Phi\left(\frac{|x_0| + |k|}{\sqrt{t}} - a\sqrt{t} \right) \right). \tag{25}$$

2) If $S_0 < 1, K > 1$, then

$$\mathsf{C}_{\text{in}} = 2pe^{\frac{\sigma_2 x_0}{2}} \left(G\left(-\frac{\sigma_1}{2}, x_0 \right) - e^{k\sigma_1} G\left(\frac{\sigma_1}{2}, x_0 \right) \right)$$

$$G(a, x_0) = \int_0^T e^{-\frac{\sigma_1^2 v}{8} - \frac{\sigma_2^2 (T-v)}{8}} e^{\frac{\sigma_1^2}{8}\left(u\left(\frac{p}{q}\right)^2 + v \right) - |x_0|\frac{pa}{q}} G_1\left((a, v, |x_0|, -\frac{p}{q} a \right) dv$$

where

$$G_1(a, v, y, w) = \int_k^\infty \int_0^\infty h(v, lp + x, a) h(T - v, lq + y, w) dl dx.$$

In turn, G_1 can be expressed in terms of the univariate standard normal distribution (20) *and a bivariate normal cdf (21) as follows*

$$
\begin{aligned}
&G_1(a,y,v,w)q\sqrt{v(T-v)}\\
&=\frac{\mathrm{n}\left(\gamma X+Y\right)\mathrm{n}(X)}{1+\gamma^2}-\frac{\gamma Y}{(1+\gamma^2)^{3/2}}\mathrm{n}\left(\frac{Y}{\sqrt{1+\gamma^2}}\right)\Phi\left(-\frac{(1+\gamma^2)X+\gamma Y}{\sqrt{1+\gamma^2}}\right)\\
&-\frac{\alpha}{\sqrt{1+\gamma^2}}\mathrm{n}\left(\frac{Y}{\sqrt{1+\gamma^2}}\right)\Phi\left(-\frac{(1+\gamma^2)X+\gamma Y}{\sqrt{1+\gamma^2}}\right)\\
&-\beta\mathrm{n}(X)\Phi(-\gamma X-Y)-\frac{\gamma}{\sqrt{2\pi(1+\gamma^2)}}\mathrm{n}\left(\frac{Y}{\sqrt{1+\gamma^2}}\right)\Phi\left(-\frac{(1+\gamma^2)X+\gamma Y}{\sqrt{1+\gamma^2}}\right)\\
&+\alpha\beta\mathcal{N}\left(-X,-\frac{Y}{\sqrt{1+\gamma^2}},-\frac{\gamma}{\sqrt{1+\gamma^2}}\right)
\end{aligned}
$$

where $\alpha=w\sqrt{T-v}$, $\beta=a\sqrt{v}$, $\gamma=\frac{p}{q}\frac{\sqrt{T-v}}{\sqrt{v}}$, $X=\frac{y+(T-v)w}{\sqrt{T-v}}$ *and* $Y=\frac{qk-py-pw(T-v)+qva}{q\sqrt{v}}$.

Theorem 3.1 is proved in Section 4.3.

4 Proofs

4.1 Proofs of Theorem 2.1 and Theorem 2.2

We prove Theorem 2.1 only. Theorem 2.2 can be proved in a similar way with straightforward modifications (see Remark 4.1).

Given $n\in\mathbb{N}$ consider a discrete time Markov chain $S_k^{(n)}\in\mathbb{Z},\ k\in\mathbb{Z}_+$, which evolves like a simple symmetric random walk with unit jumps except at 0. If $S_k^{(n)}=0$, then the chain jumps up by one with probability p and down by one with probability $q=1-p$. Define the following stochastic process

$$X_t^{(n)}=\frac{1}{\sqrt{n}}S_{[nt]}^{(n)}+\frac{nt-[nt]}{\sqrt{n}}\left(S_{1+[nt]}^{(n)}-S_{[nt]}^{(n)}\right),\ t\geq 0. \tag{26}$$

Let us define the following quantities

$$\tau_n=\max\left\{k:S_k^{(n)}=0\right\},\ V_n=\sum_{i=0}^{\tau_n}1_{\left\{S_i^{(n)}\geq 0,S_{i+1}^{(n)}\geq 0\right\}},\ L_n=\sum_{i=0}^{[Tn]}1_{\left\{S_i^{(n)}=0\right\}}. \tag{27}$$

Theorem 2.1 is implied by Lemmas 4.1 and 4.2 below.

Lemma 4.1. *Let $X_t^{(n)}$ be the process defined by (26) and let τ_n,V_n,L_n be quantities defined by (27). Then*

$$\left(\frac{\tau_n}{n},\frac{V_n}{n},\frac{L_n}{\sqrt{n}},X_T^{(n)}\right)\to\left(\tau,V,L_T^{(0)}(W^{(p)}),W_T^{(p)}\right),$$

in distribution, as $n\to\infty$.

Proof. It has been proved in Harrison and Shepp (1981) that $X_t^{(n)}$ converges in the space of continuous functions, as $n \to \infty$, to SBM $W_t^{(p)}$. This implies the claim of the lemma. $\square$

Lemma 4.2. *Let $X_t^{(n)}$ be the process defined by (26) and let τ_n, V_n and L_n be quantities defined by (27). Suppose that sequences of numbers $r_n, r_{1,n}$ and k_n are such that*

$$\frac{2r_{1,n}}{n} \to x \in [0,T],\ \frac{2(r_n - r_{1,n})}{n} \to y \in [0,T],\ \frac{k_n}{\sqrt{n}} \to l \in \mathbb{R}_+,\ \text{as } n \to \infty,$$

where $x + y = t \le T$.

1) If, in addition, $\frac{j_n}{\sqrt{n}} \to z \ge 0$, as $n \to \infty$, then

$$\lim_{n\to\infty} \frac{n^3}{8} \mathsf{P}\left(V_n = 2r_{1,n},\ \tau_n = 2r_n,\ L_n = k_n,\ X_1^{(n)} = j_n | X_0^{(n)} = 0\right)$$
$$= \frac{2p^2 q l^2 z}{[2\pi x(t-x)(T-t)]^{3/2}} e^{-\frac{z^2}{2(T-t)} - \frac{l^2}{2}\left(\frac{p^2}{x} + \frac{q^2}{t-x}\right)}.$$

2) If, in addition, $\frac{j_n}{\sqrt{n}} \to z < 0$, as $n \to \infty$, then

$$\lim_{n\to\infty} \frac{n^3}{8} \mathsf{P}\left(V_n = 2r_{1,n},\ \tau_n = 2r_n,\ L_n^{(n)} = k_n, X_1^{(n)} = j_n | X_0^{(n)} = 0\right)$$
$$= \frac{2p^2 q l^2 |z|}{[2\pi x(t-x)(T-t)]^{3/2}} e^{-\frac{z^2}{2(T-t)} - \frac{l^2}{2}\left(\frac{p^2}{x} + \frac{q^2}{t-x}\right)}.$$

Proof. Recall that $X_t^{(n)}$ is the process defined by (26).

Definition 4.1. *Given n define $t_i = \frac{i}{n}$, $i = 0, 1, \ldots, [Tn]$, and consider a sequence of states $X_{t_i}^{(n)}, i = 0, 1, \ldots, n$.*

- *A subsequence $\left(X_{t_k}^{(n)}, X_{t_{k+1}}^{(n)}, \ldots, X_{t_{k+2d}}^{(n)}\right)$ such that $X_{t_k}^{(n)} = 0$, $X_{t_{k+1}}^{(n)} > 0, \ \ldots, X_{t_{k+2d-1}}^{(n)} > 0$, $X_{t_{k+2d}}^{(n)} = 0$, is called a positive cycle of length $2d$.*

  A subsequence $\left(X_{t_k}^{(n)}, X_{t_{k+1}}^{(n)}, \ldots, X_{t_{k+2d}}^{(n)}\right)$ such that $X_{t_k}^{(n)} = 0$, $X_{t_{k+1}}^{(n)} < 0, \ \ldots, X_{t_{k+2d-1}}^{(n)} < 0$, $X_{t_{k+2d}}^{(n)} = 0$, is called a negative cycle of length $2d$.

- *Denote by R_n the number of positive cycles in a sequence $X_{t_k}^{(n)}$, $k = 0, \ldots, [Tn]$. Given $r, r_1, k, i \in \mathbb{Z}_+$, where $r_1 \le r$ and $i \le k$, let $A_{r,r_1,k,i}$ be a set of sequences $X_{t_k}^{(n)}$, $k = 0, \ldots, [Tn]$, for which $\tau_n = 2r$, $L_n = k$, $R_n = i$, $V_n = 2r_1$.*

Notice that the total number of both positive and negative cycles equals L_n. We prove the lemma only if $z \ge 0$ (the case $z < 0$ can be considered similar).

Given $j \geq 0$ denote $B_{n,r,j} = \left\{ X^{(n)}_{t_{2r+1}} > 0, \ldots, X^{(n)}_{t_{[Tn]-1}} > 0,\ X^{(n)}_T = j \right\}$. It is easy to see that

$$\begin{aligned} &\mathsf{P}\left(V_n = 2r_{1,n},\ \tau_n = 2r_n,\ L_n = k_n,\ X^{(n)}_T = j_n | X^{(n)}_0 = 0\right) \\ &= \left(\sum_{i=0}^{k_n} \mathsf{P}(A_{r_n, r_{1,n}, k_n, i}) \right) \mathsf{P}\left(B_{n,r_n,j_n} | X^{(n)}_{t_{2r_n}} = 0 \right) \end{aligned}$$

and the statement of the lemma is implied by two following propositions.

Proposition 4.1. *Under assumptions of Lemma 4.2*

$$\lim_{n\to\infty} n^2 \left(\sum_{i=0}^{k_n} \mathsf{P}(A_{r_n,r_{1,n},k_n,i}) \right) = \frac{2pql^2}{\pi(x(t-x))^{3/2}} e^{-\frac{l^2}{2}\left(\frac{p^2}{x} + \frac{q^2}{t-x}\right)} = 4h(x, pl)h(t-x, lq).$$

Proposition 4.2. *1) Under assumptions of Part 1) of Lemma 4.2,*

$$\begin{aligned} &\lim_{n\to\infty} n\mathsf{P}\left(X^{(n)}_{t_{2r_n+1}} > 0,\ \ldots, X^{(n)}_{t_{[Tn]-1}} > 0,\ X^{(n)}_T = j_n | X^{(n)}_{t_{2r_n}} = 0 \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{2pz}{(T-t)^{3/2}} e^{-\frac{z^2}{2(T-t)}} = 4ph(T-t, z). \end{aligned}$$

2) Under assumptions of Part 2) of Lemma 4.2,

$$\begin{aligned} &\lim_{n\to\infty} n\mathsf{P}\left(X^{(n)}_{t_{2r_n+1}} < 0,\ \ldots, X^{(n)}_{t_{[Tn]-1}} < 0,\ X^{(n)}_T = j_n | X^{(n)}_{t_{2r_n}} = 0 \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{2q|z|}{(T-t)^{3/2}} e^{-\frac{z^2}{2(T-t)}} = 4qh(T-t, z). \end{aligned}$$

These propositions are proved in Section 4.2. □

Remark 4.1. Theorem 2.2 can be proved by modifying appropriately the proof of Theorem 2.1. First of all, one should consider Markov chain $S^{(n)}_k \in \mathbb{Z},\ k \in \mathbb{Z}_+$, which jumps up/down by one with probabilities $\frac{1}{2}\left(1 \pm \frac{m_1}{\sqrt{n}}\right)$ in the positive half-space; with probabilities $\frac{1}{2}\left(1 \pm \frac{m_2}{\sqrt{n}}\right)$ in the negative half-space and with probabilities p and $q = 1-p$ respectively starting at the origin. Continuous time random process $X^{(n)}_t$ is defined by equation (26) as before. Convergence of $X^{(n)}_t$ to a SBM with two-valued drift (4) can be proved by a straightforward modification of the proof in Harrison and Shepp (1981) (see also Lejay (2006)) in the driftless case. Convergence implies an analogue of Lemma 4.1. It is also rather straightforward to make appropriate changes in both the statement and proof of Lemma 4.2 in the case of non-zero drift. We skip details.

Alternatively, one can combine Theorem 2.1 and the Girsanov's theorem (see Remark 3.1) to obtain Theorem 2.2.

4.2 Proofs of Proposition 4.1 and Proposition 4.2

We write $r = r_n, r_1 = r_{1,n}, k = k_n$ and $j = j_n$ throughout proofs.

Proof of Proposition 4.1. It is easy to see that probabilities of a positive cycle of length $2d$ and of a negative cycle of length $2d$, where $d \geq 1$, are equal to $2p/4^d$ and $2q/4^d$, respectively. Therefore a probability of a single path from $A_{r,r_1,k,i}$ is equal to $\frac{2^k p^i q^{k-i}}{2^{2r}}$. Denote by $N_{2d,i}$ the number of paths of length $2d$, starting and ending at the origin and formed by i cycles regardless of their signs. It is easy to see that the number of paths of length $2d$, starting and ending at the origin and formed by i cycles *of the same sign* is equal to $N_{2d,i}/2^i$. Therefore, the number of trajectories forming set $A_{r,r_1,k,i}$ is equal to $\binom{k}{i}\frac{N_{2r_1,i}}{2^i}\frac{N_{2(r-r_1),k-i}}{2^{k-i}}$. Also, notice that $\frac{N_{2d,i}}{2^{2d}} = f_{2d}^{(i)}$, where $f_{2d}^{(i)}$ is the probability that the $i-$th return of SSRW to the origin occurs at time $2d$. Summarizing all these facts, we obtain that

$$\mathsf{P}(A_{r,r_1,k,i}) = \binom{k}{i} p^i q^{k-i} f_{2r_1}^{(i)} f_{2(r-r_1)}^{(k-i)}.$$

It is known (Section 7, ch.3, Feller (1968)) that $f_{2d}^{(i)} = \frac{i}{2d-i}\frac{1}{2^{2d-i}}\binom{2d-i}{d}$. If d is large and $i^2/(2d)$ is not very large or close to zero, then the following approximations can be used (equation (7.6) in Section 7, ch.3, Feller (1968))

$$f_{2d}^{(i)} \approx \sqrt{\frac{2}{\pi}} \frac{i}{(2d-i)^{3/2}} e^{-\frac{i^2}{2(2d-i)}}.$$

Using this approximation, it can be shown that

$$\left| \sum_{i=0}^{k} \mathsf{P}(A_{r,r_1,k,i}) - \frac{2}{\pi} \sum_{i=0}^{k} \frac{\binom{k}{i} p^i q^{k-i} i(k-i) e^{-\frac{i^2}{2(2r_1-i)} - \frac{(k-i)^2}{2(2(r-r_1)-k+i)}}}{(2r_1-i)^{3/2}(2(r-r_1)-k+i)^{3/2}} \right| \to 0, \tag{28}$$

as $n \to \infty$. Under assumptions of Lemma 4.2, the second sum in the preceding display can be replaced by the following one

$$\frac{1}{n^2}\frac{2l^2}{\pi(xy)^{3/2}} \sum_{i=0}^{k} \frac{\binom{k}{i} p^i q^{k-i} i(k-i)}{k^2} e^{-\frac{l^2}{2}\left(\frac{i^2}{k^2 x} + \frac{1}{y}\left(1-\frac{i}{k}\right)^2\right)}, \tag{29}$$

which, in turn, is equal to the expectation $\mathsf{E}\left(F\left(\frac{\xi_n}{k}\right)\right)$, where ξ_n is a Binomial random variable with parameters k_n and p, and $F(z) = z(1-z)e^{-\frac{l^2}{2}\left(\frac{z^2}{x} + \frac{(1-z)^2}{y}\right)}$. By the Law of Large Numbers

$$\mathsf{E}\left(F\left(\frac{\xi_n}{k}\right)\right) \to F(p) = pqe^{-\frac{l^2}{2}\left(\frac{p^2}{x}+\frac{q^2}{y}\right)}, \quad \text{as} \quad n \to \infty, \tag{30}$$

Combining equations (28), (29) and (30), we get that

$$n^2 \sum_{i=0}^{k} \mathsf{P}(A_{r,r_1,k,i}) = n^2 \sum_{i=0}^{k} \binom{k}{i} p^i q^{k-i} f_{2r_1}^{(i)} f_{2(r-r_1)}^{(k-i)} \to \frac{2pql^2}{\pi(x(t-x))^{3/2}} e^{-\frac{l^2}{2}\left(\frac{p^2}{x}+\frac{q^2}{t-x}\right)},$$

as $n \to \infty$. $\square$

Proposition 4.2 is proved in Billingsley (1968), chapter 9, as a part of the derivation of the joint distribution of the standard BM, its last visit to the origin and the occupation time. We give the proof here for the sake of completeness and for reader's convenience.

Proof of Proposition 4.2. For simplicity of notation and without loss of generality, we assume that $[Tn]$ is an integer, so that $t_{[Tn]} = T$. It is easy to see that the probability of a single trajectory such that

$$X^{(n)}_{t_{2r}} = 0,\, X^{(n)}_{t_{2r+1}} > 0,\, \dots,\, X^{(n)}_{t_{[Tn]-1}} > 0,\, X^{(n)}_{t_{[Tn]}} = X^{(n)}_T = j > 0,$$

is equal to $p/2^{n-2r-1}$. Therefore,

$$\begin{aligned}\mathsf{P}&\left(X^{(n)}_{t_{2r+1}} > 0,\, \dots,\, X^{(n)}_{t_{Tn-1}} > 0,\, X^{(n)}_T = j | X^{(n)}_{t_{2r}} = 0\right)\\ &= 2p\,\mathsf{P}\left(S_{2r+1} > 0,\, \dots,\, S_{Tn-1} > 0,\, S_{Tn} = j | S_{2r} = 0\right),\end{aligned}$$

where S_k is the simple symmetric random walk (SSRW). If $Tn-2r$ and j have the same parity, then

$$\mathsf{P}\left(S_{2r+1} > 0,\, \dots,\, S_{n-1} > 0,\, S_{Tn} = j | S_{2r} = 0\right) = \frac{j}{Tn-2r}\mathsf{P}(S_{Tn-2r} = j | S_0 = 0).$$

It is easy to see that under assumptions of the lemma $\frac{j}{\sqrt{Tn-2r}} \to \frac{z}{\sqrt{T-t}}$, hence, by the Local Limit Theorem

$$\frac{\sqrt{Tn-2r}}{2}\mathsf{P}(S_{Tn-2r} = j | S_0 = 0) \to \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2(T-t)}}.$$

We conclude the proof by noticing that $\lim_{n\to\infty} n\frac{2j}{(Tn-2r)^{3/2}} = \frac{2z}{(T-t)^{3/2}}$. □

4.3 Proof of Theorem 3.1

Proof of Part 1) of Theorem 3.1. It is easy to see that if $S_0 > 1$ and $K > 1$, then we get the following equation for the option price

$$\mathsf{C}_{\text{in}} = \int\limits_k^\infty \int\limits_0^\infty \int\limits_{\Gamma_{T,1}} \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right) e^{-(t_0+v+s)\lambda_1 - u\lambda_2 + (x-x_0)\mu_1} h\left(t_0, x_0\right) \psi_{p,T-t_0}(u+v, v, x, l) dt_0 dx dl dv ds$$

where $\lambda_i = \frac{\sigma_i^2}{8}$, $i = 1, 2$, t_0 is the hitting time to zero, v and u are occupation times of the positive half-line and of the negative half-line respectively which are observed between t_0 and the last visit to the origin (i.e. $t_0 + v + u$), $s = T - (t_0 + v + u)$, $\Gamma_{T,1} = \{(t_0, v, u, s) : t_0 + v + u + s = T\}$ and where $\psi_{p,T-t_0}$ is given by (8), i.e. $\psi_{p,T-t_0}(u+v, v, x, l) = 2ph(v, lp)h(u, lq)h(s, x)$, because $x > 0$. Using the convolution property of hitting times, we get that

$$\int\limits_{t_0+s=t} h\left(t_0, x_0\right) h(s, x) dt_0 ds = \int\limits_0^t h\left(t-s, x_0\right) h(s, x) dt ds = h\left(t, |x_0| + |x|\right).$$

Notice that $2ph(v,lp)h(u,lq)h\left(t,|x_0|+|x|\right) = \psi_{p,T}(v+u,v,|x_0|+|x|,l)$ and rewrite the expression for C_{in} as follows

$$\mathsf{C}_{\text{in}} = \int_k^{\infty}\int_0^{\infty}\int_{\Gamma_{T,2}} \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right)\psi_{p,T}(v+u,v,|x_0|+|x|,l)e^{-(t+v)\lambda_1 - u\lambda_2}e^{\mu_1(x-x_0)}dldtdvdx,$$

where $\Gamma_{T,2} = \{(t,v,u) : t+v+u = T\}$. Denoting

$$g(u,v) = 2\int_0^{\infty} h(v,lp)h(u,lq)dl = \frac{pq}{\sqrt{2\pi}\left(p^2u+q^2v\right)^{3/2}}$$

we can rewrite

$$\mathsf{C}_{\text{in}} = p\int_k^{\infty}\int_{\Gamma_{T,2}} \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right)g(u,v)h(t,|x_0|+|x|)e^{-(t+v)\lambda_1 - u\lambda_2}e^{\mu_1(x-x_0)}dtdvdx. \tag{31}$$

Further, recalling that $\sigma_1 = -2\mu_1$ we arrive to the following expression for the price

$$\mathsf{C}_{\text{in}} = pe^{-\frac{\sigma_1 x_0}{2}}\left(F_{\text{call}}\left(\frac{\sigma_1}{2},x_0\right) - e^{\sigma_1 k}F_{\text{call}}\left(-\frac{\sigma_1}{2},x_0\right)\right),$$

where

$$F_{\text{call}}(\alpha,x_0) = \int_k^{\infty}\int_{t+v+u=T} g(u,v)e^{-v\lambda_1 - u\lambda_2}h\left(t,|x_0|+|x|\right)e^{\alpha x}e^{-t\lambda_1}dtdvdx.$$

Integrating with respect to variables u,v, provided that $u+v = T-t = s$ is fixed, we obtain the function

$$\begin{aligned}F_1(s) &= \int_{v+u=s} g(u,v)e^{-v\lambda_1 - u\lambda_2}dv\\ &= \frac{\sigma_1\sigma_2}{\sigma_1-\sigma_2}\left[\frac{2}{\sqrt{2\pi s}}\left(\frac{e^{-\frac{1}{8}s\sigma_2^2}}{\sigma_2} - \frac{e^{-\frac{1}{8}s\sigma_1^2}}{\sigma_1}\right) + \left(\Phi\left(\frac{\sqrt{s}\sigma_2}{2}\right) - \Phi\left(\frac{\sqrt{s}\sigma_1}{2}\right)\right)\right],\end{aligned}$$

defined earlier by equation (24). Integrating out variable x, we get

$$\begin{aligned}\int_k^{\infty} h\left(t,|x_0|+|x|\right)e^{\alpha x}dx &= \frac{1}{\sqrt{2\pi}\sqrt{t}}e^{k\alpha - \frac{(|x_0|+|k|)^2}{2t}}\\ &\quad + \alpha e^{\frac{t\alpha^2}{2} - \alpha|x_0|}\left(1 - \Phi\left(\frac{|x_0|+|k|-t\alpha}{\sqrt{t}}\right)\right)\\ &= F_2(\alpha,t,x_0),\end{aligned}$$

where function $F_2(\alpha, t, x_0)$ is defined by (25). Finally, we rewrite F_{call} in terms of F_1 and F_2

$$F_{\text{call}}(\alpha, x_0) = \int_0^T F_1(T-t) F_2(\alpha, t, x_0) e^{-\frac{t\sigma_1^2}{8}} dt,$$

as claimed in (23). $\square$

Proof of Part 2) of Theorem 3.1. If $S_0 < 1$ and $K > 1$, then $x_0 = \phi(S_0) = \frac{\log(S_0)}{\sigma_2} < 0$, $k = \phi(K) = \frac{\log(K)}{\sigma_1} > 0$, and we get, using notation introduced in the proof of Part 1), that

$$\mathsf{C}_{\text{in}} = \int_k^\infty \int_0^\infty \int_{\Gamma_{T,1}} \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right) h\left(t_0, x_0\right) \psi_{p,T-t_0}(v+u, v, x, l) e^{-\lambda_1(v+s)-\lambda_2(t_0+u)+\mu_1 x - \mu_2 x_0} dt_0 dx dl dv ds,$$

where, as before, $h(t_0, x_0)\psi_{p,T-t_0}(u+v, v, x, l) = 2p h(t_0, x_0) h(v, lp) h(u, lq) h(s, x)$. We use again the convolution property of hitting times as in Part 1) but now integrate products $h(v, lp)h(s, x)$ and $h(t_0, x_0)h(u, lq)$ given constraints $v + s = \mathit{const}$ and $t_0 + u = \mathit{const}$, respectively. It leads to the following expression for the price

$$\begin{aligned}
\mathsf{C}_{\text{in}} &= 2p \int_0^T \int_k^\infty \int_0^\infty \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right) h(v, lp + x) h(u, lq + |x_0|) e^{-\lambda_1 v - \lambda_2 u + \mu_1 x - \mu_2 x_0} dl dx dv \\
&= 2p e^{\frac{\sigma_2 x_0}{2}} \int_0^T e^{-\frac{\sigma_1^2 v}{8} - \frac{\sigma_2^2 u}{8}} \int_k^\infty \int_0^\infty \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right) h(v, lp + x) h(u, lq + |x_0|) e^{-\frac{\sigma_1 x}{2}} dl dx dv \\
&= 2p e^{\frac{\sigma_2 x_0}{2}} \int_0^T e^{-\frac{\sigma_1^2 v}{8} - \frac{\sigma_2^2 u}{8}} \int_k^\infty \int_0^\infty h(v, lp + x) h(u, lq + |x_0|) e^{\frac{\sigma_1 x}{2}} dl dx dv \\
&\quad - 2p e^{\frac{\sigma_2 x_0}{2} + \sigma_1 k} \int_0^T e^{-\frac{\sigma_1^2 v}{8} - \frac{\sigma_2^2 u}{8}} \int_k^\infty \int_0^\infty h(v, lp + x) h(u, lq + |x_0|) e^{-\frac{\sigma_1 x}{2}} dl dx dv
\end{aligned}$$

where $u = T - v$ and $\mu_i = -\sigma_i/2$ and $\lambda_i = \sigma_i^2/8$. Rewrite

$$\mathsf{C}_{\text{in}} = 2p e^{\frac{\sigma_2 x_0}{2}} \int_0^T e^{-\frac{\sigma_1^2 v}{8} - \frac{\sigma_2^2 (T-v)}{8}} \left(\mathrm{I}\left(-\frac{\sigma_1}{2}, |x_0|, v\right) - e^{\sigma_1 k} \mathrm{I}\left(\frac{\sigma_1}{2}, |x_0|, v\right)\right) dv, \tag{32}$$

where

$$\mathrm{I}(a, y, v) = \int_k^\infty \int_0^\infty h(v, lp + x) h(u, lq + y) e^{-ax} dl dx, \quad y \geq 0. \tag{33}$$

Notice that

$$\begin{aligned} h(v,lp+x)h(u,lq+y)e^{-ax} &= h(v,lp+x,a)e^{\frac{va^2}{2}+alp}h(u,lq+y) \\ &= e^{\frac{va^2}{2}+\frac{u}{2}\left(\frac{ap}{q}\right)^2-ay\frac{p}{q}}h(v,lp+x,a)h\left(u,lq+y,-apq^{-1}\right) \end{aligned}$$

where $h(t,x,b)$ is defined by (22), so we can rewrite

$$\mathrm{I}(a,y,v) = e^{\frac{va^2}{2}+\frac{T-v}{2}\left(\frac{ap}{q}\right)^2-ay\frac{p}{q}}G_1\left(a,v,y,-apq^{-1}\right),$$

where

$$G_1(a,v,y,w) = \int_k^\infty \int_0^\infty h(v,lp+x,a)h(u,lq+y,w)dldx. \tag{34}$$

Further, noticing that

$$h(v,lp+x,a)h(u,lq+y,w) = \frac{(lp+x)(lq+y)}{2\pi(uv)^{3/2}}e^{-\frac{(lp+x+va)^2}{2v}-\frac{(lq+y+uw)^2}{2u}}$$

and changing variables $z_1 = \frac{lp+x+av}{\sqrt{v}}$ and $z_2 = \frac{lq+y+uw}{\sqrt{u}}$, we can rewrite the expression for G_1 as follows

$$G_1(a,y,v,w) = \int_D \frac{e^{-\frac{w_1^2}{2}-\frac{w_2^2}{2}}}{2\pi}\frac{(z_1-a\sqrt{v})(z_2-w\sqrt{u})}{q\sqrt{uv}}dz_1dz_2$$

where $D = \{(z_1,z_2)\in\mathbb{R}^2 : z_2\sqrt{u} > y+uw,\ -z_2p\sqrt{u}+q\sqrt{v}w_2 > qk-py+qva-puw\}$. Denote

$$\alpha = w\sqrt{u},\ \beta = a\sqrt{v},\ X = \frac{y+uw}{\sqrt{u}},\ Y = \frac{qk-py-puw+qva}{q\sqrt{v}},\ \gamma = \frac{p}{q}\sqrt{\frac{u}{v}},$$

and $\Gamma = \{(z_1,z_2) : z_1 > Y+\gamma z_2,\ z_2 > X\}$. In these notations

$$G_1(a,y,v,w) = \int_\Gamma \frac{e^{-\frac{z_1^2}{2}-\frac{z_2^2}{2}}}{2\pi}\frac{(z_1-\beta)(z_2-\alpha)}{q\sqrt{uv}}dz_1dz_2 = \frac{1}{q\sqrt{uv}}\mathrm{J}(a,y,v,w),$$

where $\mathrm{J}(a,y,v,w) = \int_\Gamma \mathrm{n}(z_1)\mathrm{n}(z_2)(z_1-\beta)(z_2-\alpha)dz_1dz_2$, and function n is defined by (20). Notice that

$$\begin{aligned} \mathrm{J}(a,y,v,w) &= \int_\Gamma z_1z_2\mathrm{n}(z_1)\mathrm{n}(z_2)dz_1dz_2 - \alpha\int_\Gamma z_1\mathrm{n}(z_1)\mathrm{n}(z_2)dz_1dz_2 \\ &- \beta\int_\Gamma z_2\mathrm{n}(z_1)\mathrm{n}(z_2)dz_1dz_2 + \alpha\beta\int_\Gamma \mathrm{n}(z_1)\mathrm{n}(z_2)dz_1dz_2 \\ &:= \mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4. \end{aligned}$$

It can be shown (we skip intermediate computational details) that

$$\mathrm{J}_1 = \frac{\mathrm{n}\,(\gamma X + Y)\,\mathrm{n}(X)}{1+\gamma^2} - \frac{\gamma Y}{(1+\gamma^2)^{3/2}}\mathrm{n}\left(\frac{Y}{\sqrt{1+\gamma^2}}\right)\Phi\left(-\frac{(1+\gamma^2)X+\gamma Y}{\sqrt{1+\gamma^2}}\right)$$

$$\mathrm{J}_2 = -\frac{\alpha}{\sqrt{1+\gamma^2}}\,\mathrm{n}\left(\frac{Y}{\sqrt{1+\gamma^2}}\right)\Phi\left(-\frac{X(1+\gamma^2)+\gamma Y}{\sqrt{1+\gamma^2}}\right)$$

$$\mathrm{J}_3 = -\beta\mathrm{n}(X)\Phi(-\gamma X - Y) - \frac{\gamma}{\sqrt{2\pi(1+\gamma^2)}}\,\mathrm{n}\left(\frac{Y}{\sqrt{1+\gamma^2}}\right)\Phi\left(-\frac{(1+\gamma^2)X+\gamma Y}{\sqrt{1+\gamma^2}}\right)$$

$$\mathrm{J}_4 = \alpha\beta\mathcal{N}\left(-X, -\frac{Y}{\sqrt{1+\gamma^2}}, -\frac{\gamma}{\sqrt{1+\gamma^2}}\right)$$

This finishes the proof of the second part of the theorem. □

5 A Black-Scholes approximation

In this section, we derive in the special case where $S_0 = 1$ a surprisingly simple and accurate approximation for the option price which is based on the Black-Scholes (BS) formula. We use the same notation as in Sections 3.2 and 4.3.

If $S_0 = 1$ ($x_0 = 0$) and $K > 1$ ($k > 0$) then $\mathsf{C} = \mathsf{C}_{\mathrm{in}}$ and equation (31) becomes

$$\mathsf{C} = p\int_k^\infty \int_{t+u+v=T} \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right) g(u,v)h(t,x)e^{-(t+v)\lambda_1 - u\lambda_2}e^{\mu_1 x}dtdvdx. \tag{35}$$

The approximation of the call price is motivated by the following idea. As $k > 0$ we "should be mostly interested" in those trajectories of X_t that spend "most of their lifetime" in region $X_t > 0$, where $\sigma = \sigma_1$. Therefore, let us first replace function $e^{-(t+v)\lambda_1 - u\lambda_2}$ in (35) by $e^{-\lambda_1 T}$. Secondly, integrating out variables v and $u = T - t - v$ gives

$$\int_0^{T-t} g(u,v)dv = \int_0^{T-t} \frac{pq}{\sqrt{2\pi}(p^2(T-t-v)+q^2 v)^{3/2}}dv = \frac{2}{\sqrt{2\pi(T-t)}} = 2\mathrm{p}(0, T-t),$$

where $\mathrm{p}(0, T-t)$ is the value at 0 of $\mathrm{p}(y, T-t)$, the transition density of the standard BM at time $T - t$, so that the result of integration does not depend on p and q. Thus, we arrive, after expressing both λ_1 and μ_1 in terms of σ_1, to the following approximation for the option price

$$\begin{aligned}\mathsf{C} &\approx 2p \int_k^\infty \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right) e^{-\frac{\sigma_1^2 T}{8}} e^{-\frac{\sigma_1 x}{2}} \left(\int_0^T h(t,x) \mathrm{p}(0, T-t) dt \right) dx \\ &= 2p \int_k^\infty \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right) e^{-\frac{\sigma_1^2 T}{8} - \frac{\sigma_1 x}{2}} \mathrm{p}(x,T) dx = \frac{2\sigma_2}{\sigma_1 + \sigma_2} \mathsf{BSC}(\sigma_1)\end{aligned}$$

where $\mathsf{BSC}(\sigma_1)$ is the BS price of the option under the log-normal model with volatility σ_1. It is obvious that if we set $\sigma_1 = \sigma_2$ on both sides of the preceding display, then the approximation becomes the BS formula for the call option price with volatility σ_1.

Using the same argument, we obtain a similar approximation for the put option price. Namely, if $K < 1$ and $S_0 = 1$, then $\mathsf{Put} \approx 2q\mathsf{BSP}(\sigma_2) = \frac{2\sigma_1}{\sigma_1+\sigma_2}\mathsf{BSP}(\sigma_2)$, where $\mathsf{BSP}(\sigma_2)$ is the BS price of the put option with volatility σ_2. Similar to the call option case, the BS approximation provides either an upper bound (if $\sigma_1 > \sigma_2$) or a lower bound (if $\sigma_1 < \sigma_2$).

The discontinuous (at $K = 1$) curve shown on the left-hand side of Figure 1 is the implied volatility curve calculated by using the approximation. In this calculation call prices have been used, if $K > 1$, and put prices have been used, if $K < 1$. The solid curve in the middle of the left side Figure 1 is the implied volatility curve calculated by using the exact formula provided by Theorem 2.2. It is easy to see that if $\sigma_1 < \sigma_2$, then the BS approximation provides an upper (lower) bound of the price in the case of call (put) options, and, vice versa, if $\sigma_1 > \sigma_2$, then the approximation provides a lower (upper) bound for call (put) prices. In this example $\sigma_1 = 0.5 < \sigma_2 = 0.9$, therefore the approximate curve is below the exact curve, if $K < 1$, and above it, if $K > 1$, as expected. The upper dashed curve is the implied volatility curve calculated by using an approximation proposed in Lipton and Sepp (2011) for calibration of a LVM with a piecewise volatility (tiled LVM). The latter includes the two-valued LVM as a particular case.

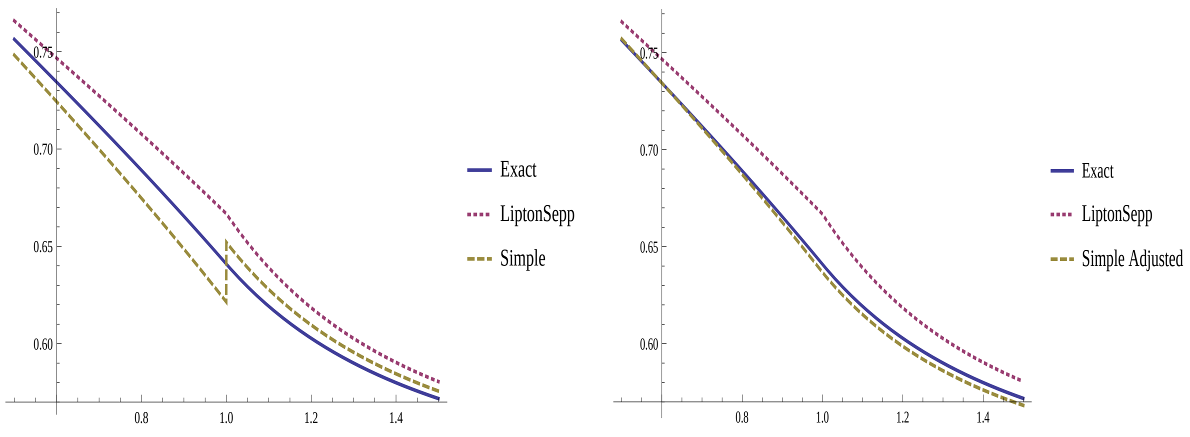

Figure 1: Implied volatility curves, $\sigma_1 = 0.5, \sigma_2 = 0.9, T = 2, S_0 = 1$. In both figures: the solid line corresponds to the two-valued LVM and the dashed upper curve corresponds to Lipton-Sepp's approximation. Implied volatility calculated by using BS approximation: without adjustment on the left and with adjustment on the right.

The BS approximation can be improved. Indeed, recall that we must have the put-call parity $\mathsf{C} - \mathsf{Put} = K - S_0$, which becomes $\mathsf{C} = \mathsf{Put}$, if $K = S_0$. The put-call parity does not hold for the approximate prices and we adjust them to restore it in the case where $K = S_0 = 1$. Namely, let us define the following adjustment factors

$$A_{\text{cl}} = \frac{p\mathsf{BSC}(\sigma_1) + q\mathsf{BSP}(\sigma_2)}{2p\mathsf{BSC}(\sigma_1)} \quad \text{and} \quad A_{\text{pt}} = \frac{p\mathsf{BSC}(\sigma_1) + q\mathsf{BSP}(\sigma_2)}{2q\mathsf{BSP}(\sigma_2)},$$

and consider adjusted approximate prices $\widetilde{\mathsf{BSC}(\sigma_1)} = A_{\text{cl}}\mathsf{BSC}(\sigma_1)$ and $\widetilde{\mathsf{BSP}(\sigma_2)} = A_{\text{pt}}\mathsf{BSP}(\sigma_2)$. By construction, the put-call parity holds for adjusted approximate prices in the case where $K = S_0 = 1$. This adjustment smooths the approximate implied volatility curve which becomes continuous everywhere. The adjustment result is shown on the right-hand side of Figure 1, where both the solid line and the upper dashed line are as before, and the new dashed curve is calculated by using adjusted prices. It is quite visible that the adjustment improves the approximation.

Finally, numerical tests showed that accuracy of the approximation improves as the time to expiration becomes smaller, which agrees with intuition.

6 A note on a displaced diffusion model with discontinuity

Our results on the joint distribution of SBM and its functionals can be also applied to derivative pricing in the following displaced model

$$dS_t = \left(\sigma_1 \left(S_t - \alpha_1\right) 1_{\{S_t \geq S^*\}} + \sigma_2 \left(S_t - \alpha_2\right) 1_{\{S_t < S^*\}}\right) dW_t, \tag{36}$$

where $\sigma_1 \neq \sigma_2$, $\alpha_i \in \mathbb{R}$, $i = 1, 2$ and $S^* > 0$. Model (36) is a particular case of the following model considered in Decamps, Goovaerts and Schoutens (2006a)

$$dS_t = \left(\sigma_1 \left(S_t - \alpha_1\right)^{\beta_1} 1_{\{S_t \geq S^*\}} + \sigma_2 \left(S_t - \alpha_2\right)^{\beta_2} 1_{\{S_t < S^*\}}\right) dW_t.$$

where, in addition, $\beta_i \geq 0$, $i = 1, 2$. In Decamps, Goovaerts and Schoutens (2006a) they derived certain semi-analytical expressions for the transition density of the underlying process. The technique of Decamps, Goovaerts and Schoutens (2006a) is an adaptation of a technique that was used in Gorovoi and Linetsky (2004). In turn, the technique of Gorovoi and Linetsky (2004) is based on a well known observation (e.g. Gikhman and Skorohod (1968)) that the transition density satisfies a partial differential equation and can be constructed by means of an eigenfunction expansion in the corresponding Sturm-Liouville problem. In general, these eigenfunction expansions for the transition densities are difficult to handle analytically and an approximation is required. It should be noticed that in Decamps, Goovaerts and Schoutens (2006a) an analytical expression for the transition density was obtained in a particular case where $\sigma_1 = \sigma_2$, $\alpha_1 \neq \alpha_2$, so that dependence of the joint density on the occupation time becomes trivial (e.g., see equation (9) or (19), where $m_1 = m_2$).

Notice also that if $\sigma_1 = \sigma_2 = \sigma$, $\beta_1 = \beta_2 = 1$ and $\alpha_1 = \alpha_2 = a$, then it is a classical case of a displaced log-normal model. The latter is just $S_t = Z_t - a$, where Z_t is the log-normal process, and it can be written in the local volatility form, namely, $dS_t = \sigma(1 - a/S_t)S_t dW_t$. The displaced diffusion is a very useful tool for approximating more complicated stochastic processes in finance. The main reason is that this model is a first-order approximation of any LVM (see Remark 7.2.14 in Andersen and Piterbarg (2010) and other examples therein). A known problem with a displaced model of any sort is that, theoretically, the underlying process can take negative values (e.g. when $\alpha_i > 0$). This problem can be dealt with by imposing some constraints. For instance, instead of the classic displaced log-normal model one can consider model (36) with $\alpha_2 = 0$. This means that the volatility is a hyperbolic function above level S^* and a constant one below level S^* and, hence, is prevented to take large values as the process approaches 0. It is rather straightforward to apply our results to the displaced log-normal model with such constraints. Let us take, for example, model (36), where $S^* = 1, \alpha_1 < 1$ and $\alpha_2 = 0$, and consider briefly the case when the process starts at $S_0 < 1$. Given $\sigma_1, \sigma_2, \alpha_1$ and strike $K > 1$ define

$$p = \frac{\sigma_2}{\sigma_2 + \sigma_1 (1 - \alpha_1)},\ q = 1 - p,\ k = \frac{1}{\sigma_1} \log\left(\frac{K - \alpha_1}{1 - \alpha_1}\right),\ x_0 = \frac{\log(S_0)}{\sigma_2}, b = \frac{q\sigma_2 - p\sigma_1}{2}.$$

Then the price of a knock-in European call option with strike K and expiration date T is given by the following integral

$$\mathsf{C}_{\text{in}} = 2p(1 - \alpha_1) \int_k^{\infty} \int_0^{\infty} \int_{\Gamma_{T,1}} \left(e^{\sigma_1 x} - e^{\sigma_1 k}\right) e^{-l\beta - \lambda_1(s+v) - x_0\mu_2 - \lambda_2(t_0+u) + \mu_1 x} R(u, v, x, l, t_0)\, dx\, dl\, dv\, dt_0$$

where $R(u, v, x, l, t_0) = h(t_0, x_0)\, \psi_{p,T-t_0}(u + v, v, x, l)$ and we used notation introduced in the proof of Part 1) of Theorem 2.2. Using the same argument as in the proof of the theorem one can show that computation of the above integral can be reduced to computation of the following integral

$$\tilde{\mathrm{I}}(b, a, v, y) = \int_k^{\infty} \int_0^{\infty} e^{-ax - bl} h(v, lp + x) h(u, lq + y)\, dl\, dx.$$

In turn, one can express, by modifying appropriately the argument applied to integral (33), the integral in the preceding display in terms of both a univariate and a bivariate normal distribution as follows

$$\tilde{\mathrm{I}}(b,a,v,y) = \frac{e^{\frac{\nu^2 u}{2}+\frac{a^2 v}{2}+\nu y}}{q\sqrt{uv}} \left(\frac{e^{-\frac{X^2+(Y+\gamma X)^2}{2}}}{2\pi\,(1+\gamma^2)} - \frac{Be^{-\frac{X^2}{2}}\Phi(-Y-\gamma X)}{\sqrt{2\pi}} \right.$$
$$+ \frac{1}{\sqrt{2\pi(1+\gamma^2)}} \left(-A + B\gamma - \frac{\gamma Y}{1+\gamma^2} \right) e^{-\frac{Y^2}{2(1+\gamma^2)}} \Phi\left(-\frac{(1+\gamma^2)X+\gamma Y)}{\sqrt{1+\gamma^2}} \right)$$
$$\left. +AB\mathcal{N}\left(-X, -\frac{Y}{\sqrt{1+\gamma^2}}, -\frac{\gamma}{\sqrt{1+\gamma^2}} \right) \right)$$

where $\nu = -\frac{ap-b}{q}$, $A = \nu\sqrt{u}$, $B = a\sqrt{v}$, $Y = \frac{q(k+av)-p(\nu u+y)}{q\sqrt{v}}$, $X = \frac{\nu u+y}{\sqrt{u}}$ and $\gamma = \frac{p}{q}\sqrt{\frac{u}{v}}$.

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