Divergent Platforms*

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Abstract

Models of electoral competition between two opportunistic, office-motivated parties typically predict that both parties become indistinguishable in equilibrium. I show that this strong connection between the office motivation of parties and their equilibrium choice of identical platforms depends on two - possibly false - assumptions: 1. Issue spaces are uni-dimensional and 2. Parties are unitary actors whose preferences can be represented by expected utilities. I provide an example of a two-party model in which parties offer substantially different equilibrium platforms even though no exogenous differences between parties are assumed. In this example, some voters' preferences over the 2-dimensional issue space exhibit non-convexities and parties evaluate their actions with respect to a set of beliefs on the electorate.

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1 Introduction

Two parties never run on the exact same platform in an electoral campaign. In contrast, two-party-models following Downs (1957) and Hotelling (1929) predict that parties choose identical platforms in equilibrium. Downs-Hotelling-type models assume that parties strive to win as many votes as possible and that voters have single peaked preferences. I show that some games of electoral competition, that maintain these two standard assumptions, have equilibria in which parties announce substantially different platforms. To obtain such

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divergent equilibria I relax two features of the Downs-Hotelling model: I neither assume uni-dimensional issue spaces nor that parties are expected utility maximizers. Instead there are multiple issues and parties hold multiple beliefs on the distribution of voter preferences. A party changes its platform only if such a change achieves a higher vote share according to all its beliefs.

Two forces drive parties to adopt identical platforms in the tradition of Downs and Hotelling : convex preferences and the fact that any party can get half the vote by adopting the platform of the opponent. Assume that Parties 1 and 2 run on two different platforms $\mathbf{x}$ and $\mathbf{y}$. In the Downs-Hotelling model Party 1 cannot loose by moving its platform from $\mathbf{x}$ to some intermediate platform $\lambda\mathbf{x} + (1 - \lambda\mathbf{y})$ (for $\lambda \in (0, 1)$) since any voter with single-peaked preferences who prefers $\mathbf{x}$ to $\mathbf{y}$ also prefers $\lambda\mathbf{x} + (1 - \lambda\mathbf{y})$ to $\mathbf{y}$. The crux of the argument is that single-peaked preferences over a uni-dimensional issue space are convex. However, single-peaked preferences over a multi-dimensional issue space need not be convex, and a voter might well prefer both platforms $\mathbf{x}$ and $\mathbf{y}$ to the intermediate $\lambda\mathbf{x} + (1 - \lambda\mathbf{y})$. With a multi-dimensional issue space, a party might lose votes by moving its platform closer to that of the opponent.[1]

For a party that obtains less than half the vote in electoral competition à la Downs-Hotelling there is yet another incentive to emulate: the party can increase its vote share by adopting the opponent's platform. If the vote is not evenly split, then one party has an incentive to copy the other. To see that the presence of multiple priors in my model dilutes this incentive, consider a model in which each party uses two different beliefs to evaluate electoral outcomes. Say that at some platform profile, Party 1 obtains 41% according to the first belief and 55% according to the second. According to the first belief Party 1 can increase its vote share from 41% to 50% by adopting of Party 2's platform, according to the second belief the same change decreases Party 1's vote share from 55% to 50%. In sum, there are platform profiles and sets of party beliefs, such that neither party has an incentive to emulate the other although the vote is unevenly split according to each belief in the set.

In Theorem 1, I show that games of electoral competition with multi-dimensional issue spaces, single-peaked preferences and two parties with multiple beliefs on the electorate can have divergent equilibria. Both forces that drive parties to adopt identical equilibrium platforms in the Downs-Hotelling model are defused in my model. Under the assumption of a multi-dimensional issue space the voters' single-peaked preferences need not be convex. Without convex preferences, the first of the two gravitational forces described above looses its strength. The assumption of multiple priors weakens the second force. In my model vote

[1]There is a vast empirical literature on the dimensionality of political issue spaces. This literature was pioneered by Poole and Rosenthal (1985); http://voteworld.berkeley.edu/ links to an extensive array of econometric analysis of political issue spaces and voter preferences in various parts of the world.

shares may be split unevenly in equilibrium. Theorem 1 then shows that these two changes of the Downs-Hotelling model are sufficient to obtain divergent equilibria. I give an example of a game of electoral competition with office-motivated parties and single-peaked voter preferences that has a divergent equilibrium. Theorem 1 furthermore shows that divergent equilibria need not be flukes. The game of electoral competition I define to prove Theorem 1 not only has a divergent equilibrium, this equilibrium is unique up to a role-reversal of parties. So divergent equilibria need not be part of large equilibrium sets. Moreover the equilibrium I find is robust to small changes in the parties' beliefs. Theorem 2 shows that the assumptions of a multi-dimensional issue space and multiple beliefs are necessary for the existence of divergent equilibria: if we only introduce one of these two features into a Downs-Hotelling type model the classic conclusion that equilibrium platforms converge persists.

Let me give two motivations for the multiple beliefs model of party preferences and behavior. Parties might not know enough about the voters' preferences to assign objective probabilities to vote shares for all possible constellations of platforms: electoral outcomes are generally hard to predict as elections are held in changing environments characterized by new and surprising issues, turnover of party elites, and an evolving electorate. It is well documented that individuals violate expected utility maximization when faced with such subjective uncertainty.[2] In Bewley's (2002) multiple prior model, a cornerstone to the literature on uncertainty aversion,[3] agents calculate expected utilities for a set of priors Ψ on a subjectively uncertain environment. An agent prefers some action f over some other action g if f yields a higher expected utility according to all priors $\psi \in \Psi$. The resulting preferences are incomplete: if some action f is associated with strictly higher expected utility than g according to some prior $\psi^f \in \Psi$ while the reverse holds true according to a different $\psi^g \in \Psi$, the agent cannot rank f and g. To determine behavior in the face of incompleteness, Bewley's framework contains an inertia assumption: a new course of action has to be strictly preferred to the status quo to be adopted.

The multiple-prior-model of party preferences and behavior can also be derived from a model of group decision making. Parties generally consist of many members or factions who have to come to some form of compromise to decide on a platform. We could assume that all these leaders share a common goal: they all hope to maximize their party's vote share. However, the leaders might have different expectations with respect to the preferences of the electorate. The model of party preferences proposed here obtains if the change of any

[2]A large range of experimental studies demonstrates a bias against uncertainty (see for instance Camerer and Weber (1992), Halevy (2007) and Ahn et al. (2014)).

[3]Other approaches towards modeling uncertainly aversion have been proposed by Gilboa and Schmeidler (1989), Schmeidler (1989), Klibnoff, Marinacci, and Mukerji (2005) and Cerreira-Vioglio, Maccheroni, Marinacci, and Montrucchio (2009) and many more.

plank of a party's platform requires the agreement of all factions. If no compromise can be reached, the status quo rules. Roemer (1999, 2001) and Levy (2004) tell similar stories of parties as non-unitary actors. Both posit that a party prefers platform profile $(\mathbf{x}, \mathbf{y})$ over another profile $(\mathbf{x}', \mathbf{y}')$ if it is preferred from the vantage point of every faction or member of that party. However, while different leaders are set apart by different beliefs in the present model, they are set apart by different goals in the models of Roemer (1999, 2001) and Levy (2004).

Formal studies of electoral competition typically simplify by assuming a uni-dimensional issue space. The Downs-Hotelling model thereby ensures that an equilibrium always exists. If one replaces the uni- with a multi-dimensional issue space, the voters' preferences have to meet stringent conditions for a Downs-Hotelling-type model to have an equilibrium (in pure strategies). Davis, Hinich, and de Groot (1972), Grandmont (1978), and Plott (1967) all provide negative results on the existence of equilibria in multi-dimensional Downs-Hotelling games. Thanks to the assumption of multiple beliefs, the games under review in this study are not plagued by the same non-existence problems.[4] With the problem of equilibrium existence resolved by the assumption of multiple priors the usually troublesome assumption of a multi-dimensional issue space turns into a an advantage. The voters' single-peaked preferences over a multi-dimensional issue space may exhibit non-convexities and thereby allow for equilibria with substantially different platforms.

Non-convexities play a major role in my arguments. To see the plausibility of non-convex preferences over multi-dimensional issue spaces take the example of a mayor who can spend 10 to improve the opera or the museum. Consider a voter V who is indifferent between either spending 10 on the opera or on the museum. Convexity would demand that V weakly prefers spending 5 on each of the institutions. This preference does not hold if V prefers one outstanding cultural institution to two mediocre ones. Besides this - standard - argument that preferences might not be convex there is another argument specific to preferences over issue spaces that lack a natural scale of measurement. While it might easy to say which one of two foreign policies is more dovish and which one of two sets of abortion rights is more liberal, it is harder to say whether one policy is twice as dovish or liberal as another. But such statements are required in the definition of convex preferences. Since agents may disagree about the natural scale there may not be a common scale with respect to which all voters' preferences are convex (even assuming that there is such a scale for each agent).

This is not the first model of electoral competition with divergent equilibrium platforms.

[4]In Bade (2010), I described conditions on the set of party beliefs Ψ, under which games of electoral competition among uncertainty averse parties have equilibria. In that paper I used Gilboa and Schmeidler's (1989) maximin expected utilities to model party-behavior. The results of that paper transfer to the alternative assumption that party behavior follows Bewley's (2002) model.

However, as far as I am aware, the emergence of different platforms is usually derived from an assumption of an exogenous ideological allegiance of parties and/or politicians, see Wittman (1973), Osborne and Slivinski (1996), Besely and Coate (1997), and Roemer (1999). Similarly the candidates in Krasa and Polborn have some immutable characteristics that exogenously differentiate them. Another set of models goes a different route by prefacing Downsian competition among two parties by a stage in which these two competitors are selected (or threatened by the entry of a third), see Palfrey (1984) and Brusco et al. (2012).

A more complex picture arises when the assumption that parties are uncertain about voter preferences is combined with the assumption that voters need not be certain about party positions. An influential paper, Glazer (1990), shows that both parties may prefer to state ambiguous policies that leave voters uncertain about their positions if the parties are uncertain about the median voter's preferred policy position. Meirowitz (2005) and Aragones and Neeman (2000) add the candidates' desire to remain flexible as an explanation of ambiguous platforms. If one allows for both voter and party uncertainty in a classical Downs-Hotelling model, equilibrium platforms might neither converge nor diverge: parties might refuse to declare precise platforms.

2 Electoral Competition

I model political competition as a two-stage game played by two different types of actors: two political parties, and a large set of voters. First, Parties 1 and 2 simultaneously choose their platforms within some (non-empty) convex issue space $X \subset \mathbb{R}^n$, $n \geq 1$. Then, the voters, whose preferences are defined over that same issue space X, cast their votes. Generic elements of X are denoted $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, profiles of platforms are denoted $(\mathbf{x}, \mathbf{y})$ with the understanding that the first platform (here $\mathbf{x}$) is Party 1's. Parties credibly commit to their platforms and voters only care about platforms. In particular, no voter has any ideological attachment or bias towards either party. So there are no a priori differences between the two parties. Any differences between their equilibrium positions arise endogenously.

2.1 Voters

Voter preferences are single-peaked in the sense that each voter has some most preferred policy in the issue space and that his utility decreases as platforms move away from this ideal point. Formally, a preference $\succsim$ is **single-peaked** if there exists an **ideal point** $\mathbf{a} \in X$, such that, for any two platforms $\mathbf{x} \neq \mathbf{y} \in X$, $\min\{a_i, y_i\} \leq x_i \leq \max\{a_i, y_i\}$ for all i implies $\mathbf{x} \succ \mathbf{y}$.[5] If $\succsim$ is single-peaked then it has a unique ideal point $\mathbf{a}(\succsim)$. This ideal $\mathbf{a}(\succsim) = \mathbf{a}$ is strictly

[5] Barbera, Gul, and Stachetti (1993) propose the same definition of single-peakedness.

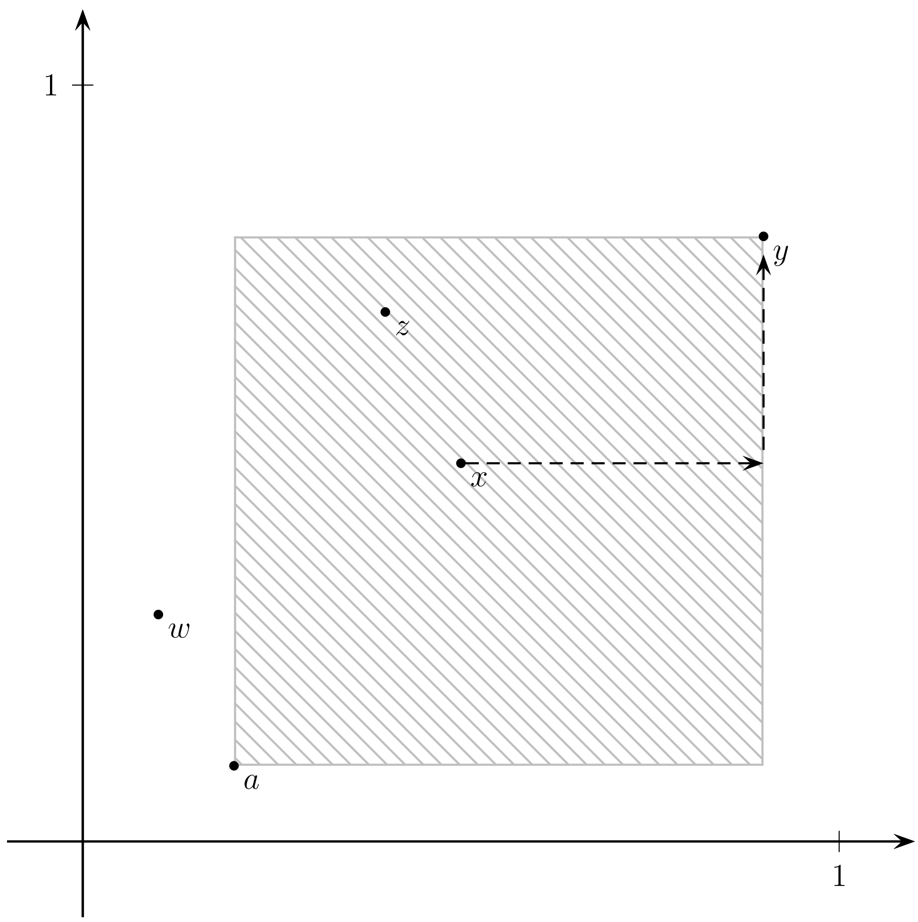

Figure 1: The Condition of single-peakedness

preferred to any other platform $\mathbf{x} \neq \mathbf{a}$ since $\min\{a_i, x_i\} \leq x_i \leq \max\{a_i, x_i\}$ trivially holds for all i. For a uni-dimensional X the present definition reduces to the standard definition of single-peakedness. Similarly a voter's preferences over a subset of platforms that differ only with respect to one component are single-peaked. If $X' = \{\mathbf{x} \in X \mid x_i = y_i \text{ for all } i \neq j\}$ for some $\mathbf{y} \in X$ and $1 \leq j \leq n$, then the restriction of $\succsim$ to X' is single-peaked. Moreover, if $\mathbf{a}(\succsim) = \mathbf{a}$ is the ideal point of $\succsim$, then a_j is the ideal point associated with the restriction of $\succsim$ to X'. So the voter's ideal point in dimension j does not depend on the policies adopted in dimensions $i \neq j$.

As an illustration consider the platforms $\mathbf{a}, \mathbf{x}, \mathbf{y}, \mathbf{w}$, and $\mathbf{z}$ in the two-dimensional issue space illustrated in Figure 1. The criterion of single-peakedness requires that a voter with the ideal point $\mathbf{a}$ prefers platforms $\mathbf{x}$ and $\mathbf{z}$ to platform $\mathbf{y}$, given that $\mathbf{x}$ and $\mathbf{z}$ lie in the hatched rectangle with $\mathbf{a}$ and $\mathbf{y}$ as its south-west and north-east corners. The two arrows indicate that $\mathbf{y}$ is more distant from $\mathbf{a}$ than $\mathbf{x}$ with respect to both issues. The same holds for the relation between $\mathbf{z}$ and $\mathbf{y}$. Conversely, $\mathbf{z}$ and $\mathbf{x}$ are not ranked by the criterion, as $a_1 < z_1 < x_1$ and $a_2 < x_2 < z_2$. Finally, the criterion does not rank $\mathbf{w}$ with respect to $\mathbf{x}, \mathbf{y}$

and $\mathbf{z}$, since $w_1 < a_1 < y_1, x_1, z_1$.

Figure 2 displays the same platforms $\mathbf{x}$, $\mathbf{y}$, $\mathbf{w}$, and $\mathbf{z}$ together with two indifference curves corresponding to different single-peaked preferences with ideal point $\mathbf{a}$. Euclidean preferences measure the disutility of a platform $\mathbf{x}$ as the Euclidean distance between the ideal point $\mathbf{a} \in X$ and $\mathbf{x}$. When X is two-dimensional, Euclidean preferences with ideal point $\mathbf{a}$ are represented by $v^a : X \to \mathbb{R}$ with $v^a(\mathbf{x}) = -(x_1 - a_1)^2 - (x_2 - a_2)^2$. See the circle in Figure 2 for a sample indifference curve. Such an agent prefers $\mathbf{w}$ to $\mathbf{x}$, which he, in turn, prefers to $\mathbf{z}$. Next, consider the preference that is represented by $u^a : X \to \mathbb{R}$ with $u^a(\mathbf{x}) = -\sqrt{|x_1 - a_1|} - \sqrt{|x_2 - a_2|}$, for which the dashed curvy star is an indifference curve. These preferences are not convex; the agent strictly prefers both $\mathbf{s}$ and $\mathbf{t}$ to the intermediate platform $\frac{1}{2}\mathbf{t} + \frac{1}{2}\mathbf{s}$. To see that the convexity of preferences depends on the scaling of the axes, consider the preferences that are represented by $u^{(0,0)}$. If one rescales both axes with the strictly monotonic function $t \mapsto t^4$ these preferences are represented by $v^{(0,0)}$.

An electorate is a distribution ψ on the set of single-peaked preferences over X, with both a finite and an infinite support allowed. Any voter who strictly prefers one of the two parties' platforms votes for his preferred platform. Indifferent agents vote for each party with probability one half. Party 1's **vote share** at the platform profile $(\mathbf{x}, \mathbf{y})$ and some given electorate ψ is denoted $\pi_\psi(\mathbf{x}, \mathbf{y}) = \psi(\succsim \mid \mathbf{x} \succ \mathbf{y}) + \frac{1}{2}\psi(\succsim \mid \mathbf{x} \sim \mathbf{y})$.

2.2 Parties

Each party chooses its platform in X to maximize its vote share. Parties take a set of electorates Ψ into account to evaluate different platform profiles. A party (weakly) prefers some profile $(\mathbf{x}, \mathbf{y})$ over another profile $(\mathbf{x}', \mathbf{y}')$ if and only if for every $\psi \in \Psi$ its vote share under $(\mathbf{x}, \mathbf{y})$ is at least as great as its vote share under $(\mathbf{x}', \mathbf{y}')$. If a party weakly prefers $(\mathbf{x}, \mathbf{y})$ to $(\mathbf{x}', \mathbf{y}')$ and $(\mathbf{x}', \mathbf{y}')$ to $(\mathbf{x}, \mathbf{y})$ then the party is indifferent between $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}', \mathbf{y}')$. In that case $\pi_\psi(\mathbf{x}, \mathbf{y}) = \pi_\psi(\mathbf{x}', \mathbf{y}')$ holds for all $\psi \in \Psi$. A party strictly prefers $(\mathbf{x}, \mathbf{y})$ to $(\mathbf{x}', \mathbf{y}')$ if it weakly prefers $(\mathbf{x}, \mathbf{y})$ to $(\mathbf{x}', \mathbf{y}')$ and if $\pi_\psi(\mathbf{x}, \mathbf{y})$ differs from $\pi_\psi(\mathbf{x}', \mathbf{y}')$ for some $\psi \in \Psi$. If there exist two priors $\psi^*, \psi' \in \Psi$ such that $\pi_{\psi'}(\mathbf{x}, \mathbf{y}) > \pi_{\psi'}(\mathbf{x}', \mathbf{y}')$ and $\pi_{\psi^*}(\mathbf{x}, \mathbf{y}) < \pi_{\psi^*}(\mathbf{x}', \mathbf{y}')$ neither party ranks $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}'\mathbf{y}')$. The standard case of vote-share-maximizing parties obtains if Ψ is a singleton $\{\psi\}$.

The simplest approach to embed these preferences in Bewley's multiple prior model would be to assume that the goal of each party is to maximize the probability that a randomly drawn voter votes for that party: with the electorate ψ and the profile $(\mathbf{x}, \mathbf{y})$ the probability of a vote for party 1 is $\pi_\psi(\mathbf{x}, \mathbf{y})$. But it is also possible to construct a tighter link between the introductory story of vote share maximizing parties and the present model of party preferences. To do so assume that a party prefers profiles with higher vote shares and holds a set of priors on electorates. So parties neither know the electorate nor do they subscribe

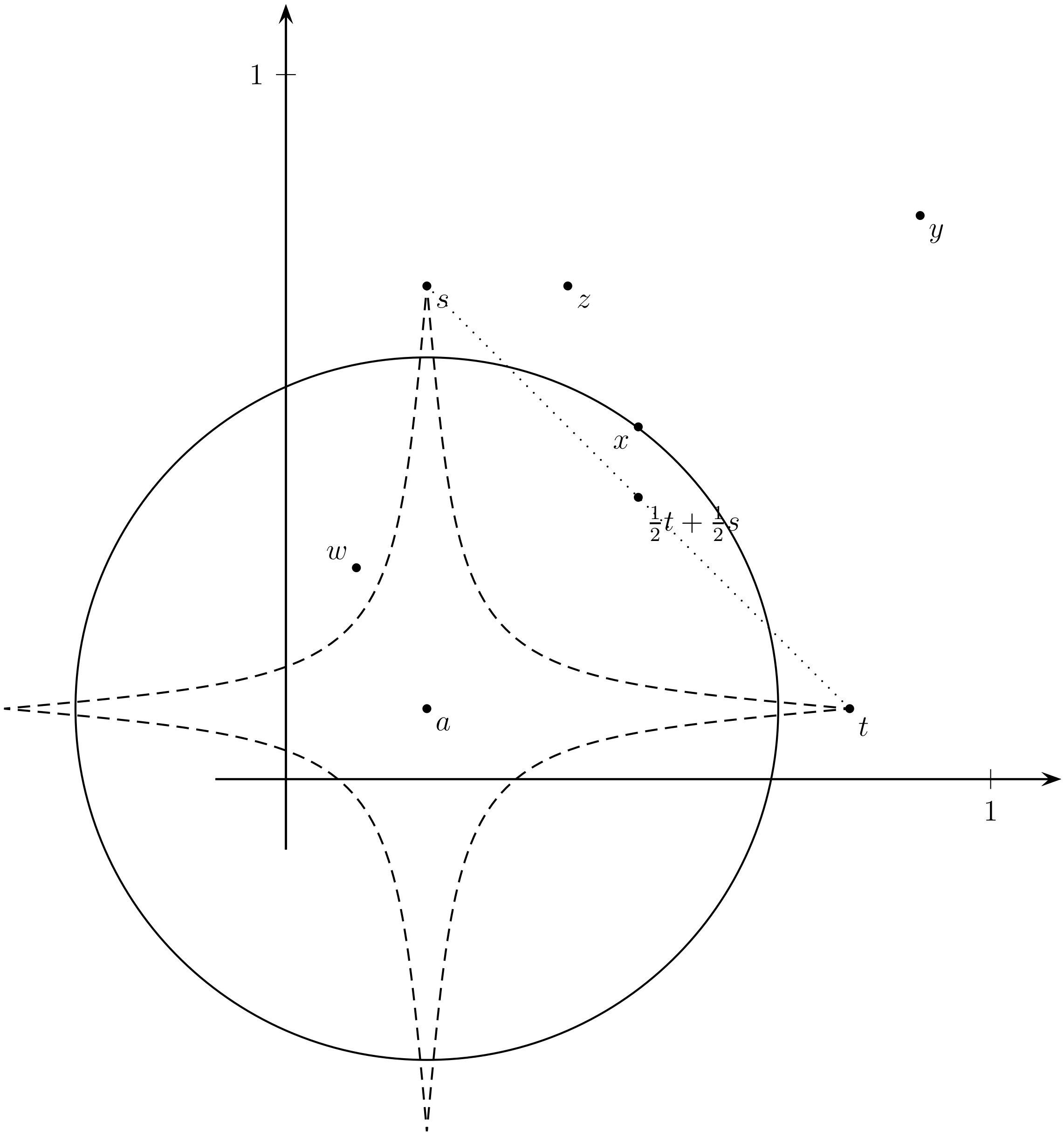

Figure 2: Examples of Single-peaked Preferences

to a fixed prior on electorates. Instead a party prefers $(\mathbf{x}, \mathbf{y})$ to $(\mathbf{x}', \mathbf{y}')$ if it obtains a higher expected vote share according to every prior in its set of priors. For any given prior p over electorates the expected vote share $E(\pi_\psi(\mathbf{x}, \mathbf{y}))$ equals the vote share according to the "expected" electorate ψ' defined by $\psi'(S) = \int \psi(S) dp(\psi)$ for any set S of preferences on X: $E(\pi_\psi(\mathbf{x}, \mathbf{y})) = \pi_{\psi'}(\mathbf{x}, \mathbf{y})$.[6] Formally speaking the "expected" electorate ψ' is itself an electorate and the calculation of expected vote shares $E(\pi_\psi(\mathbf{x}, \mathbf{y}))$ for a set of party priors on electorates is equivalent to the calculation of vote shares for a set of (fictitious) electorates. The model of party behavior postulated above arises when we interpret Ψ as such a set of fictitious electorates.

Alternatively the model of party preferences can be derived from a theory in which different factions within a party share the goal of expected vote share maximization but subscribe to different beliefs on electorates. In this case any element of Ψ represents an expected electorate arising out of some factions' prior on electorates.

I interchangeably refer to the elements of Ψ as electorates or priors. Without any formal difference between electorates and "expected electorates" it makes sense to call Ψ simply a set of electorates. On the other hand, since Ψ can be derived from a set of priors on electorates, I also refer to Ψ as a set of priors or beliefs.

To strengthen the argument that divergence arises endogenously, I assume that both parties subscribe to the same set of priors Ψ. The results can easily be modified to comprise games in which parties hold different sets of priors.

2.3 The Game of Platform Positioning

A two-player game of electoral competition is a triple (n, X, Ψ), where n is the dimension of the issue space X and Ψ is the set of priors. The two parties are the players, the issue space X is the strategy space of each party, and the parties' preferences are incomplete as described in the preceding section. Standard Downsian games are embedded in the present framework. They are the games with uni-dimensional issue spaces and singleton priors: $(1, X, \{\psi\})$.

A platform profile $(\mathbf{x}, \mathbf{y})$ is a equilibrium of the game (n, X, Ψ) if and only if no party strictly prefers changing its platform given that the other keeps its platform. The platform $\mathbf{x}$ is a best reply to $\mathbf{y}$ for Party 1 if for any $\mathbf{z} \neq \mathbf{x}$ we either have $\pi_\psi(\mathbf{z}, \mathbf{y}) = \pi_\psi(\mathbf{x}, \mathbf{y})$ for all $\psi \in \Psi$ or $\pi_{\psi'}(\mathbf{z}, \mathbf{y}) < \pi_{\psi'}(\mathbf{x}, \mathbf{y})$ for some $\psi' \in \Psi$. Under complete preferences $\mathbf{x}$ is a best reply to $\mathbf{y}$ for Party 1 if and only if Party 1 weakly prefers $(\mathbf{x}, \mathbf{y})$ to any $(\mathbf{z}, \mathbf{y})$. With incomplete preferences this equivalence between best replies and preferred actions does not hold. For $\mathbf{x}$ to be a best reply to $\mathbf{y}$, $(\mathbf{z}, \mathbf{y})$ and $(\mathbf{x}, \mathbf{y})$ can be unranked.[7]

[6] For a proof of this claim see Bade (2010).

[7] In Bade (2005), I provide a characterization of equilibrium sets of such games with incomplete preferences.

As Ψ increases, party preferences become more incomplete: If a party cannot rank two platform profiles when it holds the set of beliefs Ψ, then it also cannot rank these profiles when it subscribes to a larger set of beliefs $\Psi' \supset \Psi$. So one might expect that the equilibrium set of (n, X, Ψ) is increasing in Ψ. Indeed for any two electoral games (n, X, Ψ) and (n, X, Ψ') with $\Psi \subset \Psi'$, an equilibrium $(\mathbf{x}, \mathbf{y})$ of (n, X, Ψ) is also an equilibrium of (n, X, Ψ') if there is no $\mathbf{z}$ such that either ($\mathbf{z} \neq \mathbf{x}$ and $\pi_\psi(\mathbf{z}, \mathbf{y}) = \pi_\psi(\mathbf{x}, \mathbf{y})$ for all $\psi \in \Psi$) or ($\mathbf{z} \neq \mathbf{y}$ and $\pi_\psi(\mathbf{x}, \mathbf{z}) = \pi_\psi(\mathbf{x}, \mathbf{y})$ for all $\psi \in \Psi$). If some party had an indifferent deviation $\mathbf{z}$ under Ψ, this party might strictly prefer $\mathbf{z}$ under the larger set of beliefs Ψ'.[8]

2.4 Divergence

Even classical Downsian games can have equilibria with different platforms; any profile $(\mathbf{x}, \mathbf{y})$ is an equilibrium of $(1, X, \{\psi\})$ if both $\mathbf{x}$ and $\mathbf{y}$ are medians of the distribution of voter ideal points. To avoid such uninteresting cases, I consider a profile of platforms $(\mathbf{x}, \mathbf{y})$ to be divergent if some voter ideal points lie between $\mathbf{x}$ and $\mathbf{y}$. Let

$$\lfloor \mathbf{x}, \mathbf{y} \rceil := \{\mathbf{z} \in X \mid \min(x_i, y_i) < z_i < \max(x_i, y_i) \text{ for all } i\}$$

be the open rectangle delimited by $\mathbf{x}$ and $\mathbf{y}$ and say that platform $\mathbf{z}$ **lies between** the two platforms $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{z} \in \lfloor \mathbf{x}, \mathbf{y} \rceil$.

Definition: A profile $(\mathbf{x}, \mathbf{y})$ is **divergent** if $\psi(\{\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil\}) > 0$ for all $\psi \in \Psi$.

A profile $(\mathbf{x}, \mathbf{y})$ that is not divergent is **convergent**. This notion of divergence is robust to any rescaling of the issue space. While the Euclidean distance between two platforms $\mathbf{x}$ and $\mathbf{y}$ changes when one uses different scales on the axis of the issue space, the probability $\psi(\{\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil\})$ stays constant.

[8]Suppose $(\mathbf{x}, \mathbf{y})$ is an equilibrium of (n, X, Ψ) and there exists no $\mathbf{z}$ such that either ($\mathbf{z} \neq \mathbf{x}$ and $\pi_\psi(\mathbf{z}, \mathbf{y}) = \pi_\psi(\mathbf{x}, \mathbf{y})$ for all $\psi \in \Psi$) or ($\mathbf{z} \neq \mathbf{y}$ and $\pi_\psi(\mathbf{x}, \mathbf{z}) = \pi_\psi(\mathbf{x}, \mathbf{y})$ for all $\psi \in \Psi$). So for any $\mathbf{z} \neq \mathbf{x}$ there is a prior ψ^* such that $\pi_{\psi^*}(\mathbf{z}, \mathbf{y}) < \pi_{\psi^*}(\mathbf{x}, \mathbf{y})$. Since $\Psi \subset \Psi'$ we have that for all $\mathbf{z} \neq \mathbf{x}$ there exists some $\psi^* \in \Psi \subset \Psi^*$ such that $\pi_{\psi^*}(\mathbf{z}, \mathbf{y}) < \pi_{\psi^*}(\mathbf{x}, \mathbf{y})$. So $\mathbf{x}$ is a best reply to $\mathbf{y}$ for Party 1 under (n, X, Ψ'). By the same logic $\mathbf{y}$ is a best reply to $\mathbf{x}$ under (n, X, Ψ') and $(\mathbf{x}, \mathbf{y})$ is an equilibrium of (n, X, Ψ'). To see that the conclusion need not hold if for some $\mathbf{z} \neq \mathbf{x}$, $\pi_\psi(\mathbf{x}, \mathbf{y}) = \pi_\psi(\mathbf{z}, \mathbf{y})$ holds for all $\psi \in \Psi$ consider the example $(1, [0, 1], \{\psi^*\})$ where ψ^* is an electorate with two voters who respectively have their ideal points at 0 and 1. The profile $(0, 1)$ is an equilibrium of $(1, [0, 1], \{\psi^*\})$ and we have $\pi_{\psi^*}(0, 1) = \pi_{\psi^*}(1, 1) = \frac{1}{2}$. Now consider $(1, [0, 1], \{\psi^*, \psi'\})$ where all voters have their ideal point at .5 according to ψ'. The profile $(0, 1)$ is not an equilibrium in $(1, [0, 1], \{\psi^*, \psi'\})$.

3 Essential Uniqueness and Robustness

The symmetry of games of electoral competition implies that $(\mathbf{x},\mathbf{y})$ is an equilibrium of (n,X,Ψ) if and only if $(\mathbf{y},\mathbf{x})$ is an equilibrium of (n,X,Ψ). A game (n,X,Ψ) with a divergent equilibrium therefore has at least two equilibria. An equilibrium $(\mathbf{x},\mathbf{y})$ of some game (n,X,Ψ) is **essentially unique** if $(\mathbf{y},\mathbf{x})$ is the only other equilibrium of (n,X,Ψ).

Predictions should not be flukes: they should hold for broad sets of parameters and small perturbations of the parameters of a model should have only a small impact on its equilibria. To define a robust equilibrium fix a game (n,X,Ψ), a set Ψ' of priors on X, and a function $f:\Psi\to\Psi'$. Let $\Psi^{f,\epsilon}:=\{\psi' \mid \psi'=(1-\epsilon)\psi+\epsilon f(\psi) \text{ for some } \psi\in\Psi\}$ and note that the set of priors $\Psi^{f,\epsilon}$ continuously approaches $\Psi^{f,0}=\Psi$ as ϵ goes to 0. As ϵ goes to 0 every prior $\psi\in\Psi$ more resembles its "sister" $(1-\epsilon)\psi+\epsilon f(\psi)\in\Psi^{f,\epsilon}$. Define the difference between two platform profiles $(\mathbf{x},\mathbf{y})$ and $(\mathbf{x}',\mathbf{y}')$ as $d^*\big((\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}')\big):=d(\mathbf{x},\mathbf{x}')+d(\mathbf{y},\mathbf{y}')$ where d is the Euclidean distance. An equilibrium $(\mathbf{x},\mathbf{y})$ of some game (n,X,Ψ) is **robust** if for every $\delta>0$, every set of electorates Ψ' and every function $f:\Psi\to\Psi'$, there exists an $\overline{\epsilon}>0$ such that $(n,X,\Psi^{f,\epsilon})$ for $\epsilon<\overline{\epsilon}$ has an equilibrium $(\mathbf{x}',\mathbf{y}')$ such that $d^*\big((\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}')\big)<\delta$.

In Theorem 1 I show not only that some games of electoral competition (n,X,Ψ) have divergent equilibria but also that these equilibria can be essentially unique and robust. Divergence is therefore not a knife-edge result. Moreover divergence does not arise as a side effect of large equilibrium sets.

4 The Existence of Divergent Equilibrium

Multi-dimensional games with uncertainty averse parties can have divergent equilibria. The set of equilibria generally increases with the set of beliefs Ψ. To establish the existence of divergent equilibria, one could try to assume a large set Ψ and hope to find some divergent ones among the many equilibria. Theorem 1 goes a different route: I show that even games (n,X,Ψ) with only two priors ($\mid\Psi\mid=2$) can have an equilibrium that is divergent and essentially unique. This is not a knife edge result: small perturbations of the model I construct to prove Theorem 1 have the same equilibrium.

Theorem 1: *A game of electoral competition (n,X,Ψ) with only two priors can have an equilibrium that is divergent, robust and essentially unique.*

Proof: Take the game $(2, [0,1]^2, \Psi)$ with $\Psi = \{\psi^1, \psi^2\}$ defined by

$$\begin{aligned}
\psi^1(\{\succsim \mid \mathbf{a}(\succsim) = (0,0)\}) &= \psi^2(\{\succsim \mid \mathbf{a}(\succsim) = (1,1)\}) = 0.25, \\
\psi^1(\{\succsim \mid \mathbf{a}(\succsim) = (1,1)\}) &= \psi^2(\{\succsim \mid \ \mathbf{a}(\succsim) = (0,0)\}) = 0.05, \\
\psi^1(\{\succsim \mid \mathbf{a}(\succsim) = (1,0)\}) &= \psi^2(\{\succsim \mid \mathbf{a}(\succsim) = (0,1)\}) = 0.15, \\
\psi^1(\{\succsim \mid \mathbf{a}(\succsim) = (0,1)\}) &= \psi^2(\{\succsim \mid \mathbf{a}(\succsim) = (1,0)\}) = 0.45, \\
\psi^i(\{\succsim \mid \mathbf{a}(\succsim) = (.5,.5)\}) &= .1 \text{ for } i = 1,2.
\end{aligned}$$

Let the preferences of voters with ideal points $(1,0)$ and $(0,1)$ be represented by the non-convex utility functions u^a, as defined in Section 2.1. All other voters have Euclidean preferences. Figure 3 illustrates the electorate. The dashed lines represent the voters' indifference curves of the platforms $(0,0)$ and $(1,1)$. The fractions next to the ideal points denote the probability of voters with the respective ideal points, where the label $0.15 + [.3]_1$ of point $(0,1)$ means that under either distribution, at least 0.15 of the electorate is expected to have their ideal point at $(0,1)$; under distribution ψ^1, the probability mass of voters with ideal point $(0,1)$ increases to $0.15 + 0.3$. All voters with ideal points at $(0,1)$ and $(1,0)$ and $(.5,.5)$ are indifferent between the "extreme policies" $(0,0)$ and $(1,1)$. The preferences of voters with their ideal point at $(0,1)$ and $(1,0)$ are not convex; these voters prefer platforms $(0,0)$ and $(1,1)$ to any of the intermediate policies (x,x) with $x \in (0,1)$. The hatched area represents the set of platforms that voters with ideal point $(0,1)$ prefer to platforms $(0,0)$ and $(1,1)$.

Claim 1: $((0,0);(1,1))$ is an equilibrium of $(2, [0,1]^2, \{\psi^1, \psi^2\})$.

Under $((0,0);(1,1))$ Party 1 obtains the vote shares $\pi_{\psi^1}((0,0),(1,1)) = 0.25 + \frac{1}{2}(0.1 + 0.15 + .45) = 0.6$ and $\pi_{\psi^2}((0,0),(1,1)) = 0.05 + \frac{1}{2}(0.1 + 0.15 + .45) = 0.4$ according to ψ^1 and ψ^2 respectively. To see that $(0,0)$ is a best reply to $(1,1)$, I show that for any deviation $\mathbf{x} \neq (0,0)$ Party 1 decreases its vote share according to ψ^1 or ψ^2 (or both).

Since $\pi_{\psi^1}((0,0),(1,1)) = 0.6 > 0.5 = \pi_{\psi^1}((1,1),(1,1))$ Party 1 would reduce its vote share according to ψ^1 if it was to deviate to $\mathbf{x} = (1,1)$.

For any $\mathbf{x} \notin \{(0,0),(1,1)\}$, either voters with their ideal point at $(0,1)$ or the voters with their ideal point at $(1,0)$ (or both) strictly prefer $(1,1)$ to $\mathbf{x}$. Suppose there was a platform $\mathbf{x}$ such that voters with one of these two ideal points strictly prefer $\mathbf{x}$ to $(1,1)$ while voters with the other ideal point at least weakly prefer $\mathbf{x}$ to $(1,1)$. Such a platform $\mathbf{x}$ would have to satisfy

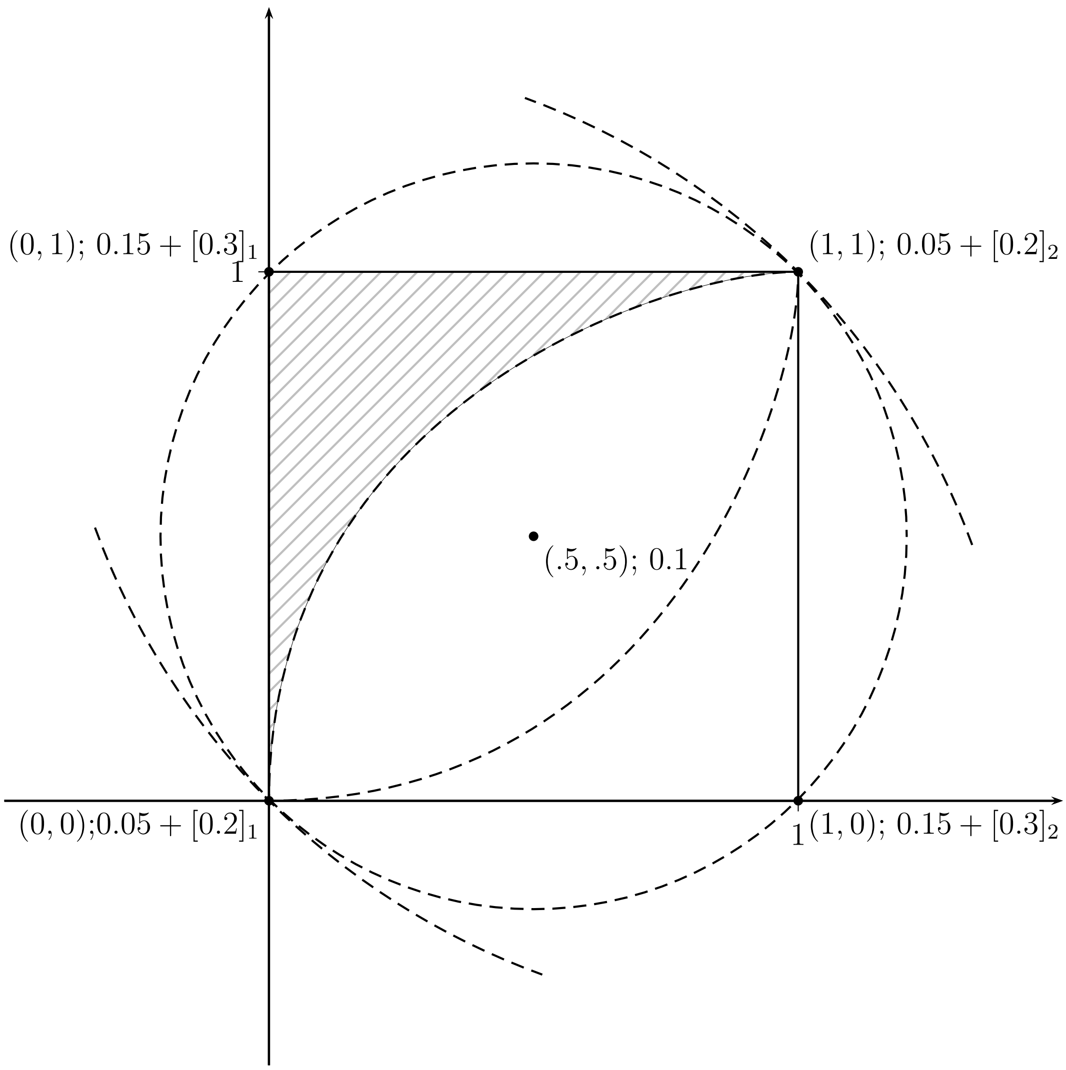

Figure 3: The Electorate

$$-\sqrt{|x_1-1|}-\sqrt{|x_2-0|} \geq -\sqrt{|1-1|}-\sqrt{|1-0|} \Leftrightarrow |x_1-1|+2\sqrt{|x_1-1|\,|x_2|}+|x_2| \leq 1$$

and

$$-\sqrt{|x_1-0|}-\sqrt{|x_2-1|} \geq -\sqrt{|1-0|}-\sqrt{|1-1|} \Leftrightarrow |x_1|+2\sqrt{|x_1|\,|x_2-1|}+|x_2-1| \leq 1$$

with at least one of the two inequalities holding strictly Adding both inequalities and taking into account that at least one holds strictly we obtain the contradiction

$$2\sqrt{|x_1-1|\,|x_2|}+2\sqrt{|x_1|\,|x_2-1|} < 2-|x_1-1|-|x_1|-|x_2|-|x_2-1| = 0.$$

Voters with ideal point $(1,1)$ vote for Party 2's platform $(1,1)$ when Party 1 runs on any $\mathbf{x} \neq (1,1)$. Party 1 can therefore at best either gain the support of the voters with ideal points $(0,1)$, $(.5,.5)$ and $(0,0)$ or of the voters with the ideal points $(0,1)$, $(.5,.5)$ and $(0,0)$. If $\mathbf{x}$ obtains the favor of (a subset of) voters with ideal points $(0,1)$, $(.5,.5)$ and $(0,0)$ we have

$$\pi_{\psi^2}(\mathbf{x},(1,1)) \leq \psi^2(\succsim|\ \mathbf{a}(\succsim) \in \{(0,1),(.5,.5),(0,0)\}) = 0.15+0.1+0.05 = .3$$

which is smaller than $\pi_{\psi^2}((0,0),(1,1)) = .4$, and Party 1 does not prefer to deviate to $\mathbf{x}$. Similarly, Party 1 does not prefer to deviate to some $\mathbf{x}$ that would win the favor of (a subset of) voters with ideal points at $(1,0)$, $(.5,.5)$ and $(0,0)$, since

$$\psi^1(\succsim|\ \mathbf{a}(\succsim) \in \{(1,0),(.5,.5),(0,0)\}) = 0.15+0.1+0.25 = .5$$

whereas $\pi_{\psi^1}((0,0),(1,1)) = .6$. So $(0,0)$ is indeed a best reply to $(1,1)$. Due to the symmetry of the example $(1,1)$ is a also best reply to $(0,0)$, and consequently $((0,0),(1,1))$ is an equilibrium.

Claim 2: $((0,0);(1,1))$ is divergent in $(2,[0,1]^2,\{\psi^1,\psi^2\})$.

Claim 2 follows from $(.5,.5) \in \lfloor(0,0),(1,1)\rceil$ and $\psi^i(\{\succsim|\ \mathbf{a}(\succsim) = (.5,.5)\}) = .1 > 0$ for both $i=1$ and $i=2$.

Claim 3: $((0,0);(1,1))$ is a robust equilibrium of $(2,[0,1]^2,\{\psi^1,\psi^2\})$.

Fix $\delta > 0$ and any two distributions ρ^1,ρ^2 on $[0,1]^2$. For any $\epsilon \geq 0$ define $(2,[0,1]^2,\Psi^\epsilon)$ with $\Psi^\epsilon = \{(1-\epsilon)\psi^1+\epsilon\rho^1,(1-\epsilon)\psi^2+\epsilon\rho^2\}$.[9] Since all inequalities in the proof that

[9]Given that $\{\psi^1,\psi^2\}$ has only two elements it is without loss of generality to assume that $f(\psi^1) = \rho^1$ and $f(\psi^2) = \rho^2$ for some ρ^1,ρ^2. I therefore dropped f from the description of the perturbed set of beliefs.

$((0,0);(1,1))$ is an equilibrium of $(2,[0,1]^2,\Psi)$ hold by a margin of 0.1 the same inequalities are also satisfied in $(2,[0,1]^2,\Psi^\epsilon)$ for $\epsilon \leq 0.01$. So $((0,0);(1,1))$ is an equilibrium of $(2,[0,1]^2,\Psi^\epsilon)$ for any $\epsilon \leq 0.01$, and $((0,0);(1,1))$ is a robust equilibrium of $(2,[0,1]^2,\{\psi^1,\psi^2\})$.

Claim 4: $((0,0);(1,1))$ is an essentially unique equilibrium of $(2,[0,1]^2,\{\psi^1,\psi^2\})$.

Assume that some other equilibrium $(\mathbf{x},\mathbf{y}) \notin \{((0,0),(1,1))((1,1),(0,0))\}$ existed.

Case 1. $\mathbf{y} \in (0,1)^2$. Assume w.l.o.g that $y_1 + y_2 \leq 1$ (due to the symmetry of $(2,[0,1]^2,\{\psi^1,\psi^2\})$ the case $y_1 + y_2 \geq 1$ is covered by the same arguments mutatis mutandis).

Case 1.1 $\mathbf{x} = \mathbf{y}$. Define $\epsilon > 0$ such that $(y_1 - \epsilon, y_2 - \epsilon) \in [0,1]^2$. Voters with ideal point $(0,1)$ prefer this new platform to $\mathbf{y}$ as we have

$$
\begin{aligned}
u^{(0,1)}(\mathbf{y}) < u^{(0,1)}(y_1-\epsilon,y_2-\epsilon) &\Leftrightarrow \\
-\sqrt{y_1} - \sqrt{1-y_2} < -\sqrt{y_1-\epsilon} - \sqrt{1-(y_2-\epsilon)} &\Leftrightarrow \\
y_1 + 1 - y_2 + 2\sqrt{y_1(1-y_2)} > y_1 - \epsilon + 1 - y_2 + \epsilon + 2\sqrt{(y_1-\epsilon)(1-y_2+\epsilon)} &\Leftrightarrow \\
\epsilon(\epsilon + 1 - y_2 - y_1) > 0&,
\end{aligned}
$$

where the last line holds since $y_1 + y_2 \leq 1$. A similar calculation shows that voters with ideal point $(1,0)$ also prefer $(y_1-\epsilon,y_2-\epsilon)$ to $\mathbf{y}$. Voters with ideal point at $(0,0)$ prefer $(y_1-\epsilon,y_2-\epsilon)$ to $\mathbf{y}$ by single-peakedness. In sum, Party 1's vote share at $((y_1-\epsilon,y_2-\epsilon),\mathbf{y})$ according to ψ^i is at least $\psi^i(\{\succsim \mid \mathbf{a}(\succsim) \in \{(0,0),(1,0),(0,1)\}\})$. Since voters with ideal points $(.5,.5)$ and $(1,1)$ prefer $\mathbf{y}$ to $(y_1-\epsilon,y_2-\epsilon)$ by single-peakedness, Party 1's vote share at $((y_1-\epsilon,y_2-\epsilon),\mathbf{y})$ is $0.25+0.45+0.15 = 0.85$ and $0.05+0.45+0.15 = 0.65$ according to ψ^1 and ψ^2 respectively. The deviation from $\mathbf{x}$ to $(y_1-\epsilon,y_2-\epsilon)$ increases Party 1's vote share according to both priors and $(\mathbf{x},\mathbf{y})$ cannot be an equilibrium.

Case 1.2. $\mathbf{x} \neq \mathbf{y}$. Since the agents with ideal points $(0,1)$ and $(1,0)$ have preferences that are symmetric around the line $\{\mathbf{z} \mid z_1 + z_2 = 1\}$, these agents are indifferent between $(y_1-\epsilon,y_2-\epsilon)$, as defined in Case 1.1, and $(1-y_2+\epsilon,1-y_1+\epsilon)$.[10] Voters with ideal point at $(1,1)$ prefer $(1-y_2+\epsilon,1-y_1+\epsilon)$ to $\mathbf{y}$ by single-peakedness. By the symmetry of $(2,[0,1]^2,\{\psi^1,\psi^2\})$, Party 1's vote share at $((1-y_2+\epsilon,1-y_1+\epsilon),\mathbf{y})$ is 0.65 and 0.85 according to ψ^1 and ψ^2 respectively.

For $(\mathbf{x},\mathbf{y})$ to be an equilibrium, Party 2 must obtains at least half the vote according to one prior ψ^i, otherwise it should to deviate to $\mathbf{x}$. For Party 1 not to have an incentive to deviate to $(y_1-\epsilon,y_2-\epsilon)$ or $(1-y_2+\epsilon,1-y_1+\epsilon)$, Party 1 must obtain more than .85 of the vote according to the other prior ψ^j. In sum $(\mathbf{x},\mathbf{y})$ must be such that Party 1 $\pi_{\psi^i}(\mathbf{x},\mathbf{y}) < 0.5$ and

[10] Since ϵ was chosen such that $y_i - \epsilon \geq 0$ for $i = 1,2$ we have $(1-y_2+\epsilon,1-y_1+\epsilon) \in [0,1]^2$.

$\pi_{\psi^j}(\mathbf{x},\mathbf{y}) > .85$ with $\{i,j\} = \{1,2\}$. But there is no $S \subset \{(0,0),(0,1),(1,0),(1,1),(0.5,0.5)\}$ such that $\psi^i(\{\succsim |\ \mathbf{a}(\succsim) \in S\}) < 0.5$ and $\psi^j(\{\succsim |\ \mathbf{a}(\succsim) \in S\}) \geq 0.85$ for $\{i,j\} = \{1,2\}$ implying that there is no equilibrium $(\mathbf{x},\mathbf{y})$ with $\mathbf{x} \neq \mathbf{y} \in (0,1)^2$.

Case 2. $\mathbf{y} \notin (0,1)^2 \cup \{(0,0),(1,1)\}$. Assume w.l.o.g that $y_1 = 0 < y_2$ (due to the symmetry of $(2,[0,1]^2,\{\psi^1,\psi^2\})$ all arguments pertaining the present case apply mutatis mutandis to the alternative cases that $y_2 = 0 < y_1$, $y_1 = 1 > y_2$, and $y_2 = 1 > y_1$).

Case 2.1. $\mathbf{x} = \mathbf{y}$. Consider the platform $(\epsilon, y_2 - \epsilon) \in [0,1]^2$ with $\frac{y_2}{2}\epsilon > 0$ and observe that voters with ideal points at $(0,0)$ and $(1,1)$ and $(.5,.5)$ prefer the platform $(\epsilon, y_2-\epsilon)$ to $\mathbf{y}$ as

$$\begin{aligned} v^{(\alpha,\alpha)}(\mathbf{y}) < v^{(\alpha,\alpha)}(\epsilon, y_2-\epsilon) &\Leftrightarrow \\ -\alpha^2 - (\alpha - y_2)^2 < -(\alpha-\epsilon)^2 - (\alpha - y_2 + \epsilon)^2 &\Leftrightarrow \\ -\alpha^2 - (\alpha - y_2)^2 < -\alpha^2 + 2\epsilon\alpha - \epsilon^2 - (\alpha - y_2)^2 - 2\epsilon(\alpha - y_2) - \epsilon^2 &\Leftrightarrow \\ 0 < 2\epsilon\alpha - 2\epsilon^2 - 2\epsilon(\alpha - y_2) \Leftrightarrow 0 < \epsilon(y_2 - \epsilon)&. \end{aligned}$$

Single-peakedness implies that voters with ideal point $(1,0)$ prefer $(\epsilon, y_2-\epsilon)$ to $\mathbf{y}$ and voters with ideal point $(0,1)$ hold the opposite preference. Party 1's vote share at $((\epsilon, y_2-\epsilon),\mathbf{y})$ is $\psi^i(\{\succsim\colon \mathbf{a}(\succsim) \in \{(0,0),(1,0),(1,1),(.5,.5)\}\}) > \frac{1}{2}$ for $i = 1,2$ and $\mathbf{x} \neq \mathbf{y}$ must hold in equilibrium $(\mathbf{x},\mathbf{y})$.

Case 2.2. $\mathbf{x} \neq \mathbf{y}$, and voters with ideal point $(0,1)$ strictly prefer one of the two platforms. The analysis of Case 1 implies that also $\mathbf{x}$ is on the boundary of $[0,1]^2$. Assume that $\mathbf{y}$ is the preferred platform for voters with their ideal point at $(0,1)$. Case 2.1 implies that all other voters prefer $(\epsilon, y_2-\epsilon)$ to $\mathbf{y}$. In the conjectured equilibrium $(\mathbf{x},\mathbf{y})$ all voters who don't have their ideal point at $(0,1)$ must prefer $\mathbf{x}$ to $\mathbf{y}$ (otherwise $(\epsilon, y_2-\epsilon)$ would be a better reply to $\mathbf{y}$). So $\pi_{\psi^i}(\mathbf{x},\mathbf{y}) = \psi^i(\{\succsim |\ \mathbf{a}(\succsim) \in \{(0,0),(1,0),(1,1),(.5,.5)\}\})$ must hold for $i = 1,2$. Since $\psi^i(\{\succsim |\ \mathbf{a}(\succsim) \in \{(0,0),(1,0),(1,1),(.5,.5)\}\}) > \frac{1}{2}$ holds for $i = 1,2$, $(\mathbf{x},\mathbf{y})$ cannot be an equilibrium - Party 2 would be better off to adopt platform $\mathbf{x}$.

Case 2.3 $\mathbf{x} \neq \mathbf{y}$ and voters with ideal point $(0,1)$ are indifferent between $\mathbf{x}$ and $\mathbf{y}$, which means that $\mathbf{x}$ equals $(1-y_2, 1)$. Voters with their ideal point at $(1,0)$ and at $(.5,.5)$ are also indifferent. By single-peakedness voters with ideal point $(1,1)$ strictly prefer $\mathbf{x}$, whereas voters with ideal point $(0,0)$ strictly prefer $\mathbf{y}$. So Party 1's vote shares according to ψ^i for $i = 1,2$ is

$$\begin{aligned} \psi^i(\{\succsim |\ \mathbf{a}(\succsim) = (1,1)\}) + \frac{1}{2}\psi^i(\{\succsim |\ \mathbf{a}(\succsim) \in \{(0,1),(1,0),(.5,.5)\}\}) \leq \\ \psi^i(\{\succsim |\ \mathbf{a}(\succsim) \in \{(0,0),(1,0),(1,1),(.5,.5)\}\}). \end{aligned}$$

By the analysis of Case 2.2 Party 1 obtains the latter vote shares if it deviates to $(\epsilon, y_2-\epsilon)$. So $(\mathbf{x},\mathbf{y})$ cannot be an equilibrium.

Case 3. $\mathbf{x} = \mathbf{y} \in \{(0,0),(1,1)\}$. Assume w.l.o.g that $\mathbf{y} = (0,0)$ (due to the symmetry of $(2,[0,1]^2,\{\psi^1,\psi^2\})$ all arguments pertaining the case that $\mathbf{y} = (0,0)$ apply mutatis mutandis to the alternative case that $\mathbf{y} = (1,1)$). Consider the profile $((0,.5),(0,0))$. Voters with ideal points $(0,1),(.5,.5)$ and $(1,1)$ all prefer $(0,.5)$ to $(0,0)$, the remaining voters have the inverse preference - all by single-peakedness. Now, observe that $\psi^1(\{\succsim: \mathbf{a}(\succsim) \in \{(0,1)(.5,.5),(1,1)\}\}) = 0.45+0.1+0.05 = 0.6$ and $\psi^2(\{\succsim: \mathbf{a}(\succsim) \in \{(0,1)(.5,.5),(1,1)\}\}) = 0.15+0.1+0.25 = 0.5$. Therefore, $(0,.5)$ is a better reply for Party 1 and $((0,0),(0,0))$ cannot be an equilibrium. ■

The notion of divergence is already quite strong in that a divergent profile $(\mathbf{x},\mathbf{y})$ must have $\psi(\{\succsim|\ \mathbf{a}(\succsim) \in \lfloor\mathbf{x},\mathbf{y}\rceil\}) > 0$ for every $\psi \in \Psi$. But Theorem 1 can be strengthened further. To do so identify the area between $\mathbf{x}$ and $\mathbf{y}$ with some subset $A(\mathbf{x},\mathbf{y})$ of $\lfloor\mathbf{x},\mathbf{y}\rceil$ and call $(\mathbf{x},\mathbf{y})$ A-divergent if $\psi(\{\succsim|\ \mathbf{a}(\succsim) \in A(\mathbf{x},\mathbf{y})\}) > 0$ holds for all $\psi \in \Psi$. If the area $A(\mathbf{x},\mathbf{y})$ contains the line segment between $\mathbf{x}$ and $\mathbf{y}$ then Theorem 1 remains valid if we strengthen the notion of divergence to A-divergence. The game $(2,[0,1]^2,\{\psi^1,\psi^2\})$ also serves to prove the strengthened version of Theorem 1. The line segment between $(0,0)$ and $(1,1)$ contains the point $(.5,.5)$ and $\psi^i(\{\succsim|\ \mathbf{a}(\succsim) \in A(\mathbf{x},\mathbf{y})) \geq \psi^i(\{\succsim|\ \mathbf{a}(\succsim) = (.5,.5)\}) = 0.1$ holds for $i = 1,2$.

Theorem 1 continues to hold if we require that voter preferences are representable by an additively separable and symmetric utility function. The voters in the proof of Theorem 1 have such preferences, implying that this proof applies unchanged to the stronger version of Theorem 1. So divergence is driven by a single non-standard feature of voter preferences: non-convexities.

The robustness of the equilibrium in the proof of Theorem 1 is striking when one contrasts it with the case of multi-dimensional games with a known prior. Davis, Hinich, and de Groot (1972), Grandmont (1978), and Plott (1967) have all shown that (pure strategy) equilibria of multi-dimensional voting games are typically fragile.

Divergent equilibria also exist if parties aim to maximize their respective probabilities of winning. To see this modify the game $(2,[0,1]^2,\{\psi^1,\psi^2\})$ in the proof of Theorem 1 such that a party (weakly) prefers $(\mathbf{x},\mathbf{y})$ to $(\mathbf{x}',\mathbf{y}')$ if its winning probability is (weakly) higher under ψ^1 and under ψ^2. A party's winning probability is 1 if it obtains more than half the vote, it is $\frac{1}{2}$ if it obtains exactly half the vote, and it is 0 if it obtains less of the vote.[11] The profile $((0,0),(1,1))$ is also an equilibrium of the modified game. Under this profile Party

[11] In this context ψ has to be interpreted as an electorate. While a party's problem of maximizing its expected vote share is equivalent to maximizing its vote share according to the expected electorate as discussed in Section 2.3 this equivalence does not extend to winning probabilities. A party's (expected) winning probability does generally not coincide with its winning probability according to the expected electorate.

1's vector of winning-probabilities is $(1, 0)$. For Party 1 to prefer $(\mathbf{x}, (1, 1))$ to $((0, 0), (1, 1))$, Party 1's vector of winning probabilities with $(\mathbf{x}, (1, 1))$ would have to be $(1, \frac{1}{2})$ or $(1, 1)$. For Party 1 to win with $(\mathbf{x}, (1, 1))$ according to ψ^1, voters with ideal point $(0, 1)$ must at least weakly prefer $\mathbf{x}$ to $(1, 1)$. If $\mathbf{x}$ satisfies this criterion, then Party 1 obtains at most 0.4 of the vote according to ψ^2 with $(\mathbf{x}, (1, 1))$. Consequently the winning probabilities vectors $(1, \frac{1}{2})$ and $(1, 1)$ are out of reach for Party 1 given that Party 2 runs on the platform $(1, 1)$ and $(0, 0)$ is a best reply to $(1, 1)$ for Party 1. Mutatis mutandis one sees that $(1, 1)$ is a best reply to $(0, 0)$ for Party 2, and $((0, 0), (1, 1))$ is an equilibrium of the modified game.

In the classic Downs-Hotelling model, the assumptions of vote share and winning-probability maximization are equivalent, in the sense that a profile is an equilibrium of a game with vote share maximizing parties if and only if it is an equilibrium of the modified game when parties maximize winning probabilities. This equivalence does not extend to the current model as $((0, 0), (0, 0))$ is an equilibrium of of the modified game in which parties maximize winning probabilities but not of the original $(2, [0, 1]^2, \{\psi^1, \psi^2\})$.

If one drops the assumption of single peakedness it becomes considerably easier to obtain divergent equilibria. To see this consider a game of electoral competition with voters $\{1, 2, 3\}$ and the issue space $[0, 2]$. For all $x \notin \{0, 1, 2\}$, let the voters' preferences be given by $0 \succ_1 2 \succ_1 x \succ_1 1$, $2 \succ_2 0 \succ_2 x \succ_2 1$ and $1 \succ_3 0 \sim_3 2 \succ_3 x$. The profile $(0, 2)$ is an equilibrium: both parties obtain exactly half the vote. Moreover for any deviation other than to the opponents platform a party obtains at most one vote (that of voter 3). The profile is divergent given that voter 3's ideal policy lies between 0 and 2.[12]

5 Necessary Conditions for Divergence

There are two differences between my model and the Downs-Hotelling model: the issue space is multi-dimensional and parties hold multiple beliefs on the electorate. Both features are essential for the existence of divergent equilibria. Theorem 2 below extends the convergence-part of the Downs-Hotelling theorem to games of electoral competition with convex preferences and to games of electoral competition without uncertainty. All voters in some game (n, X, Ψ) are said to have convex preferences if all preferences in the support of any prior in the parties belief set $(\bigcup_{\psi \in \Psi} supp(\psi))$ are convex. In the classical Downs-Hotelling model $(1, X, \{\psi\})$ all equilibria converge. This convergence also holds for all equilibria of games (n, X, Ψ) if either Ψ is a singleton or if all voters have convex preferences.

Theorem 2: *Consider a game of electoral competition (n, X, Ψ). If all voters have convex preferences or if Ψ is a singleton, then (n, X, Ψ) does not have any divergent equilibria.*

[12]I would like to thank a referee for this example.

Proof: Suppose some game (n, X, Ψ) in which any $\succsim \in \bigcup_{\psi\in\Psi} supp(\psi)$ is convex had a divergent equilibrium $(\mathbf{x}, \mathbf{y})$, implying that $\psi(\{\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil\}) > 0$ holds for all $\psi \in \Psi$. Assume w.l.o.g that

$$\psi^*(\{\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil \text{ and } \mathbf{y} \succ \mathbf{x}\}) + \frac{1}{2}\psi^*(\{\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil \text{ and } \mathbf{y} \sim \mathbf{x}\}) :\; = \beta > 0$$

for some $\psi^* \in \Psi$. Since any $\succsim \in \bigcup_{\psi\in\Psi} supp(\psi)$ is convex the following equality holds for all $\lambda \in (0, 1)$ and $\psi \in \Psi$:

$$\begin{aligned}
&\pi_\psi\big(\lambda\mathbf{y} + (1-\lambda)\mathbf{x}, \mathbf{y}\big) - \pi_\psi\big(\mathbf{x}, \mathbf{y}\big) = \\
&\psi\big(\succsim \mid \mathbf{y} \succ \mathbf{x} \text{ and } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \succ \mathbf{y}\big) + \\
&\frac{1}{2}\psi\big(\succsim \mid \mathbf{y} \sim \mathbf{x} \text{ and } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \succ \mathbf{y}\big) + \frac{1}{2}\psi\big(\succsim \mid \mathbf{y} \succ \mathbf{x} \text{ and } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \sim \mathbf{y}\big).
\end{aligned}$$

It follows that Party 1 cannot be any worse off by moving its platforms from $\mathbf{x}$ to $\lambda\mathbf{y}+(1-\lambda)\mathbf{x}$ for any $\lambda \in (0, 1)$. To show that that Party 1 is strictly better off for some $\lambda^*\mathbf{y}+(1-\lambda^*)\mathbf{x}$ consider the prior ψ^*. For any $\lambda \in (0, 1)$ we have

$$\begin{aligned}
&\psi^*\big(\succsim \mid \mathbf{y} \succ \mathbf{x} \text{ and } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \succ \mathbf{y}\big) + \\
&\frac{1}{2}\psi^*\big(\succsim \mid \mathbf{y} \sim \mathbf{x} \text{ and } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \succ \mathbf{y}\big) + \frac{1}{2}\psi^*\big(\succsim \mid \mathbf{y} \succ \mathbf{x} \text{ and } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \sim \mathbf{y}\big) \geq \\
&\psi^*\big(\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil, \mathbf{y} \succ \mathbf{x} \text{ and } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \succ \mathbf{y}\big) + \\
&\frac{1}{2}\psi^*\big(\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil, \mathbf{y} \sim \mathbf{x} \text{ and } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \succ \mathbf{y}\big) + \\
&\frac{1}{2}\psi^*\big(\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil, \mathbf{y} \succ \mathbf{x} \text{ and } \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \sim \mathbf{y}\big) :\; = \beta(\lambda)
\end{aligned}$$

Since all $\succsim \in \bigcup_{\psi\in\Psi} supp(\psi)$ are single-peaked and since $\lfloor \mathbf{x}, \mathbf{y} \rceil$ is the open rectangle delimited by $\mathbf{x}$ and $\mathbf{y}$, we have $\lim_{\lambda\to 1} \beta(\lambda) = \beta$. Since $\beta > 0$ there exists a λ^* such that $\beta(\lambda^*) > 0$ and we obtain that

$$\pi_{\psi^*}(\lambda^*\mathbf{y} + (1-\lambda^*)\mathbf{x}, \mathbf{y}) - \pi_{\psi^*}(\mathbf{x}, \mathbf{y}) \geq \beta(\lambda^*) > 0.$$

In sum Party 1 is strictly better off when deviating to $\lambda^*\mathbf{y} + (1-\lambda^*)\mathbf{x}$ from $\mathbf{x}$. If all voters' preferences are convex (n, X, Ψ) cannot have divergent equilibria.

Now fix any $(n, X, \{\psi\})$ together with an equilibrium $(\mathbf{x}, \mathbf{y})$. If one party obtains less than half the vote under $(\mathbf{x}, \mathbf{y})$ it has an incentive to adopt the platform of the opponent. So $\pi_\psi(\mathbf{x}, \mathbf{y}) = \frac{1}{2}$ must hold. Let ϕ be the distribution of voter ideal points in the j-th dimension: for any $S \in \mathbb{R}$ let $\phi(S) = \psi(\succsim \mid \mathbf{a}(\succsim) = \mathbf{a} \text{ with } a_j \in S)$. If $\phi(a_j \mid a_j < x_j) > \frac{1}{2}$ then Party 2 can increase its vote share above $\frac{1}{2}$ by adopting a policy $\mathbf{y}'$ with $y'_i = x_i$ for $i \neq j$ and y'_j slightly below x_j. Similarly Party 2 has a beneficial deviation if $\phi(a_j \mid a_j > x_j) > \frac{1}{2}$. In sum

x_j must be a median of ϕ. By the same token y_j must be a median of ϕ. The definitions of ϕ and $\lfloor \mathbf{x}, \mathbf{y} \rceil$ then imply that

$$0 = \phi(\{a_j \mid \min\{x_j, y_j\} < a_j < \max\{x_j, y_j\}\}) =$$
$$\psi(\{\succsim \mid \mathbf{a}(\succsim) = \mathbf{a} \text{ with } \min\{x_j, y_j\} < a_j < \max\{x_j, y_j\}\}) \geq$$
$$\psi(\{\succsim \mid \mathbf{a}(\succsim) = \mathbf{a} \text{ with } \min\{x_i, y_i\} < a_i < \max\{x_i, y_i\} \text{ for all } i\}) =$$
$$\psi(\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil).$$

and $(\mathbf{x}, \mathbf{y})$ is convergent as $\psi(\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil)$ holds for the unique belief of the two parties. ■

A stronger version of Theorem 2 that builds on a weaker notion of divergence is possible. Even if we require $\psi(\{\succsim : a(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil\}) > 0$ to hold only for some $\psi \in \Psi$ for $(\mathbf{x}, \mathbf{y})$ to be considered divergent, Theorem 2 remains valid. Since the proof of Theorem 2 builds on one particular prior ψ^* with $\psi^*(\{\succsim \mid \mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil\}) > 0$, the same proof applies to the strengthening of Theorem 2.

Theorem 2 can also be strengthened to cover partially divergent equilibria. The notion of the area between two platforms $\lfloor \mathbf{x}, \mathbf{y} \rceil = \{\mathbf{z} \mid \min\{x_i, y_i\} < z_i < \max\{x_i, z_i\} \text{ for all } i\}$ used in the definition of divergence, implies that two platforms $\mathbf{x}, \mathbf{y}$ have to differ with respect to every issue to be considered divergent. Indeed if $x_i = y_i$ holds for $\mathbf{x}, \mathbf{y}$ and i, then $\lfloor \mathbf{x}, \mathbf{y} \rceil$ is the empty set and $\psi(\succsim \mid \{\mathbf{a}(\succsim) \in \lfloor \mathbf{x}, \mathbf{y} \rceil\}) = 0$ holds for any $\psi \in \Psi$. Say that two platforms $\mathbf{x} \neq \mathbf{y}$ **diverge partially** if

$$\psi\big(\succsim \mid \mathbf{a}(\succsim) = \mathbf{a} \text{ with } x_i \neq y_i \Rightarrow \min\{x_i, y_i\} < a_i < \max\{x_i, y_i\}\big) > 0$$

for all $\psi \in \Psi$. A stronger version of Theorem 2, which states that (n, X, Ψ) does not have any partially divergent equilibria if all voters have convex preferences or if there is no uncertainty also holds. To prove this stronger version of Theorem 2, fix a partially divergent profile $(\mathbf{x}, \mathbf{y})$ and amend the proof of Theorem 2 to only consider the subset of platforms $\{\mathbf{z} \in X \mid z_j = x_j \text{ if } x_j = y_j\}$.

If all voters' preferences are convex, Theorem 2 implies that all equilibria of (n, X, Ψ) converge - no matter the size of Ψ. In Section 2.3 I argued that equilibrium sets generally increase in the size of Ψ. Indeed there is a simple trick to define a set of beliefs Ψ for which $\mathbf{x}$ is an equilibrium platform: assume that Ψ contains a prior ψ^* with $\psi^*(\mathbf{x}) = 1$. However, as long as voter preferences are convex any equilibrium $(\mathbf{x}, \mathbf{y})$ must be convergent. To obtain divergent equilibria some of the voters' preferences must exhibit non-convexities.

The multi-dimensionality of the issue space is a necessary condition for the existence of divergent equilibria since any single-peaked preference over a uni-dimensional issue space is convex. So the bane of models à la Downs-Hotelling - multi-dimensional issue spaces -

turns into a boon in the present context. Single-peaked preferences over multi-dimensional issue spaces may exhibit non-convexities, which in turn can be used to explain divergent equilibria.

6 Conclusion

I have shown that the equilibrium platforms in games of electoral competition with office-seeking parties and voters with single-peaked preferences may diverge. To obtain such equilibria I dropped the assumptions that voter preferences are convex and that parties are expected utility maximizers.

One crucial argument for convergence is that moving closer to the other party's platform can never hurt. But this argument requires convexity. Without convexity, a party might decrease its vote share by moving its platform closer to that of the opponent. The dimensionality of the issue space comes into play since single-peaked preferences over a uni-dimensional issue space are automatically convex.

The convexity argument also applies to parties that know the electorate. However when parties are expected utility maximizers there is yet another force that drives them to announce the same platform: each party has to obtain half of the (expected) vote in equilibrium. A party who obtains less than that should match the platform of the opponent. This argument for convergence does not apply in my model given that parties are subjectively uncertain about the electorate.

This paper studies the *existence* of divergent equilibria in games of electoral competition. It sets the stage for interesting and difficult questions on the characterization of the equilibrium sets in such games. How can the connection between party uncertainty and equilibrium divergence be quantified? Can the extent of the non-convexities in voter preferences be related to the divergence of equilibrium platforms? When does a game (n, X, Ψ) have multiple equilibria? In the classical Downs-Hotelling model both parties announce a median of the distribution of voter ideal points in equilibrium. Is there a similar link between distributions of voter ideal points implied by the set of priors Ψ and the equilibrium platforms of (n, X, Ψ)?

7 References

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