A NORM INEQUALITY FOR PAIRS OF COMMUTING POSITIVE SEMIDEFINITE MATRICES*

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Abstract. For $i = 1, \ldots, k$, let A_i and B_i be positive semidefinite matrices such that, for each i, A_i commutes with B_i. We show that, for any unitarily invariant norm,

$$|||\sum_{i=1}^k A_i B_i||| \le |||(\sum_{i=1}^k A_i)\ (\sum_{i=1}^k B_i)|||.$$

The $k = 2$ case was recently conjectured by Hayajneh and Kittaneh and proven by them for the trace norm and the Hilbert-Schmidt norm. A simple application of this norm inequality answers a question of Bourin in the affirmative.

Key words. Matrix Inequality, Unitarily Invariant Norm, Positive semidefinite matrix

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1. Preliminaries. In this paper, we denote the vectors of eigenvalues and singular values of a matrix A by $\lambda(A)$ and $\sigma(A)$, respectively. We adhere to the convention to sort singular values, and eigenvalues as well whenever they are real, in non-increasing order. In general, for a real vector x, we will write $x^\downarrow$ for the vector with the same components as x but sorted in non-increasing order.

For real n-dimensional vectors x and y, we say that x is *weakly majorised* by y, denoted $x \prec_w y$, if and only if for $k = 1, \ldots, n$, $\sum_{i=1}^k x_i^\downarrow \le \sum_{i=1}^k y_i^\downarrow$. We say that x is *majorised* by y, denoted $x \prec y$, if and only if $x \prec_w y$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. If, moreover, x and y are non-negative, we say that x is weakly *log-majorised* by y, denoted $x \prec_{w,\log} y$, if and only if for $k = 1, \ldots, n$, $\prod_{i=1}^k x_i^\downarrow \le \prod_{i=1}^k y_i^\downarrow$.

According to Weyl's Majorant Theorem ([1] Theorem II.3.6, or [4], Theorem 2.4), the vector of singular values of any matrix log-majorises the vector of the absolute values of its eigenvalues: $|\lambda(A)| \prec_{\log} \sigma(A)$. As $x \prec_{w,\log} y$ implies $x^r \prec_w y^r$ for any

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$r > 0$, Weyl's Majorant Theorem can in slightly weaker form be stated as

$$|\lambda(A)|^r \prec_w \sigma^r(A), \text{ for all } r > 0. \tag{1.1}$$

The sum of the k largest singular values of a matrix defines a norm, known as the k-th Ky Fan norm. The convexity of the Ky Fan norms can be expressed as a majorisation relation: for any p such that $0 \le p \le 1$,

$$\sigma(pA + (1-p)B) \prec_w p\sigma(A) + (1-p)\sigma(B).$$

When A and B are positive semidefinite, their singular values coincide with their eigenvalues and we have

$$\lambda(pA + (1-p)B) \prec p\lambda(A) + (1-p)\lambda(B). \tag{1.2}$$

For positive semidefinite matrices A and B, the eigenvalues of AB are real and non-negative. Furthermore $\lambda(AB) \prec_{\log} \lambda(A) \circ \lambda(B)$ ([4] eq. (2.4)). Hence, we also have

$$\lambda(AB) \prec_w \lambda(A) \circ \lambda(B). \tag{1.3}$$

2. A Majorisation Relation for Singular Values. We start with a rather technical result concerning a majorisation relation for singular values. For any matrix A, we denote by $\operatorname{diag}(A)$ the matrix obtained from A by setting all its off-diagonal elements equal to zero.

LEMMA 2.1. *Let S be an $n \times m$ complex matrix, and let L and M be diagonal, positive semidefinite $m \times m$ matrices. Then*

$$\sigma(SL\operatorname{diag}(S^*S)MS^*) \prec_w \sigma((S(LM)^{1/2}S^*)^2) \prec_w \sigma(SLS^*SMS^*). \tag{2.1}$$

Proof. Let us begin with the first majorisation inequality. Since L, M, and $\operatorname{diag}(S^*S)$ are diagonal, they commute, and we can write

$$SL\operatorname{diag}(S^*S)MS^* = S(LM)^{1/2}\operatorname{diag}(S^*S)(LM)^{1/2}S^*.$$

This is a positive semidefinite matrix, hence its singular values are equal to its eigenvalues. The same is true for $(S(LM)^{1/2}S^*)^2$. Let us introduce $X = S(LM)^{1/4}$. Then we have to show that

$$\lambda(X\operatorname{diag}(X^*X)X^*) \prec \lambda(XX^*XX^*).$$

In terms of the matrix $T = X^*X \ge 0$, this is equivalent to

$$\lambda(T\operatorname{diag}(T)) \prec \lambda(T^2).$$

Now note that there exist some number m of unitary matrices U_j such that $\mathrm{diag}(T) = \sum_{j=1}^m (U_j T U_j^*)/m$. Exploiting inequalities (1.2) and (1.3) in turn, we obtain

$$\begin{aligned}
\lambda(T\,\mathrm{diag}(T)) &= \lambda(T^{1/2}\,\mathrm{diag}(T) T^{1/2}) \\
&= \lambda\left(T^{1/2} \ \sum_{j=1}^m \frac{1}{m}(U_j T U_j^*) \ T^{1/2}\right) \\
&\prec \sum_{j=1}^m \frac{1}{m}\lambda(T^{1/2} U_j T U_j^* T^{1/2}) \\
&= \sum_{j=1}^m \frac{1}{m}\lambda(T U_j T U_j^*) \\
&\prec_w \sum_{j=1}^m \frac{1}{m}\lambda(T)\lambda(U_j T U_j^*) \\
&= \sum_{j=1}^m \frac{1}{m}\lambda^2(T) = \lambda(T^2),
\end{aligned}$$

which proves the first inequality of (2.1).

For the second inequality, note that, since $(LM)^{1/2}$ and S^*S are both positive semidefinite, their product has real, non-negative eigenvalues. Thus,

$$\lambda^2((LM)^{1/2}S^*S) = |\lambda(L^{1/2}S^*SM^{1/2})|^2 \prec_w \sigma^2(L^{1/2}S^*SM^{1/2}),$$

by Weyl's Majorant Theorem (eq. (1.1) with $r = 2$). This implies that

$$\begin{aligned}
\sigma((S(LM)^{1/2}S^*)^2) &= \lambda((LM)^{1/2}S^*S(LM)^{1/2}S^*S) \\
&= \lambda^2((LM)^{1/2}S^*S) \\
&\prec_w \sigma^2(L^{1/2}S^*SM^{1/2}) \\
&= \lambda^2((M^{1/2}S^*SLS^*SM^{1/2})^{1/2}) \\
&= \lambda(M^{1/2}S^*SLS^*SM^{1/2}) \\
&= \lambda(SLS^*SMS^*) \\
&= |\lambda(SLS^*SMS^*)| \\
&\prec_w \sigma(SLS^*SMS^*),
\end{aligned}$$

where in the last line we again exploit Weyl's Majorant Theorem (eq. (1.1) with $r = 1$). This proves the second inequality of (2.1). ∎

3. Main Result. We can now state and prove the main result of this paper.

Theorem 3.1. *For $i = 1, \ldots, k$, let A_i and B_i be positive semidefinite $d \times d$ matrices such that, for each i, A_i commutes with B_i. Then for all unitarily invariant norms*

$$|||\sum_{i=1}^k A_i B_i||| \le |||(\sum_{i=1}^k A_i^{1/2} B_i^{1/2})^2||| \le |||(\sum_{i=1}^k A_i)\,(\sum_{i=1}^k B_i)|||. \tag{3.1}$$

Proof. Let A_i and B_i have eigenvalue decompositions

$$A_i = U_i L_i U_i^*, \quad B_i = U_i M_i U_i^*,$$

where the U_i are unitary matrices, and L_i and M_i are positive semidefinite diagonal matrices. Let

$$L = \bigoplus_{i=1}^k L_i, \quad M = \bigoplus_{i=1}^k M_i, \quad S = (U_1|U_2|\cdots|U_k).$$

Then

$$\sum_{i=1}^k A_i = SLS^*, \quad \sum_{i=1}^k B_i = SMS^*, \quad \sum_{i=1}^k A_i B_i = SLMS^*.$$

In addition, the diagonal elements of S^*S are 1 since all columns of S are normalised. Hence, $\mathrm{diag}(S^*S) = I$. By Lemma 2.1, we then have

$$\sigma\left(\sum_{i=1}^k A_i B_i\right) \prec_w \sigma\left(\left(\sum_{i=1}^k A_i^{1/2} B_i^{1/2}\right)^2\right) \prec_w \sigma\left((\sum_{i=1}^k A_i)\,(\sum_{i=1}^k B_i)\right)$$

which is equivalent to (3.1). $\square$

The case $k = 2$ is an inequality recently conjectured by Hayajneh and Kittaneh (Conjecture 1.2 in [3]) and proven by them for the trace norm and the Hilbert-Schmidt norm.

A simple consequence of Theorem 3.1 is that for any set of k positive semidefinite matrices A_i, all positive functions f and g, and all unitarily invariant norms,

$$|||\sum_{i=1}^k f(A_i) g(A_i)||| \le |||(\sum_{i=1}^k f(A_i))\,(\sum_{i=1}^k g(A_i))|||. \tag{3.2}$$

Setting $k = 2$, $f(x) = x^p$ and $g(x) = x^q$ yields the inequality

$$|||A^{p+q} + B^{p+q}||| \le |||(A^p + B^p)(A^q + B^q)|||, \tag{3.3}$$

which was conjectured by Bourin [2].

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REFERENCES

[1] R. Bhatia. *Matrix Analysis.* Springer, Berlin, 1997.
[2] J.-C. Bourin. Matrix subadditivity inequalities and block-matrices. *Internat. J. Math.*, 20:679–691, 2009.
[3] S. Hayajneh and F. Kittaneh. Trace inequalities and a question of Bourin. *Bull. Austral. Math. Soc.*, 88(3):384–389, 2013.
[4] X. Zhan. *Matrix Inequalities.* Springer, Berlin, 2002.