Counting points of fixed degree and bounded height on linear varieties

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Abstract

We count points of fixed degree and bounded height on a linear projective variety defined over a number field k. If the dimension of the variety is large enough compared to the degree we derive asymptotic estimates as the height tends to infinity. This generalizes results of Thunder, Christensen and Gubler and special cases of results of Schmidt and Gao.

1 Introduction and results

Let $N > 1$ be an integer and let k be a number field. For a point $P = (x_1 : ... : x_N)$ in projective space $\mathbb{P}^{N-1}$ over an algebraic closure $\overline{k}$ we define $k(P) = k(..., x_i/x_j, ..)$; $1 \leq i, j \leq N, x_j \neq 0$. Then P has a natural degree over k, namely $[k(P) : k]$. Let $\mathbb{V}$ be a projective linear subvariety of $\mathbb{P}^{N-1}$ defined over k. We count points on $\mathbb{V}$ of given degree e over k and bounded height.

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Special cases of our result have already appeared in the literature. The case $\mathbb{V} = \mathbb{P}^{N-1}, e = 1$ is known as Schanuel's Theorem [15] (with a slightly different choice of the height) and partial results for $\mathbb{V} = \mathbb{P}^{N-1}, e > 1$ have been obtained by Schmidt [19], Gao [5], Masser and Vaaler [12] and the author [25]. Thunder's Theorem 1 in [22] settles the case for arbitrary linear subvarieties $\mathbb{V}$ but only for $e = 1$. Christensen and Gubler [4] refined Thunder's Theorem by giving a more explicit error term.

In an early work Schmidt [16] gave asymptotic estimates for the number of subspaces of $\mathbb{Q}^N$ of arbitrary but fixed dimension and bounded height. This result was generalized to the affine space over arbitrary number fields by Thunder in [20] (see also [21] for a further generalization). The relation with our work becomes clearer using a more intrinsic formulation of our counting problem; consider the $\overline{k}$-vector space, say S, induced by $\mathbb{V}$. Then we are counting the one-dimensional subspaces $\overline{k}\mathbf{x}$ of S of fixed degree over k with $H_2(\mathbf{x}) \leq X$, where the degree of $\overline{k}\mathbf{x}$ is interpreted as the degree of the projective point $(x_1 : ... : x_N)$. The definition of the height $H_2(\cdot)$ is given in Section 2.

To state our result we choose a slightly different formulation, which is more appropriate in the context of the work of Franke, Manin, Tschinkel [6], Peyre [14], Salberger, Thunder [21] and many others. These authors usually take a projective variety $\mathbb{X}$ (of Fano type) defined over $\mathbb{Q}$ or k and then count points of bounded height in $\mathbb{X}(\mathbb{Q})$ or $\mathbb{X}(k)$. Let us point out that any variety defined over k has a Zariski-dense set of points over $\overline{k}$ of sufficiently large degree, whereas the points over $\mathbb{Q}$ or k are necessarily restricted via diophantine constraints like Faltings' Theorem or the various conjectural generalizations.

For positive integers e, n we define

$$\mathbb{P}^n(k; e) = \{P \in \mathbb{P}^n(\overline{k}); [k(P) : k] = e\}.$$

Now for a subvariety $\mathbb{V}$ of $\mathbb{P}^{N-1}$, defined over the number field k, we define

$$\mathbb{V}(k; e) = \mathbb{P}^{N-1}(k; e) \cap \mathbb{V}(\overline{k}). \tag{1.1}$$

Denote by $Z_{H_2}(\mathbb{V}(k; e), X)$ the associated counting function abbreviated to $Z_2(\mathbb{V}(k; e), X)$ so that

$$Z_2(\mathbb{V}(k; e), X) = |\{P \in \mathbb{V}(k; e); H_2(P) \leq X\}|. \tag{1.2}$$

Furthermore we define the formal sum α to be

$$\alpha = \alpha(k,e,n) = \sum_K (2^{-r_K}\pi^{-s_K})^{n+1} V(n+1)^{r_K} V(2n+2)^{s_K} S_K(n) \quad (1.3)$$

where the sum runs over all extensions K of k with relative degree e, r_K is the number of real embeddings of K, s_K is the number of pairs of distinct complex conjugate embeddings of K, $V(p)$ denotes the volume of the euclidean ball in $\mathbb{R}^p$ with radius one and the Schanuel constant $S_K(n)$ is given by

$$S_K(n) = \frac{h_K R_K}{w_K \zeta_K(n+1)} \left(\frac{2^{r_K}(2\pi)^{s_K}}{\sqrt{|\Delta_K|}} \right)^{n+1} (n+1)^{r_K+s_K-1}. \quad (1.4)$$

Here h_K is the class number, R_K the regulator, w_K the number of roots of unity in K, ζ_K the Dedekind zeta-function of K, Δ_K the discriminant and, as above, r_K is the number of real embeddings of K and s_K is the number of pairs of distinct complex conjugate embeddings of K.

A vector space S defined over a number field k has also a height (see (3.3) for its definition). Suppose $\mathbb{V}$ is a linear projective subvariety of $\mathbb{P}^{N-1}$. Let $S \subseteq \overline{k}^N$ be the vector space induced by $\mathbb{V}$. Then we define $H_2(\mathbb{V}) = H_2(S)$. For the precise definition we refer to Section 3.

It will be convenient to use Landau's O-notation. For non-negative real functions $f(X), g(X), h(X)$ we say that $f(X) = g(X) + O(h(X))$ as $X > 0$ tends to infinity if there is a constant C_0 such that $|f(X) - g(X)| \leq C_0 h(X)$ for each $X > 0$.

The main result of this article is the following theorem.

Theorem 1.1. *Let k be a number field of degree m, let n, e and $N \geq n+1$ be positive integers, and let $\mathbb{V}$ be a linear subvariety of $\mathbb{P}^{N-1}$ of dimension n defined over k. Suppose that either $e = 1$ or*

$$n > 5e/2 + 4 + 2/(me).$$

Then the sum in (1.3) converges and as $X > 0$ tends to infinity we have

$$Z_2(\mathbb{V}(k;e), X) = \alpha H_2(\mathbb{V})^{-me} X^{me(n+1)} + O(X^{me(n+1)-1}\mathfrak{L}_0).$$

Here $\mathfrak{L}_0 = \log\max\{2, 2X\}$ if $(me, n) = (1,1)$ and $\mathfrak{L}_0 = 1$ otherwise, and the constant in O depends only on k, e, n.

Note that the error term does not depend on the variety $\mathbb{V}$. We expect that Theorem 1.1 holds for all $n > e$ but we were unable to prove this. However, the order of magnitude of $Z_2(\mathbb{V}(k;e),X)$ is the same as those for $Z_2(\mathbb{P}^n(k;e),X)$ which, because of Schmidt's lower bound in [18], is at least $X^{me(e+1)}$. This implies that Theorem 1.1 cannot hold for $e > 1$ and $n < e$.

Next we consider a few simple examples. Let $\mathbb{V}$ be given by the equation $2x_1 + 3x_2 + 5x_3 = 0$. Then we get $H_2(\mathbb{V}) = \sqrt{38}$. The constant α is $2^{-2}V(2)S_{\mathbb{Q}}(1) = 3/\pi$. So we have asymptotically

$$\frac{3}{\pi\sqrt{38}}X^2 + O(X\log X)$$

rational points on this projective variety. The example above is already covered by Thunder's result.

The novelty in Theorem 1.1 is that we can count also points of fixed degree provided the dimension is much larger than the degree. In fact the remark at the end of Section 6 means that we could probably obtain the asymptotics for counting points on lines of fixed degree over any k despite $n = 1$ being so small; and this even for arbitrary lines in $\mathbb{P}^{N-1}$. What about the simple equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} = 0$$

defined over $\mathbb{Q}$? We compute $H_2(\mathbb{V}) = \sqrt{13}$. Using the formula $V(p) = \pi^{p/2}/\Gamma(p/2+1)$ for $p = n+1$ and $p = 2n+2$ where $n+1 = N - M = 13 - 1 = 12$ we obtain: there are

$$\frac{1}{13}\left(\sum_{\substack{K \\ [K:\mathbb{Q}]=2}} \left(\frac{\pi^6}{2949120}\right)^{r_K}\left(\frac{1}{479001600}\right)^{s_K} S_K(11)\right) X^{24} + O(X^{23})$$

pairwise non-proportional solutions of degree 2 over $\mathbb{Q}$ with height less or equal X.

If we increase the ground field k then we can sometimes even decrease the number of variables. Here is an example actually with rather a large field:

$$\sqrt{1}x_1 + \sqrt{2}x_2 + \sqrt{3}x_3 + \sqrt{4}x_4 + \sqrt{5}x_5 + \sqrt{5}x_6$$
$$+\sqrt{7}x_7 + \sqrt{8}x_8 + \sqrt{9}x_9 + \sqrt{10}x_{10} + \sqrt{11}x_{11} + \sqrt{12}x_{12} = 0$$

defined over the field

$$\begin{aligned} k &= \mathbb{Q}(\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{9}, \sqrt{10}, \sqrt{11}, \sqrt{12}) \\ &= \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}). \end{aligned}$$

So we have $m = [k : \mathbb{Q}] = 32$. We find $H_2(\mathbb{V}) = \sqrt{78}$ and setting $e = 2$ we get $H_2(\mathbb{V})^{-me} = 78^{-32}$. As in the previous example we find

$$\frac{1}{78^{32}} \left(\sum_{\substack{K \\ [K:k]=2}} (\frac{\pi^5}{332640})^{r_K} (\frac{1}{39916800})^{s_K} S_K(10) \right) X^{704} + O(X^{703})$$

pairwise non-proportional solutions of degree 2 over k with height less or equal X.

We can also count the number of solutions on an affine equation such as

$$\begin{aligned} &\sqrt{2}x_1 + \sqrt{3}x_2 + \sqrt{5}x_3 + \sqrt{7}x_4 + \sqrt{11}x_5 + \sqrt{13}x_6 + \sqrt{17}x_7 \\ &+\sqrt{19}x_8 + \sqrt{23}x_9 + \sqrt{29}x_{10} + \sqrt{31}x_{11} + \sqrt{37}x_{12} = \sqrt{41} \end{aligned}$$

defined over the field

$$k = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}, \sqrt{17}, \sqrt{19}, \sqrt{23}, \sqrt{29}, \sqrt{31}, \sqrt{37}, \sqrt{41}).$$

To count the solutions of degree 2 over k we first count the points on the projective variety given by the homogenized equation

$$\begin{aligned} &\sqrt{2}x_1 + \sqrt{3}x_2 + \sqrt{5}x_3 + \sqrt{7}x_4 + \sqrt{11}x_5 + \sqrt{13}x_6 + \sqrt{17}x_7 \\ &+\sqrt{19}x_8 + \sqrt{23}x_9 + \sqrt{29}x_{10} + \sqrt{31}x_{11} + \sqrt{37}x_{12} - \sqrt{41}x_{13} = 0. \end{aligned}$$

Then we subtract the number of points given by the subvariety defined by the additional equation $x_{13} = 0$. In this way we find that there are asymptotically

$$\frac{1}{238^{8192}} \left(\sum_{\substack{K \\ [K:k]=2}} (\frac{\pi^6}{2949120})^{r_K} (\frac{1}{479001600})^{s_K} S_K(11) \right) X^{196608} + O(X^{196607})$$

solutions $(x_1, ..., x_{12})$ of degree 2 over k with $H_2((1 : x_1 : ... : x_{12})) \leq X$.

Actually there is no problem to obtain similar results using the l^∞-height H_∞ (see end of Section 2 for its definition) on $\mathbb{V}(k;e)$. For example with rational points on $2x_1+3x_2+5x_3=0$ as above we get a main term $12/(5\pi)^2X^2$. But already the volume computations for $x_1+...+x_{13}=0$ are more intricate, and in general the dependence on $\mathbb{V}$ will probably not be expressible as any recognizable height function of $\mathbb{V}$.

We close the introduction with a few remarks about the structure of our paper.

In Section 2 we introduce the height $H_2(\cdot)$ on $\mathbb{P}^{N-1}(\overline{k})$. In Section 3 we define the height of a projective linear subvariety of $\mathbb{P}^{N-1}$. In the previous papers [24] and [25] the author introduced adelic-Lipschitz heights on a number field and on collections of number fields; we reproduce these definitions and the corresponding notation in Section 4 and Section 5. In Section 6 we recall the main result of [25] which asymptotically estimates the number of elements in $\mathbb{P}^n(k;e)$ with bounded adelic-Lipschitz height. The use of the latter result becomes clear in Section 7; here we show that $Z_2(\mathbb{V}(k;e),X)$ can be interpreted as $Z_{H_{\mathcal{N}}}(\mathbb{P}^n(k;e),X)$ for a certain adelic-Lipschitz height $H_{\mathcal{N}}(\cdot)$. Finally in Section 8 we are in position to prove Theorem 1.1.

2 Definition of the height

Let K be a finite extension of $\mathbb{Q}$ of degree $d=[K:\mathbb{Q}]$. By a place v of K we mean an equivalence class of non-trivial absolute values on K. The set of all places of K will be denoted by M_K. For each v in M_K we write K_v for the completion of K at the place v and d_v for the local degree defined by $d_v=[K_v:\mathbb{Q}_v]$ where $\mathbb{Q}_v$ is a completion with respect to the place which extends to v. A place v in M_K corresponds either to a non-zero prime ideal $\mathfrak{p}_v$ in the ring of integers $\mathcal{O}_K$ or to an embedding σ of K into $\mathbb{C}$. If v comes from a prime ideal we call v a finite or non-archimedean place and denote this by $v\nmid\infty$ and if v corresponds to an embedding we say v is an infinite or archimedean place and denote this by $v\mid\infty$. For each place in M_K we choose a representative $|\cdot|_v$, normalized in the following way: if v is finite and $\alpha\neq 0$ we set by convention

$$|\alpha|_v=N\mathfrak{p}_v^{-\frac{\mathrm{ord}_{\mathfrak{p}_v}(\alpha\mathcal{O}_K)}{d_v}}$$

where $N\mathfrak{p}_v$ denotes the norm of $\mathfrak{p}_v$ from K to $\mathbb{Q}$ and $\mathrm{ord}_{\mathfrak{p}_v}(\alpha\mathcal{O}_K)$ is the power of $\mathfrak{p}_v$ in the prime ideal decomposition of the fractional ideal $\alpha\mathcal{O}_K$.

Moreover we set

$$|0|_v = 0.$$

And if v is infinite and corresponds to an embedding $\sigma : K \hookrightarrow \mathbb{C}$ we define

$$|\alpha|_v = |\sigma(\alpha)|.$$

If α is in $K^* = K\backslash\{0\}$ then $|\alpha|_v \neq 1$ holds for only a finite number of places v.

Throughout this article n will denote a positive rational integer. The l^2-height on K^{n+1} is defined by

$$H_2((\alpha_0, ..., \alpha_n)) = \prod_{v|\infty} (\sqrt{|\sigma_v(\alpha_0)|_v^2 + ... + |\sigma_v(\alpha_n)|_v^2})^{d_v/d} \prod_{v\nmid\infty} \max\{|\sigma_v(\alpha_0)|_v, ..., |\sigma_v(\alpha_n)|_v\}^{d_v/d} \tag{2.1}$$

where the v run over the set of places M_K, $[K : \mathbb{Q}] = d$, $[K_v : \mathbb{Q}_v] = d_v$ is the local degree with respect to the place v and σ_v is the canonical embedding of K into K_v. Due to the remark above this is in fact a finite product. Furthermore this definition is independent of the field K containing the coordinates (see [1] Lemma 1.5.2 or [7] p.51,52) and therefore defines a height on $\overline{\mathbb{Q}}^{n+1}$. The well-known *product formula* (see [1] Proposition 1.4.4) says that

$$\prod_{M_K} |\sigma_v(\alpha)|_v^{d_v} = 1 \text{ for each } \alpha \text{ in } K^*.$$

This has important consequences, two of them are: for $\boldsymbol{\alpha} \in \overline{\mathbb{Q}}^{n+1}\backslash\{\mathbf{0}\}$ we have $H_2(\boldsymbol{\alpha}) \geq 1$, and the value of the height in (2.1) does not change if we multiply each coordinate with a fixed element of K^*. Therefore one can define a height on points $P = (\alpha_0 : ... : \alpha_n)$ in $\mathbb{P}^n(\overline{\mathbb{Q}})$ by

$$H_2(P) = H_2((\alpha_0, ..., \alpha_n)). \tag{2.2}$$

This is the absolute non-logarithmic projective l^2-height or just l^2-height. By choosing maximum norms at all places we get the l^∞-height on K^{n+1}

$$H_\infty((\alpha_0, ..., \alpha_n)) = \prod_{M_K} \max\{|\sigma_v(\alpha_0)|_v, ..., |\sigma_v(\alpha_n)|_v\}^{d_v/d}$$

and just as for the l^2-height this gives rise to a height H_∞ on $\mathbb{P}^n(\overline{\mathbb{Q}})$.

3 Definition of $H_2(\mathbb{V})$

Let $\mathbb{V}$ be a non-empty projective linear subvariety of $\mathbb{P}^{N-1}(\overline{k})$. If $\mathbb{V} \subsetneq \mathbb{P}^{N-1}(\overline{k})$ then there exists a positive integer M and a system of $M < N$ linearly independent linear homogeneous equations, defined over a number field k,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1N}x_N &= 0 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{M1}x_1 + a_{M2}x_2 + ... + a_{MN}x_N &= 0 \end{aligned} \tag{3.1}$$

defining $\mathbb{V}$. Now $\mathbb{V}$ induces a $\overline{k}$-vector space, namely $\overline{k}^N$ if $\mathbb{V} = \mathbb{P}^{N-1}(\overline{k})$ and the solution space in $\overline{k}^N$ of the system (3.1) if $\mathbb{V} \subsetneq \mathbb{P}^{N-1}(\overline{k})$. We start by defining the height of a subspace S of $\overline{k}^N$ and then we define $H_2(\mathbb{V})$ as $H_2(S)$ where S is the vector space induced by $\mathbb{V}$.

So let S be a subspace of $\overline{k}^N$ of dimension $n+1$ over $\overline{k}$ and let $v_1, ..., v_{n+1}$ be a basis of S over $\overline{k}$. We form the wedge-product ([17] paragraph 5)

$$v_1 \wedge ... \wedge v_{n+1} \in \overline{k}^{\binom{N}{n+1}}. \tag{3.2}$$

Since the vectors are linearly independent it is non-zero ([17] Lemma 5C). Let $v_1', ..., v_{n+1}'$ be linearly independent vectors in $\overline{k}^N$ then $v_1 \wedge ... \wedge v_{n+1}$ is proportional to $v_1' \wedge ... \wedge v_{n+1}'$ if and only if $v_1', ..., v_{n+1}'$ and $v_1, ..., v_{n+1}$ span the same space ([17] p.14 Lemma 5D). Hence we may define

$$H_2(S) = H_2(v_1 \wedge ... \wedge v_{n+1}). \tag{3.3}$$

Moreover we set

$$H_2(\{\mathbf{0}\}) = H_2(\overline{k}^N) = 1.$$

The wedge product in (3.2) is called a tuple of Grassmann coordinates for S. Up to a non-zero scalar multiple and permutations of the coordinates such a tuple is determined by S.

For our purpose the following equivalent definition using the matrix A of (3.1) with entries a_{ij} $(1 \leq i \leq M, 1 \leq j \leq N)$ is more convenient. We denote by A_0 the various maximal minors of A. Let K be any number field containing k. Then with v as in (2.1)

$$H^{fin}(A) = \prod_{v \nmid \infty} \max_{A_0} |\sigma_v(\det A_0)|_v^{\frac{d_v}{d}},$$
$$H^{inf}(A) = \prod_{v | \infty} (\sum_{A_0} |\sigma_v(\det A_0)|_v^2)^{\frac{d_v}{2d}}$$

and as always $d = [K : \mathbb{Q}]$ and $d_v = [K_v : \mathbb{Q}_v]$. Using an analogue of the Cauchy-Binet formula over number fields one can prove (see [2] p.15 (iv)) that

$$H^{inf}(A) = \prod_{v|\infty} |\det(\sigma_v(A)\overline{\sigma_v(A)}^t)|_v^{\frac{d_v}{2d}} \tag{3.4}$$

where over-line means complex conjugation, $\overline{\sigma_v(A)}^t$ is the transpose of $\overline{\sigma_v(A)}$ and σ_v acts on each entry. Multiplying the finite and infinite parts yields the height of the matrix

$$H_2(A) = H^{fin}(A)H^{inf}(A).$$

Notice that a tuple of determinants of all maximal minors is a tuple of Grassmann coordinates of S. Hence $H_2(A)$ is nothing else but $H_2(S)$.

Now recall that we define the height $H_2(\mathbb{V})$ of a projective linear subvariety $\mathbb{V}$ of $\mathbb{P}^{N-1}$ by

$$H_2(\mathbb{V}) = H_2(S)$$

where S is the $\overline{k}$-vector space induced by $\mathbb{V}$.

4 Adelic-Lipschitz heights over a number field

This section is plainly contained in [25] Section 2. However, in order to help the reader and to recall the most important notation we are reproducing all basic notation here. Let r be the number of real embeddings and s the number of pairs of complex conjugate embeddings of K so that $d = r + 2s$. Recall that M_K denotes the set of places of K. For every place v we fix a completion K_v of K at v. The value set of v, $\Gamma_v := \{|\alpha|_v; \alpha \in K_v\}$ is equal to $[0, \infty)$ if v is archimedean, and to

$$\{0, (N\mathfrak{p}_v)^0, (N\mathfrak{p}_v)^{\pm 1/d_v}, (N\mathfrak{p}_v)^{\pm 2/d_v}, ...\}$$

if v is non-archimedean. For $v \mid \infty$ we identify K_v with $\mathbb{R}$ or $\mathbb{C}$ respectively and we identify $\mathbb{C}$ with $\mathbb{R}^2$ via $\xi \longrightarrow (\Re(\xi), \Im(\xi))$ where we used $\Re$ for the real and $\Im$ for the imaginary part of a complex number.

Before we can introduce adelic-Lipschitz systems we have to give a technical definition. For a vector $\mathbf{x}$ in $\mathbb{R}^n$ we write $|\mathbf{x}|$ for the euclidean length of $\mathbf{x}$.

Definition 4.1. *Let M and $D > 1$ be positive integers and let L be a non-negative real. We say that a set S is in Lip(D, M, L) if S is a subset of $\mathbb{R}^D$, and if there are M maps $\phi_1, ..., \phi_M : [0,1]^{D-1} \longrightarrow \mathbb{R}^D$ satisfying a Lipschitz condition*

$$|\phi_i(\mathbf{x}) - \phi_i(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}| \text{ for } \mathbf{x}, \mathbf{y} \in [0,1]^{D-1}, i = 1, ..., M \tag{4.1}$$

such that S is covered by the images of the maps ϕ_i.

We call L a Lipschitz constant for the maps ϕ_i. By definition the empty set lies in $\mathrm{Lip}(D, M, L)$ for any positive integers M and $D > 1$ and any non-negative L.

Definition 4.2 (Adelic-Lipschitz system)**.** *An adelic-Lipschitz system (ALS) $\mathcal{N}_K$ or simply $\mathcal{N}$ on K (of dimension n) is a set of continuous maps*

$$N_v : K_v^{n+1} \to \Gamma_v \quad v \in M_K$$

such that for $v \in M_K$ we have

(i) $N_v(\mathbf{z}) = 0$ *if and only if* $\mathbf{z} = \mathbf{0}$,

(ii) $N_v(\omega \mathbf{z}) = |\omega|_v N_v(\mathbf{z})$ *for all ω in K_v and all $\mathbf{z}$ in K_v^{n+1},*

(iii) if $v \mid \infty$: $\{\mathbf{z}; N_v(\mathbf{z}) = 1\}$ is in Lip$(d_v(n+1), M_v, L_v)$ for some M_v, L_v,

(iv) if $v \nmid \infty$: $N_v(\mathbf{z}_1 + \mathbf{z}_2) \leq \max\{N_v(\mathbf{z}_1), N_v(\mathbf{z}_2)\}$ for all $\mathbf{z}_1, \mathbf{z}_2$ in K_v^{n+1}.

Moreover we assume that

$$N_v(\mathbf{z}) = \max\{|z_0|_v, ..., |z_n|_v\} \tag{4.2}$$

for all but a finite number of $v \in M_K$. If we consider only the functions N_v for $v \mid \infty$ then we get an (r, s)-Lipschitz system (of dimension n) in the sense of Masser and Vaaler [12]. With M_v and L_v from (iii) we define

$$M_{\mathcal{N}} = \max_{v \mid \infty} M_v,$$

$$L_{\mathcal{N}} = \max_{v \mid \infty} L_v.$$

We say that $\mathcal{N}$ is an *ALS* with associated constants $M_{\mathcal{N}}, L_{\mathcal{N}}$. The set defined in (*iii*) is the boundary of the set $\mathbf{B}_v = \{\mathbf{z}; N_v(\mathbf{z}) < 1\}$ and therefore $\mathbf{B}_v$ is a bounded symmetric open star-body in $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$ (see also [12] p.431). In particular $\mathbf{B}_v$ has a finite volume V_v.

Remark 1. *Let us consider the system where N_v is as in (4.2) for all finite v places and N_v is the l^2-norm for the infinite places. If v is an infinite place then $\mathbf{B}_v$ is a unit ball in $\mathbb{R}^{d_v(n+1)}$. Its boundary is clearly in Lip$(d_v(n+1), M_v, L_v)$ for example with $M_v = 1$ map and $L_v = 2\pi d_v(n+1)$ (for a proof see Lemma 7.2).*

We return to arbitrary adelic-Lipschitz systems. We claim that for any $v \in M_K$ there is a c_v in the value group $\Gamma_v^* = \Gamma_v \backslash \{0\}$ with

$$N_v(\mathbf{z}) \geq c_v \max\{|z_0|_v, ..., |z_n|_v\} \tag{4.3}$$

for all $\mathbf{z} = (z_0, ..., z_n)$ in K_v^{n+1}. For if v is archimedean then $\mathbf{B}_v$ is bounded open and it contains the origin. Since Γ_v^* contains arbitrary small positive numbers the claim follows by (*ii*). Now for v non-archimedean N_v and $\max\{|z_0|_v, ..., |z_n|_v\}$ define norms on the vector space K_v^{n+1} over the complete field K_v. But on a finite dimensional vector space over a complete field all norms are equivalent ([3] Corollary 5. p.93) hence (4.3) remains true for a suitable choice of c_v.

So let $\mathcal{N}$ be an *ALS* on K of dimension n. For every v in M_K let c_v be an element of Γ_v^*, such that $c_v \leq 1$ and (4.3) holds. Due to (4.2) we can assume that $c_v = 1$ for all but a finite number of places v. We define

$$C_{\mathcal{N}}^{fin} = \prod_{v \nmid \infty} c_v^{-\frac{d_v}{d}} \geq 1 \tag{4.4}$$

and

$$C_{\mathcal{N}}^{inf} = \max_{v|\infty}\{c_v^{-1}\} \geq 1. \tag{4.5}$$

Multiplying the finite and the infinite part gives rise to another constant

$$C_{\mathcal{N}} = C_{\mathcal{N}}^{fin} C_{\mathcal{N}}^{inf}. \tag{4.6}$$

It will turn out that besides $M_{\mathcal{N}}$ and $L_{\mathcal{N}}$ this is another important quantity for an *ALS*. So we say that $\mathcal{N}$ *is an ALS with associated constants* $C_{\mathcal{N}}, M_{\mathcal{N}}, L_{\mathcal{N}}$.

Remark 2. *Let v be an infinite place. Suppose $N_v : K_v^{n+1} \longrightarrow [0,\infty)$ defines a norm, so that $N_v(\mathbf{z}_1+\mathbf{z}_2) \leq N_v(\mathbf{z}_1)+N_v(\mathbf{z}_2)$. Then $\mathbf{B}_v$ is convex and (4.3) combined with (4.4), (4.5) and (4.6) shows that $\mathbf{B}_v$ lies in $B_0(C_{\mathcal{N}}\sqrt{n+1})$. This implies (see Theorem A.1 in [23]) that $\partial\mathbf{B}_v$ lies in $Lip(d_v(n+1), 1, 8{d_v}^2(n+1)^{5/2}C_{\mathcal{N}})$.*

We denote by $\sigma_1,...,\sigma_d$ the embeddings from K to $\mathbb{R}$ or $\mathbb{C}$ respectively, ordered such that $\sigma_{r+s+i} = \overline{\sigma}_{r+i}$ for $1 \leq i \leq s$. We define

$$\sigma : K \longrightarrow \mathbb{R}^r \times \mathbb{C}^s$$
$$\sigma(\alpha) = (\sigma_1(\alpha),...,\sigma_{r+s}(\alpha)).$$

Sometimes it will be more readable to omit the brackets and simply to write $\sigma\alpha$. We identify $\mathbb{C}$ in the usual way with $\mathbb{R}^2$ and extend σ componentwise to get a map

$$\sigma : K^{n+1} \longrightarrow \mathbb{R}^D$$

where $D = d(n+1)$. On $\mathbb{R}^D$ we use $|\cdot|$ for the usual euclidean norm. For $v \in M_K$ let σ_v be the canonical embedding of K in K_v, again extended componentwise on K^{n+1}.

Definition 4.3. *Let $\mathfrak{D} \neq 0$ be a fractional ideal in K and $\mathcal{N}$ an ALS of dimension n. We define*

$$\Lambda_{\mathcal{N}}(\mathfrak{D}) = \{\sigma(\boldsymbol{\alpha}); \boldsymbol{\alpha} \in K^{n+1}, N_v(\sigma_v\boldsymbol{\alpha}) \leq |\mathfrak{D}|_v \text{ for all finite } v\} \tag{4.7}$$

where $|\mathfrak{D}|_v = N\mathfrak{p}_v^{-\frac{\mathrm{ord}_{\mathfrak{p}_v}\mathfrak{D}}{d_v}}$.

It is easy to see that $\Lambda_{\mathcal{N}}(\mathfrak{D})$ is an additive subgroup of $\mathbb{R}^D$. Now assume $B \geq 1$ and $|\sigma(\boldsymbol{\alpha})| \leq B$; then (4.3) implies $H_\infty(\boldsymbol{\alpha})^d \leq (BC_{\mathcal{N}}^{fin})^d N\mathfrak{D}^{-1}$ and by Northcott's Theorem we deduce that $\Lambda_{\mathcal{N}}(\mathfrak{D})$ is discrete. The same argument as for (4.3) yields positive real numbers C_v, one for each non-archimedean place $v \in M_K$, with $N_v(\mathbf{z}) \leq C_v \max\{|z_0|_v,...,|z_n|_v\}$ for all $\mathbf{z} = (z_0,...,z_n)$ in K_v^{n+1} and $C_v = 1$ for all but finitely many non-archimedean $v \in M_K$. Thus there exists an ideal $\mathfrak{C}_1 \neq 0$ in $\mathcal{O}_K$ with $|\mathfrak{C}_1|_v \leq 1/C_v$ for all non-archimedean places $v \in M_K$. This means that $\sigma(\mathfrak{C}_1\mathfrak{D})^{n+1} \subseteq \Lambda_{\mathcal{N}}(\mathfrak{D})$. It is well-known that the additive group $\sigma(\mathfrak{C}_1\mathfrak{D})^{n+1}$ has maximal rank in $\mathbb{R}^D$. Therefore $\Lambda_{\mathcal{N}}(\mathfrak{D})$ is a discrete additive subgroup of $\mathbb{R}^D$ of maximal rank. Hence $\Lambda_{\mathcal{N}}(\mathfrak{D})$ is a lattice. Notice that for ε in K^* one has

$$\det \Lambda_{\mathcal{N}}((\varepsilon)\mathfrak{D}) = |N_{K/\mathbb{Q}}(\varepsilon)|^{n+1} \det \Lambda_{\mathcal{N}}(\mathfrak{D}).$$

Therefore

$$\Delta_{\mathcal{N}}(\mathcal{D}) = \frac{\det \Lambda_{\mathcal{N}}(\mathfrak{D})}{N\mathfrak{D}^{n+1}} \tag{4.8}$$

is independent of the choice of the representative $\mathfrak{D}$ but depends only on the ideal class $\mathcal{D}$ of $\mathfrak{D}$. Let Cl_K denote the ideal class group of K. We define

$$V_{\mathcal{N}}^{fin} = 2^{-s(n+1)}|\Delta_K|^{\frac{n+1}{2}} h_K^{-1} \sum_{\mathcal{D}\in Cl_K} \Delta_{\mathcal{N}}(\mathcal{D})^{-1} \tag{4.9}$$

for the finite part, where as usual, s denotes the number of pairs of complex conjugate embeddings of K, h_K the class number of K and Δ_K the discriminant of K. The infinite part is defined by

$$V_{\mathcal{N}}^{inf} = \prod_{v|\infty} V_v.$$

We multiply the finite and the infinite part to get a global volume

$$V_{\mathcal{N}} = V_{\mathcal{N}}^{inf} V_{\mathcal{N}}^{fin}. \tag{4.10}$$

An *ALS* $\mathcal{N}$ on K of dimension n defines a height $H_{\mathcal{N}}$ on K^{n+1} by setting

$$H_{\mathcal{N}}(\boldsymbol{\alpha}) = \prod_{v\in M_K} N_v(\sigma_v(\boldsymbol{\alpha}))^{\frac{d_v}{d}}.$$

Thanks to the product formula and (ii) from Definition 4.2 $H_{\mathcal{N}}(\boldsymbol{\alpha})$ does not change if we multiply each coordinate of $\boldsymbol{\alpha}$ with a fixed element of K^*. Therefore $H_{\mathcal{N}}$ is well-defined on $\mathbb{P}^n(K)$ by setting

$$H_{\mathcal{N}}(P) = H_{\mathcal{N}}(\boldsymbol{\alpha})$$

where $P = (\alpha_0 : ... : \alpha_n) \in \mathbb{P}^n(K)$ and $\boldsymbol{\alpha} = (\alpha_0, ..., \alpha_n) \in K^{n+1}$.

Remark 3. *Multiplying (4.3) over all places with suitable multiplicities yields*

$$H_{\mathcal{N}}(P) \geq C_{\mathcal{N}}^{-1} H_\infty(P) \tag{4.11}$$

for $P \in \mathbb{P}^n(K)$. *Thanks to Northcott's Theorem it follows that* $\{P \in \mathbb{P}^n(K); H_{\mathcal{N}}(P) \leq X\}$ *is a finite set for each* X *in* $[0, \infty)$.

5 Adelic-Lipschitz heights over a collection of number fields

We are now going to define adelic Lipschitz heights on collections of K^{n+1} for number fields K. Let k be a number field of degree m and $\overline{k}$ an algebraic closure of k. We fix k and $\overline{k}$ throughout and assume finite extensions of k to lie in $\overline{k}$. Let $\mathcal{C}$ be a collection of finite extensions of k. We are especially interested in the set of all extensions of fixed relative degree. We denote it by

$$\mathcal{C}_e = \mathcal{C}_e(k) = \{K \subseteq \overline{k}; [K : k] = e\}.$$

Let $\mathcal{N}$ be a collection of adelic-Lipschitz systems $\mathcal{N}_K$ of dimension n - one for each K of $\mathcal{C}$. Then we call $\mathcal{N}$ an *adelic-Lipschitz system* (*ALS*) *on* $\mathcal{C}$ *of dimension* n. We say $\mathcal{N}$ is a *uniform ALS* on $\mathcal{C}$ of dimension n with associated constants $C_{\mathcal{N}}, M_{\mathcal{N}}, L_{\mathcal{N}}$ in $\mathbb{R}$ if the following holds: for each *ALS* $\mathcal{N}_K$ of the collection $\mathcal{N}$ we can choose associated constants $C_{\mathcal{N}_K}, M_{\mathcal{N}_K}, L_{\mathcal{N}_K}$ satisfying

$$C_{\mathcal{N}_K} \le C_{\mathcal{N}}, \quad M_{\mathcal{N}_K} \le M_{\mathcal{N}}, \quad L_{\mathcal{N}_K} \le L_{\mathcal{N}}.$$

Notice that a uniform *ALS* $\mathcal{N}$ (of dimension n) on the collection consisting only of a single field K with associated constants $C_{\mathcal{N}}, M_{\mathcal{N}}, L_{\mathcal{N}}$ is simply an *ALS* $\mathcal{N}$ (of dimension n) on K with associated constants $C_{\mathcal{N}}, M_{\mathcal{N}}, L_{\mathcal{N}}$ in the sense of the Definition 4.2 of Section 4.

Let $\mathcal{C}$ be a collection of finite extensions of k and let $\mathcal{N}$ be an *ALS* of dimension n on $\mathcal{C}$. Now we can define heights on $\mathbb{P}^n(K/k)$ (the set of points P in $\mathbb{P}^n(K)$ with $k(P) = K$) for K in $\mathcal{C}$. Let $P = (\alpha_0 : ... : \alpha_n) \in \mathbb{P}^n(K/k)$, so that $k(P) = K$. According to the previous section we know that $H_{\mathcal{N}_K}(\cdot)$ defines a projective height on $\mathbb{P}^n(K)$. Now we define

$$H_{\mathcal{N}}(P) = H_{\mathcal{N}_K}(P). \tag{5.1}$$

Set $\boldsymbol{\alpha} = (\alpha_0, ..., \alpha_n)$. From the previous section we know

$$H_{\mathcal{N}_K}(P) = \prod_{v \in M_K} N_v(\sigma_v(\boldsymbol{\alpha}))^{\frac{d_v}{d}} \tag{5.2}$$

for the functions N_v of $\mathcal{N}_K$ and $[K : \mathbb{Q}] = d$, $[K_v : \mathbb{Q}_v] = d_v$.

Remark 4. *An important ALS $\mathcal{N}$ on $\mathcal{C}_e$ (of dimension n) in the context of this work is given by choosing l^2-norms at all infinite places and N_v as in (4.2) for all finite places. From the definition of c_v in (4.3) we see that we can choose $c_v = 1$ for all places v and thus by (4.6) $C_{\mathcal{N}} = 1$. Together with Remark 1 this implies that $\mathcal{N}$ is a uniform ALS with associated constants $C_{\mathcal{N}} = 1, M_{\mathcal{N}} = 1, L_{\mathcal{N}} = 4\pi(n+1)$. The induced height $H_{\mathcal{N}}$ is of course just the l^2-height H_2 on $\mathbb{P}^n(k;e)$.*

6 Asymptotics of $\mathbb{P}^n(k;e)$ with respect to adelic-Lipschitz heights

In this section we recall the main result of [25], which is essential to deduce Theorem 1.1. Let $\mathcal{N}$ be an *ALS* on $\mathcal{C}_e$ of dimension n. Then $H_{\mathcal{N}}(\cdot)$ defines a height on $\mathbb{P}^n(k;e)$, the set of points $P = (\alpha_0 : ... : \alpha_n)$ in $\mathbb{P}^n(\overline{k})$ with $[k(P) : k] = e$ where $k(P) = k(..., \alpha_i/\alpha_j, ...)$ for $0 \leq i, j \leq n$; $\alpha_j \neq 0$. Assume $\mathcal{N}$ is a uniform *ALS* on $\mathcal{C}_e$ (of dimension n). Then due to Northcott's Theorem and (4.11) the number of points P in $\mathbb{P}^n(k;e)$ with $H_{\mathcal{N}}(P) \leq X$ is finite for all X in $[0, \infty)$. Let us denote this number by $Z_{\mathcal{N}}(\mathbb{P}^n(k;e), X)$, so that $Z_{\mathcal{N}}(\mathbb{P}^n(k;e), X)$ is the counting function of $\mathbb{P}^n(k;e)$ with respect to $H_{\mathcal{N}}$. Recall from (1.4) the definition of the Schanuel constant $S_K(n)$ and from (4.10) those of $V_{\mathcal{N}_K}$. By r_K we denote the number of real embeddings of K and s_K is the number of pairs of distinct complex conjugate embeddings of K. Now we define the sum

$$D_{\mathcal{N}} = D_{\mathcal{N}}(k, e, n) = \sum_K 2^{-r_K(n+1)}\pi^{-s_K(n+1)} V_{\mathcal{N}_K} S_K(n) \tag{6.1}$$

where the sum runs over all extensions of k with relative degree e. The following theorem is the main result of [25].

Theorem 6.1. *Let e, n be positive integers and k a number field of degree m. Suppose $\mathcal{N}$ is a uniform adelic-Lipschitz system of dimension n on $\mathcal{C}_e$, the collection of all finite extensions of k of relative degree e, with associated constants $C_{\mathcal{N}}, M_{\mathcal{N}}$ and $L_{\mathcal{N}}$. Write*

$$A_{\mathcal{N}} = M_{\mathcal{N}}^{me}(C_{\mathcal{N}}(L_{\mathcal{N}} + 1))^{me(n+1)-1}.$$

Suppose that either $e = 1$ or

$$n > 5e/2 + 4 + 2/(me).$$

Then the sum in (6.1) converges and as $X > 0$ tends to infinity we have

$$Z_{\mathcal{N}}(\mathbb{P}^n(k;e),X) = D_{\mathcal{N}} X^{me(n+1)} + O(A_{\mathcal{N}} X^{me(n+1)-1}\mathfrak{L}),$$

where $\mathfrak{L} = \log\max\{2, 2C_{\mathcal{N}}X\}$ if $(me,n) = (1,1)$ and $\mathfrak{L} = 1$ otherwise. The constant in O depends only on k, e and n.

It is likely that the Theorem 6.1 is valid for $n > e$. Gao showed, at least for his definition of height (see [5] or [23] Appendix B), that for $k = \mathbb{Q}$ the bound $n > e$ suffices. On the other hand Schmidt's lower bound in [18] implies that the Theorem 6.1 cannot hold for $e > 1$ and $n < e$. However, there is a good possibility of obtaining the asymptotics for $e > 1$ and $n = 1$ using a kind of generalized Mahler measure and following Masser and Vaaler's strategy in [12].

7 A reformulation of Theorem 1.1

Let $\mathbb{V} \subseteq \mathbb{P}^{N-1}$ be a linear subvariety of dimension $n > 0$ and defined over k. Suppose $n = N - 1$ i.e. $\mathbb{V} = \mathbb{P}^{N-1}$. In this case Theorem 1.1 follows easily from Theorem 6.1 and the observation in Remark 4. Therefore we are entitled to assume $n < N - 1$. Thus there are coefficients $a_{11}, ..., a_{MN}$ in k such that a system as in (3.1) of $M > 0$ linearly independent equations defines $\mathbb{V}$. Then the dimension n of $\mathbb{V}$ is given by

$$n = N - M - 1 \geq 1. \tag{7.1}$$

Let S be the $\overline{k}$-vector space induced by $\mathbb{V}$, so that $(x_1 : ... : x_N)$ lies in $\mathbb{V}(\overline{k})$ if and only if $(x_1, ..., x_N)$ lies in $S \backslash \{\mathbf{0}\}$. Thus the dimension of the vector space S (over $\overline{k}$) is $n+1$. Since S is defined over k, there are (homogeneous) linear forms $L_1, ..., L_N$ in $k[\mathbf{z}]$ such that there is a $(1:1)$-correspondence between $\overline{k}^{n+1}$ and S given by

$$\mathbf{z}^t = (z_0, ..., z_n) \longleftrightarrow (L_1(\mathbf{z}), ..., L_N(\mathbf{z})). \tag{7.2}$$

Now (7.2) implies a $(1:1)$-correspondence between the sets $\mathbb{P}^n(\overline{k})$ and $\mathbb{V}(\overline{k})$

$$(z_0 : ... : z_n) \longleftrightarrow (L_1(\mathbf{z}) : ... : L_N(\mathbf{z})). \tag{7.3}$$

If we permute the coordinates in $(L_1(\mathbf{z}) : ... : L_N(\mathbf{z}))$ we will probably no longer parameterize $\mathbb{V}$ but we will parameterize a linear subvariety with the same number of points of fixed degree and bounded height. Therefore we may assume

$$L_j(\mathbf{z}) = z_{j-1} \quad \text{for } 1 \leq j \leq n+1. \tag{7.4}$$

Lemma 7.1. *The counting function $Z_2(\mathbb{V}(k;e),X)$ is given by the number of*

$$(z_0 : ... : z_n) \in \mathbb{P}^n(k;e)$$

with

$$H_2((L_1(\mathbf{z}) : ... : L_N(\mathbf{z}))) \leq X.$$

Proof. By (7.3) we see that $Z_2(\mathbb{V}(k;e),X)$ is the number of projective points $(z_0 : ... : z_n)$ in $\mathbb{P}^n(\overline{k})$ with

$$[k((L_1(\mathbf{z}) : ... : L_N(\mathbf{z}))) : k] = e \tag{7.5}$$

$$H_2((L_1(\mathbf{z}) : ... : L_N(\mathbf{z}))) \leq X. \tag{7.6}$$

Moreover the linear forms L_j have coefficients in k. By (7.4) we see that

$$k((L_1(\mathbf{z}) : ... : L_N(\mathbf{z}))) = k((z_0 : ... : z_n)).$$

So the number of points of $\mathbb{V}$ of degree e over k and with l^2-height not exceeding X is the number of $(z_0 : ... : z_n) \in \mathbb{P}^n(k;e)$ with $H_2((L_1(\mathbf{z}) : ... : L_N(\mathbf{z}))) \leq X$. □

7.1 The corresponding adelic-Lipschitz system

The previous lemma shows that we shall count $(z_0 : ... : z_n) \in \mathbb{P}^n(k;e)$ with $H_2((L_1(\mathbf{z}) : ... : L_N(\mathbf{z}))) \leq X$. The strategy is to choose a uniform *ALS* $\mathcal{N}$ on $\mathcal{C}_e = \mathcal{C}_e(k)$ of dimension n to obtain

$$H_{\mathcal{N}}(\mathbf{z}) = H_2((L_1(\mathbf{z}) : ... : L_N(\mathbf{z}))). \tag{7.7}$$

We define $\mathcal{N}$ as follows: for each K of $\mathcal{C}_e$ we define an *ALS* $\mathcal{N}_K$ on K (of dimension n) by

$$N_v(\mathbf{z}) = \max\{|(\sigma_v L_1)(\mathbf{z})|_v, ..., |(\sigma_v L_N)(\mathbf{z})|_v\} : v \nmid \infty \tag{7.8}$$

$$N_v(\mathbf{z}) = \sqrt{|(\sigma_v L_1)(\mathbf{z})|_v^2 + ... + |(\sigma_v L_N)(\mathbf{z})|_v^2} \ : v \mid \infty. \tag{7.9}$$

Here σ_v acts on the coefficients of the linear forms L_i. Note that the right hand-side of (7.8) differs from (4.2) only for a finite number of finite places. With this definition of $\mathcal{N}$, and having (5.1) and (5.2) in mind, we see that equation (7.7) holds. For $v \nmid \infty$ the ultrametric inequality $|(\sigma_v L_1)(\mathbf{z}_1 + \mathbf{z}_2)|_v \leq \max\{|(\sigma_v L_1)(\mathbf{z}_1)|_v, |(\sigma_v L_1)(\mathbf{z}_2)|_v\}$ implies that condition (iv) of Section 4 is satisfied. For $v \mid \infty$ it is not so obvious that (iii) holds and we postpone the proof. But let us describe the set $\mathbf{B}_v = \{\mathbf{z}; N_v(\mathbf{z}) < 1\}$ and its boundary $\partial \mathbf{B}_v = \{\mathbf{z}; N_v(\mathbf{z}) = 1\}$. We write

$$b_v(\mathbf{z}, \mathbf{z}') = (\sigma_v L_1)(\mathbf{z})\overline{(\sigma_v L_1)(\mathbf{z}')} + ... + (\sigma_v L_N)(\mathbf{z})\overline{(\sigma_v L_N)(\mathbf{z}')}.$$

Let $e_1, ..., e_{n+1}$ be the canonical basis of $\mathbb{R}^{n+1}$ if v is real and of $\mathbb{C}^{n+1}$ if v is non-real. Let $Q = Q_v$ be the matrix with entries q_{ij} $(1 \leq i, j \leq n+1)$ where

$$q_{ij} = b_v(e_i, e_j).$$

At this stage where matrices enter the game we should point out that $\mathbf{z}$ is a column. Using the definition (7.9) of $N_v(\cdot)$ we see that

$$N_v(\mathbf{z})^2 = \sum_{j=1}^{N}\sum_{r=0}^{n}\sum_{p=0}^{n} z_r \overline{z_p} (\sigma_v L_j)(e_{r+1})\overline{(\sigma_v L_j)(e_{p+1})} = \overline{\mathbf{z}}^t Q \mathbf{z}.$$

Thus $\mathbf{B}_v = \{\mathbf{z}; \overline{\mathbf{z}}^t Q\mathbf{z} < 1\}$ and $\partial\mathbf{B}_v = \{\mathbf{z}; \overline{\mathbf{z}}^t Q\mathbf{z} = 1\}$. In fact we need that $\mathcal{N}$ defines even a uniform ALS on $\mathcal{C}_e$. This will be verified in the next subsection.

Now notice that according to our choice of $L_1, ..., L_N$

$$b_v(\mathbf{z}, \mathbf{z}') = z_0\overline{z_0'} + ... + z_n\overline{z_n'} \\ + (\sigma_v L_{n+1})(\mathbf{z})\overline{(\sigma_v L_{n+1})(\mathbf{z}')} + ... + (\sigma_v L_N)(\mathbf{z})\overline{(\sigma_v L_N)(\mathbf{z}')}. \tag{7.10}$$

Equation (7.10) shows that $Q = E + R$ where E is the identity matrix and R is a hermitian positive semidefinite matrix. Hence all the eigenvalues of R are non-negative reals. There is a unitary matrix U with $\overline{U}^t J' U = R$ for a diagonal matrix J' whose diagonal entries are the eigenvalues of R. Now $\overline{U}^t (J' + E)U = Q$ and so the eigenvalues $\lambda_0, ..., \lambda_n$ of Q are real numbers satisfying

$$\lambda_i \geq 1 \qquad (0 \leq i \leq n). \tag{7.11}$$

7.2 $\mathcal{N}$ is a uniform adelic-Lipschitz system

Suppose v is infinite then we just have seen that $Q = Q_v$ is a positive definite matrix so that N_v is a norm on K_v^{n+1}. We could apply the observation in Remark 2 to deduce that $\partial\mathbf{B}_v$ is Lipschitz parameterizable. But Remark 2 refers to Appendix A [23] and so we prefer to give a direct proof here. More precisely we will show that $\partial\mathbf{B}_v$ lies in $\mathrm{Lip}(d_v(n+1), 1, 2\pi d_v(n+1))$. But first we need a simple lemma.

Lemma 7.2. *Suppose $p > 1$. Then the $(p-1)$-dimensional unit sphere $\partial B_0(1)$ lies in $Lip(p, 1, 2\pi p)$.*

Proof. Let

$$\varphi : [0, 2\pi] \times [0, \pi]^{p-2} \longrightarrow \partial B_0(1)$$

be the standard parameterization of $\partial B_0(1)$ via polar coordinates $\boldsymbol{\theta} = (\theta_1, ..., \theta_{p-1})$ such that

$$\begin{aligned} x_1 =& \cos\theta_1 \cos\theta_2 \cos\theta_3 ... \cos\theta_{p-1} \\ x_2 =& \sin\theta_1 \cos\theta_2 \cos\theta_3 ... \cos\theta_{p-1} \\ x_3 =& \qquad\quad\ \sin\theta_2 \cos\theta_3 ... \cos\theta_{p-1} \\ \vdots& \\ x_p =& \qquad\qquad\qquad\qquad\quad \sin\theta_{p-1}. \end{aligned}$$

Using the maximum norm $|\cdot|_\infty$ we have $|\partial x_i/\partial\theta_j|_\infty \leq 1$ for $1 \leq i \leq p$ and $1 \leq j \leq p-1$. Applying the Mean-Value Theorem we get $|\varphi(\boldsymbol{\theta}) - \varphi(\boldsymbol{\theta'})| \leq \sqrt{p(p-1)}|\boldsymbol{\theta} - \boldsymbol{\theta'}| \leq p|\boldsymbol{\theta} - \boldsymbol{\theta'}|$. Now normalizing to get a map as in (4.1) with parameter domain $[0, 1]^{p-1}$ gives an additional factor 2π and thereby proves the lemma. □

Now we can prove that the set $\partial\mathbf{B}_v$ is Lipschitz parameterizable with the mentioned parameter values.

Lemma 7.3. *Suppose $v \mid \infty$. Then the set $\partial\mathbf{B}_v = \{\mathbf{z}; N_v(\mathbf{z}) = 1\}$ lies in $Lip(d_v(n+1), 1, 2\pi d_v(n+1))$.*

Proof. The set $\partial\mathbf{B}_v$ is defined by the equation $\overline{\mathbf{z}}^t Q\mathbf{z} = 1$. Since Q is hermitian there is a unitary matrix U with $\overline{U}^t JU = Q$ for a diagonal matrix J whose diagonal entries $\lambda_0, ..., \lambda_n$ are the eigenvalues of Q. Set $\mathbf{y} = U\mathbf{z}$. Then $\partial\mathbf{B}_v = \{\overline{U}^t\mathbf{y}; \overline{\mathbf{y}}^t J\mathbf{y} = 1\} = \overline{U}^t\{\mathbf{y}; \overline{\mathbf{y}}^t J\mathbf{y} = 1\}$. Now $|\overline{U}^t(\mathbf{y}) - \overline{U}^t(\mathbf{y}')| = |\mathbf{y} - \mathbf{y}'|$ so it suffices to check that $\{\mathbf{y}; \overline{\mathbf{y}}^t J\mathbf{y} = 1\}$ lies in $\mathrm{Lip}(d_v(n+1), 1, 2\pi d_v(n+1))$. But the latter set is the image of the unit sphere in $K_v^{n+1} = \mathbb{R}^{d_v(n+1)}$ centered at the origin under the K_v^{n+1}-endomorphism say ϕ, defined by

$$\phi((w_0, ..., w_n)) = (\lambda_0^{-1/2} w_0, ..., \lambda_n^{-1/2} w_n).$$

By the previous lemma we know already that the unit sphere lies in $\mathrm{Lip}(d_v(n+1), 1, 2\pi d_v(n+1))$. So let φ be the corresponding parameterizing map of the sphere then $\phi(\varphi)$ is a parameterization of $\{\mathbf{y}; \overline{\mathbf{y}}^t J\mathbf{y} = 1\}$. We compute a Lipschitz constant:

$$\begin{aligned}|\phi(\varphi(\mathbf{t})) - \phi(\varphi(\mathbf{t}'))| &= |\phi(\varphi(\mathbf{t}) - \varphi(\mathbf{t}'))| \\ &\leq \sup_{|\boldsymbol{w}|=1} |\phi(\boldsymbol{w})||\varphi(\mathbf{t}) - \varphi(\mathbf{t}')| \\ &= \sup_{|\boldsymbol{w}|=1} (\sum_{i=0}^{n} \frac{|w_i|^2}{\lambda_i})^{1/2} |\varphi(\mathbf{t}) - \varphi(\mathbf{t}')|.\end{aligned}$$

Since by (7.11) $\lambda_i \geq 1$ $(0 \leq i \leq n)$ we see that the latter is

$$\begin{aligned}&\leq \sup_{|\boldsymbol{w}|=1} (\sum_{i=0}^{n} |w_i|^2)^{1/2} |\varphi(\mathbf{t}) - \varphi(\mathbf{t}')| \\ &= |\varphi(\mathbf{t}) - \varphi(\mathbf{t}')| \\ &\leq 2\pi d_v (n+1)|\mathbf{t} - \mathbf{t}'|.\end{aligned}$$

This completes the proof of the lemma. □

We write

$$\begin{aligned}C_{\mathcal{N}} &= 1, \\ M_{\mathcal{N}} &= 1, \\ L_{\mathcal{N}} &= 4\pi(n+1).\end{aligned}$$

Lemma 7.4. *The system $\mathcal{N}$ defines a uniform ALS on $\mathcal{C}_e$ of dimension n with associated constants $C_{\mathcal{N}}, M_{\mathcal{N}}, L_{\mathcal{N}}$.*

Proof. Let $\mathcal{N}_K$ be an *ALS* of the collection $\mathcal{N}$. For non-archimedean v in M_K we have

$$\begin{aligned}N_v(\mathbf{z}) &= \max\{|(\sigma_v L_1)(\mathbf{z})|_v, ..., |(\sigma_v L_N)(\mathbf{z})|_v\} \\ &\geq \max\{|(\sigma_v L_1)(\mathbf{z})|_v, ..., |(\sigma_v L_{n+1})(\mathbf{z})|_v\} \\ &= \max\{|z_0|_v, ..., |z_n|_v\}.\end{aligned}$$

The l^2-norm is at least as big as the maximum norm and so we get also for v archimedean

$$N_v(\mathbf{z}) \geq \max\{|z_0|_v, ..., |z_n|_v\}.$$

So for the finite part we may choose according to (4.3) and (4.4)

$$C_{\mathcal{N}_K}^{fin} = 1.$$

For the infinite part we get according to (4.5)

$$C_{\mathcal{N}_K}^{inf} = 1.$$

Combining both parts we end up with

$$C_{\mathcal{N}_K} = C_{\mathcal{N}_K}^{fin} C_{\mathcal{N}_K}^{inf} = 1 = C_{\mathcal{N}}.$$

Lemma 7.3 implies that the sets $\partial \mathbf{B}_v = \{\mathbf{z}; N_v(\mathbf{z}) = 1\}$ lie in $\mathrm{Lip}(d_v(n+1), 1, 4\pi(n+1))$ and thus we may choose $M_{\mathcal{N}_K} = 1, L_{\mathcal{N}_K} = 4\pi(n+1)$. This shows that $\mathcal{N}$ is a uniform ALS of dimension n on $\mathcal{C}_e$ with associated constants $C_{\mathcal{N}} = 1$, $M_{\mathcal{N}} = 1$, $L_{\mathcal{N}} = 4\pi(n+1)$. $\square$

8 Proof of Theorem 1.1

We have a uniform ALS $\mathcal{N}$ of dimension n on $\mathcal{C}_e$ with

$$H_{\mathcal{N}}((z_0 : ... : z_n)) = H_{\mathcal{N}}(\mathbf{z}) = H_2((L_1(\mathbf{z}) : ... : L_N(\mathbf{z}))).$$

By Lemma 7.1 we conclude that $Z_2(\mathbb{V}(k;e), X)$ is given by the number of $(z_0 : ... : z_n)$ in $\mathbb{P}^n(k;e)$ with $H_{\mathcal{N}}((z_0 : ... : z_n)) \leq X$, which we denote by $Z_{\mathcal{N}}(\mathbb{P}^n(k;e), X)$. Furthermore we have the hypothesis $e = 1$ or $n > 5e/2 + 4 + 2/(me)$ in Theorem 1.1. This is exactly the situation where we can apply Theorem 6.1. So we find

$$Z_2(\mathbb{V}(k;e), X) = Z_{\mathcal{N}}(\mathbb{P}^n(k;e), X) = \sum_{\substack{K \\ [K:k]=e}} 2^{-r_K(n+1)} \pi^{-s_K(n+1)} V_{\mathcal{N}_K} S_K(n) X^{me(n+1)}$$
$$+ O(A_{\mathcal{N}} X^{me(n+1)-1} \mathfrak{L}),$$

where $A_{\mathcal{N}} = M_{\mathcal{N}}^{me} (C_{\mathcal{N}}(L_{\mathcal{N}} + 1))^{me(n+1)-1}$ with

$$C_{\mathcal{N}} = 1, \quad M_{\mathcal{N}} = 1, \quad L_{\mathcal{N}} = 4\pi(n+1)$$

and $\mathfrak{L} = \log \max\{2, 2C_{\mathcal{N}} X\}$ if $(me, n) = (1,1)$ and $\mathfrak{L} = 1$ otherwise. Moreover the constant in O depends only on k, e, n and

$$A_{\mathcal{N}} = (4\pi(n+1) + 1)^{me(n+1)-1}$$

depends only on m, e, n. All that remains is to compare the main terms. Therefore we are finished once we have shown that

$$V_{\mathcal{N}_K} = \frac{V(n+1)^{r_K} V(2n+2)^{s_K}}{H_2(S)^d}. \tag{8.1}$$

At this point we make a simple but crucial remark. Recall the general Definition 4.2. Given a positive rational number l and a ALS $\mathcal{N}$ we can define a new ALS $l\mathcal{N}$ by changing each N_v to $|\sigma_v(l)|_v N_v$. However, the volume $V_{l\mathcal{N}} = V_{\mathcal{N}}$ is independent of l. This can be computationally verified from the definitions using the product formula. More intuitively it is clear that the height $H_{l\mathcal{N}} = H_{\mathcal{N}}$ is independent of l, and since $V_{l\mathcal{N}}, V_{\mathcal{N}}$ occur in their respective counting functions, their equality follows.

For the purposes of evaluating $V_{\mathcal{N}_K}$ in (8.1) we are therefore entitled to use $l\mathcal{N}$. Note that changing $\mathcal{N}$ into $l\mathcal{N}$ (with a positive rational number l) changes $L_1, ..., L_N$ from (7.8) into $lL_1, ..., lL_N$. We choose a positive rational integer l such that $lL_1, ..., lL_N$ have coefficients in $\mathcal{O}_K$. In order to keep the notation simple we will redefine $\mathcal{N}_K$ as $l\mathcal{N}_K$ and $L_1, ..., L_N$ as $lL_1, ..., lL_N$; this will cause no confusion. So from (7.4) we get

$$L_j(\mathbf{z}) = lz_{j-1} \quad \text{for } 1 \le j \le n+1. \tag{8.2}$$

And clearly

$$\mathbf{z}^t = (z_0, ..., z_n) \longleftrightarrow (L_1(\mathbf{z}), ..., L_N(\mathbf{z})) \tag{8.3}$$

remains a $(1:1)$-correspondence between $\overline{k}^{n+1}$ and S. For the rest of this paper $\mathcal{N}_K$ will be fixed. Therefore it is convenient to drop the index and simply to write $\mathcal{N}$.

Recall from the definition that $V_{\mathcal{N}}$ splits into a finite and an infinite part. Let $S^{\perp}$ be the orthogonal complement of S consisting of all $\mathbf{y} \in \overline{k}^N$ with $\mathbf{x}^t\mathbf{y} = x_1y_1 + ... + x_Ny_N = 0$ for all $\mathbf{x}$ in S or equivalently $(L_1(\mathbf{z}), ..., L_N(\mathbf{z}))\mathbf{y} = 0$ for all $\mathbf{z}$ in $\overline{k}^{n+1}$. Let $A^{\perp}$ be the $(N-M) \times N$ matrix with columns formed by the coefficients of $L_1, ..., L_N$. Writing $L_r(\mathbf{z}) = \sum_{j=1}^{n+1} l_{j-1}^{(r)} z_{j-1}$ and not forgetting (8.2) we have

$$A^{\perp} = \begin{pmatrix} l & & & l_0^{(n+2)} & \cdots & l_0^{(N)} \\ & \ddots & & \vdots & & \vdots \\ & & l & l_n^{(n+2)} & \cdots & l_n^{(N)} \end{pmatrix}. \tag{8.4}$$

So the first $n+1$ columns of $A^{\perp}$ are given by $(l, 0, ..., 0)^t$, ...,$(0, ..., 0, l)^t$. The equations

$$(L_1(\mathbf{z}), ..., L_N(\mathbf{z}))\mathbf{y} = ((A^{\perp})^t\mathbf{z})^t\mathbf{y} = \mathbf{z}^t A^{\perp}\mathbf{y}$$

show that $S^\perp$ is given by the equation $A^\perp \mathbf{y} = \mathbf{0}$. Now by definition $H_2(S^\perp) = H_2(A^\perp)$ and later on we will use a duality for the height of subspaces (see [17] p.28), telling us that $H_2(S^\perp) = H_2(S)$. This is no surprise since changing signs of certain coordinates of a tuple of Grassmann coordinates of S yields a tuple of Grassmann coordinates of $S^\perp$.

8.1 Computing $V_{\mathcal{N}}^{inf}$

Suppose $v \mid \infty$. The volume V_v is that of the set defined by $\overline{\mathbf{z}}^t Q\mathbf{z} < 1$ where the (i,j) entry of Q is given by

$$b_v(e_i, e_j) = (\sigma_v L_1)(e_i)\overline{(\sigma_v L_1)(e_j)} + ... + (\sigma_v L_N)(e_i)\overline{(\sigma_v L_N)(e_j)}.$$

For v real this is

$$\frac{V(n+1)}{\sqrt{\det Q}}$$

and for v non-real it is

$$\frac{V(2n+2)}{\det Q}.$$

Recalling from above that $L_r(\mathbf{z}) = \sum_{j=1}^{n+1} l_{j-1}^{(r)} z_{j-1}$ we get $b_v(e_i, e_j) = \sum_{r=1}^{N} \sigma_v(l_{i-1}^{(r)})\overline{\sigma_v(l_{j-1}^{(r)})}$ which is the (i,j) entry of $\sigma_v(A^\perp)\overline{\sigma_v(A^\perp)}^t$. So

$$Q = \sigma_v(A^\perp)\overline{\sigma_v(A^\perp)}^t.$$

Therefore the denominator is just the local part of the height $H_2(A^\perp)^d$ (see (3.4) or [10] p.13). This is one of the reasons why it is convenient to choose the l^2-height here. Multiplying V_v over all archimedean places yields

$$V_{\mathcal{N}}^{inf} = \frac{V(n+1)^{r_K} V(2n+2)^{s_K}}{H^{inf}(A^\perp)^d}. \tag{8.5}$$

8.2 Computing $V_{\mathcal{N}}^{fin}$

The finite part is more troublesome. Here we will use the fact that the coefficients $l_{j-1}^{(r)}$ of the linear forms $L_1, ..., L_N$ are algebraic integers. Recall also the definition of $A^\perp$. The matrix $A^\perp$ defines two maps. One from K^{n+1} to K^N by multiplication on the right $\mathbf{z} \longrightarrow (\mathbf{z}^t A^\perp)^t$. Now (8.3) is a $(1:1)$-correspondence, which tells us that this map is injective. The second map comes from multiplication on the left $\mathbf{x} \longrightarrow A^{\perp}\mathbf{x}$, which sends the column $\mathbf{x}$ from K^N to K^{n+1}.

Recall the lattice $\Lambda_{\mathcal{N}}$ from (4.7).

Lemma 8.1. *Let $\mathfrak{A} \neq 0$ be a fractional ideal in K. Then*

$$\Lambda_{\mathcal{N}}(\mathfrak{A}) = \sigma(\mathfrak{A}^N A^{\perp^{-1}}) \tag{8.6}$$

where $A^{\perp}$ is considered as a map $K^{n+1} \longrightarrow K^N$ defined by $\mathbf{z}^t \longrightarrow (\mathbf{z}^t A^{\perp})^t$ and $A^{\perp^{-1}}$ denotes the set-theoretical inverse.

Proof. The lattice $\Lambda_{\mathcal{N}}(\mathfrak{A})$ is given by the $\sigma\boldsymbol{\alpha}$ where

$$\max\{|(\sigma_v L_1)(\boldsymbol{\alpha})|_v, ..., |(\sigma_v L_N)(\boldsymbol{\alpha})|_v\} \leq |\mathfrak{A}|_v \tag{8.7}$$

for all finite v. But (8.7) is equivalent to

$$L_1(\boldsymbol{\alpha}), ..., L_N(\boldsymbol{\alpha}) \in \mathfrak{A}$$

and this in turn means nothing else but $(\boldsymbol{\alpha}^t A^{\perp})^t \in \mathfrak{A}^N$. So the $\boldsymbol{\alpha}$'s are exactly the elements of the set $\mathfrak{A}^N A^{\perp^{-1}}$ and therefore $\Lambda_{\mathcal{N}}(\mathfrak{A}) = \sigma(\mathfrak{A}^N A^{\perp^{-1}})$.□

Now $A^{\perp}$ takes $\mathfrak{A}^{n+1}$ to $\mathfrak{A}^N$, thanks to the integrality of its entries, and so

$$\sigma\mathfrak{A}^{n+1} \subseteq \sigma(\mathfrak{A}^N A^{\perp^{-1}}) = \Lambda_{\mathcal{N}}(\mathfrak{A}). \tag{8.8}$$

We know $\det \sigma\mathfrak{A}^{n+1} = (2^{-s} N\mathfrak{A}\sqrt{|\Delta_K|})^{n+1}$ (see [13] p.33 (5.2) Satz) and so to calculate $\det \Lambda_{\mathcal{N}}(\mathfrak{A})$ it suffices to calculate the index $[\mathfrak{A}^N A^{\perp^{-1}} : \mathfrak{A}^{n+1}]$. This can be done by using some "duality" where the set-up is as follows.

Let W be a finite dimensional $\mathbb{Q}$-vector space and let $b : W \times W \longrightarrow \mathbb{Q}$ be a non-degenerate, symmetric $\mathbb{Q}$-bilinear form. An additive subgroup $G \subset W$ has a dual

$$\widetilde{G} = \{\boldsymbol{w} \in W; b(\boldsymbol{w}, \boldsymbol{g}) \in \mathbb{Z} \text{ for all } \boldsymbol{g} \in G\}, \tag{8.9}$$

which is also an additive subgroup. Suppose $\dim W = D$ and from now on assume G is a free $\mathbb{Z}$-module of rank D so that there exist $\boldsymbol{g}_1, ..., \boldsymbol{g}_D$ in W with

$$G = \boldsymbol{g}_1\mathbb{Z} + ... + \boldsymbol{g}_D\mathbb{Z}.$$

Since $\boldsymbol{g}_1, ..., \boldsymbol{g}_D$ are $\mathbb{Z}$-linearly independent we have

$$W = \boldsymbol{g}_1\mathbb{Q} + ... + \boldsymbol{g}_D\mathbb{Q}. \tag{8.10}$$

The following four lemmas are well-known but for the sake of completeness we include the simple proofs.

Lemma 8.2. *There are $\widetilde{\boldsymbol{g}_1}, ..., \widetilde{\boldsymbol{g}_D}$ in W linearly independent with*

$$b(\widetilde{\boldsymbol{g}}_i, \boldsymbol{g}_j) = \delta_{ij} \tag{8.11}$$

for $1 \leq i, j \leq D$.

Proof. By solving a homogeneous linear system of $D - 1$ equations in D variables we may find a non-zero $\widetilde{\boldsymbol{g}_1}$ in W with $b(\widetilde{\boldsymbol{g}_1}, \boldsymbol{g}_j) = 0$ for $2 \leq j \leq D$. Now suppose $b(\widetilde{\boldsymbol{g}_1}, \boldsymbol{g}_1) = 0$. Then by (8.10) $b(\widetilde{\boldsymbol{g}_1}, \boldsymbol{w}) = 0$ for all $\boldsymbol{w}$ in W but b is non-degenerate and we conclude $\widetilde{\boldsymbol{g}_1} = \mathbf{0}$ - a contradiction. Hence $b(\widetilde{\boldsymbol{g}_1}, \boldsymbol{g}_1) \neq 0$ and so after multiplying $\widetilde{\boldsymbol{g}_1}$ with a suitable rational number we get $b(\widetilde{\boldsymbol{g}_1}, \boldsymbol{g}_1) = 1$. In this way we obtain $\widetilde{\boldsymbol{g}_1}, ..., \widetilde{\boldsymbol{g}_D}$ with (8.11). The linear independence is immediately implied by (8.11). □

Lemma 8.3. *We have*

$$\widetilde{G} = \widetilde{\boldsymbol{g}_1}\mathbb{Z} + ... + \widetilde{\boldsymbol{g}_D}\mathbb{Z}. \tag{8.12}$$

Proof. Clearly the right set is contained in the left one. To prove the other inclusion let $\boldsymbol{w}$ be an element of $\widetilde{G}$. There are $\mu_1, ..., \mu_D$ in $\mathbb{Q}$ with

$$\boldsymbol{w} = \mu_1\widetilde{\boldsymbol{g}_1} + ... + \mu_D\widetilde{\boldsymbol{g}_D}.$$

By definition of $\widetilde{G}$ we have $b(\boldsymbol{w}, \boldsymbol{g})$ in $\mathbb{Z}$ for every $\boldsymbol{g}$ in G. In particular

$$b(\boldsymbol{w}, \boldsymbol{g}_i) \in \mathbb{Z}.$$

But by (8.11) we see that $b(\boldsymbol{w}, \boldsymbol{g}_i) = \mu_i$, which proves the second inclusion. □

Lemma 8.4. *We have*

$$\widetilde{\widetilde{G}} = G.$$

Proof. We have $b(\boldsymbol{g}_i, \widetilde{\boldsymbol{g}}_j) = b(\widetilde{\boldsymbol{g}}_j, \boldsymbol{g}_i) = \delta_{ji}$ for $1 \leq i, j \leq D$. So by Lemma 8.2 and Lemma 8.3 $\widetilde{\widetilde{G}} = \boldsymbol{g}_1\mathbb{Z} + ... + \boldsymbol{g}_D\mathbb{Z} = G$. □

Now let $H \subseteq G$ be a submodule of G also of rank D.

Lemma 8.5. *The indices* $[G : H]$, $[\widetilde{H} : \widetilde{G}]$ *are finite and equal.*

Proof. Using the elementary divisors Theorem (see [8] p.153 Th.7.8) we find a basis $\boldsymbol{g}_1, ..., \boldsymbol{g}_D$ of G such that there are rational integers $a_1, ..., a_D$ with $a_1\boldsymbol{g}_1, ..., a_D\boldsymbol{g}_D$ is a basis of H. So $[G : H] = |a_1...a_D|$. Let $\widetilde{\boldsymbol{g}_1}, ..., \widetilde{\boldsymbol{g}_D}$ be as in Lemma 8.2. By Lemma 8.3 it is a basis of $\widetilde{G}$. Due to (8.11) we have ${a_1}^{-1}\widetilde{\boldsymbol{g}_1}\mathbb{Z} + ... + {a_D}^{-1}\widetilde{\boldsymbol{g}_D}\mathbb{Z} \subseteq \widetilde{H}$. On the other hand every element in $\widetilde{H}$ is of the form $\mu_1\widetilde{\boldsymbol{g}_1} + ... + \mu_D\widetilde{\boldsymbol{g}_D}$ where $\mu_1, ..., \mu_D$ are in $\mathbb{Q}$. Now $a_1\boldsymbol{g}_1 + ... + a_D\boldsymbol{g}_D$ lies in H and therefore by the definition of the dual group and (8.11) we see that in fact $\mu_i a_i \in \mathbb{Z}$ for $1 \leq i \leq D$ and so ${a_1}^{-1}\widetilde{\boldsymbol{g}_1}\mathbb{Z} + ... + {a_D}^{-1}\widetilde{\boldsymbol{g}_D}\mathbb{Z} \supseteq \widetilde{H}$ so ${a_1}^{-1}\widetilde{\boldsymbol{g}_1}\mathbb{Z} + ... + {a_D}^{-1}\widetilde{\boldsymbol{g}_D} = \widetilde{H}$. And now we get $[\widetilde{H} : \widetilde{G}] = |a_1...a_D| = [G : H]$. □

From the general context let us return to our specific situation. We define

$$W = K^{n+1} \text{ and} \tag{8.13}$$

$$G = \mathfrak{A}^N {A^{\perp}}^{-1}, \quad H = \mathfrak{A}^{n+1} \tag{8.14}$$

Clearly $\mathfrak{A}^{n+1} \subseteq G$ and using just the first $n+1$ columns of $A^{\perp}$ we get $G \subseteq l^{-1}\mathfrak{A}^{n+1}$. Now $\mathfrak{A}^{n+1}$ is a free $\mathbb{Z}$-module of rank $D = d(n+1) = \dim_{\mathbb{Q}} K^{n+1}$. Thus G and H are both free $\mathbb{Z}$-modules of rank D. For $b(\cdot, \cdot)$ we choose

$$b(\boldsymbol{w}, \boldsymbol{w}') = Tr_{K/\mathbb{Q}}(\boldsymbol{w}^t\boldsymbol{w}'). \tag{8.15}$$

Here $\boldsymbol{w}^t\boldsymbol{w}' = w_1w_1' + ... + w_{n+1}w_{n+1}'$ means just the scalar product and $Tr_{K/\mathbb{Q}}$ denotes the trace of K relative to $\mathbb{Q}$, from now on abbreviated to Tr. It is well-known ([9] p.214 Satz 3) that $b(\cdot, \cdot)$ defines a non-degenerate, symmetric $\mathbb{Q}$-bilinear form from K^{n+1} to $\mathbb{Q}$ at least for "$n = 0$"; but the extension to any n is clear. Suppose $\boldsymbol{w}$ is such that $Tr(\boldsymbol{w}^t\boldsymbol{w}') \in \mathbb{Z}$ for all $\boldsymbol{w}'$ in $\mathfrak{A}^{n+1}$ and moreover assume $\lambda \in \mathcal{O} = \mathcal{O}_K$. Then $Tr((\lambda\mathbf{x})^t\mathbf{y}) \in \mathbb{Z}$ for all $\mathbf{y}$ in $\mathfrak{A}^{n+1}$. Hence $\widetilde{H}$ is the $(n+1)$-th power of a non-zero fractional ideal, say $\mathfrak{B}$. So

$$\widetilde{H} = \mathfrak{B}^{n+1}. \tag{8.16}$$

Lemma 8.6. *We have*

$$\widetilde{G} = A^{\perp}\mathfrak{B}^N. \tag{8.17}$$

Here $A^{\perp}$ *is considered as a map from* K^N *to* K^{n+1}.

Proof. We abbreviate $A^\perp\mathfrak{B}^N$ to G_0. Again using the first $n+1$ columns of $A^\perp$ it is easily seen that $l\mathfrak{B}^{n+1} \subseteq A^\perp\mathfrak{B}^N$ and clearly $A^\perp\mathfrak{B}^N \subseteq \mathfrak{B}^{n+1}$. Therefore G_0 is a $\mathbb{Z}$-module of rank D. Now let us calculate the dual group of G_0

$$\widetilde{G_0} = \{\boldsymbol{\alpha} \in K^{n+1}; Tr(\boldsymbol{\alpha}^t A^\perp \boldsymbol{\beta}) \in \mathbb{Z} \text{ for all } \boldsymbol{\beta} \in \mathfrak{B}^N\} \tag{8.18}$$

where of course $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both columns. First consider $\widetilde{G_0}A^\perp$ being the set of $(\boldsymbol{\alpha}^t A^\perp)^t$ with $\boldsymbol{\alpha} \in K^{n+1}$ and $Tr(\boldsymbol{\alpha}^t(A^\perp\boldsymbol{\beta})) \in \mathbb{Z}$ for all $\boldsymbol{\beta} \in \mathfrak{B}^N$. Clearly one has $\widetilde{G_0}A^\perp \subseteq K^{n+1}A^\perp$ and since

$$Tr(((\boldsymbol{\alpha}^t A^\perp)^t)^t \boldsymbol{\beta}) = Tr((\boldsymbol{\alpha}^t A^\perp)\boldsymbol{\beta}) = Tr(\boldsymbol{\alpha}^t(A^\perp\boldsymbol{\beta})) \tag{8.19}$$

we see that $\widetilde{G_0}A^\perp \subseteq \widetilde{\mathfrak{B}^N}$ where $\widetilde{\mathfrak{B}^N}$ denotes the dual of $\mathfrak{B}^N$ in K^N with respect to the analogue of (8.15). Hence $\widetilde{G_0}A^\perp \subseteq \widetilde{\mathfrak{B}^N} \cap K^{n+1}A^\perp$. On the other hand suppose $\boldsymbol{\alpha} \in K^{n+1}$ with $(\boldsymbol{\alpha}^t A^\perp)^t \in \widetilde{\mathfrak{B}^N}$. Then (8.19) implies that $\boldsymbol{\alpha}$ is in $\widetilde{G_0}$ and so $(\boldsymbol{\alpha}^t A^\perp)^t \in \widetilde{G_0}A^\perp$. So we conclude $\widetilde{G_0}A^\perp \supseteq \widetilde{\mathfrak{B}^N} \cap K^{n+1}A^\perp$. Combining both inclusions yields

$$\widetilde{G_0}A^\perp = \widetilde{\mathfrak{B}^N} \cap K^{n+1}A^\perp.$$

It follows easily using the injectivity of the map $\mathbf{z} \longrightarrow (\mathbf{z}^t A^\perp)^t$ that

$$\widetilde{G_0} = \widetilde{\mathfrak{B}^N} {A^\perp}^{-1}.$$

By (8.14) and (8.16) it is clear that $\widetilde{\mathfrak{A}^N} = \mathfrak{B}^N$ where again $\widetilde{\mathfrak{A}^N}$ denotes the dual of $\mathfrak{A}^N$ in K^N. By Lemma 8.4 we have $\widetilde{\widetilde{\mathfrak{A}^N}} = \mathfrak{A}^N$. So $\widetilde{\mathfrak{B}^N} = \mathfrak{A}^N$ and it follows that $\widetilde{G_0} = \mathfrak{A}^N {A^\perp}^{-1} = G$. Appealing once more to Lemma 8.4 we obtain $G_0 = \widetilde{G}$ - exactly the claim. □

We are now in a position to compute the determinant of the lattice $\Lambda_{\mathcal{N}}(\mathfrak{A})$.

Lemma 8.7. *Let $\mathfrak{A} \neq 0$ be a fractional ideal in K. Then we have*

$$\det \Lambda_{\mathcal{N}}(\mathfrak{A}) = (2^{-s} N\mathfrak{A}\sqrt{|\Delta_K|})^{n+1} H^{fin}(A^\perp)^d.$$

Proof. By Lemma 8.6 and Lemma 8.5 we get

$$[\mathfrak{A}^N {A^\perp}^{-1} : \mathfrak{A}^{n+1}] = [\mathfrak{B}^{n+1} : A^\perp\mathfrak{B}^N].$$

Now

$$H^{fin}(A^{\perp})^d = [\mathcal{O}^{n+1} : A^{\perp}\mathcal{O}^N]^{-1}$$

by Lemma 2.1 (p.111) in [11]. But here $\mathcal{O}$ can be replaced by any non-zero fractional ideal, which comes out of [11] immediately after equation (2.25) (p.114). We conclude

$$\begin{aligned} \det \Lambda_{\mathcal{N}}(\mathfrak{A}) &= \frac{\det \sigma(\mathfrak{A}^{n+1})}{[\mathfrak{A}^N A^{\perp -1} : \mathfrak{A}^{n+1}]} \\ &= (2^{-s} N\mathfrak{A}\sqrt{|\Delta_K|})^{n+1} H^{fin}(A^{\perp})^d. \end{aligned}$$

□

So the $\Delta_{\mathcal{N}}(\mathcal{D})$ in (4.8) are all equal to

$$(2^{-s}\sqrt{|\Delta_K|})^{n+1} H^{fin}(A^{\perp})^d.$$

Hence by (4.9)

$$V_{\mathcal{N}}^{fin} = H^{fin}(A^{\perp})^{-d}.$$

8.3 Completion of the proof

Now the height of $\mathbb{V}$ is defined as the height of the vector space S. But $H_2(S) = H_2(A)$ and $H_2(S^{\perp}) = H_2(A^{\perp})$ also by definition, and the duality for heights of subspaces says $H_2(S^{\perp}) = H_2(S)$. Finally (8.1) comes out after we recall $H_2(A^{\perp}) = H^{fin}(A^{\perp})H^{inf}(A^{\perp})$. This finishes the proof of Theorem 1.1.

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