A TRACE BOUND FOR POSITIVE DEFINITE CONNECTED INTEGER SYMMETRIC MATRICES

JAMES MCKEE AND PAVLO YATSYNA

Abstract. Let A be a connected integer symmetric matrix, i.e., $A = (a_{ij}) \in M_n(\mathbb{Z})$ for some n, $A = A^T$, and the underlying graph (vertices corresponding to rows, with vertex i joined to vertex j if $a_{ij} \neq 0$) is connected. We show that if all the eigenvalues of A are strictly positive, then $\mathrm{tr}(A) \geq 2n - 1$.

There are two striking corollaries. First, the analogue of the Schur-Siegel-Smyth trace problem is solved for characteristic polynomials of connected integer symmetric matrices. Second, we find new examples of totally real, separable, irreducible, monic integer polynomials that are not *minimal* polynomials of integer symmetric matrices.

1. Introduction and statement of results

Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix, with entries in $\mathbb{Z}$ (an *integer symmetric matrix*). It will be convenient to associate to such a matrix its *underlying graph*, whose vertices $1, \ldots, n$ correspond to the rows of A, and with vertex i joined to vertex j ($j \neq i$) if $a_{ij} \neq 0$. We call the matrix A *connected* if this underlying graph is connected. If the matrix is not connected, then for some permutation matrix P, the matrix P^TAP is in block diagonal form, with blocks corresponding to connected components of the underlying graph: we call these blocks the *components* of A. We write $\mathrm{tr}(A)$ for the trace of A. The eigenvalues of A are the roots of its characteristic polynomial, $p_A(x) = \det(xI - A)$. These are all real, since A is a real symmetric matrix (with integer entries).

The product of the eigenvalues of A is an integer. If all the eigenvalues of A are strictly positive, then this integer is at least 1, and the inequality of arithmetic and geometric means implies that $\mathrm{tr}(A) \geq n$. Our main theorem is a best-possible sharpening of this inequality.

Theorem 1. *Let A be an $n \times n$, connected, integer symmetric matrix. If all the eigenvalues of A are strictly positive, then* $\mathrm{tr}(A) \geq 2n - 1$.

Although we require A to be connected in the statement of the theorem, we do not require that the characteristic polynomial $p_A(x)$ is irreducible. On the other hand, if $p_A(x)$ is irreducible, then A is certainly connected.

The *absolute trace* of a totally positive algebraic integer (one whose galois conjugates are all real and strictly positive) is defined to be its trace divided by its degree. The Schur-Siegel-Smyth trace problem (so-called in [3]; see [2] for a survey) asks if 2 is the smallest limit point of the set of absolute traces of totally positive algebraic integers. It is known that the smallest limit point is at least 1.79193 ([8]; and see [13, 14, 15, 7, 11, 1, 2, 6, 9] for earlier records).

1991 *Mathematics Subject Classification.* 15A42.

Key words and phrases. integer symmetric matrices; trace problem; minimal polynomials.

By analogy, one can define the absolute trace of an $n\times n$ integer symmetric matrix A to be $\mathrm{tr}(A)/n$. In the case where the characteristic polynomial is irreducible, and all eigenvalues are strictly positive, this is just the absolute trace of any one of the eigenvalues. The conclusion of Theorem 1 is equivalent to the assertion that A has absolute trace at least $2 - 1/n$. We then easily get the following corollary.

Corollary 2. *Let X be the set of absolute traces of connected integer symmetric matrices having all their eigenvalues strictly positive. Then the smallest limit point of X is 2. Thus the integer symmetric matrix analogue of the Schur-Siegel-Smyth trace problem is settled.*

Estes and Guralnick [5] considered the question of which separable monic totally real integer polynomials are *minimal* polynomials of integer symmetric matrices. They showed that for degree up to 4, all such polynomials arise as minimal polynomials, and conjectured that this might be true for all larger degrees too. Dobrowolski [4] showed that if the discriminant of a polynomial is too small (compared to a function of its degree), then it cannot be the minimal polynomial of an integer symmetric matrix. He showed the existence of infinitely many counterexamples, derived from cyclotomic polynomials. The smallest degree of any of these counterexamples is 2880. The first author [9] gave a classification of all integer symmetric matrices whose *span* (the difference between the largest and smallest eigenvalues) is strictly less than 4. This led to the discovery of some counterexamples of much lower degree, including three examples of degree 6. Theorem 1 shows that if the trace of an irreducible polynomial is too small (compared to a function of its degree), then it cannot be the characteristic polynomial of an integer symmetric matrix. It is not hard to deduce the following corollary, which can be used to give more counterexamples to the Estes-Guralnick conjecture.

Corollary 3. *Let m be a monic, irreducible integer polynomial of degree n, with all roots real and strictly positive. If the trace of m is strictly less than $2n - 1$, then m is not the* minimal *polynomial of an integer symmetric matrix.*

The smallest degree to which this corollary can usefully be applied is 10. The three degree 10, trace 18 polynomials

$$x^{10} - 18x^9 + 134x^8 - 537x^7 + 1265x^6 - 1798x^5 + 1526x^4 - 743x^3 + 194x^2 - 24x + 1\,,$$
$$x^{10} - 18x^9 + 134x^8 - 538x^7 + 1273x^6 - 1822x^5 + 1560x^4 - 766x^3 + 200x^2 - 24x + 1\,,$$
$$x^{10} - 18x^9 + 135x^8 - 549x^7 + 1320x^6 - 1920x^5 + 1662x^4 - 813x^3 + 206x^2 - 24x + 1\,,$$

which appeared first in [10], give counterexamples to the Estes-Guralnick conjecture that do not yield to the methods of [4] or [9] (the discriminants and spans are too large). For all d, there exists n such that there is a totally positive irreducible polynomial of degree n and trace $2n - d$: see [11]. The smallest degree n known for which the trace is as small as $2n - 3$ is the degree-27, trace-51 example given in [9].

2. Proofs

2.1. **Proof of Theorem 1.** Suppose that the theorem is false. Let $A = (a_{ij})$ be an $n \times n$ counterexample with n minimal, and with the trace minimal for this value of n. In particular, A is symmetric, connected, has all eigenvalues strictly positive, and $\mathrm{tr}(A) \le 2n - 2$.

Let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$. Having chosen a basis, the matrix A defines a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n \times \mathbb{R}^n$ via $\langle e_i, e_j \rangle = a_{ij}$. Since all eigenvalues of

A are strictly positive, this bilinear form is positive definite, and hence $a_{ii} > 0$ for each i.

As A is supposed to be a counterexample, $\mathrm{tr}(A) \leq 2n - 2$. With the diagonal entries being strictly positive integers, at least two of them must equal 1. (In particular, $n \geq 2$.) Shuffling rows and columns (i.e., relabelling the basis vectors), we may assume that $\langle e_1, e_1 \rangle = 1$.

Being a counterexample, the underlying graph of A is connected, so we must have $\langle e_1, e_j \rangle \neq 0$ for some $j > 1$: relabelling we may suppose that $\langle e_1, e_2 \rangle \neq 0$. (Here we use $n \geq 2$. Were it not for this detail, the proof would work for $\mathrm{tr}(A) = 2n - 1$.)

Let $e_2' = e_2 - a_{12}e_1$. Then e_1, e_2', e_3, $\ldots$, e_n is another basis of $\mathbb{R}^n$, and the matrix $A' = (a_{ij}')$ of our bilinear form with respect to this new basis still has integer entries, and all eigenvalues are strictly positive. We have $a_{12}' = a_{21}' = 0$, and $a_{22}' = a_{22} - a_{12}^2$. The matrices A and A' agree in all rows and columns except the second.

The trace of A' is strictly smaller than that of A, since $a_{22}' < a_{22}$, so by the minimality assumptions on our counterexample, we must have that A' is not connected. We now show that A' has exactly two components.

Take any j between 3 and n, and let x_1, $\ldots$, x_r be a *path* from 2 to j in the underlying graph of A (so $x_1 = 2$, $x_r = j$, and for $1 \leq i < r$ we have $a_{x_i x_{i+1}} \neq 0$, and the x_i are distinct; a *walk* is similar to a path, but without the requirement of having distinct vertices). If this path remains a path in the underlying graph of A', then j is in the same component as 2 in A'. Otherwise, $a_{2x_2} \neq 0$, but $a_{2x_2}' = 0$ (this first edge is the only edge in the path that can have been deleted in moving from A to A', as it is the only one incident with vertex 2). Since $a_{2x_2}' = a_{2x_2} - a_{12}a_{1x_2}$, we must have $a_{1x_2} \neq 0$. Replacing $x_1 = 2$ by $x_1 = 1$, we get a walk from 1 to j in the underlying graph of A', so j is in the same component as 1 in A'. Thus A' has at most (and hence exactly) two components: the component containing 1 and the component containing 2.

Let the components of A' have r_1 and r_2 rows, so that $r_1 + r_2 = n$. Let the traces of these components be t_1 and t_2. Since $t_1 + t_2 = \mathrm{tr}(A') < \mathrm{tr}(A) < 2n - 1$, we have $t_1 + t_2 \leq 2(r_1 + r_2) - 3$, and hence either $t_1 < 2r_1 - 1$ or $t_2 < 2r_2 - 1$, or both. So at least one of the two components would give a smaller counterexample, giving a contradiction.

Hence the theorem cannot be false.

2.2. **Proof of Corollary 2.** The tridiagonal $n \times n$ matrix

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

is totally positive (one way to see this is to pick an orthonormal basis e_1, $\ldots$, e_n for $\mathbb{R}^n$, and note that the above matrix is the Gram matrix for the basis e_1, $e_1 + e_2$, $e_2 + e_3$, $\ldots$, $e_{n-1} + e_n$) and has trace $2n - 1$. Connectedness is plain: the underlying graph is a path. Hence 2 is a limit point of the set of absolute traces of connected totally positive integer symmetric matrices.

Take any $\epsilon > 0$. By Theorem 1, any connected totally positive $n \times n$ integer symmetric matrix of absolute trace below $2 - \epsilon$ has $n \leq 1/\epsilon$, so there are finitely many possibilities for n. For each n, the bound on the absolute trace gives a finite set of possibilities for the diagonal entries of the matrix. Then since for any $i < j$ the submatrix

$$\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{pmatrix}$$

must be positive definite, the off-diagonal a_{ij} are bounded. Hence, for any $\epsilon > 0$, the number of connected totally positive $n \times n$ integer matrices of absolute trace below $2 - \epsilon$ is finite, and the corollary follows.

2.3. **Proof of Corollary 3.** Let m be a totally positive irreducible monic integer polynomial of degree n and trace at most $2n - 2$. Suppose that m is the minimal polynomial of an integer symmetric matrix. Then the characteristic polynomial of that matrix is a power of m. If the matrix were not connected, then each component would have m as its minimal polynomial. Hence the smallest integer symmetric matrix A for which m is the minimal polynomial would be connected. Then A would be $rn \times rn$ for some r, with characteristic polynomial m^r having trace at most $2nr - 2r < 2nr - 1$. This would contradict Theorem 1

References

[1] J. Aguirre, M. Bilbao, J.C Peral, The trace of totally positive algebraic integers, *Math. Comp.* **75** (2006), no. 253, 385–393.

[2] J. Aguirre, J.C Peral,The trace problem for totally positive algebraic integers, *Number Theory and Polynomials*, London Mathematical Society Lecture Note Series **352**, 1–19, Cambridge University Press, Cambridge 2008.

[3] P. Borwein, *Computational excursions in analysis and number theory*, CMS Books in Mathematics, **10**, Springer-Verlag, New York 2002.

[4] E. Dobrowolski, A note on integer symmetric matrices and Mahler's measure, *Canad. Math. Bull.* **51** (2008), 57–59.

[5] D.R. Estes, R.M. Guralnick, Minimal Polynomials of Integral Symmetric Matrices, *Linear Algebra and its Applications* **192** (1993), 83–99.

[6] V. Flammang, Trace of totally positive algebraic integers and integer transfinite diameter, *Math. Comp.* **78** (2009), no. 266, 1199–1125.

[7] V. Flammang, M. Grandcolas, G. Rhin, Small Salem numbers, *Number theory in progress*, Vol. 1 (Zakopane-Kościelisko 1997), 165–168, de Gruyter, Berlin, 1999.

[8] Y. Liang, Q. Wu, The trace problem for totally positive algebraic integers, *J. Aust. Math. Soc.* **90** (2011), 341–354.

[9] J. McKee, Computing totally positive algebraic integers of small trace, *Math. Comp.* **80** (2011), no. 274, 1041–1052.

[10] J. McKee, C.J. Smyth, Salem numbers of trace −2 and traces of totally positive algebraic integers, *Algorithmic number theory*, Lecture Notes in Computer Science **3076**, 327–337, Springer Verlag, Berlin, 2004.

[11] ———, There are Salem numbers of every trace, *Bull. London Math. Soc* **37** (2005), 25–36.

[12] I. Schur, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, *Math. Z.* **1** (1918), no. 4, 377–402.

[13] C.L. Siegel, The trace of totally positive and real algebraic integers, *Ann. of Math.* (2) **46** (1945), 302–312.

[14] C.J. Smyth, Totally positive algebraic integers of small trace, *Ann. Inst. Fourier (Grenoble)* **34** (1984), no. 3, 1–28.

[15] ———, The mean values of totally real algebraic integers, *Math. Comp.* **42** (1984), no. 166, 663–681.

Department of Mathematics, Royal Holloway, University of London, Egham Hill, Egham, Surrey, TW20 0EX, England, UK.

E-mail address: james.mckee@rhul.ac.uk, pavlo.yatsyna.2012@live.rhul.ac.uk