

# SOME $H_1\mathfrak{F}$ -GROUPS WITH UNBOUNDED TORSION AND A CONJECTURE OF KROPHOLLER AND MISLIN

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ABSTRACT. Kropholler and Mislin conjectured that groups acting admissibly on a finite-dimensional  $G$ -CW-complex with finite stabilisers admit a finite-dimensional model for  $E_{\mathfrak{F}}G$ , the classifying space for proper actions. This conjecture is known to hold for groups with bounded torsion. In this note we consider a large class of groups  $\mathfrak{U}$  containing the above and many known examples with unbounded torsion. We show that the conjecture holds for a large subclass of  $\mathfrak{U}$ .

## 1. INTRODUCTION

Let  $G$  be a group and denote by  $\mathfrak{F}$  the class of finite groups. A  $G$ -CW-complex  $X$  is a *classifying space for proper actions of  $G$* , or a model for  $E_{\mathfrak{F}}G$ , if the fixed point subcomplex  $X^K$  is contractible for every finite subgroup  $K$  of  $G$ , and empty otherwise. Examples for  $E_{\mathfrak{F}}G$  abound, see for example Lück's survey article [41].

Generalisations of the constructions of Milnor [45] and Segal [54] yield a model for  $E_{\mathfrak{F}}G$  for every group  $G$ . However, these constructions lead to infinite models. The minimal dimension of a model for  $E_{\mathfrak{F}}G$ , denoted  $\mathrm{gd}_{\mathfrak{F}}G$  is called *the Bredon geometric dimension of  $G$* . The most natural algebraic counterpart is Bredon cohomology. In Bredon cohomology there is a well-defined notion of cohomological dimension. For a group  $G$ , *the Bredon cohomological dimension*,  $\mathrm{cd}_{\mathfrak{F}}G$  plays a role analogous to that of the integral cohomological dimension  $\mathrm{cd}G$  in ordinary group cohomology. In particular,  $\mathrm{cd}_{\mathfrak{F}}G$  is finite if and only if  $\mathrm{gd}_{\mathfrak{F}}G$  is finite [39]. Since both dimensions are often very difficult to compute, various authors have proposed alternative geometric and algebraic invariants to guarantee the finiteness of  $\mathrm{cd}_{\mathfrak{F}}G$ , and partial results have been proved, see for example [24, 4, 50].

The first of these invariants are  $\mathrm{silp}G$  and  $\mathrm{spli}G$ , the supremum of the injective lengths of the projective modules and the supremum of the projective lengths of injective modules respectively, introduced by Gedrich and Gruenberg [23] in a different context.

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In 1993 Kropholler introduced the class  $\mathbf{H}\mathfrak{F}$  of *hierarchically decomposable groups* [29]. The class  $\mathbf{H}\mathfrak{F}$  is defined as the smallest class of groups containing the class  $\mathfrak{F}$  and which contains a group  $G$  whenever there is an admissible action of  $G$  on a finite-dimensional contractible cell complex for which all isotropy groups already belong to  $\mathbf{H}\mathfrak{F}$ . The class  $\mathbf{H}\mathfrak{F}$  can also be defined transfinitely. Classes of groups with a hierarchical decomposition defined in terms of suitable actions on finite-dimensional complexes appeared previously in the literature, see for examples [2, 27]. The class of main importance for this note is the class  $\mathbf{H}_1\mathfrak{F}$  [29], the first step in the hierarchy. A group belongs to  $\mathbf{H}_1\mathfrak{F}$  if there is a finite-dimensional contractible  $G$ -CW-complex  $X$  with cell stabilisers in  $\mathfrak{F}$ . Clearly every group with finite Bredon geometric dimension lies in  $\mathbf{H}_1\mathfrak{F}$ , yet it is still unknown whether the converse holds:

**Conjecture 1** (Kropholler-Mislin, [24, 46]). Every  $\mathbf{H}_1\mathfrak{F}$ -group  $G$  admits a finite-dimensional model for  $E_{\mathfrak{F}}G$ .

From a result proved independently by Bouc [9] and Kropholler-Wall [35] it follows that the augmented cellular chain complex  $C_*(X)$  of a finite-dimensional contractible  $G$ -CW-complex with finite stabilisers splits whenever restricted to a finite subgroup of  $G$ .

Group cohomology relative to a  $G$ -set  $\Delta$ , called  $\mathfrak{F}$ -cohomology was introduced in order to algebraically mimic the splitting property of  $\mathbf{H}_1\mathfrak{F}$ -groups [49]. Let  $\Delta$  be a  $G$ -set satisfying

$$\Delta^H \neq \emptyset \iff H \in \mathfrak{F}.$$

Relative cohomology with respect to all finite subgroups is obtained via resolutions of  $\mathbb{Z}$  using direct summands of modules of the form  $M \otimes_{\mathbb{Z}} \Delta$ , which split when restricted to every finite subgroup. One can now define the  $\mathfrak{F}$ -cohomological dimension of a group, denoted  $\mathfrak{F} \text{cd } G$ , as the shortest length of such a resolution. In particular, it turns out that  $\mathfrak{F} \text{cd } G = n$  if and only if there is a resolution by permutation modules with finite stabilisers, which splits when restricted to all finite subgroups. Hence, whenever a group  $G$  admits a finite-dimensional model for  $E_{\mathfrak{F}}G$  or belongs to  $\mathbf{H}_1\mathfrak{F}$ , it has finite  $\mathfrak{F} \text{cd } G$ . For detail the reader is referred to [49, 50]. It is still an open question whether groups of finite  $\mathfrak{F}$ -cohomological dimension belong to  $\mathbf{H}_1\mathfrak{F}$ .

A related invariant is the Gorenstein dimension of the group. A  $\mathbb{Z}G$ -module  $M$  is said to be Gorenstein projective if it admits a complete resolution in the strong sense, i.e. an acyclic complex of projective modules

$$\mathbf{P}_* : \quad \cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots$$

where  $P_0 \twoheadrightarrow M \hookrightarrow P_{-1}$ , and  $\text{Hom}_{\mathbb{Z}G}(\mathbf{P}_*, Q)$  is acyclic for every  $\mathbb{Z}G$ -projective module  $Q$ . We say a group has finite Gorenstein dimension,  $\text{Gcd } G < \infty$ , if the trivial module  $\mathbb{Z}$  has a finite length resolution by Gorenstein-projectives. This notion goes back to Auslander [3] and was recently developed in [4, 13] in the context we are considering here.

**Theorem 1.1.** [23, 4, 19, 50] *For every group  $G$  we have*

$$\mathfrak{F} \text{cd } G < \infty \implies \text{silp } G = \text{spli } G < \infty \iff \text{Gcd } G < \infty.$$

Lück [40] defined a further invariant: Let  $d \geq 0$  be an integer. We say a group satisfies the property  $b(d)$ , if whenever a  $\mathbb{Z}G$ -module  $M$  is projective when restricted to a finite subgroup, then  $\text{pd}_{\mathbb{Z}G} M \leq d$ . A group satisfies  $B(d)$  if for every finite subgroup  $K$ , the Weyl-group  $WK = N_G(K)/K$  satisfies  $b(d)$ . This ties in very closely with the ring  $B(G, \mathbb{Z})$  of bounded functions from  $G$  to  $\mathbb{Z}$  having finite projective dimension as a  $\mathbb{Z}G$ -module. In particular,  $B(G, \mathbb{Z})$  is projective when restricted to a finite subgroup of  $G$ , [34]. Furthermore,

**Proposition 1.2.** [33, Lemma 7.3] [5, Theorem 5.7] *Let  $G$  be a group and let  $d \geq 0$  be an integer.*

- (1)  *$G$  satisfies  $B(d)$  implies that  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) \leq d$ .*
- (2) *For every finite subgroup  $K$  of  $G$ , we have  $\text{pd}_{\mathbb{Z}WK} B(WK, \mathbb{Z}) \leq \text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z})$ .*
- (3) *If  $G \in \mathbf{HF}$  then  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) \leq d$  implies  $G$  satisfies  $B(d)$ .*

This links with the above invariants as follows:

**Proposition 1.3.** [50, Theorem 4.4] [11, Theorem C]

- (1)  $\mathfrak{F} \text{cd } G \leq d \implies G \text{ satisfies } B(d)$
- (2) *If  $G \in \mathbf{HF}$ , then  $\text{silp } G < \infty \implies \text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) < \infty$ .*

The validity of Kropholler and Mislin's Conjecture 1 has been verified in several cases, but a general approach is still missing. In particular, it is known to hold for groups with a bound on the orders of its finite subgroups.

**Theorem 1.4.** [33, Theorem B] *Let  $G \in \mathbf{HF}$  such that  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) < \infty$  and that  $G$  has a bound on the orders of its finite subgroups. Then  $G$  admits a finite-dimensional model for  $E_{\mathfrak{F}}G$ .*

*In particular, let  $G \in \mathbf{H}_1\mathfrak{F}$  with a bound on the lengths of its finite subgroups. Then  $G$  admits a finite-dimensional model for  $E_{\mathfrak{F}}G$ .*

The bound given by Kropholler and Mislin is exponential in  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z})$ . Lück gives a linear bound in  $d$  for groups satisfying  $B(d)$  for some  $d$ . The length of a finite subgroup  $H$  of  $G$  is the supremum of the lengths of all nested sequences  $\{1\} = H_0 < H_1 < H_2 < \dots < H_l = H$  of finite subgroups such that  $H_i \neq H_{i+1}$  for all  $i = 0, \dots, l-1$ .

**Theorem 1.5.** [40, Theorem 1.10] *Let  $G$  be a group satisfying  $B(d)$  for some integer  $d \geq 0$ . Furthermore, suppose there is a bound  $l$  on the lengths of all finite subgroups of  $G$ . Then  $G$  admits a model for  $E_{\mathfrak{F}}G$  of dimension less or equal to  $\max\{3, d\} + l(d+1)$ .*

Proposition 1.3 and Theorem 1.5 led the second author to make the following conjecture:

**Conjecture 2.** [50] *Every group of finite  $\mathfrak{F}$ -cohomological dimension admits a finite-dimensional model for  $E_{\mathfrak{F}}G$ .*

By considering invariants related to  $\text{silp } G$  and  $\text{spli } G$ , see also Theorem 1.1 above, Talelli made the following conjecture:

**Conjecture 3.** [56] Every group with  $\text{silp } G$  finite admits a finite-dimensional model for  $E_{\mathfrak{F}} G$ .

Theorems 1.4 and 1.5 imply, that to verify any of the above conjectures it remains to consider groups with unbounded torsion. There are plenty of examples of groups with unbounded torsion satisfying the above conjecture. For example, every countable locally finite group acts on a tree with finite stabilisers. Furthermore, lamplighter groups are covered by the work of Flores and the second author on elementary amenable groups [20]. On the other hand, fix a prime  $p$ , and then let  $\{P_i\}_{i \in I}$  be an infinite family of finite  $p$ -groups of strictly increasing orders, then the group  $G = *_{i \in I} P_i$  has no bound on the length of its finite subgroups and is non-amenable. Nonetheless,  $G$  acts on a tree with finite stabilisers. In this note we introduce a class of groups that contains all of the above, and discuss the validity of Kropholler and Mislin's conjecture as well as those by Talelli and the second author within this class.

**Definition 1.6.** We define  $\mathcal{U}$  to be the smallest class of groups containing all groups satisfying condition  $B(d)$  for some integer  $d \geq 0$  and having a bound on the orders of their finite subgroups, which is closed under taking extensions and fundamental groups of graphs of groups.

The class  $\mathcal{U}$  contains all groups of finite virtual cohomological dimension, Gromov hyperbolic groups, Burnside groups of large odd exponent, and more generally, all groups of finite Bredon cohomological dimension with a bound on the orders of their finite subgroups. Furthermore it contains all countable locally finite groups, lamplighter groups, Houghton's groups [26], all countable elementary amenable groups, countable free products of finite groups and Dunwoody's inaccessible group [17].

We begin by showing that the class  $\mathcal{U}$  admits a natural hierarchical decomposition over the ordinals and establish some of its basic properties. We prove that Kropholler and Mislin's conjecture as well as those of Talelli and Nucinkis hold within a subclass  $\mathfrak{U}_{\omega_0}^*$  of  $\mathcal{U}$  containing all the above examples.

**Theorem.** *Every  $\mathbf{H}_1 \mathfrak{F}$ -group contained in the class  $\mathfrak{U}_{\omega_0}^*$  admits a finite-dimensional classifying space for proper actions.*

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## 2. THE CLASS $\mathfrak{U}$ AND ITS HIERARCHY

Given a class of groups  $\mathfrak{X}$ , define  $\mathbf{F}_1 \mathfrak{X}$  to be the class consisting of those groups, which are isomorphic to a fundamental group of a graph of groups in  $\mathfrak{X}$ . Let  $\mathfrak{J}$

be the class consisting of the trivial group, then  $\mathbf{F}_1\mathfrak{J}$  is the class of free groups. Furthermore, if  $\mathfrak{X} \subseteq \mathfrak{Y}$  then  $\mathfrak{X} \subseteq \mathbf{F}_1\mathfrak{X} \subseteq \mathbf{F}_1\mathfrak{Y}$ .

Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be classes of groups. We denote by  $\mathfrak{X}\mathfrak{Y}$  the class of groups, which are extensions of groups in  $\mathfrak{X}$  by groups in  $\mathfrak{Y}$ .

**Definition 2.1.** Let  $\mathfrak{B}$  denote the class of groups satisfying condition  $B(d)$  for some integer  $d \geq 0$ , which have a bound on the orders of their finite subgroups. We define:

- $\mathfrak{U}_0 = \mathfrak{J}$
- $\mathfrak{U}_\alpha = (\mathbf{F}_1 \mathfrak{U}_{\alpha-1}) \mathfrak{B}$  if  $\alpha$  is a successor ordinal,
- $\mathfrak{U}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{U}_\beta$  if  $\alpha$  is a limit ordinal.

The class  $\mathfrak{U}$  is defined as  $\mathfrak{U} = \bigcup_{\alpha \geq 0} \mathfrak{U}_\alpha$ .

**Remark 2.2.**  $\mathfrak{U}_1 = (\mathbf{F}_1\mathfrak{J})\mathfrak{B} = \mathfrak{B}$ .

This follows from the fact that any finite extension of a free group admits a 1-dimensional model for  $E_{\mathfrak{F}}G$ . Now apply [40, Theorem 3.1] to give a finite-dimensional model for  $E_{\mathfrak{F}}G$  for any group in  $\mathfrak{U}_1$ , implying that it satisfies  $B(d')$  for some  $d'$ . Since  $G$  is an extension of a torsion-free group by a group with bounded torsion, it has a bound on the orders of its finite subgroups.

**Lemma 2.3.** *The class  $\mathfrak{U}$  coincides with the class  $\mathcal{U}$ .*

*Proof.* Clearly  $\mathfrak{U} \subseteq \mathcal{U}$  and  $\mathfrak{U}$  is closed under taking fundamental groups of graphs of groups. In order to show  $\mathcal{U} \subseteq \mathfrak{U}$  we need only to verify that the class  $\mathfrak{U}$  is extension closed. By Bass-Serre theory it follows that if  $G/N$  acts on a tree  $T$ , then  $G$  has an action on  $T$  such that  $N$  acts trivially. Hence, if  $\mathfrak{X}$  and  $\mathfrak{Y}$  are two classes of groups then  $\mathfrak{X}(\mathbf{F}_1\mathfrak{Y}) \subseteq \mathbf{F}_1(\mathfrak{X}\mathfrak{Y})$ . We argue by induction on  $\beta$  to show  $\mathfrak{U}_\alpha \mathfrak{U}_\beta \subseteq \mathfrak{U}_{\alpha+\beta}$ . If  $\beta = 1$  then by Lemma 2.2  $\mathfrak{U}_\alpha \mathfrak{U}_1 = \mathfrak{U}_\alpha \mathfrak{B} \subseteq (\mathbf{F}_1 \mathfrak{U}_\alpha) \mathfrak{B} = \mathfrak{U}_{\alpha+1}$ .

- Suppose  $\beta$  is a successor ordinal,  $\beta = \gamma + 1$ .

$$\begin{aligned} \mathfrak{U}_\alpha \mathfrak{U}_\beta &= \mathfrak{U}_\alpha((\mathbf{F}_1 \mathfrak{U}_\gamma) \mathfrak{B}) \\ &\subseteq (\mathfrak{U}_\alpha(\mathbf{F}_1 \mathfrak{U}_\gamma)) \mathfrak{B} && \text{(by universality, [53, pg. 2])} \\ &\subseteq (\mathbf{F}_1(\mathfrak{U}_\alpha \mathfrak{U}_\gamma)) \mathfrak{B} && \text{(by the above)} \\ &\subseteq (\mathbf{F}_1(\mathfrak{U}_{\alpha+\gamma})) \mathfrak{B} && \text{(by induction)} \\ &= \mathfrak{U}_{\alpha+\beta}. \end{aligned}$$

- Suppose  $\beta$  be a limit ordinal, then  $\mathfrak{U}_\beta = \bigcup_{\gamma < \beta} \mathfrak{U}_\gamma$ .

$$\begin{aligned} \mathfrak{U}_\alpha \mathfrak{U}_\beta &= \mathfrak{U}_\alpha\left(\bigcup_{\gamma < \beta} \mathfrak{U}_\gamma\right) \\ &= \bigcup_{\gamma < \beta} \mathfrak{U}_\alpha \mathfrak{U}_\gamma \\ &\subseteq \bigcup_{\gamma < \beta} \mathfrak{U}_{\alpha+\gamma} && \text{(by induction)} \\ &= \mathfrak{U}_{\alpha+\beta}. \end{aligned}$$

□

**Proposition 2.4.** *The class  $\mathfrak{U}$  is closed under taking free products with amalgamation, HNN-extensions, countable directed unions, extensions and subgroups.*

*Proof.* It is obvious that  $\mathfrak{U}$  is closed under taking free products with amalgamation and HNN-extensions. If  $G$  is a countable directed union of groups in  $\mathfrak{U}$  then  $G$  acts on a tree with stabilisers conjugate to groups in the directed system [30, Lemma 3.2.3] and so  $G \in \mathfrak{U}$ . In Lemma 2.3 it is shown that the class  $\mathfrak{U}$  is closed under taking extensions.

For any class of groups  $\mathfrak{X}$  we write  $G \in \mathfrak{sX}$  if  $G \leq K \in \mathfrak{X}$ . Note that  $\mathfrak{sF}_1\mathfrak{X} \subseteq \mathfrak{F}_1\mathfrak{sX}$ ; in fact if  $G \in \mathfrak{sF}_1\mathfrak{X}$ ,  $G$  is a subgroup of a group  $K$  that acts on a tree  $T$  with stabilisers in  $\mathfrak{X}$  and so  $T$  is a  $G$ -tree with stabilisers that are subgroups of the stabilisers of the  $K$ -tree  $T$ . Clearly if a class  $\mathfrak{X}$  is subgroup closed then  $\mathfrak{sF}_1\mathfrak{X} \subseteq \mathfrak{F}_1\mathfrak{X}$  and note that the class  $\mathfrak{B}$  is subgroup closed. A transfinite induction as in Lemma 2.3 yields that  $\mathfrak{sU} \subseteq \mathfrak{U}$ . The main observation here is the following: Let  $G \in \mathfrak{U}_\alpha$ ; then any subgroup of  $G$  is an extension of a subgroup of a group in  $\mathfrak{F}_1\mathfrak{U}_{\alpha-1}$  by a subgroup of a group in  $\mathfrak{B}$ . □

For a class of groups  $\mathfrak{X}$ , let  $\mathfrak{F}\mathfrak{X}$  be the smallest class of groups containing the class  $\mathfrak{X}$ , and which contains a group  $G$  whenever  $G$  can be realised as the fundamental group of a graph of groups already in  $\mathfrak{F}\mathfrak{X}$ . The class  $\mathfrak{F}\mathfrak{F}$  was considered by Richard J. Platten in his PhD thesis. Note that the classes  $\mathfrak{F}\mathfrak{B}$  and  $\mathfrak{F}\mathfrak{F}$  differ. Obviously, any non-trivial group of finite cohomological dimension with Serre's property FA does not belong to  $\mathfrak{F}\mathfrak{F}$  but it lies in  $\mathfrak{B}$ ; examples of such groups appear in [52].

**Lemma 2.5.**  *$\mathfrak{U} \subseteq \mathfrak{H}\mathfrak{F}$ ,  $\mathfrak{F}\mathfrak{B} \subsetneq \mathfrak{U}$  and  $\mathfrak{U}$  is not closed under taking quotients.*

*Proof.*  $\mathfrak{B} \subset \mathfrak{H}_1\mathfrak{F}$ . Any group acting on a tree with stabilisers in  $\mathfrak{H}\mathfrak{F}$  obviously lies in  $\mathfrak{H}\mathfrak{F}$ , and  $\mathfrak{H}\mathfrak{F}$  is extension closed by [29, 2.3]. In particular  $\mathfrak{U} \subseteq \mathfrak{H}\mathfrak{F}$ .

Let  $H$  be a non-trivial finite group and let  $P$  be Pride's group of cohomological dimension equal to two with Serre's property FA [52]. Clearly the group  $G = H \wr P$  has no bound on the lengths of its finite subgroups and it lies in  $\mathfrak{U}_2 \setminus \mathfrak{U}_1$ . By [12] the group  $G$  has Serre's property FA, it does not lie in  $\mathfrak{B}$  and so  $G \notin \mathfrak{F}\mathfrak{B}$ . Note that  $\text{cd}_{\mathbb{Q}} G \leq 3$ . We recall that the rational cohomological dimension of a group  $G$ , denoted by  $\text{cd}_{\mathbb{Q}} G$ , is defined as the projective dimension over  $\mathbb{Q}G$  of the trivial module  $\mathbb{Q}$ .

Since the class of free groups  $\mathfrak{Fr}$  is contained in  $\mathfrak{B}$  and  $\mathfrak{U} \subseteq \mathfrak{H}\mathfrak{F}$  it is enough to give an example of a finitely generated group that does not belong to  $\mathfrak{H}\mathfrak{F}$ . The first Grigorchuk group is a 3-generator group but it does not lie in  $\mathfrak{H}\mathfrak{F}$  [21]. Another classical example of such a group is given by the Thompson group  $\mathbf{F}$  [7, 29]. □

### 3. GROUPS IN $\mathfrak{U}$ SATISFYING $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) < \infty$

We start by recalling some basic facts about Bredon cohomology. Let  $G$  be a group and denote by  $\mathcal{O}_{\mathfrak{F}}G$  the orbit category, that is the category having as objects

transitive  $G$ -sets with finite stabilisers and as morphisms the  $G$ -maps. A Bredon-module is a contravariant functor  $M(-) : \mathcal{O}_{\mathfrak{F}}G \rightarrow \mathfrak{Ab}$ . The category of Bredon-modules is an abelian category and has enough projectives. In particular, projectives are direct summands of direct sums of Bredon-modules of the form  $\mathbb{Z}[-, G/K]$ , where  $K \in \mathfrak{F}$  and  $[G/H, G/K]$  denotes the set of all morphisms  $G/H \rightarrow G/K$ . Bredon cohomology groups  $H_{\mathfrak{F}}^n(G, -)$  are now computed via projective resolutions of the constant Bredon module  $\mathbb{Z}(-) = \mathbb{Z}$ . For background the reader is referred to [39], and for a survey to [46].

**Theorem 3.1.** [42, Theorem 0.1] *Let  $G$  be a group of finite Bredon cohomological dimension. Then  $G$  admits a model for  $E_{\mathfrak{F}}G$  of dimension  $\max\{3, \text{cd}_{\mathfrak{F}}G\}$ .*

There now follows a basic lemma in Bredon cohomology, which has also been proved in [16, Corollary 4.7] using spectral sequences. Let  $H$  be a subgroup of  $G$ . We then have a functor

$$I_H : \begin{array}{ccc} \mathcal{O}_{\mathfrak{F}, H}H & \rightarrow & \mathcal{O}_{\mathfrak{F}, G}G \\ H/K & \mapsto & G/K, \end{array}$$

where  $\mathfrak{F}, H$  and  $\mathfrak{F}, G$  denote the families of finite subgroups of  $H$  and  $G$  respectively.

**Lemma 3.2.** *Let  $T$  be a  $G$ -tree with edge set  $E = \bigsqcup_{i \in I} L_i \setminus G$  and vertex set  $V = \bigsqcup_{j \in J} N_j \setminus G$ . Then there is a Mayer-Vietoris sequence:*

$$\begin{aligned} \cdots \rightarrow H_{\mathfrak{F}}^n(G, -) &\rightarrow \bigoplus_{j \in J} H_{\mathfrak{F}}^n(N_j, \text{res}_{I_{N_j}} -) \\ &\rightarrow \bigoplus_{i \in I} H_{\mathfrak{F}}^n(L_i, \text{res}_{I_{L_i}} -) \rightarrow H_{\mathfrak{F}}^{n+1}(G, -) \rightarrow \cdots \end{aligned}$$

*Proof.* By Corollary 3.4 in [32] the augmented Bredon cell complex

$$\mathbb{Z}[-, E] \hookrightarrow \mathbb{Z}[-, V] \rightarrow \mathbb{Z}$$

is a short exact sequence of  $\mathcal{O}G$ -modules. Now applying the long exact sequence in Bredon cohomology we obtain

$$\begin{aligned} \cdots \rightarrow \text{Ext}_{\mathfrak{F}}^n(\mathbb{Z}, -) &\rightarrow \bigoplus_{j \in J} \text{Ext}_{\mathfrak{F}}^n(\mathbb{Z}[-, N_j \setminus G], -) \\ &\rightarrow \bigoplus_{i \in I} \text{Ext}_{\mathfrak{F}}^n(\mathbb{Z}[-, L_i \setminus G], -) \rightarrow \text{Ext}_{\mathfrak{F}}^{n+1}(\mathbb{Z}, -) \rightarrow \cdots \end{aligned}$$

We show that  $\text{Ext}_{\mathfrak{F}, G}^n(\mathbb{Z}[-, H \setminus G], -) \cong \text{Ext}_{\mathfrak{F}, H}^n(\mathbb{Z}, -)$ .

[55, Lemma 2.7] implies that  $\mathbb{Z}[-, H \setminus G] \cong \text{ind}_{I_H} \mathbb{Z}$ . From the adjoint isomorphism it follows that induction along  $I_H$  is left adjoint to restriction along  $I_H$ :

$$\text{mor}_{\mathfrak{F}, G}(\text{ind}_{I_H} \mathbb{Z}, -) \cong \text{mor}_{\mathfrak{F}, H}(\mathbb{Z}, \text{res}_{I_H} -).$$

Hence the result follows.  $\square$

**Corollary 3.3.** *Let  $T$  be a  $G$ -tree with vertex set  $V = \bigsqcup_{j \in J} N_j \setminus G$ . If there is a non-negative integer  $n$  such that  $\text{cd}_{\mathfrak{F}} N_i \leq n$  for all  $i$ , then  $\text{cd}_{\mathfrak{F}} G \leq n + 1$ .*

*Proof.* It is an immediate consequence of Lemma 3.2.  $\square$

Let  $\mathfrak{X}$  be a class of groups admitting a finite-dimensional model for  $E_{\mathfrak{F}}G$ , and let  $\mathbf{F}_{\mathbf{B}}\mathfrak{X}$  be the class of groups consisting of those groups which are isomorphic to a fundamental group of a graph of groups in  $\mathfrak{X}$ , such that, for all vertex groups  $G_\lambda$ , there is a finite bound  $\mathbf{B}$  on the differences between  $\text{pd}_{\mathbb{Z}G} B(G_\lambda, \mathbb{Z})$  and  $\text{cd}_{\mathfrak{F}} G_\lambda$ . Note that for every integer  $m \geq 1$  there are examples of groups such that  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) = 2m$  yet  $\text{cd}_{\mathfrak{F}} G = 3m$  [38].

**Definition 3.4.** We now define a subclass of  $\mathfrak{U}$  using the above closure operation. We define:

- $\mathfrak{U}_0^* = \mathfrak{I}$
- $\mathfrak{U}_\alpha^* = (\mathbf{F}_{\mathbf{B}} \mathfrak{U}_{\alpha-1}^*) \mathfrak{B}$  if  $\alpha < \omega_0$ ,
- $\mathfrak{U}_{\omega_0}^* = \bigcup_{\beta < \omega_0} \mathfrak{U}_\beta^*$ .

**Theorem 3.5.** *Let  $G$  be a group in  $\mathfrak{U}_{\omega_0}^*$  such that  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) < \infty$ . Then  $G$  admits a finite-dimensional model for  $E_{\mathfrak{F}}G$ .*

*In particular, every  $\mathbf{H}_1$   $\mathfrak{F}$ -group contained in the class  $\mathfrak{U}_{\omega_0}^*$  admits a finite-dimensional model for  $E_{\mathfrak{F}}G$ .*

*Proof.* In light of Lück's Theorem 3.1 it suffices to show that  $G$  has finite Bredon cohomological dimension.

Suppose  $G \in \mathfrak{U}_\alpha^*$  and  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) = n$ . If  $\alpha$  is finite, then  $G$  is an extension  $N \hookrightarrow G \twoheadrightarrow Q$  with  $N \in \mathbf{F}_{\mathbf{B}} \mathfrak{U}_{\alpha-1}^*$  and  $Q \in \mathfrak{B}$ . It follows from [11] that  $\text{pd}_{\mathbb{Z}N} B(N, \mathbb{Z}) \leq \text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) = n$ . Hence every vertex group  $N_\lambda$  of the graph of groups associated to  $N$  has  $\text{pd}_{\mathbb{Z}N_\lambda} B(N_\lambda, \mathbb{Z}) \leq n$ , and Bredon cohomological dimension bounded by  $n + \mathbf{B}$ . By Corollary 3.3 we have that  $\text{cd}_{\mathfrak{F}} N \leq n + \mathbf{B} + 1$ . Since  $Q$  is in  $\mathfrak{B}$ , we have that  $\text{cd}_{\mathfrak{F}} Q < \infty$  and that  $Q$  has a bound  $t$  on the orders of its finite subgroups. An application of [40, Theorem 3.1] gives that  $\text{cd}_{\mathfrak{F}} G \leq t(n + \mathbf{B} + 1) + \text{cd}_{\mathfrak{F}} Q < \infty$ .

Now suppose  $\alpha = \omega_0$  and  $G \in \mathfrak{U}_{\omega_0}^* = (\bigcup_{\beta < \omega_0} \mathfrak{U}_\beta^*)$ . Hence  $G \in \mathfrak{U}_\beta^*$  for some finite  $\beta$  and we apply the above.  $\square$

The examples in [38] show, that in general  $t \neq 1$ .

**Remark 3.6.** One might be tempted to define a slightly larger class of groups by replacing “bound on the orders of the finite subgroups” by “bound on the lengths of the finite subgroups” throughout. Let  $\mathfrak{B}'$  denote the class of groups satisfying condition  $B(d)$  for some integer  $d \geq 0$ , which have a bound on the lengths of their finite subgroups. We then define  $\mathfrak{U}'$  as above. This gives a class equivalent to  $\mathfrak{U}'$  also defined analogously to the above. But for this class, we cannot use an analogous argument to that of Theorem 3.5, as Lück's Theorem [40, Theorem 3.1] requires the quotient group to have a bound on the orders of the finite subgroups. We have the following analogue to Remark 2.2:

**Lemma.** *Let  $H \hookrightarrow G \twoheadrightarrow Q$  be a group extension, where  $H$  is torsion-free of finite cohomological dimension  $\text{cd} H = n$  and  $Q$  satisfies  $B(d)$  for some integer  $d \geq 0$  and has a bound  $l$  on the lengths of its finite subgroups. Then  $G$  admits a finite-dimensional model for  $E_{\mathfrak{F}}G$  of dimension  $\max\{3, l + n + d\}$ .*

In particular,  $\mathfrak{U}'_1 = (\mathbf{F}_1\mathfrak{J})\mathfrak{B}' = \mathfrak{B}'$ .

*Proof.* The first assertion is basically a special case of [43, Corollary 5.2]. Every finite subgroup of  $G$  is isomorphic to a finite subgroup of  $Q$  with length bounded by  $l$ . Hence, every finite extension  $\Gamma$  of  $H$  is virtually torsion-free and the lengths of the finite subgroups are bounded by  $l$ . Hence, by [40, Theorem 6.4]  $\Gamma$  admits a  $\max\{3, n\} + l$ -dimensional model for  $E_{\mathfrak{F}}\Gamma$ . By 1.5  $Q$  admits a  $\max\{3, d\} + l(d + 1)$ -dimensional model for  $E_{\mathfrak{F}}Q$ . Now apply [43, Corollary 5.2] giving a  $\max\{3, n\} + \max\{3, d\} + l(d + 2)$ -dimensional model for  $E_{\mathfrak{F}}G$ .

This bound can be improved to that of the claim as follows: [4, Theorem 2.7] and [56, Proposition 2.3] imply that  $\text{pd}_{\mathbb{Z}Q} B(Q, \mathbb{Z}) = \text{Gcd } Q = m \leq d$ . Now one can use [4, Theorem 2.8], which states that  $\text{Gcd } G \leq \text{Gcd } H + \text{Gcd } Q = \text{cd } H + \text{Gcd } Q$ , hence  $\text{Gcd } G \leq n + m$ . Since, by the above,  $\text{cd}_{\mathfrak{F}} G < \infty$ , and in particular  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) < \infty$ , a further application of [4, Theorem 2.7] yields that  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) = \text{Gcd } G$ . We now apply [44, Theorem 3.10], to get that  $\text{cd}_{\mathfrak{F}} G \leq \text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) + l \leq n + d + l$  as required.

Now let  $G \in \mathfrak{U}_1$ . Hence it is an extension of a free group by a group  $Q$  in  $\mathfrak{B}$ . By the above,  $G$  admits a finite-dimensional model for  $E_{\mathfrak{F}}G$ , hence satisfies  $B(d')$  for some  $d'$ , and has a bound on the lengths of its finite subgroups.  $\square$

**Theorem 3.7.** *Suppose that there exists a function  $\rho : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\text{gd}_{\mathfrak{F}} G \leq \rho(\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}))$  for every group  $G$  of finite Bredon cohomological dimension. Then every group in  $\mathfrak{U}$  such that  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) < \infty$  admits a finite-dimensional model for  $E_{\mathfrak{F}}G$ .*

*In particular, in this case, Kropholler and Mislin's conjecture holds inside  $\mathfrak{U}$ .*

*Proof.* This can be proved similarly to Theorem 3.5, using transfinite induction. Let  $G \in \mathfrak{U}$  such that  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) = n$ . We prove that  $\text{cd}_{\mathfrak{F}} G \leq \rho(n)$ . The claim is obviously true for  $G \in \mathfrak{U}_0$  and  $G \in \mathfrak{U}_1$ . Now let  $G \in \mathfrak{U}_\alpha$ . If  $\alpha$  is a successor ordinal, we have, as above that  $G$  is an extension  $N \hookrightarrow G \twoheadrightarrow Q$  with  $N$  a graph of groups  $N_\lambda \in \mathfrak{U}_{\alpha-1}$ . Now by induction,  $\text{cd}_{\mathfrak{F}} N_\lambda \leq \rho(n)$  and hence, as above,  $\text{cd}_{\mathfrak{F}} G \leq t(\rho(n) + 1) + \text{cd}_{\mathfrak{F}} Q < \infty$ . Hence, by assumption  $\text{cd}_{\mathfrak{F}} G \leq \rho(n)$ .

For  $\alpha$  a limit ordinal we have that  $G \in \mathfrak{U}_\beta$  for some  $\beta < \alpha$  and we are done.  $\square$

**Remark 3.8.** Every group admitting a finite model for  $E_{\mathfrak{F}}G$  lies in  $\mathfrak{B}$ , as these have finitely many conjugacy classes of finite subgroups [40]. This includes free groups, Gromov-hyperbolic groups [48],  $\text{Out}(F_n)$ , where  $F_n$  is a free group of rank  $n$  [57], mapping class groups [47] and elementary amenable groups of type  $\text{FP}_\infty$  [32].

**Examples 3.1.** We list further examples of groups in  $\mathfrak{U}$ .

- (1) Countable locally free groups belong to  $\mathfrak{U}_1$  as they are torsion-free of Bredon-cohomological dimension  $\leq 2$ .
- (2) Free Burnside groups  $B(m, n)$  of large odd exponent lie in  $\mathfrak{U}_1 \setminus \mathfrak{U}_0$ . It is known by [1] that  $B(m, n)$  are infinite for large enough exponent and that

they have a bound on the orders of their finite subgroups. By [28] they admit an action on a contractible 2-dimensional CW-complex with cyclic stabilisers and hence are contained in  $\mathfrak{B} \setminus \mathfrak{J}$ .

Hence Petrosyan's class  $\mathcal{N}^{cell}(\mathcal{P}_6) \neq \mathfrak{U}$ . If  $G \in \mathcal{N}^{cell}(\mathcal{P}_6)$ , then either it contains a free subgroup on two generators or it is countable elementary amenable [51, Theorem 3.9]. A finitely generated infinite periodic group cannot be elementary amenable, therefore free Burnside groups of large odd exponent are not contained in  $\mathcal{N}^{cell}(\mathcal{P}_6)$  but they belong to  $\mathfrak{B}$ .

- (3)  $\mathfrak{U}_2$  contains all countable abelian groups, and  $\mathfrak{U}$  contains all countable elementary amenable groups. Every finitely generated abelian group lies in  $\mathfrak{B}$  and so every countable abelian group  $G$  can be realised as a group acting on a tree with finitely generated abelian stabilisers and so  $G \in \mathfrak{U}_2$ . Clearly  $\mathfrak{B}$  contains all finite groups and  $\mathfrak{U}$  is closed under taking countable directed unions and so it contains all countable locally finite groups. By Proposition 2.4,  $\mathfrak{U}$  is closed under taking extensions and so it contains all countable elementary amenable groups.
- (4) Let  $\{F_i\}_{i \in I}$  be an infinite countable ordered family of finite subgroups such that  $|F_i| < |F_{i+1}|$ . If  $G = *_{i \in I} F_i$ , then  $G \in \mathfrak{U}_2 \setminus \mathfrak{U}_1$ .  $G$  has no bound on the orders of its finite subgroups but it is realised as the fundamental group of a graph of finite groups, and so  $G \in \mathfrak{U}_2 \setminus \mathfrak{U}_1$ .
- (5) For every  $n$ , Houghton's groups  $\mathfrak{H}_n$  lie in  $\mathfrak{U}_2 \setminus \mathfrak{U}_1$ . The group  $\mathfrak{H}_n$  is isomorphic to an extension of the infinite countable finitary symmetric group  $\Theta$  by  $\mathbb{Z}^{n-1}$ . The group  $\Theta$  is countable [15, Exercise 8.1.3] and locally finite. Hence  $\Theta \in \mathbf{F}_1 \mathfrak{U}_1$  and  $\mathfrak{H}_n \in \mathfrak{U}_2$ .
- (6) Dunwoody's inaccessible group  $\mathfrak{D}$  lies in  $\mathfrak{U}_3$  [17].

The group  $\mathfrak{D}$  is the fundamental group of a graph  $X$  of groups. Every edge group is finite and the only non-finite vertex group is isomorphic to a free product with amalgamation  $Q_n *_{H_\omega} H$ . Where  $Q_n$  is the fundamental group of an infinite graph of groups with all finite edge and vertex groups,  $H_\omega$  is an infinite countable locally finite group, hence lies in  $\mathfrak{U}_2 \setminus \mathfrak{U}_1$ .  $H$  is isomorphic to a semidirect product of the infinite finitary symmetric group on a countable set by an infinite cyclic group, hence  $H \in \mathfrak{U}_2 \setminus \mathfrak{U}_1$ . It is clear from the construction that  $\mathfrak{D} \in \mathfrak{U}_3$  and  $\text{cd}_{\mathbb{Q}} \mathfrak{D} \leq 4$ . Note that  $\mathfrak{D}$  has no bound on the orders of its finite subgroups, which follows from its construction or from Linnell's theorem on inaccessible groups [36].

With the exception of elementary amenable groups, it can be seen directly that all the above groups lie in  $\mathfrak{U}_{\omega_0}^*$ , since for finite groups  $K$ ,  $\text{cd}_{\mathfrak{F}} K = \text{pd}_{\mathbb{Z}K} B(K, \mathbb{Z}) = 0$  and for finitely generated abelian groups  $A$ ,  $\text{cd}_{\mathfrak{F}} A = \text{pd}_{\mathbb{Z}A} B(A, \mathbb{Z}) = h(A)$ , the Hirsch-length of  $A$ .

**Lemma 3.9.** *Countable elementary amenable groups lie in  $\mathfrak{U}_{\omega_0}^*$ .*

*Proof.* Suppose first that  $G$  is an elementary amenable group with  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) = d$ . [4, Theorem 2.7] and [56, Proposition 2.1] imply that  $\text{spli } G \leq d + 1$ . It now follows

from [23], that  $h(G) \leq d + 1$ . Furthermore, for countable elementary amenable groups,  $\text{cd}_{\mathfrak{F}} G \leq h(G) + 1$  [20]. Thus  $\text{cd}_{\mathfrak{F}} G \leq \text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) + 2$ . The general case now follows directly using the hierarchical description of elementary amenable groups given in [31]: Let  $\mathfrak{Y}$  denote the class of all finitely generated abelian-by-finite groups. For each ordinal  $\alpha$  the class  $\mathfrak{X}_\alpha$  is defined by

$$\begin{aligned} \mathfrak{X}_0 &= \mathfrak{F} \\ \mathfrak{X}_\alpha &= (L \mathfrak{X}_{\alpha-1}) \mathfrak{Y} \quad \text{if } \alpha \text{ is a successor ordinal,} \\ \mathfrak{X}_\alpha &= \bigcup_{\beta < \alpha} \mathfrak{X}_\beta \quad \text{if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

The class of all elementary amenable groups is defined by setting

$$\mathfrak{X} = \bigcup_{\alpha \geq 0} \mathfrak{X}_\alpha.$$

Note that countable elementary amenable groups lie in  $\mathfrak{X}_{\omega_0}$ . Let  $\mathfrak{J}$  be a class of groups and let  $G \in L\mathfrak{J}$  be a countable group, for which there is a finite  $\mathbf{B}$  such that for every finitely generated subgroup  $H$  of  $G$ , we have  $\text{cd}_{\mathfrak{F}} H \leq \text{pd}_{\mathbb{Z}H} B(H, \mathbb{Z}) + \mathbf{B}$ . Then automatically,  $G \in \mathbf{F}_\mathbf{B}\mathfrak{J}$ . Since finitely generated abelian-by-finite groups lie in  $\mathfrak{B}$  and by the above, for every countable group in  $\mathfrak{X}$ , there is such a  $\mathbf{B} = 2$ , the claim follows.  $\square$

Also note that elementary amenable groups of finite Hirsch-length lie in  $\mathfrak{U}_2^*$ . By a Theorem of Hillman and Linnell [25] they are locally finite-by-virtually (torsion-free soluble). Since virtually torsion-free soluble groups of finite Hirsch length belong to  $\mathfrak{B}$  and finite groups satisfy  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) = \text{cd}_{\mathfrak{F}} G = 0$ , the above claim follows.

Let  $\underline{G}$  be the finitely generated group constructed in [14] admitting the remarkable decomposition  $\underline{G} = A *_\mathbb{Z} \underline{G}$ .

**Proposition 3.10.** *The group  $\underline{G}$  has a bound on the orders of its finite subgroups and belongs to  $\mathfrak{B}$ .*

*Proof.* In [18] it is shown that the group  $\underline{G}$  can be realised as the fundamental group of a graph of groups  $Y$  with two orbits of vertices  $VY$  and two orbits of edges  $EY$ . Each finite subgroup of  $\underline{G}$  must lie in one of the conjugates of the  $A$  factors. Since  $A = \langle a; b \mid b^3 = 1; a^{-1}ba = b^{-1} \rangle$  every finite subgroup has order bounded by 3. Moreover,  $\underline{G} \cong G_1 *_L G_2$  where  $G_1$  and  $G_2$  are fundamental groups of graphs of groups with all virtually cyclic stabilisers, and  $L$  is locally infinite cyclic. Hence, by Lemma 3.2 and [40, Theorem 3.1]  $\underline{G} \in \mathfrak{B}$ .  $\square$

This is in contrast with finitely generated infinite groups of the form  $G = A \times G$  with  $A$  infinite. By an argument similar to [21, Theorem 4.11], we see that these groups must have infinite rational cohomological dimension. Tensoring with  $\mathbb{Q}$  over  $\mathbb{Z}$  the augmented cellular chain complex of a classifying space for proper actions leads to a  $\mathbb{Q}G$ -projective resolution of the trivial module  $\mathbb{Q}$ , therefore if  $G = A \times G$  (with  $A$  infinite) then  $G$  has infinite Bredon cohomological dimension.

**Remark 3.11.** Note that if there is a countable (periodic)  $\mathbf{H}_1 \mathfrak{F}$ -group that does not lie in  $\mathfrak{U}$ , then there exists a finitely generated (periodic)  $\mathbf{H}_1 \mathfrak{F}$ -group with no bound on the orders of its finite subgroups. To see this, suppose that  $G$  is a countable (periodic)  $\mathbf{H}_1 \mathfrak{F}$ -group not belonging to  $\mathfrak{U}$ . Then  $G$  is the directed union of its finitely generated (periodic) subgroups, which are  $\mathbf{H}_1 \mathfrak{F}$ -groups. Hence  $G$  acts on a tree with stabilisers conjugate to groups in the directed union. If every stabiliser was in  $\mathfrak{U}$  so would be  $G$ , giving a contradiction. Hence, in particular, at least one of the stabilisers does not lie in  $\mathfrak{B}$ , hence it can't have a bound on the orders of its finite subgroups.

**Question 3.12.** Are there countable  $\mathbf{H}_1 \mathfrak{F}$ -groups not contained in the class  $\mathfrak{U}$ ?

The easiest example we know of, which lies in  $\mathbf{H}_1 \mathfrak{F}$ , yet it is unknown whether it belongs to  $\mathfrak{U}$ , is  $SL_3(\mathbb{F}_p[t])$ . Hence it might be advisable to consider the class of  $S$ -arithmetic groups over global function fields. We start by fixing some notation. Let  $S$  be a finite non-empty set of pairwise inequivalent valuations on a global function field  $K$ . Let  $\mathcal{O}_S \leq K$  be the ring of  $S$ -integers and let  $\mathbf{G}$  be a reductive  $K$ -group. Given a valuation  $v$  of  $K$ ,  $K_v$  is the completion of  $K$  with respect to  $v$ . If  $L/K$  is a field extension, the  $L$ -rank of  $\mathbf{G}$ ,  $\text{rank}_L \mathbf{G}$  is the dimension of a maximal  $L$ -split torus of  $\mathbf{G}$ . The  $K$ -group  $\mathbf{G}$  is  $L$ -isotropic if  $\text{rank}_L \mathbf{G} \neq 0$ . For any connected non-commutative absolutely almost simple  $K$ -isotropic  $K$ -group  $\mathbf{G}$ , the  $S$ -arithmetic subgroup  $\mathbf{G}(\mathcal{O}_S)$  acts on a building  $X$  of dimension equal to  $k(\mathbf{G}, S) := \sum_{v \in S} \text{rank}_{K_v} \mathbf{G}$  [10]. It turns out that the building  $X$  is a classifying space for proper actions for  $\mathbf{G}(\mathcal{O}_S)$ , see the theorem below. From the homological properties of  $\mathbf{G}(\mathcal{O}_S)$  we are able to easily determine most of its algebraic and geometric dimensions.

We introduce the notion of *Kropholler dimension*, denoted  $\mathfrak{K}(G)$ : given a group  $G$ , it is the minimal dimension of a contractible  $G$ -CW-complex with finite stabilisers. By definition,  $G \in \mathbf{H}_1 \mathfrak{F}$  if and only if  $\mathfrak{K}(G) < \infty$ .

**Theorem 3.13.** *Let  $\mathbf{H}$  be a connected non-commutative absolutely almost simple  $K$ -isotropic  $K$ -group. Then  $\text{gd}_{\mathfrak{F}} \mathbf{H}(\mathcal{O}_S) = \mathfrak{K}(\mathbf{H}(\mathcal{O}_S)) = \mathfrak{F} \text{cd} \mathbf{H}(\mathcal{O}_S) = \text{cd}_{\mathbb{Q}} \mathbf{H}(\mathcal{O}_S) = k(\mathbf{H}, S)$ .*

*Proof.* Let  $H$  be  $\prod_{v \in S} \mathbf{H}(K_v)$ , then there is a  $k(\mathbf{H}, S)$ -dimensional Euclidean building  $X$  associated to  $H$ . Since  $\mathbf{H}(\mathcal{O}_S)$  acts with finite stabilisers on  $X$ , and this admits a CAT(0)-metric, by [6, Proposition 5] we obtain that  $\text{gd}_{\mathfrak{F}} \mathbf{H}(\mathcal{O}_S) \leq k(\mathbf{H}, S)$ . By [8] the group  $\mathbf{H}(\mathcal{O}_S)$  is of type  $\text{FP}_{k(\mathbf{H}, S)-1}$ . Note that  $\mathbf{H}(\mathcal{O}_S)$  has no bound on the orders of its finite subgroups as remarked in [22]. If the rational cohomological dimension of  $\mathbf{H}(\mathcal{O}_S)$  was less than  $k(\mathbf{H}, S)$ , an application of [37, Proposition 1] would give that  $\mathbf{H}(\mathcal{O}_S)$  has a bound on the orders of its finite subgroups, a contradiction. Therefore we have  $\text{cd}_{\mathbb{Q}} \mathbf{H}(\mathcal{O}_S) = k(\mathbf{H}, S)$  and the result follows.  $\square$

#### 4. SOME CONSEQUENCES

Let us begin by recording that Proposition 1.3 and Theorem 3.5 imply that both Talelli's Conjecture 3 and the second author's Conjecture 2 hold within  $\mathfrak{U}_{\omega_0}^*$ .

Furthermore, if we assume that there are functions  $\rho_1, \rho_2 : \mathbb{N} \rightarrow \mathbb{N}$  such that for every group of finite Bredon cohomological dimension we have  $\text{cd}_{\mathfrak{F}} G \leq \rho_1(\mathfrak{F} \text{cd} G)$  and  $\text{cd}_{\mathfrak{F}} G \leq \rho_2(\text{silp} G)$  respectively, then arguments analogous to those in Theorem 3.7 imply that both conjectures hold in  $\mathfrak{U}$ .

**Lemma 4.1.** *Let  $T$  be a  $G$ -tree with edge set  $E = \bigsqcup_{i \in I} L_i \setminus G$  and vertex set  $V = \bigsqcup_{j \in J} N_j \setminus G$ . Then  $\mathfrak{F} \text{cd}(G) \leq \sup\{\mathfrak{F} \text{cd} N_j \mid j \in J\} + 1$ .*

*Proof.* This is an immediate consequence of the Mayer-Vietoris sequence in  $\mathfrak{F}$ -cohomology associated to the short exact  $\mathfrak{F}$ -split sequence:  $\mathbb{Z} E \hookrightarrow \mathbb{Z} V \twoheadrightarrow \mathbb{Z}$  [50].  $\square$

**Corollary 4.2.** *Suppose Conjecture 2 holds. Then there exists a function  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\text{gd}_{\mathfrak{F}} G \leq \phi(\mathfrak{F} \text{cd}(G))$  for every group  $G$  of finite  $\mathfrak{F}$ -cohomological dimension.*

*Proof.* Assume by contradiction that there is no such function. Then there exists an  $n > 1$  and a family of groups  $\{G_i\}_{i \in \mathbb{N}}$  such that  $\mathfrak{F} \text{cd} G_i \leq n$  for every  $i$  and  $\lim_{i \rightarrow \infty} \text{gd}_{\mathfrak{F}} G_i = \infty$

The group  $G = \ast_{i \in \mathbb{N}} G_i$  has  $\mathfrak{F} \text{cd} G \leq n + 1$  but  $\text{gd}_{\mathfrak{F}} G = \infty$  a contradiction. The inequality  $\mathfrak{F} \text{cd} G \leq n + 1$  follows from Lemma 4.1. Since  $G$  contains subgroups of arbitrarily large Bredon geometric dimension we have  $\text{gd}_{\mathfrak{F}} G = \infty$ .  $\square$

**Remark 4.3.** We can also make a statement in the same manner as in Corollary 4.2, involving  $\text{silp} G = \text{spli} G$  or  $\text{pd}_{\mathbb{Z}G}(B(G, \mathbb{Z}))$  and Conjecture 3. The proof is analogous to the above. In particular: Let  $\lambda(G)$  denote either  $\text{silp} G = \text{spli} G$  or  $\text{pd}_{\mathbb{Z}G}(B(G, \mathbb{Z}))$ . If Conjecture 3 holds, then there is a function  $\psi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\text{gd}_{\mathfrak{F}} G \leq \psi(\lambda(G))$  for every group  $G$  with  $\lambda(G) < \infty$ .

**Theorem 4.4.** *Let  $T$  be a  $G$ -tree with edge set  $E = \bigsqcup_{i \in I} L_i \setminus G$  and vertex set  $V = \bigsqcup_{j \in J} N_j \setminus G$ . Then  $\mathfrak{K}(G) \leq \sup\{\mathfrak{K}(N_j)_{j \in J}\} + 1$ . In particular  $G \in \mathbf{H}_1 \mathfrak{F}$  if and only if there is a bound on the Kropholler dimensions of the edge and vertex groups.*

*Proof.* Replace the edge and vertex groups with suitable  $\mathbf{H}_1 \mathfrak{F}$ -spaces of minimal dimension and proceed as in [41] to obtain an  $\mathbf{H}_1 \mathfrak{F}$ -space for  $G$  of dimension equal to  $\sup\{\mathfrak{K}(N_j)_{j \in J}\} + 1$ .  $\square$

**Corollary 4.5.** *If Conjecture 1 holds, then there exists a function  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\text{gd}_{\mathfrak{F}} G \leq \gamma(\mathfrak{K}(G))$  for every  $\mathbf{H}_1 \mathfrak{F}$ -group  $G$ .*

*Proof.* This follows from Theorem 4.4, and can be proved analogously to Corollary 4.2.  $\square$

**Question 4.6.** Does there exist a function  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\text{gd}_{\mathfrak{F}} G \leq \gamma(\mathfrak{K}(G))$ ? for all groups  $G$  admitting a finite-dimensional model for  $E_{\mathfrak{F}} G$ .

Can we replace  $\mathfrak{K}(G)$  above by  $\mathfrak{F} \text{cd} G, \text{spli} G = \text{silp} G$  or  $\text{pd}_{\mathbb{Z}G} B(G, \mathbb{Z})$ ?

The only known examples where this function is not the identity are those of [38], mentioned above.

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