CHARACTERIZATIONS OF VERTEX PANCYCLIC AND PANCYCLIC ORDINARY COMPLETE MULTIPARTITE DIGRAPHS

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Abstract. A digraph obtained by replacing each edge of a complete multipartite graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a complete multipartite digraph. Such a digraph D is called ordinary if for any pair X, Y of its partite sets the set of arcs with end vertices in $X \cup Y$ coincides with $X \times Y = \{x, y) : x \in X,\ y \in Y\}$ or $Y \times X$ or $X \times Y \cup Y \times X$. We characterize all the pancyclic and vertex pancyclic ordinary complete multipartite digraphs. Our characterizations admit polynomial time algorithms.

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1. Introduction

A digraph D on m disjoint vertex classes (*partite sets*) is called a *complete m-partite or multipartite digraph* (abbreviated to CMD) if for any two vertices u, v in different partite sets either (u, v) or (v, u) (or both) is an arc of D and there are no arcs between vertices which are found in a same partite set. Such a digraph D is called *ordinary* if for any pair X, Y of its partite sets, the set of arcs with end vertices in $X \cup Y$ coincides with $X \times Y = \{(x, y) : x \in Y,\ y \in Y\}$ or $Y \times X$ or $X \times Y \cup Y \times X$. A complete m-partite digraph is called an *m-partite or multipartite tournament* if it has no cycles of length two. A digraph H is *pancyclic* if it contains a simple cycle of length i (*i-cycle*) for any $3 \leq i \leq n$, where n is the order of H. H is *vertex pancyclic* if it has an i-cycle containing v for any $v \in V(H)$, $3 \leq i \leq n$. We assume that every digraph with one or two vertices is pancyclic and vertex pancyclic. Even pancyclicity and vertex even pancyclicity are defined analogously: in this case we only require cycles of all possible lengths $i \equiv 0 (\mathrm{mod}\, 2)$. Characterizations of even pancyclic and vertex even pancyclic bipartite tournaments were derived in [1,10]: a bipartite tournament is even pancyclic as well as vertex even pancyclic if and only if it is hamiltonian and is not isomorphic to the bipartite tournament $F_{4r} (r = 2, 3, \ldots)$. F_{4r} has two partite sets $\{x_1, x_2, \ldots, x_{2r}\}$, $\{y_1, y_2, \ldots, y_{2r}\}$ and its arc set is $\{(x_i, y_j) : i \equiv j (\mathrm{mod}\, 2),\ 1 \leq i, j \leq 2r\} \cup \{(y_j, x_i) : i \equiv j + 1 (\mathrm{mod}\, 2),\ 1 \leq i, j \leq 2r\}$. Observe that a characterization of even pancyclic (and vertex even pancyclic) complete bipartite digraphs coincides with the above-mentioned one. Indeed, the result follows from the fact that any bipartite tournament obtained by the reorientation of an arc of F_{4r} is hamiltonian, and so, vertex even pancyclic. Combining these results with the known necessary and sufficient conditions for the existence of a hamiltonian cycle in a complete bipartite digraph [3,5,7] we obtain a polynomial characterization for the above properties.

A characterization of pancyclic (and vertex pancyclic) ordinary m-partite $(m \geq 3)$ tournaments was established in [4]. As opposed to the characterization of even pancyclic bipartite graphs the last one cannot imply immediately a characterization of pancyclic (or vertex pancyclic) ordinary complete m-partite digraphs. Indeed, there exist vertex pancyclic ordinary CMD's which contain no hamiltonian ordinary multipartite tournaments as spanning subgraphs. Such examples are complete symmetric m-partite digraphs

$S_{m,r}$ with r vertices in each partite set but one and $(m-1)r$ vertices in the last one $(r \geq 1,\ m \geq 3)$. A complete m-partite digraph is called *symmetric* if it has the arcs (u,v), (v,u) for any pair u,v in distinct partite sets. $S_{m,r}$ is vertex pancyclic by Theorem 1 (see below) and it has no hamiltonian ordinary m-partite tournament as a spanning subgraph since any hamiltonian cycle of $S_{m,r}$ must alternate between the largest partite set and the other partite sets and hence it cannot be a subgraph of an *ordinary* multipartite tournament.

In this work we derive characterizations of pancyclic and vertex pancyclic ordinary CMD's. These results differ from the corresponding ones for ordinary multipartite tournaments.

A complete m-partite digraphs is called a *complete digraph* if its order is m. Moon [9] derived the following characterization of vertex pancyclic complete digraphs which we shall apply extensively in this paper: every strongly-connected complete digraph is vertex pancyclic. Some generalizations of Moon's Theorem were recently obtained in [2,6].

2. Notation and Terminology

Let D be an ordinary CMD. $V(D), A(D)$ are the sets of vertices and arcs of D. For $W \subseteq V(D)$, $D\langle W\rangle$ denotes the subgraph of D induced on W. For $X, Y \subseteq V(D)$, $A(X,Y) = \{(x,y) \in A(D) : x \in X, y \in Y\}$. Let $V_1, V_2, \dots, V_m$ be the partite sets of D; then for $v \in V_i$ we shall write $S(v) = V_i$. For $W \subseteq V(D)$, $S(W) = \{S(v) : v \in W\}$. For a subgraph H of D, we shall sometimes write instead of $|V(H)|, D\langle V(H)\rangle$ and $S(V(H))$ the abbreviations $|H|$, $D\langle H\rangle$ and $S(H)$, respectively. By a *cycle (path)* we mean a directed simple cycle (path, respectively). An *m-cycle* (*m-path*) is a cycle (path) which has m arcs. An m-cycle in D is called a *hamiltonian (prehamiltonian)* if $m = |V(D)|$ $(m = |V(D)| - 1)$. A subgraph H of D is a 1-*difactor* of D if H is a spanning subgraph of D and $d^+_H(x) = d^-_H(x) = 1$ for any $x \in V(H)$. Obviously, any 1-difactor H of D is a collection of vertex disjoint cycles $C_1, C_2, \dots, C_t (t \geq 1)$, i.e. $H = C_1 \cup C_2 \cup \cdots \cup C_t$. Denote by $G(F)$ the undirected graph with the vertex set $\{C_1, C_2, \dots, C_t\}$ and the edge set $\{C_iC_j : S(C_i) \cap S(C_j) \neq \emptyset,\ 1 \leq i < j \leq t\}$. A sequence of vertices $v_0, v_1, \dots, v_p$ of a digraph D is called a *tour of length* p if (v_i, v_{i+1}) is in D for every $i = 0, 1, \ldots, p$ and $v_0 = v_p$. An ordinary CMD D is called a *zigzag digraph* if it has more than four vertices and $k(\geq 3)$ partite sets $V_1, V_2, V_3, \ldots, V_k$ such that $A(V_2, V_1) = A(V_i, V_2) = A(V_1, V_i) = \emptyset$ for any $i \in \{3, 4, \ldots, k\}$, $|V_1| = |V_2| = |V_3| + |V_4| + \cdots + |V_k|$.

Observe that any cycle in such a graph has the same number, say s, of vertices from V_1 and V_2 and at least s vertices from $V_3 \cup \cdots \cup V_k$. Therefore, H has no prehamiltonian cycle. Observe that a 4-partite tournament with more than four vertices is not a pancyclic digraph too. Indeed, the single (up to isomorphism) strongly connected tournament with four vertices has no tour of length five.

3. Main Theorem

The aim of this paper is to obtain Theorem 1, two first parts of which immediately follow from Lemmas 8,9 proved below.

Theorem 1. *(1) An ordinary complete k-partite digraph ($k \geq 3$) D is pancyclic if and only if*

i) D is strongly connected;

ii) it has a 1-difactor;

iii) it is neither a zigzag digraph nor a 4-partite tournament with at least five vertices.

(2) A pancyclic ordinary complete k-partite digraph D is vertex pancyclic if and only if either

i) $k > 3$ or

ii) $k = 3$ and D has two 2-cycles Z_1, Z_2 such that $|S(Z_1 \cup Z_2)| = 3$.

(3) There exists an $O\left(|V(D)|^{2.5}\right)$ algorithm for determining whether an ordinary complete k-partite ($k \geq 3$) digraph D is pancyclic (vertex pancyclic).

The third part of Theorem 1 follows from the equivalence of the problem of finding a 1-difactor in a digraph and the problem of finding a 1-factor in an appropriate bipartite graph. The last problem for a graph with n vertices may be solved using $O\left(n^{2.5}\right)$ algorithm for construction of maximum bipartite matching [8].

Here is a brief outline of the proof of the first two parts of Theorem 1. Let D be an ordinary CMD satisfying the conditions of Theorem 1. Then D contains an 1-difactor

$F = C_1 \cup \cdots \cup C_t$ which has the following two properties: every $D\langle C_i\rangle$ is a complete digraph and the graph $G(F)$ is connected. This claim is proved as Lemma 2. By Moon's Theorem each $D\langle C_i\rangle$ with at least three vertices is vertex pancyclic. If $D\langle C_i\rangle$ has two vertices it is vertex pancyclic by definition. Next we show that there are always two cycles C_i, C_j which are adjacent in $G(F)$ so that $D\langle C_i \cup C_j\rangle$ is pancyclic. Repeated iteration of this process yields the desired result. This second part of the proof is established in Lemmas 8,9 which apply Lemmas 3-7.

4. Lemmas

In the statements and the proofs of the lemmas, we use the following additional notation: $csp(x)$ is the set of the lengths of all cycles of D containing a vertex $x \in V(D)$; from now on D is an ordinary complete k-partite digraph ($k \geq 3$); $C = (x_1, x_2, \ldots, x_\ell, x_1)$, and $Z(y_1, y_2, \ldots, y_m, y_1)$ ($\ell, m \geq 2$) are vertex disjoint cycles of D such that $S(x_1) = S(y_1)$ and the digraphs $D\langle C\rangle$, $D\langle Z\rangle$ are vertex pancyclic.

We shall make a trivial but an important observation.

Remark 1. *If $S(v) = S(u)$ and v lies on a cycle $Q = (v, w_1, \ldots, w_q, v)$, then u lies on the cycle $(u, w_1, \ldots, w_q, u)$.*

Lemma 2. *If D is strongly connected and has a 1-difactor, then it contains a 1-difactor $F = C_1 \cup C_2 \cup \cdots \cup C_t$ such that $G(F)$ is connected and each $D\langle C_i\rangle$ is a complete digraph ($1 \leq i \leq t$).*

Proof: Suppose that

$$F = C_1 \cup C_2 \cup \cdots \cup C_t \qquad (1)$$

is an arbitrary 1-difactor of D. Assume that $C_1 = (v_1, v_2, \ldots, v_p, v_1)$ and $S(v_i) = S(v_j)$, $1 \leq i \neq j \leq p$. Since $(v_i, v_{j+1}), (v_j, v_{i+1}) \in A(D)$, we obtain the new 1-difactor $F' = C_1' \cup C_2' \cup C_2 \cup \cdots \cup C_t$, where $C_1' = (v_{i+1},\ v_{i+2}, \ldots, v_j, v_{i+1})$, $C_2' = (v_{j+1}, v_{j+2}, \ldots, v_i, v_{j+1})$, which contains more cycles. Therefore, this process must terminate and we may assume that the 1-difactor (1) is such that each $D\langle C_i\rangle$ is a complete digraph ($1 \leq i \leq t$).

Suppose now that the graph $G(F)$ is disconnected. Then $G(F)$ has $c \geq 2$ components: $G_1, G_2, \ldots, G_c$. Assume that there exist two cycles Z_1, Z_2 of F which are found in different components and such that $D\langle Z_1 \cup Z_2\rangle$ is strongly connected. By Moon's Theorem, $D\langle Z_1 \cup Z_2\rangle$ is a hamiltonian complete digraph. Hence the replacement of Z_1, Z_2 by a hamiltonian cycle of $D\langle Z_1 \cup Z_2\rangle$ in F leads to a new 1-difactor F with $(c-1)$ components. We may execute such amalgamation of the components of $G(F)$ until we get either a connected $G(F)$ or $G(F)$ such that for each pair $Z_1 \cup Z_2$ of different cycles of F the digraph $D\langle Z_1 \cup Z_2\rangle$ is not strongly connected. Consider the second case, and denote for simplicity the cycles of F by $C_1, \ldots, C_t$ as in (1). Clearly for any pair C_i, C_j of the 1-difactor F either $A(C_i, C_j) = \emptyset$ or $A(C_j, C_i) = \emptyset$.

Since D is an ordinary complete multipartite digraph, for any pair of the components G_f, G_h of $G(F)$, we obtain that either

$$A\left(\bigcup_{Z \in V(G_f)} Z, \bigcup_{Z \in V(G_h)} Z\right) = \emptyset \text{ } or \text{ } A\left(\bigcup_{Z \in V(G_h)} Z, \bigcup_{Z \in V(G_f)} Z\right) = \emptyset.$$

Construct tournament T with $V(T) = \{G_1, G_2, \ldots, G_c\}$ and $A(T) = \{(G_i, G_j) : A\left(\bigcup_{Z \in V(G_i)} Z, \bigcup_{Z \in V(G_j)} Z\right) \neq \emptyset,\ 1 \leq i \neq j \leq c\}$. As D is strongly connected, T is also strongly connected. Pick out from each component G_i any cycle Z_i. Then the tournament, constructed analogously to T on the vertex set $\{Z_1, \ldots, Z_c\}$, is hamiltonian. Hence $D\langle Z_1 \cup \cdots \cup Z_c\rangle$ is also hamiltonian. Let H be a hamiltonian cycle of the complete digraph $D\langle Z_1 \cup \cdots \cup Z_c\rangle$. Then the replacement of $Z_1, \ldots, Z_c$ by H in F leads to a new F such that $G(F)$ is connected. □

Lemma 3. *If H is a prehamiltonian cycle of a strongly connected digraph G and the vertex of G, which is not in H, is adjacent with all vertices of H, then G is hamiltonian.*

The trivial proof is omitted. The following lemma was proved in [3].

Lemma 4. *Let x be a vertex of C.*

(1) If $\ell \geq 2,\ m \geq 3$, then $csp(x) \geq \{3, 4, \ldots, \ell\} \cup \{\ell+3, \ell+4, \ldots, \ell+m\}$;

(2) If $\ell \geq 4,\ m \geq 3$, then $(\ell+2) \in csp(x)$;

(3) If $\ell \geq 5,\ m \geq 3$, then $(\ell+1) \in csp(x)$;

(4) If $\ell = 4,\ m \geq 3$, and $S(Z) \nsubseteq S(C)$, then $5 \in csp(x)$;

(5) If $\ell = 3,\ m \geq 3$, and $|S(Z) \backslash S(C)| \geq 2$, then $4, 5 \in csp(x)$.

Lemma 5. *Suppose $m = 2$.If either*

(1) $\ell \geq 4$ *or*

(2) $\ell \in \{2, 3\}$ *and* $|S(Z) \cap S(C)| = 1$, *then* $D\langle C \cup Z\rangle$ *is vertex pancyclic.*

Proof: **Case 1:** $\ell \geq 3$.

By the conditions of the lemma $csp(x_i) \supseteq \{3, 4, \dots, \ell\}$ $(1 \leq i \leq \ell)$. Since $S(x_1) = S(y_1)$, $csp(y_1) \supseteq \{3, 4, \dots, \ell\}$ by Remark 1. By Lemma 4(1) we obtain $(\ell + 2) \in$ csp $(x_i), csp(y_j)$ (for all $1 \leq i \leq \ell;\ j = 1, 2)$.

Subcase 1.1. $S(y_2) = S(x_i)$ for some i.

Then $\ell \geq 4$ and $csp(y_2) \supseteq \{3, 4, \dots, \ell\}$. Pick out from $D\langle C\rangle$ any $(\ell - 1)$-cycle C_1 containing x_1. Let C_2 be an $(\ell - 1)$-cycle containing the vertex of C which is not in C_1; then $D\langle\{y_1, y_2\} \cup V(C_j)\rangle$ is hamiltonian by Lemma 4(1) for $j = 1, 2$. Hence $(\ell + 1) \in csp(x_i), csp\ (y_j)$ $(1 \leq i \leq \ell;\ j = 1, 2)$.

Subcase 1.2. $S(y_2)$ is not in $S(C)$.

Let C_t be a t-cycle of $D\langle C\rangle$ including x_1 $(3 \leq t \leq \ell)$. Since (y_2, x_1), (x_1, y_2) are in $A(D)$, $D\langle y_2 \cup V(C_t)\rangle$ is hamiltonian by Lemma 3. So, $csp(y_2) \supseteq \{4, 5, \dots, \ell + 1\}$ and $(\ell + 1) \in csp(x_i)$, $csp(y_1)$ $(1 \leq i \leq \ell)$.

It remains to prove that $3 \in csp(y_2)$. Consider y_2 and C_3 defined above. Suppose $C_3 = (x_1, x_f, x_g, x_1)$. It is easy to see that if $(x_f, y_2) \in A(D)$ or $(y_2, x_g) \in A(D)$, then y_2 lies on a 3-cycle which includes x_1.

On the other hand, if (y_2, x_f), $(x_g, y_2) \in A(D)$, then (y_2, x_f, x_g, y_2) is a 3-cycle containing y_2.

Case 2. $\ell = 2$.

Without loss of generality, we may assume that $(x_2, y_2) \in A(D)$. Hence D has the following cycles: $(x_1, x_2, y_1, y_2, x_1)$, (x_2, y_2, x_1, x_2), (x_2, y_2, y_1, x_2). □

Lemma 6. *If* $\ell \geq 5$ *and either*

(1) $S(Z) \subseteq S(C)$ *or*

(2) $m \geq 5$ *or*

(3) $m = 4$, $|S(C)| \geq |S(Z)|$ *or*

(4) $m = 3$, $|S(C)| > |S(Z)|$, *then* $D\langle C \cup Z\rangle$ *is vertex pancyclic.*

Proof: By Lemma 4(1),(2),(3), csp $(x_i) \supseteq \{3, 4, \ldots, \ell+m\}$ $(1 \leq i \leq \ell)$. If $S(Z) \subseteq S(C)$ or $m \geq 5$, then csp $(y_j) \supseteq \{3, 4, \ldots, \ell+m\}$ too $(1 \leq j \leq m)$. Thus we may assume $S(Z) \not\subseteq S(C)$ and $3 \leq m \leq 4$. Consider first the case $m = 4$. Lemma 4 (1),(2) implies csp $(y_j) \supseteq \{3, 4, 6, 7, \ldots, \ell+4\}$ for every $1 \leq j \leq m$. As $|S(C)| \geq |S(Z)|$ and $S(Z) \not\subseteq S(C)$, we obtain $S(C) \not\subseteq S(Z)$ and so $5 \in \mathrm{csp}(y_j)$ $(1 \leq j \leq m)$ by Lemma 4(4).

Consider now the case $m = 3$. By Lemma 4(1), csp $(y_j) \supseteq \{3, 6, 7, \ldots, \ell+3\}$ for each $1 \leq j \leq m$. If $|S(C)\backslash S(Z)| \geq 2$, then $4, 5 \in \mathrm{csp}(y_j)$ $(1 \leq j \leq m)$ by Lemma 4(5). It remains to consider case $|S(C)\backslash S(Z)| = 1$. In this case $S(C) \supseteq S(Z)$ which is impossible.
$\square$

Let Z_1, Z_2 be cycles of an ordinary CMD H such that $S(Z_1) \cap S(Z_2) \neq \emptyset$. It is easy to see that $H\langle Z_1 \cup Z_2\rangle$ is hamiltonian. This fact and Lemma 2 imply the following result which was also proved in [5] using a different approach.

Lemma 7. *D is hamiltonian if and only if it is strongly connected and has a* 1*-difactor.*

Lemma 8. *Let $|S(D)| \geq 4$, and $|V(D)| \geq 5$. The following conditions are equivalent:*

(1) D is vertex pancyclic;

(2) D is pancyclic;

(3) D is strongly connected, has a 1*-difactor, and is neither a zigzag digraph nor a* 4*-partite tournament.*

Proof: Show first that (3) implies (1).

Suppose that (3) holds. Let $F = C_1 \cup C_2 \cup \cdots \cup C_t$ be a 1-difactor of D satisfying the conditions of Lemma 2 and let C_1 be a cycle of F containing the maximum number of vertices. Consider the following four possible cases.

Case 1. $|C_1| \geq 5$.

Pick out any cycle C_i $(i \neq 1)$ which is adjacent to C_1 in $G(F)$. By Lemmas 5,6, $D\langle C_1 \cup C_i\rangle$ is vertex pancyclic. Similarly, consider a hamiltonian cycle of $D\langle C_1 \cup C_i\rangle$ and the rest of the cycles of F. Repeating the same arguments, we conclude that D is vertex pancyclic.

Case 2. $|C_1| = 4$.

Choose any cycle $C_i (i \neq 1)$ which is adjacent to C_1 in $G(F)$, and such that if $|S(D)| \geq 5$, then $S(C_i) \not\subseteq S(C_1)$. Let x, y be any vertices of C_1 and C_i, respectively.

Subcase 2.1. $|C_i| = 2$.

Then $D\langle C_1 \cup C_i \rangle$ is vertex pancyclic by Lemma 5. Hence, as in Case 1, we convince that D is vertex pancyclic by repeated applications of Lemmas 5,6.

Subcase 2.2. $|C_i| = 3$.

If $|S(D)| \geq 5$, then $D\langle C_1 \cup C_i \rangle$ is vertex pancyclic according to Lemma 4. Indeed, by Lemma 4(1), $csp(x) \supseteq \{3, 4, 7\}$, $csp(y) \supseteq \{3, 6, 7\}$. Further, $5, 6 \in csp(x)$, by Lemma 4(4) and (2), respectively. Finally, $4, 5 \in csp(y)$, by Lemma 4(5).

Assume $|S(D)| = 4$. Then D has a 2-cycle, since D is not a 4-partite tournament. If there exists a 2-cycle B of D such that $V(B) \subset V(C_i)$, then $D\langle C_1 \cup B \rangle$ is vertex pancyclic by Lemma 5. Moreover $D\langle C_1 \cup C_i \rangle$ is hamiltonian by Lemma 4(1). Hence, $D\langle C_1 \cup C_i \rangle$ is vertex pancyclic (note that the vertex of $C_i \backslash B$ lies on cycles of all possible lengths according to Remark 1). If there is no 2-cycle B satisfying $V(B) \subset V(G_i)$, then there exists a 2-cycle B such that $V(B) \subset V(G_1)$ and $|S(B) \cap S(C_i)| = 1$. By Lemma 4(1) and Remark 1 (for C_1, C_i), $csp(x), csp(y) \supseteq \{3, 4, 6, 7\}$ Moreover, $D\langle C_i \cup B \rangle$ is hamiltonian by Lemma 4(1) and hence $5 \in csp(x), csp(y)$ (by Remark 1). Therefore, $D\langle C_1 \cup C_i \rangle$ is vertex pancyclic. Hence, by Lemmas 5,6, D is vertex pancyclic.

Subcase 2.3. $|C_i| = 4$.

If $|S(D)| \geq 5$, then, since $S(C_1) \neq S(C_i)$, and $|S(C_1)| = |S(C_i)| = 4$, we get that $D\langle C_1 \cup C_i \rangle$ is vertex pancyclic by Lemma 4 (1),(2),(4). Otherwise, $|S(D)| = 4$ and then there exists a 3-cycle T such that $V(T) \subset V(C_i)$. Also, $D\langle C_1 \cup T \rangle$ is vertex pancyclic by Case 2.2, $D\langle C_1 \cup C_i \rangle$ is hamiltonian by Lemma 4(1), and so the last digraph is vertex pancyclic. Therefore, by Lemmas 5,6, D is vertex pancyclic.

Case 3. $|C_1| = 3$.

Subcase 3.1. Assume that there exists a pair C_i, C_j ($|C_j| \geq |C_i|$) of the cycles of F such that

$$\max\{(C_i|, \ |C_j|\} = 3, \quad |S(C_i) \cap S(C_j)| = 1 \ . \qquad (2)$$

If $|C_i| = 2$, then $D\langle C_i \cup C_j\rangle$ is vertex pancyclic by Lemma 5. If $|C_i| = 3$, then $D\langle C_i \cup C_j\rangle$ is vertex pancyclic according to Lemma 4(1), (5). So, in both cases, D is vertex pancyclic by Lemmas 5,6.

Subcase 3.2. Assume that there is no pair satisfying the second equality of (2). Then there exists a cycle C_i such that $S(C_1) \not\supseteq S(C_i)$ and $|S(C_1) \cap S(C_i)| > 1$. Hence, $|S(C_1) \cap S(C_i)| = 2$ and $|C_i| = 3$. Put $S(C_1) \cap S(C_i) = \{V_1, V_2\}$.

We call two cycles C_j, C_k inconsistent if they have pairs of vertices $v_1, v_2 (\in C_j)$, $u_1, u_2 (\in C_k)$, such that $S(v_m) = S(u_m)$, $i = 1, 2$ and $(v_1, v_2) \in A(C_j)$, $(u_2, u_1) \in A(C_k)$.

We start with the case when F has a pair of inconsistent 3-cycles $C_j = (v_1, v_2, v_3, v_1)$, $C_k = (u_3, u_2, u_1, u_3)$ such that $S(v_m) = S(u_m)$, $m = 1, 2$. By Lemma 4(1), $3, 6 \in csp(z)$ for every $z \in V(C_j \cup C_k)$. Moreover, $D\langle C_j \cup C_k\rangle$ contains the following cycles $(v_3, v_1, u_3, v_2, v_3)$, $(v_3, u_1, u_3, u_2, v_3)$, $(v_3, v_1, v_2, u_1, u_2, v_3)$, $(u_3, v_2, v_1, u_2, u_1, u_3)$. Hence, $D\langle C_j \cup C_k\rangle$ is vertex pancyclic. Therefore, D is vertex pancyclic as well (by Lemmas 5,6).

Consider now the case when F has no pairs of inconsistent 3-cycles. Then since $G(F)$ is connected, $\{V_1, V_2\} \subseteq S(C_f)$ for every $C_f \in F$.

Suppose that F has no 2-cycles. Since D is not a zigzag digraph, it contains a 2-cycle B with $S(B) \cap \{V_1, V_2\} \neq \emptyset$. If $S(B) \neq \{V_1, V_2\}$, then there exists a 3-cycle C_f in F such that $S(C_f) \supseteq S(B)$. Without loss of generality, we may assume that $V(C_i) \supseteq V(B)$; $v = V(C_i) \backslash V(B)$. Then, since $S(C_1) \neq S(C_i)$, the pair C_1, B satisfies (2) and hence $D - v$ is vertex pancyclic by Subcase 3.1. It remains to note that D is hamiltonian by Lemma 7. Suppose now that $S(B) = \{V_1, V_2\}$. Without loss of generality, we assume $V(B) \subseteq V(C_1)$. Let $v = V(C_1) \backslash V(B)$. Obviously, the complete digraph $D\langle v \cup V(C_i)\rangle$ is strongly connected. Hence, it contains a hamiltonian cycle H. Then $B \cup H$ is a 1-difactor of $D\langle C_i \cup C_1\rangle$ and so $F' = F \cup B \cup H \backslash \{C_1, C_i\}$ is a 1-difactor of D. Therefore, D is vertex pancyclic by Case 2.

Suppose that F has a 2-cycle. By the assumption of the subcase there exists a 2-cycle B such that $S(B) = \{V_1, V_2\}$. Therefore, D is vertex pancyclic as has been proved above.

Case 4. $|C_1| = 2$.

Since $|S(D)| \geq 3$ and $G(F)$ is connected, then F has two 2-cycles C_i, C_j such that $|S(C_i) \cap S(C_j)| = 1$. So, D has a 4-cycle by Lemma 5, and according to Lemma 5, D is vertex pancyclic.

We thus showed that (3) implies (1).

Clearly, (1) implies (2), the fact that (2) implies (3) is easy. Indeed, every pancyclic digraph is hamiltonian and so it is strongly connected, and has a 1-difactor. Besides, zigzag digraphs and 4-partite tournaments with at least five vertices are not pancyclic as already has been proved. □

Lemma 9. *Suppose D is strongly connected, has a 1-difactor, $|S(D)| = 3$, and $|V(D)| \geq 4$. Then*

(1) D is vertex pancyclic if and only if it has 2-cycles Z_1, Z_2 such that $|S(Z_1) \cup S(Z_2)| = 3$;

(2) D is pancyclic if and only if it is not a zigzag digraph.

Proof: Let V_1, V_2, V_3 be the partite sets of D, F be a 1-difactor of D, and $s_{ij},\ s'_{ij}$ be the number of the 2-cycles in F and in D, respectively, with the vertices from $V_i \cup V_j$ $(1 \leq i < j \leq 3)$. Suppose that $|V(D)| \geq 4$, $|S(D)| = 3$ and D is not a zigzag digraph; then it has a 2-cycle and so $max\{s'_{12}, s'_{13}, s'_{23}\} \geq 1$. Without loss of generality, assume that $s_{12} \geq s_{13} \geq s_{23}$ and $s'_{12} \geq 1$. Consider the following four possible cases.

Case 1. $s_{12} + s_{13} + s_{23} \geq 3,\ s_{13} \geq 1$.

Let C_1, C_2, C_3 be 2-cycles of F such that $S(C_1) = \{V_1, V_2\},\ S(C_2) = \{V_1, V_3\}$. One can show that $D\langle C_1 \cup C_2 \cup C_3\rangle$ is vertex pancyclic (by the arguments similar to that in the proof of Case 4 of Lemma 8. Hence, D is vertex pancyclic by Lemma 6(1).

Case 2. $s_{13} = s_{12} = 1,\ s_{23} = 0$.

If $|V(D)| = 4$, then obviously D is vertex pancyclic. If $|V(D)| > 4$, then F has a 3-cycle. We may assume that F has a 3-cycle $Q = (v_1, v_2, v_3, v_1)$, where $v_i \in V_i$. It is easy to see that $D - v_3$ has the 1-difactor $F \cup (v_1, v_2, v_1) \backslash Q$ and so $D - v_3$ is vertex pancyclic by Case 1. D is hamiltonian according to Lemma 7. Therefore, D is vertex pancyclic by Remark 1.

Case 3. $s_{23} = s'_{23} = s_{13} = s'_{13} = 0$.

Assume that $A(V_3, V_2) = A(V_1, V_3) = \emptyset$. Then F can have cycles only of the following two forms:

$$(v_1, v_2, v_1),\ (v_1, v_2, v_3, v_1), \quad \text{where} \quad v_i \in V_i\ , \quad 1 \le i \le 3\ .$$

Hence $|V_1| = |V_2| = t + s_{12}$, $|V_3| = t$, where t is the number of 3-cycles of F. D contains a $2i$-cycle

$$\left(v_1^{(1)}, v_2^{(1)}, v_1^{(2)}, v_2^{(2)}, \ldots, v_1^{(i)}, v_2^{(i)}, v_1^{(1)}\right) \tag{3}$$

for each $1 \le i \le s_{12} + t$, where $v_\ell^{(j)} \in V_\ell,\ 1 \le \ell \le 2$.

To obtain a $(2i+1)$-cycle from the above $2i$-cycle , we replace an arc $\left(v_2^{(j)}, v_1^{(j+1)}\right)$ by a path $\left(v_2^{(j)}, v_3, v_1^{(j+1)}\right)$, where v_3 is a vertex of V_3. To obtain a $(2(s_{12}+t)+j)$-cycle from the $2(s_{12}+t)$-cycle, we replace j arcs, directed from V_2 to V_1, by j 2-paths, each of which includes a vertex of V_3 $(1 \le j \le t)$. We have proved that D is pancyclic. But, in this case, D is not vertex pancyclic as it has no 4-cycle containing a vertex of V_3.

Case 4. $s_{23} = s_{13} = 0$, $max\{s'_{23}, s'_{13}\} \ge 1$.

Assume that $s'_{13} \ge 1$. In the present case, we have also $|V_1| = |V_2| = t + s_{12}$, $|V_3| = t$. But in the case, in contrast to Case 3, D has a $2i$-cycle (for each $1 \le i \le s_{12} + t$) meeting all partite sets. To obtain such a cycle from the $2i$-cycle (3), we replace a subpath $(v_1^{(j)}, v_2^{(j)}, v_1^{(j+1)})$ of (3) by a path $(v_1^{(j)}, v_3, v_1^{(j+1)})$, where $v_3 \in V_3$. Hence, D is vertex pancyclic.

Therefore, D is ever pancyclic, and it is vertex pancyclic only in Cases 1,2,4, this implies Lemma 9. $\square$

Theorem 1 follows immediately from Lemma 8 and 9.

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